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THE DISTRIBUTION AND PROPERTIES OF A WEIGHTED SUM OF CHI SQUARES

by A. H. Feiveson and F. C. Delaney

Manned Spacecraft Center
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ABSTRACT

A study of some of the properties of a weighted sum of chi-square random variables is presented, including the derivation of approximations to the distribution of this sum and an evaluation of the Welch approximation for the distribution of the test statistic in the Behrens-Fisher Problem. The study indicates that if equal sample sizes are selected, the Welch approximation to the Behrens-Fisher Problem may be safely used.

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SUMMARY

A study of some of the properties of a weighted sum of chi-square random variables is presented. Derivations of approximations to the distribution of this sum and an evaluation of the Welch approximation for the distribution of the test statistic in the Behrens-Fisher Problem are included. The study indicates that if equal sample sizes are selected, the Welch approximation to the Behrens-Fisher Problem may be safely used, even for sample sizes as small as 5.

INTRODUCTION

The density function for the distribution of a weighted sum, for example, Z^2, of independent chi-square random variables cannot be represented by elementary analytic functions. In many cases, it has been found feasible to approximate the distribution of Z^2 by that of a gamma distribution, the first two moments of which are equal to the first two moments of Z^2. This paper is divided into two main sections. The first section describes the actual distribution of Z^2, and the second section evaluates the approximation described previously, particularly as it is used in the Behrens-Fisher Problem of testing for the difference between the two samples when the variances are not assumed to be equal.

SYMBOLS

a	constant greater than 1
a_H	maximum (a_1, a_2)
a_j	positive constant

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a_L	minimum (a_1, a_2)
b_k	term in infinite series
$F_{Z^2}(x), F_{\nabla}(x)$	cumulative distribution functions of the random variables Z^2 and ∇
f	degrees of freedom for Welch t statistic
$f_o(v)$	density function for $\chi^2(n) + a\chi^2(m)$
$f_U(u)$	density function for random variable $\tilde{U}$
$f_V(v)$	density function for random variable $\tilde{V}$
$G_{Z^2}(x)$	approximation to $F_{Z^2}(x)$
k	index of summation
m, n, p, q, R, ϵ	constants
N_j	independent random variables $(j = 1, 2, \ldots, k)$ distributed $N(0, 1)$
$N(\mu, \sigma^2)$	normally distributed with mean μ and variance σ^2
n_1, n_2	sample sizes
n_H	$\begin{cases} n_1 & \text{if } a_1 > a_2 \\ n_2 & \text{if } a_1 < a_2 \end{cases}$
n_L	$\begin{cases} n_2 & \text{if } a_1 > a_2 \\ n_1 & \text{if } a_1 < a_2 \end{cases}$
r_ϵ	upper limit of integration
s_1^2, s_2^2	sample estimates of variance
t	referring to Student's t distribution

$\tilde{U}$	random variable distributed as $N(0, 1)$
u, v, z	variables of integration
$\tilde{V}$	random variable distributed as $\alpha\chi^2(n_1 - 1) + (1 - \alpha)\chi^2(n_2 - 1)$
w	complex number
w_j	complex number equal to $1 - 2ia_j t$ (see eq. (5))
$\bar{x}_1, \bar{x}_2$	sample averages
x_{ij}	observations
Z^2	weighted sum of independent chi-square random variables
α	constant between 0 and 1
$\hat{\alpha}$	estimate of α, equal to $\frac{s_1^2}{s_1^2 + s_2^2}$
$\beta(r, s)$	beta function with arguments r and s
$\Gamma(\cdot)$	gamma function
E	expected value operator
ϵ	constant > 0
μ_1, μ_2	means
σ_1^2, σ_2^2	variances
$\Phi_{Z^2}(t), \Phi_{N_j^2}(t), \ldots$	characteristic functions for the random variables Z^2, N_j^2, and so forth
$\Phi(\cdot)$	standard normal cumulative distribution function (c. d. f.)
$\chi^2(n)$	chi-square distribution with n degrees of freedom
∇	Behrens-Fisher statistic

DISTRIBUTION OF Z^2

General Case

Let N_j ($j = 1, 2, \ldots, k$) be independent, normally distributed, random variables with mean $\mu = 0$ and variance $\sigma^2 = 1$, and let $Z^2 = \sum_{j=1}^{k} a_j N_j^2$ where the a_j are real, positive constants. The characteristic function of Z^2, denoted by $\Phi_{Z^2}(t)$, is

$$\Phi_{Z^2}(t) = Ee^{itZ^2} = E\left\{\exp\left[it \sum_{j=1}^{k} a_j N_j^2\right]\right\} = \prod_{j=1}^{k} Ee^{ita_jN_j^2} = \prod_{j=1}^{k} \Phi_{a_jN_j^2}(t) \tag{1}$$

where $\Phi_{a_jN_j^2}(t)$ is the characteristic function of $a_jN_j^2$. Since N_j^2 is distributed as $\chi^2(1)$, its characteristic function $\Phi_{N_j^2}(t)$ is $(1 - 2it)^{-1/2}$; hence

$$\Phi_{a_jN_j^2}(t) = Ee^{ia_jtN_j^2} = \Phi_{N_j^2}\left(a_jt\right) = \left(1 - 2ia_jt\right)^{-1/2} \tag{2}$$

therefore

$$\Phi_{Z^2}(t) = \prod_{j=1}^{k} \left(1 - 2ia_jt\right)^{-1/2} \tag{3}$$

The cumulative distribution function (c. d. f.) of Z^2, to be denoted by $F_{Z^2}(x)$, is obviously zero for $x \le 0$; hence, it can be obtained by setting $h = x$ in the following relation (taken from ref. 1), provided that $\Phi_{Z^2}(t)$ is integrable over the real line.

$$\frac{F_{Z^2}(x + h) - F_{Z^2}(x - h)}{2h} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin ht}{ht} e^{-itx} \Phi_{Z^2}(t)dt \tag{4}$$

To make $\Phi_{Z^2}(t)$ continuous, hence integrable, for all real t, let $-\pi < \arg w \leq \pi$ for all complex numbers w, and let $w^{-1/2}$ be equal to $\frac{1}{\sqrt{|w|}} e^{-(i/2)\arg w}$. Let $w_j = 1 - 2ia_jt$. Then $\Phi_{Z^2}(t)$ can be written

$$\Phi_{Z^2}(t) = \prod_{j=1}^{k} |w_j|^{-1/2} e^{-(i/2)\arg w_j} = \exp\left[-\frac{i}{2} \sum_{j=1}^{k} \arg w_j\right] \prod_{j=1}^{k} |w_j|^{-1/2} \tag{5}$$

Since $-\pi/2 < \arg w_j < \pi/2$, the function $\arg w_j$ can be defined as

$$\arg w_j = \tan^{-1}\left(-2a_jt\right) = -\tan^{-1}2a_jt \tag{6}$$

Therefore

$$\Phi_{Z^2}(t) = \exp\left[\frac{i}{2} \sum \tan^{-1}2a_jt\right] \prod_{j=1}^{k} |w_j|^{-1/2} \tag{7}$$

and using equation (4) with $h = x$

$$\frac{F_{Z^2}(2x) - 0}{2x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin tx}{tx} \exp\left[i\left(-tx + \frac{1}{2} \sum \tan^{-1}2a_jt\right)\right] \prod_{j=1}^{k} |w_j|^{-1/2} dt \tag{8}$$

that is

$$F_{Z^2}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{tx}{2}}{t} \exp\left[i\left(-\frac{tx}{2} + \frac{1}{2} \sum \tan^{-1}2a_jt\right)\right] \prod_{j=1}^{k} \left| 1 + 4a_j^2t^2\right|^{-1/4}. \tag{9}$$

Since the imaginary part of the integrand

$$\frac{\sin t\frac{x}{2}}{t}\sin\left(-tx+\frac{1}{2}\sum\tan^{-1}2a_jt\right)\prod_{j=1}^{k}|w_j|^{-1/2} \tag{10}$$

is an odd function, its integral over the real line is zero; thus

$$F_{Z^2}(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\frac{tx}{2}}{t}\cos\left(-tx+\frac{1}{2}\sum\tan^{-1}2a_jt\right)\prod_{j=1}^{k}\left(1+4a_j^2t^2\right)^{-1/4}dt$$

$$=\int_0^{\infty}\frac{2}{\pi}\frac{\sin\frac{tx}{2}}{t}\cos\left(-tx+\frac{1}{2}\sum\tan^{-1}2a_jt\right)\prod_{j=1}^{k}\left(1+4a_j^2t^2\right)^{-1/4}dt$$

$$=\int_0^{\infty}g(t,x)dt \tag{11}$$

where $g(t,x)$ represents the integrand in the preceding equation.

Computation of $F_{Z^2}(x)$

It was decided to calculate $F_{Z^2}(x)$ by numerically integrating equation (11) and to compare the results with the c.d.f. of the approximating gamma distribution. To achieve reasonable accuracy in a reasonable amount of time, both a step size s and an upper limit of integration b must be determined. A rough upper bound for b can be obtained by noting that $\left|\int_b^{\infty}g(t,x)dt\right|$ is less than

$$\frac{2}{\pi}\int_b^{\infty}\left|\frac{\sin\frac{tx}{2}}{t}\right|\left|\cos\left(-\frac{tx}{2}+\frac{1}{2}\sum\tan^{-1}2a_jt\right)\right|\prod_{j=1}^{k}\left(1+4a_j^2t^2\right)^{-1/4}dt$$

$$<\frac{2}{\pi}\int_b^{\infty}\frac{dt}{t\prod_{j=1}^{k}\left(1+4a_j^2t^2\right)^{1/4}}<\frac{2}{\pi}\int_b^{\infty}\frac{dt}{4^{k/4}t^{(k/2)+1}\prod_{j=1}^{k}a_j^{1/2}}=\frac{4}{(2b)^{k/2}\pi k\prod_{j=1}^{k}a_j^{1/2}} \tag{12}$$

Thus, if the absolute error from incomplete integration is desired to be less than some positive number ϵ, then b must be chosen such that equation (12) is less than ϵ. The step size should be taken proportional to $1/x$, since the function to be integrated is roughly periodic with frequency $4\pi/x$.

After trial and error, $1/10x$ was determined to be a good step size to use. The results were able to be checked in some cases (for example, where all the a_j were equal), and in those instances, at least four decimal places of accuracy were obtained. There is no reason to suspect that the accuracy of the numerical integration would be materially affected if the a_j were not equal.

Gamma Approximation $F_{Z^2}(x)$

One of the common approximations to $F_{Z^2}(x)$, denoted by $G_{Z^2}(x)$, is a gamma distribution having the same first two moments as that of Z^2. Thus

$$G_{Z^2}(x) = \int_0^x \frac{\alpha^\lambda}{\Gamma(\lambda)} e^{-\alpha t} t^{\lambda-1} dt \tag{13}$$

where $\alpha = (1/2)\left(\sum a_i \Big/ \sum a_i^2\right)$ and $\lambda = (1/2)\left[\left(\sum a_i\right)^2 \Big/ \sum a_i^2\right]$.

Accuracy of Gamma Approximation

Results indicate that the mean of the weights $\left(\text{the } a_i\right)$ does not affect the accuracy of the approximation.

A statistic with one of the most measurable effects on the approximation is the standard deviation of the weights. As the standard deviation of the weights increases, the approximation tends to be an overestimation of the true functional value in the right tail of the distribution. This tendency is illustrated in tables I to XII.

Because the parameters α and λ of $G_{Z^2}(x)$ depend only on the first two moments of the weights, the approximation yields the same value for any set of weights with the same mean and standard deviation. This value is shown in tables I to XII under the heading $G(x)$. The tables also give the true values $F_{Z^2}(x)$ listed under $F(x)$ in columns I, II, and III, where the mean and standard deviation of the weights are the same for I, II, and III, but in I, the weights are equally spaced; in II, one half of the weights are equal to a constant and the other half are equal to another constant;

and in III, all of the weights except one are equal to one constant while the remaining weight is equal to a different constant. An inspection of tables I to XII indicates that as the number of weights increases, the accuracy of the approximation improves.

A marked difference was noted between the approximation and the true functional value in the left tail of the distribution.

DISTRIBUTION OF Z^2 FOR ONLY TWO DISTINCT SETS OF a's

The Behrens-Fisher Problem

The Behrens-Fisher Problem can be stated as follows: consider two independent samples $x_{11}, x_{12}, \ldots, x_{1n_1}$, and $x_{21}, x_{22}, \ldots, x_{2n_2}$ from normal distributions with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2. It is desired to test the null hypothesis that $\mu_1 = \mu_2$, when no knowledge of σ_1^2 or σ_2^2 exists.

Since the sample averages $\bar{x}_1$ and $\bar{x}_2$ are normally distributed with means μ_1 and μ_2, respectively, and variances σ_1^2/n_1 and σ_2^2/n_2, respectively, the random variable

$$\frac{\left(\bar{x}_1 - \bar{x}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1) \tag{14}$$

Let the Behrens-Fisher statistic $\triangledown$ be the random variable obtained by inserting sample estimates for σ_1^2 and σ_2^2 in equation (14). That is

$$\triangledown = \frac{\left(\bar{x}_1 - \bar{x}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \tag{15}$$

where $s_i^2 = \sum_j \frac{(x_{ij} - \bar{x}_i)^2}{n_i - 1}$ with $i = 1, 2$. The problem is to find the distribution of ∇. It is well known that

$$s_i^2 \sim \frac{\sigma_i^2 \chi^2(n_i - 1)}{n_i - 1} \tag{16}$$

and that s_i^2 is independent of $\bar{x}_i$ for $i = 1, 2$. Therefore, under the hypothesis $\mu_1 = \mu_2$

$$\nabla \sim \frac{N\left(0, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}{\sqrt{\frac{\sigma_1^2 \chi^2(n_1 - 1)}{n_1(n_1 - 1)} + \frac{\sigma_2^2 \chi^2(n_2 - 1)}{n_2(n_2 - 1)}}} \tag{17}$$

which is equivalent to

$$\begin{aligned} \nabla &\sim \frac{N(0, 1)}{\sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^{-1}\left[\frac{\sigma_1^2 \chi^2(n_1 - 1)}{n_1(n_1 - 1)} + \frac{\sigma_2^2 \chi^2(n_2 - 1)}{n_2(n_2 - 1)}\right]}} \\ &= \frac{N(0, 1)}{\sqrt{\frac{\frac{\sigma_1^2}{n_1}}{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \frac{\chi^2(n_1 - 1)}{n_1 - 1} + \frac{\frac{\sigma_2^2}{n_2}}{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \frac{\chi^2(n_2 - 1)}{n_2 - 1}}} \end{aligned} \tag{18}$$

It is no loss of generality to assume $\frac{\sigma_2}{n_2} \geq \frac{\sigma_1}{n_1}$. Therefore

$$\nabla \sim \frac{N(0, 1)}{\sqrt{\frac{\alpha\chi^2(n_1 - 1)}{n_1 - 1} + \frac{(1 - \alpha)\chi^2(n_2 - 1)}{n_2 - 1}}} = \frac{\tilde{U}}{\sqrt{\tilde{V}}} \tag{19}$$

where

$$\alpha = \frac{\frac{\sigma_1^2}{n_1}}{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \left(0 < \alpha \leq \frac{1}{2}\right) \tag{20}$$

and $\tilde{U}$ and $\sqrt{\tilde{V}}$ are the numerator and denominator, respectively, of equation (19).

Let $f_U(u)$ and $f_V(v)$ be the probability density functions of $\tilde{U}$ and $\tilde{V}$, respectively. Then $F_\nabla(x)$, the c.d.f. of ∇, can be obtained by

$$F_\nabla(x) = \int_{u<xv/2} f_U(u) f_V(v) du\, dv = \int_0^\infty f_V(v)\Phi\left(\frac{xv}{2}\right) dv$$

$$= 1 - \int_0^\infty f_V(v)\left[1 - \Phi\left(\frac{xv}{2}\right)\right] dv \tag{21}$$

where Φ represents the standard normal distribution function. To facilitate numerical integration of equation (21), an expression for $f_V(v)$ which does not involve a numerical integration[1] is needed. A rapidly converging infinite-series representation of $f_V(v)$ may be obtained in the following manner. Let $f_o(v)$ be the density function of $\chi^2(n) + a\chi^2(m)$ where $a < 1$. Then

$$f_o(v) = \int_0^{v/a} f_m(t) f_n(v - at)dt \tag{22}$$

where f_m and f_n are chi-square density functions with degrees of freedom m and n. Thus

$$f_o(v) = \frac{1}{\Gamma\left(\frac{m}{2}\right)2^{m/2}\Gamma\left(\frac{n}{2}\right)2^{n/2}} \int_0^{v/a} t^{(m/2)-1}(v - at)^{(n/2)-1} \exp\left[-\frac{1}{2}(t + v - at)\right] dt \tag{23}$$

which, upon making the transformation $z = at/v$, gives

$$f_o(v) = R\int_0^1 z^{p-1}(1 - z)^{q-1} e^{(zv/2)[1-(1/a)]} dz \tag{24}$$

where $p = m/2$, $q = n/2$, and

$$R = \frac{v^{(m/2)+(n/2)-1} e^{-v/2}}{a^{m/2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)2^{(m+n)/2}} \tag{25}$$

[1]Ray and Pitman (ref. 2) give an expression for $f_V(v)$ involving $(n_1 + n_2 - 2)/2$ terms when n_1 and n_2 are odd; however, it requires the summing of alternating positive and negative terms which can seriously impair numerical accuracy on the computer.

Expanding the exponential term in a power series gives

$$f_o(v) = R\int_0^1 z^{p-1}(1 - z)^{q-1} \sum_{k=0}^{\infty} \frac{z^k}{k!} \left[\frac{v\left(1 - \frac{1}{a}\right)}{2}\right]^k dz \tag{26}$$

Since the exponential series is uniformly convergent on the interval [0, 1], the integration and summation operations may be interchanged; hence

$$f_o(v) = R \sum_{k=0}^{\infty} \frac{v^k\left(1 - \frac{1}{a}\right)^k}{k!\, 2^k} \int_0^1 z^{k+p-1}(1 - z)^{q-1} dz$$

$$= R \sum_{k=0}^{\infty} \frac{v^k\left(1 - \frac{1}{a}\right)^k}{k!\, 2^k} \beta(k + p, q)$$

$$= \frac{a^{-p} e^{-v/2}}{\Gamma(p)} \sum_{k=0}^{\infty} b_k \tag{27}$$

where

$$b_k = \frac{v^{k+p+q-1}\left(1 - \frac{1}{a}\right)^k}{2^{p+q+k} k!\, (k + p)(k + p + 1) \ldots (k + p + q - 1)} \tag{28}$$

Note that

$$\frac{b_{k + 1}}{b_k} = \frac{v(k + p)\left(1 - \frac{1}{a}\right)}{z(k + 1)(k + p + q)} \tag{29}$$

and that for $a > 1$, $b_k > 0$.

The two properties mentioned make the series easily summable on a computer; hence, $f_o(v)$ can be approximated to a high degree of accuracy by summing only a few terms of equation (27).

Once $f_o(v)$ is obtainable, it is easy to compute $f_V(v)$. Let

$$a_1 = \frac{\alpha}{n_1 - 1} \tag{30}$$

and

$$a_2 = \frac{1 - \alpha}{n_2 - 1} \tag{31}$$

Then

$$\tilde{V} \sim a_1 \chi^2(n_1 - 1) + a_2 \chi^2(n_2 - 1) \tag{32}$$

Let

$$\left.\begin{aligned} a_H &= \max(a_1, a_2) \\ a_L &= \min(a_1, a_2) \\ n_H &= \begin{cases} n_1 & \text{if } a_1 > a_2 \\ n_2 & \text{if } a_1 < a_2 \end{cases} \\ n_L &= \begin{cases} n_2 & \text{if } a_1 > a_2 \\ n_1 & \text{if } a_1 < a_2 \end{cases} \end{aligned}\right\} \tag{33}$$

Then

$$V \sim a_H \chi^2(n_H - 1) + a_L \chi^2(n_L - 1) \sim a_L \left[\frac{a_H}{a_L} \chi^2(n_H - 1) + \chi^2(n_L - 1)\right] \tag{34}$$

Taking $a = a_H/a_L$, $m = n_H - 1$, and $n = n_L - 1$, the distribution of $(1/a_L)\tilde{V}$ (that is, $f_o(v)$) can be found from equation (27), from which $f_V(v)$ may be derived using

$$f_V(v) = \frac{1}{a_L} f_o\left(\frac{v}{a_L}\right) \tag{35}$$

Computation of $F_\nabla(x)$

In order to integrate equation (21) numerically, an upper limit and a step size must be determined. Let ϵ be an upper bound of error in $F_\nabla(x)$ resulting from incomplete integration of equation (21), and let r_ϵ be a number such that $1 - \Phi\left(x\sqrt{r_\epsilon}\right) < \epsilon$. Since $1 - \Phi(\cdot)$ is a monotonically decreasing function and since $\int_0^\infty f_V(v)dv = 1$, it follows that

$$\int_{r_\epsilon}^\infty f_V(v)\left[1 - \Phi\left(xv^{\frac{1}{2}}\right)\right]dv < \left[1 - \Phi\left(xr_\epsilon^{\frac{1}{2}}\right)\right]\int_{r_\epsilon}^\infty f_V(v)dv < \epsilon \tag{36}$$

In most cases, $\int_{r_\epsilon}^\infty f_V(v)dv$ will be very small, so that the actual error may be on the order of ϵ^2; however, the crude bound in equation (36) is adequate for most purposes.

The simplest way to choose a step size was determined to be by trial and error, looking at the changes in the computed $F_\nabla(x)$ for various sizes.

Welch Approximation to $F_{\nabla}(x)$

The approximating distribution to $F_{\nabla}(x)$ (ref. 3) is a t distribution having f degrees of freedom, where f is defined by the relation

$$\frac{1}{f} = \frac{\alpha^2}{n_1 - 1} + \frac{(1 - \alpha)^2}{n_2 - 1} \tag{37}$$

The accuracy of the approximation is shown in table XIII. In this table, F(x) represents $F_{\nabla}(x)$. Since, in general, α is unknown, it must be estimated by

$$\hat{\alpha} = \frac{\frac{s_1^2}{n_1}}{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \tag{38}$$

In the notation used, A-HIGH represents the 90-percent point of the distribution[2] of $\hat{\alpha}$, A-LOW represents the 10-percent point of $\hat{\alpha}$, and A-TRUE represents the true value of α. Similarly, $G_{HIGH}(x)$ and $G_{LOW}(x)$ are the approximating distributions based on A-HIGH and A-LOW, respectively, while G(x) is the approximate distribution based on A-TRUE.

An inspection of table II discloses that the Welch approximation is remarkably accurate, even for small n_1 and n_2, provided that n_1 and n_2 are equal or nearly equal. In most other cases, the approximation underestimates $F_{\nabla}(x)$.

Note that if a test were actually performed for the difference of two means, using this procedure, α would have to be estimated by equation (38). However, it is disclosed in the tables that if $n_1 = n_2$, the distribution of ∇ is remarkably insensitive to α, so that practically any estimate such as equation (37) would give satisfactory results.

[2]It is not difficult to show that $\hat{\alpha}$ is distributed as $\dfrac{\left(\frac{\alpha}{1 - \alpha}\right)\tilde{F}}{\left(\frac{\alpha}{1 - \alpha}\right)\tilde{F} + 1}$ where $\tilde{F}$ has the Fisher's F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.

CONCLUSION

Information concerning the distribution and approximate distribution of a weighted sum of independent chi-square random variables has been presented. The information has indicated that if equal sample sizes are selected, the Welch approximation to the Behrens-Fisher Problem may be safely used, even for sample sizes as small as 5.

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National Aeronautics and Space Administration
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TABLE I. - MEAN OF WEIGHTS, 10; NUMBER OF WEIGHTS, 4

Standard deviation of weights	Basic weight of x		Spacing of weights		
			I (a)	II (b)	III (c)
		G(x)	F(x)	F(x)	F(x)
5	71.62	0.85745	0.86245	0.86259	0.86687
	134.87	.98457	.98344	.98326	.98289
	166.49	.99514	.99420	.99417	.99350
	198.11	.99850	.99795	.99797	.99747
	229.74	.99954	.99926	.99929	.99900
	261.36	.99986	.99973	.99975	.99960
4	70.46	0.85650	0.86069	0.86082	0.86318
	131.39	.98502	.98407	.98397	.98382
	161.85	.99541	.99462	.99460	.99421
	192.32	.99862	.99817	.99818	.99788
	222.78	.99959	.99937	.99939	.99921
	253.24	.99988	.99978	.99979	.99970
3	69.53	0.85575	0.85865	0.85871	0.85977
	128.59	.98540	.98474	.98470	.98465
	158.12	.99562	.99507	.99506	.99488
	187.65	.99872	.99841	.99841	.99827
	217.18	.99963	.99948	.99949	.99940
	246.67	.99989	.99983	.99984	.99979
1	68.43	0.85487	0.85528	0.85528	0.85532
	125.28	.98586	.98576	.98576	.98576
	153.70	.99587	.99580	.99580	.99579
	182.13	.99883	.99878	.99878	.99878
	210.55	.99967	.99965	.99965	.99965
	238.98	.99991	.99990	.99990	.99990

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE II. - MEAN OF WEIGHTS, 10; NUMBER OF WEIGHTS, 5

Standard deviation of weights	Basic weight of x	Spacing of weights I (a) G(x)	I (a) F(x)	II (b) F(x)	III (c) F(x)
5	85.36	0.85476	0.85927	0.85776	0.86446
	156.07	.98591	.98473	.98491	.98396
	191.42	.99591	.99504	.99529	.99418
	226.78	.99884	.99839	.99856	.99784
	262.13	.99968	.99949	.99957	.99919
	297.49	.99991	.99984	.99988	.99970
4	84.06	0.85396	0.85772	0.85689	0.86071
	152.18	.98635	.98536	.98545	.98496
	186.24	.99614	.99543	.99557	.99491
	220.29	.99894	.99858	.99868	.99824
	254.35	.99972	.99957	.99962	.99939
	288.41	.99992	.99988	.99990	.99979
3	83.01	0.85332	0.85594	0.85557	0.85731
	149.05	.98671	.98603	.98607	.98586
	182.06	.99633	.99584	.99591	.99559
	215.08	.99902	.99878	.99883	.99862
	248.09	.99974	.99966	.99968	.99957
	281.10	.99993	.99991	.99992	.99987
1	81.78	0.85258	0.85296	0.85295	0.85302
	145.34	.98715	.98707	.98707	.98706
	177.12	.99655	.99650	.99503	.99649
	208.90	.99911	.99910	.99910	.99909
	240.68	.99978	.99978	.99978	.99978
	272.46	.99994	.99996	.99996	.99996

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE III. - MEAN OF WEIGHTS, 50; NUMBER OF WEIGHTS, 4

Standard deviation of weights	Basic weight of x	Spacing of weights			
		I (a)		II (b)	III (c)
		G(x)	F(x)	F(x)	F(x)
25	358.11	0.85745	0.86245	0.86259	0.86687
	674.34	.98457	.98344	.98326	.98290
	832.46	.99514	.99420	.99417	.99350
	990.57	.99850	.99795	.99796	.99747
	1148.68	.99954	.99926	.99929	.99900
	1306.80	.99986	.99973	.99975	.99960
20	352.31	0.85650	0.86069	0.86082	0.86318
	656.95	.98503	.98407	.98397	.98382
	809.26	.99541	.99461	.98397	.99421
	961.58	.99862	.99817	.99818	.99788
	1113.89	.99959	.99937	.99939	.99921
	1266.21	.99988	.99978	.99979	.99970
10	344.22	0.85520	0.85670	0.85671	0.85703
	632.67	.98568	.98534	.98533	.98532
	776.89	.99578	.99549	.99549	.99543
	921.11	.99879	.99863	.99863	.99858
	1065.33	.99966	.99958	.99958	.99956
	1209.55	.99990	.99987	.99987	.99986
1	341.45	0.85477	0.85478	0.85478	0.85478
	624.35	.98591	.98591	.98591	.98591
	765.80	.99591	.99590	.98591	.99590
	907.25	.99884	.99884	.99590	.99884
	1404.87	.99968	.99968	.99884	.99968
	1190.15	.99991	.99991	.99991	.99991

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE IV. - MEAN OF WEIGHTS, 50; NUMBER OF WEIGHTS, 5

Standard deviation of weights	Basic weight of x	Spacing of weights			
		I (a)		II (b)	III (c)
		G(x)	F(x)	F(x)	F(x)
25	426.78	0.85476	0.85927	0.85776	0.86245
	780.33	.98591	.98473	.98491	.98344
	957.11	.99591	.99504	.99529	.99412
	1133.88	.99884	.99839	.99856	.99794
	1310.07	.99968	.99949	.99957	.99926
	1487.44	.99991	.99984	.99988	.99973
20	420.29	0.85396	0.85772	0.85689	0.86071
	760.88	.98635	.98536	.98545	.98496
	931.18	.99614	.99543	.99557	.99491
	1101.47	.99894	.99858	.99868	.99824
	1271.76	.99972	.99957	.99962	.99939
	1442.06	.99993	.99988	.99990	.99979
10	411.25	0.85286	0.85422	0.85410	0.85465
	733.74	.98698	.98664	.98665	.98659
	894.98	.99647	.99622	.99625	.99614
	1056.22	.99908	.99896	.99897	.99891
	1217.47	.99976	.99973	.99974	.99970
	1378.72	.99994	.99994	.99994	.99993
1	408.14	0.85249	0.85252	0.85252	0.85252
	724.43	.98720	.98722	.98722	.98722
	882.58	.99658	.99660	.99660	.99660
	1040.72	.99912	.99914	.99914	.99914
	1198.87	.99978	.99980	.99980	.99980
	1357.02	.99995	.99997	.99997	.99997

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE V. - MEAN OF WEIGHTS, 50; NUMBER OF WEIGHTS, 6

Standard deviation of weights	Basic weight of x	Spacing of weights I (a)		II (b)	III (c)
		G(x)	F(x)	F(x)	F(x)
25	493.65	0.85287	0.85696	0.85687	0.86269
	880.95	.98697	.98576	.98566	.98480
	1074.60	.99646	.99566	.99567	.99468
	1268.25	.99907	.99870	.99873	.99810
	1461.90	.99976	.99962	.99965	.99932
	1655.54	.99994	.99991	.99992	.99976
20	486.55	0.85217	0.85558	0.85560	0.85892
	859.64	.98739	.98639	.98632	.98585
	1046.19	.99667	.99602	.99602	.99543
	1232.74	.99916	.99886	.99888	.99850
	1419.29	.99979	.99969	.99970	.99951
	1605.83	.99995	.99993	.99994	.99985
10	476.64	0.85121	0.85246	0.85247	0.85296
	829.91	.98799	.98765	.98765	.98758
	1006.54	.99696	.99675	.99675	.99665
	1183.18	.99927	.99919	.99919	.99913
	1359.81	.99983	.99982	.99982	.99979
	1536.45	.99996	.99998	.99998	.99996
1	473.24	0.85089	0.85093	0.85093	0.85093
	819.72	.98820	.98823	.98823	.98823
	992.96	.99706	.99709	.99709	.99709
	1166.20	.99303	.99933	.99933	.99933
	1339.43	.99984	.99987	.99987	.99987
	1512.68	.99996	.99999	.99999	.99999

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE VI. - MEAN OF WEIGHTS, 50; NUMBER OF WEIGHTS, 7

Standard deviation of weights	Basic weight of x	Spacing of weights			
		I (a)		II (b)	III (c)
		G(x)	F(x)	F(x)	F(x)
25	559.16	0.85147	0.85521	0.85417	0.86132
	977.49	.98783	.98662	.98677	.98547
	1186.66	.99689	.99614	.99632	.99506
	1395.83	.99924	.99891	.99902	.99828
	1604.99	.99982	.99971	.99976	.99940
	1814.16	.99996	.99994	.99996	.99980
20	551.50	0.85084	0.85395	0.85339	0.85755
	954.48	.98823	.98722	.98731	.98657
	1155.98	.99708	.99647	.99658	.99582
	1357.47	.99931	.99906	.99912	.99867
	1558.97	.99984	.99976	.99980	.99958
	1760.47	.99996	.99995	.99997	.99988
10	540.79	0.85000	0.85113	0.85106	0.85168
	922.36	.98880	.98458	.98847	.98836
	1113.15	.99734	.99714	.99715	.99703
	1303.39	.99940	.99933	.99934	.99927
	1494.73	.99987	.99986	.99987	.99984
	1685.51	.99997	.99999	.99999	.99998
1	537.12	0.84971	0.84974	0.84974	0.84974
	911.36	.98900	.98902	.98902	.98902
	1098.48	.99742	.99745	.99745	.99745
	1285.60	.99943	.99945	.99945	.99945
	1472.72	.99988	.99990	.99990	.99990
	1659.84	.99998	.99999	.99999	.99999

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE VII. - MEAN OF WEIGHTS, 50; NUMBER OF WEIGHTS, 8

Standard deviation of weights	Basic weight of x	Spacing of weights			
		I (a)		II (b)	III (c)
		G(x)	F(x)	F(x)	F(x)
25	623.61	0.85038	0.85381	0.85361	0.86023
	1070.82	.98854	.98732	.98726	.98602
	1294.42	.99722	.99651	.99654	.99536
	1518.03	.99936	.99907	.99910	.99842
	1741.64	.99986	.99976	.99978	.99945
	1965.25	.99997	.99995	.99995	.99981
20	615.41	0.84982	0.85266	0.85262	0.85645
	1046.22	.98893	.98792	.98788	.98716
	1261.63	.99739	.99682	.99683	.99612
	1477.03	.99942	.99919	.99921	.99880
	1692.44	.99988	.99980	.99981	.99963
	1907.85	.99997	.99996	.99996	.99989
10	603.96	0.84905	0.85009	0.85009	0.85067
	1011.88	.98947	.98912	.98911	.98900
	1215.84	.99763	.99743	.99743	.99732
	1419.80	.99950	.99943	.99943	.99937
	1623.76	.99990	.99988	.99989	.99986
	1827.73	.99998	.99998	.99998	.99997
1	600.03	0.84880	0.84882	0.84882	0.84882
	1000.12	.98966	.98967	.98967	.98967
	1200.16	.99771	.99772	.99772	.99771
	1400.20	.99953	.99953	.99953	.99953
	1600.24	.99991	.99992	.99992	.99992
	1800.28	.99998	.99999	.99999	.99999

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE VIII. - MEAN OF WEIGHTS, 50; NUMBER OF WEIGHTS, 9

Standard deviation of weights	Basic weight of x	Spacing of weights			
		I (a)		II (b)	III (c)
		G(x)	F(x)	F(x)	F(x)
25	687.17	0.84951	0.85268	0.85188	0.85933
	1161.51	.98915	.98793	.98807	.98649
	1398.68	.99749	.99682	.99696	.99561
	1635.85	.99945	.99918	.99925	.99853
	1873.02	.99989	.99979	.99982	.99949
	2110.19	.99998	.99995	.99996	.99982
20	678.47	0.84900	0.85162	0.85119	0.85556
	1135.41	.98952	.98851	.98858	.98766
	1363.90	.99765	.99710	.99718	.99637
	1592.37	.99951	.99929	.99933	.99890
	1820.84	.99990	.99983	.99985	.99966
	2049.31	.99998	.99996	.99996	.99989
10	666.33	0.84830	0.84925	0.84919	0.84986
	1099.00	.99004	.98967	.98969	.98954
	1315.33	.99786	.99767	.99768	.99755
	1531.67	.99957	.99950	.99951	.99945
	1748.00	.99992	.99989	.99990	.99987
	1964.33	.99998	.99997	.99998	.99997
1	662.17	0.84807	0.84808	0.84808	0.84808
	1086.52	.99021	.99021	.99021	.99021
	1298.70	.99793	.99793	.99793	.99793
	1510.87	.99960	.99959	.99959	.99959
	1723.05	.99993	.99992	.99992	.99992
	1935.22	.99999	.99998	.99998	.99998

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE IX. - MEAN OF WEIGHTS, 50; NUMBER OF WEIGHTS, 10

Standard deviation of weights	Basic weight of x	Spacing of weights			
		I (a)		II (b)	III (c)
		G(x)	F(x)	F(x)	F(x)
25	750.00	0.84880	0.85174	0.85151	0.85858
	1250.00	.98966	.98847	.98843	.98690
	1500.00	.99771	.99704	.99710	.99582
	1750.00	.99953	.99927	.99930	.99862
	2000.00	.99991	.99982	.99983	.99952
	2250.00	.99998	.99995	.99995	.99983
20	740.83	0.84833	0.85076	0.85069	0.85482
	1222.50	.99002	.98902	.98900	.99809
	1463.33	.99785	.99734	.99735	.99658
	1704.16	.99957	.99937	.99938	.99898
	1944.99	.99992	.99985	.99985	.99968
	2185.82	.99998	.99995	.99996	.99989
10	728.04	0.84769	0.84857	0.84857	0.84920
	1184.10	.99052	.99015	.99015	.99000
	1412.14	.99805	.99786	.99786	.99774
	1640.18	.99963	.99955	.99956	.99950
	1868.21	.99993	.99990	.99990	.99988
	2096.25	.99999	.99997	.99997	.99996
1	723.65	0.84748	0.84748	0.84748	0.84748
	1170.95	.99069	.99067	.99067	.99067
	1394.60	.99812	.99810	.99810	.99810
	1618.26	.99965	.99964	.99964	.99964
	1841.90	.99994	.99992	.99992	.99992
	2065.56	.99999	.99997	.99997	.99997

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE X. - MEAN OF WEIGHTS, 50; NUMBER OF WEIGHTS, 11

Standard deviation of weights	Basic weight of x	Spacing of weights			
		I (a)		II (b)	III (c)
		G(x)	F(x)	F(x)	F(x)
25	812.20	0.84820	0.85096	0.85031	0.85795
	1336.61	.99011	.98894	.98906	.98727
	1598.81	.99789	.99729	.99740	.99601
	1861.01	.99958	.99935	.99940	.99870
	2123.21	.99992	.99984	.99986	.99955
	2385.42	.99999	.99995	.99996	.99984
20	802.59	0.84777	0.85004	0.84970	0.85420
	1307.76	.99045	.98948	.98955	.98848
	1560.35	.99803	.99754	.99760	.99676
	1812.93	.99962	.99944	.99947	.99905
	2065.52	.99993	.99986	.99988	.99971
	2318.11	.99999	.99996	.99996	.99990
10	789.17	0.84718	0.84800	0.84795	0.84865
	1267.50	.99093	.99058	.99059	.99041
	1506.67	.99821	.99803	.99804	.99790
	1745.83	.99968	.99960	.99961	.99955
	1984.99	.99995	.99991	.99991	.99989
	2224.16	.99999	.99997	.99997	.99996
1	784.57	0.84698	0.84698	0.84698	0.84698
	1253.70	.99110	.99108	.99108	.99108
	1488.27	.99827	.99826	.99826	.99826
	1722.84	.99969	.99967	.99967	.99967
	1957.41	.99995	.99993	.99993	.99993
	2191.97	.99999	.99997	.99997	.99997

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE XI. - MEAN OF WEIGHTS, 50; NUMBER OF WEIGHTS, 20

Standard deviation of weights	Basic weight of x	Spacing of weights I (a) G(x)	Spacing of weights I (a) F(x)	II (b) F(x)	III (c) F(x)
25	1353.55	0.84538	0.84715	0.84693	0.85476
	2060.66	.99254	.99160	.99162	.98934
	2414.21	.99877	.99841	.99844	.99702
	2767.77	.99982	.99972	.99974	.99913
	3121.32	.99998	.99996	.99996	.99974
	3474.87	.99999	.99999	.99999	.99992
20	1340.59	0.84512	0.84657	0.84647	0.85112
	2021.76	.99280	.99203	.99203	.99065
	2362.35	.99884	.99856	.99858	.99773
	2702.94	.99984	.99977	.99977	.99943
	3043.53	.99998	.99997	.99997	.99985
	3384.11	.99999	.99999	.99999	.99996
10	1322.49	0.84476	0.84529	0.84528	0.84600
	1967.47	.99316	.99289	.99289	.99265
	2289.96	.99896	.99886	.99886	.99873
	2612.45	.99986	.99984	.99984	.99980
	2934.94	.99998	.99998	.99998	.99997
	3257.43	.99999	.99999	.99999	.99999
1	1316.29	0.84464	0.84465	0.84465	0.84465
	1948.87	.99328	.99328	.99328	.99328
	2265.16	.99899	.99899	.99899	.99899
	2581.45	.99987	.99987	.99987	.99987
	2897.75	.99998	.99999	.99999	.99999
	3214.04	.99999	.99999	.99999	.99999

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE XII. - MEAN OF WEIGHTS, 50; NUMBER OF WEIGHTS, 21

Standard deviation of weights	Basic weight of x	Spacing of weights			
		I (a)		II (b)	III (c)
		G(x)	F(x)	F(x)	F(x)
25	1412.28	0.84521	0.84691	0.84656	0.85455
	2136.85	.99271	.99179	.99187	.98949
	2499.14	.99882	.99848	.99853	.99708
	2861.42	.99983	.99974	.99976	.99916
	3223.71	.99998	.99996	.99996	.99975
	3585.99	.99999	.99999	.99999	.99992
20	1399.00	0.84495	0.84635	0.84617	0.85091
	2097.00	.99296	.99221	.99226	.99080
	2446.00	.99890	.99863	.99865	.99779
	2795.00	.99985	.99978	.99979	.99945
	3144.00	.99998	.99998	.99997	.99986
	3493.00	.99999	.99999	.99999	.99996
10	1380.45	0.84461	0.84512	0.84510	0.84583
	2041.36	.99331	.99305	.99305	.99281
	2371.82	.99900	.99891	.99891	.99878
	2702.27	.99987	.99985	.99985	.99981
	3032.73	.99999	.99998	.99998	.99997
	3363.18	.99999	.99999	.99999	.99999
1	1374.10	0.84450	0.84450	0.84450	0.84450
	2022.31	.99343	.99343	.99343	.99343
	2346.40	.99904	.99904	.99904	.99904
	2670.51	.99988	.99988	.99988	.99988
	2994.61	.99998	.99998	.99999	.99999
	3318.71	.99999	.99999	.99999	.99999

[a]Weights are equally spaced.

[b]One half of the weights are equal to a constant while the other half are equal to another constant.

[c]All of the weights except one are equal.

TABLE XIII. - COMPARISON OF WELCH APPROXIMATION $[G(x),\ G_{LOW}(x),\ G_{HIGH}(x)]$ WITH TRUE VALUE [F(x)]

N1= 5 N2=20

A-TRUE=.010 A-LOW=.005 A-HIGH=.022

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3268	.9000	.9000	.8999	.9002
1,7274	.9500	.9500	.9499	.9502
2,0903	.9750	.9750	.9749	.9751
2,5350	.9900	.9900	.9899	.9901
3,5689	.9990	.9990	.9990	.9990

A-TRUE=.050 A-LOW=.023 A-HIGH=.103

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3279	.9007	.9007	.9004	.9012
1,7291	.9507	.9507	.9504	.9512
2,0948	.9757	.9757	.9754	.9761
2,5520	.9907	.9907	.9905	.9910
3,8278	.9995	.9995	.9995	.9995

A-TRUE=.100 A-LOW=.048 A-HIGH=.196

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3209	.9000	.9000	.8995	.9002
1,7166	.9500	.9500	.9495	.9502
2,0730	.9750	.9750	.9746	.9752
2,5070	.9900	.9900	.9897	.9901
3,5020	.9990	.9990	.9989	.9990

A-TRUE=.200 A-LOW=.102 A-HIGH=.354

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3196	.9000	.9000	.8998	.8986
1,7142	.9500	.9500	.9498	.9485
2,0692	.9750	.9750	.9748	.9737
2,5008	.9900	.9900	.9899	.9891
3,4873	.9990	.9990	.9990	.9987

A-TRUE=.300 A-LOW=.163 A-HIGH=.485

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3238	.9001	.9000	.9007	.8964
1,7219	.9501	.9500	.9507	.9463
2,0814	.9751	.9750	.9756	.9717
2,5206	.9901	.9900	.9904	.9876
3,5343	.9990	.9990	.9991	.9983

A-TRUE=.400 A-LOW=.233 A-HIGH=.594

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3444	.9021	.9000	.9021	.8944
1,7593	.9521	.9500	.9522	.9442
2,1485	.9772	.9749	.9768	.9700
2,6577	.9921	.9898	.9910	.9864
4,5063	.9999	.9979	.9980	.9976

A-TRUE=.500 A-LOW=.313 A-HIGH=.687

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3490	.9007	.9000	.9039	.8931
1,7687	.9509	.9500	.9540	.9428
2,1568	.9759	.9749	.9783	.9686
2,6443	.9908	.9899	.9922	.9853
3,8372	.9993	.9989	.9994	.9975

N1= 5 N2=30

A-TRUE=.010 A-LOW=.005 A-HIGH=.021

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3109	.8999	.9000	.9000	.9001
1,6981	.9500	.9500	.9499	.9501
2,0444	.9750	.9750	.9750	.9751
2,4612	.9900	.9900	.9900	.9901
3,4341	.9991	.9991	.9991	.9991

A-TRUE=.050 A-LOW=.025 A-HIGH=.099

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3090	.8999	.9000	.8998	.9002
1,6947	.9500	.9500	.9498	.9502
2,0403	.9751	.9751	.9749	.9753
2,4555	.9901	.9901	.9900	.9902
3,4114	.9991	.9991	.9991	.9991

A-TRUE=.100 A-LOW=.050 A-HIGH=.189

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3120	.9006	.9007	.9005	.9005
1,6998	.9507	.9507	.9505	.9505
2,0479	.9757	.9757	.9755	.9755
2,4749	.9907	.9907	.9905	.9906
3,6149	.9995	.9995	.9995	.9995

A-TRUE=.200 A-LOW=.107 A-HIGH=.343

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3093	.9000	.9000	.9003	.8982
1,6952	.9500	.9500	.9503	.9481
2,0390	.9750	.9750	.9752	.9733
2,4521	.9900	.9900	.9902	.9888
3,3733	.9990	.9990	.9990	.9987

A-TRUE=.300 A-LOW=.170 A-HIGH=.473

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3158	.9001	.9000	.9012	.8962
1,7072	.9502	.9500	.9513	.9460
2,0580	.9752	.9750	.9761	.9715
2,4826	.9902	.9900	.9908	.9874
3,4444	.9991	.9990	.9992	.9982

A-TRUE=.400 A-LOW=.242 A-HIGH=.583

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3383	.9022	.9017	.9044	.8961
1,7478	.9523	.9517	.9545	.9459
2,1277	.9773	.9766	.9789	.9715
2,6207	.9921	.9915	.9930	.9880
4,3423	.9999	.9997	.9998	.9993

A-TRUE=.500 A-LOW=.323 A-HIGH=.677

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,3447	.9008	.9000	.9043	.8932
1,7608	.9511	.9500	.9545	.9429
2,1439	.9761	.9750	.9788	.9687
2,6231	.9910	.9900	.9925	.9855
3,7841	.9993	.9990	.9995	.9976

TABLE XIII. - COMPARISON OF WELCH APPROXIMATION $\left[G(x),\ G_{LOW}(x),\ G_{HIGH}(x)\right]$ WITH TRUE VALUE $[F(x)]$ - Continued

N1= 2 N2= 2

A-TRUE=.010 A-LOW=.000 A-HIGH=.287

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
3.0133	.9071	.9000	.8981	.9423
6.1060	.9605	.9500	.9484	.9811
12.1268	.9860	.9750	.9738	.9940
29.8306	.9971	.9900	.9894	.9987
285.2239	.9993	.9990	.9989	1.0000

A-TRUE=.050 A-LOW=.001 A-HIGH=.677

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.7841	.9154	.9000	.8905	.9383
5.3873	.9681	.9497	.9415	.9787
10.1778	.9896	.9747	.9687	.9927
23.3928	.9977	.9897	.9862	.9981
188.1644	.9992	.9986	.9979	.9996

A-TRUE=.100 A-LOW=.003 A-HIGH=.816

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.5521	.9164	.9000	.8817	.9135
4.6944	.9679	.9500	.9337	.9612
8.3896	.9888	.9748	.9625	.9826
18.0083	.9974	.9899	.9824	.9940
152.7870	.9996	.9990	.9976	.9995

A-TRUE=.200 A-LOW=.006 A-HIGH=.909

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.2244	.9115	.9000	.8667	.8826
3.7797	.9622	.9500	.9189	.9342
6.1859	.9846	.9750	.9500	.9628
11.6428	.9954	.9900	.9735	.9824
55.9752	.9996	.9990	.9946	.9973

A-TRUE=.300 A-LOW=.011 A-HIGH=.945

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.0260	.9056	.9000	.8561	.8645
3.2649	.9560	.9500	.9075	.9160
5.0342	.9799	.9750	.9395	.9471
8.7451	.9930	.9901	.9653	.9711
38.4552	.9995	.9992	.9923	.9944

A-TRUE=.400 A-LOW=.016 A-HIGH=.964

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.9193	.9015	.8991	.8494	.8531
3.0012	.9516	.9491	.9001	.9039
4.4773	.9763	.9742	.9323	.9358
7.3758	.9908	.9891	.9588	.9617
25.3605	.9991	.9982	.9878	.9892

A-TRUE=.500 A-LOW=.024 A-HIGH=.976

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.8856	.9000	.9000	.8494	.8494
2.9200	.9500	.9500	.8998	.8998
4.3027	.9750	.9750	.9319	.9319
6.9636	.9900	.9900	.9584	.9584
22.3327	.9989	.9990	.9877	.9877

N1= 3 N2= 3

A-TRUE=.010 A-LOW=.001 A-HIGH=.083

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.8693	.9014	.9000	.8989	.9086
2.8810	.9520	.9497	.9487	.9575
4.2342	.9774	.9749	.9741	.9806
6.8241	.9923	.9899	.9894	.9931
24.1867	.9998	.9990	.9989	.9994

A-TRUE=.050 A-LOW=.006 A-HIGH=.321

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.8092	.9041	.9000	.8947	.9231
2.7471	.9556	.9502	.9451	.9705
3.9539	.9807	.9751	.9712	.9893
6.2049	.9945	.9901	.9877	.9974
19.7881	.9999	.9991	.9987	.9998

A-TRUE=.100 A-LOW=.012 A-HIGH=.500

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.7683	.9071	.9012	.8912	.9232
2.6544	.9586	.9514	.9419	.9708
3.7974	.9831	.9763	.9689	.9895
6.0465	.9959	.9911	.9768	.9972
28.5031	1.0000	.9990	.9986	.9991

A-TRUE=.200 A-LOW=.027 A-HIGH=.692

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.6466	.9039	.9000	.8826	.9073
2.3727	.9547	.9498	.9326	.9567
3.2187	.9795	.9748	.9604	.9801
4.6204	.9932	.9898	.9800	.9929
11.0565	.9997	.9989	.9964	.9993

A-TRUE=.300 A-LOW=.045 A-HIGH=.794

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5819	.9020	.9000	.8781	.8937
2.2337	.9524	.9500	.9280	.9438
2.9607	.9773	.9750	.9562	.9699
4.1008	.9916	.9900	.9766	.9866
8.4689	.9994	.9990	.9948	.9982

A-TRUE=.400 A-LOW=.069 A-HIGH=.857

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5451	.9005	.9000	.8767	.8842
2.1566	.9506	.9498	.9262	.9339
2.8208	.9756	.9748	.9544	.9613
3.8310	.9904	.9898	.9750	.9803
7.4729	.9991	.9987	.9937	.9959

A-TRUE=.500 A-LOW=.100 A-HIGH=.900

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5332	.9000	.9000	.8786	.8786
2.1319	.9500	.9500	.9283	.9283
2.7765	.9750	.9750	.9563	.9563
3.7468	.9900	.9900	.9765	.9765
7.1742	.9990	.9990	.9946	.9946

TABLE XIII. - COMPARISON OF WELCH APPROXIMATION $[G(x),\ G_{LOW}(x),\ G_{HIGH}(x)]$ WITH TRUE VALUE $[F(x)]$ - Continued

N1= 4 N2= 4

A-TRUE=.010 A-LOW=.002 A-HIGH=.052

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.6367	.9013	.9008	.9001	.9044
2.3570	.9518	.9508	.9501	.9542
3.1994	.9768	.9759	.9752	.9784
4.6419	.9918	.9907	.9904	.9923
13.7289	.9998	.9994	.9993	.9995

A-TRUE=.050 A-LOW=.010 A-HIGH=.221

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5966	.9012	.9000	.8965	.9117
2.2650	.9523	.9500	.9466	.9611
3.0180	.9775	.9750	.9723	.9835
4.2262	.9922	.9901	.9883	.9950
9.4329	.9997	.9991	.9988	.9998

A-TRUE=.100 A-LOW=.020 A-HIGH=.375

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5613	.9023	.8931	.8917	.9117
2.1903	.9529	.9481	.9417	.9611
2.8817	.9787	.9731	.9679	.9830
3.9476	.9925	.9881	.9846	.9937
7.8963	.9996	.9970	.9962	.9978

A-TRUE=.200 A-LOW=.044 A-HIGH=.574

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5061	.9019	.8981	.8973	.9062
2.0760	.9524	.9481	.9373	.9560
2.6772	.9774	.9731	.9639	.9794
3.5607	.9919	.9781	.9818	.9919
6.5945	.9995	.9971	.9954	.9977

A-TRUE=.300 A-LOW=.074 A-HIGH=.698

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4685	.9019	.9000	.8869	.9001
2.0003	.9512	.9500	.9367	.9501
2.5450	.9762	.9750	.9635	.9751
3.3184	.9919	.9900	.9518	.9900
5.7473	.9993	.9990	.9965	.9990

A-TRUE=.400 A-LOW=.110 A-HIGH=.782

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4469	.9003	.8793	.8861	.8927
1.9572	.9503	.9493	.9358	.9426
2.4708	.9753	.9743	.9626	.9686
3.1851	.9903	.9793	.9809	.9853
5.3732	.9991	.9983	.9958	.9973

A-TRUE=.500 A-LOW=.156 A-HIGH=.844

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4398	.9000	.9000	.8890	.8890
1.9432	.9500	.9500	.9387	.9387
2.4504	.9751	.9751	.9654	.9654
3.1520	.9901	.9901	.9832	.9832
5.3295	.9991	.9991	.9972	.9972

N1= 5 N2= 5

A-TRUE=.010 A-LOW=.002 A-HIGH=.040

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5274	.9003	.9000	.8995	.9020
2.1198	.9504	.9500	.9495	.9520
2.7549	.9754	.9750	.9746	.9766
3.7061	.9904	.9900	.9897	.9910
7.0321	.9992	.9990	.9989	.9992

A-TRUE=.050 A-LOW=.013 A-HIGH=.178

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5055	.9010	.9000	.8975	.9071
2.0750	.9513	.9500	.9476	.9569
2.6755	.9764	.9750	.9730	.9805
3.5575	.9912	.9900	.9887	.9934
6.5245	.9994	.9990	.9987	.9996

A-TRUE=.100 A-LOW=.026 A-HIGH=.314

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4814	.9013	.9000	.8955	.9090
2.0277	.9518	.9501	.9455	.9590
2.5931	.9769	.9731	.9713	.9822
3.4091	.9916	.9901	.9875	.9944
6.1345	.9995	.9991	.9965	.9998

A-TRUE=.200 A-LOW=.057 A-HIGH=.507

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4433	.9011	.9000	.8925	.9065
1.9502	.9514	.9500	.9424	.9565
2.4508	.9765	.9750	.9685	.9803
3.1637	.9912	.9900	.9855	.9933
5.2716	.9994	.9989	.9976	.9996

A-TRUE=.300 A-LOW=.094 A-HIGH=.638

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4171	.9006	.9000	.8912	.9015
1.8989	.9508	.9500	.9409	.9515
2.3718	.9758	.9750	.9671	.9762
3.0105	.9906	.9900	.9844	.9908
4.8217	.9992	.9989	.9973	.9991

A-TRUE=.400 A-LOW=.140 A-HIGH=.733

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4018	.9002	.9000	.8915	.8968
1.8692	.9502	.9500	.9412	.9467
2.3234	.9753	.9750	.9674	.9723
2.9274	.9902	.9900	.9846	.9881
4.6400	.9991	.9990	.9974	.9985

A-TRUE=.500 A-LOW=.196 A-HIGH=.804

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3968	.9000	.9000	.8934	.8934
1.8596	.9500	.9500	.9432	.9432
2.3060	.9750	.9750	.9691	.9691
2.8965	.9900	.9900	.9858	.9858
4.5012	.9990	.9989	.9977	.9977

TABLE XIII. - COMPARISON OF WELCH APPROXIMATION [G(x), G_{LOW}(x), G_{HIGH}(x)]

WITH TRUE VALUE [F(x)] - Continued

N1=10 N2=10

A-TRUE=.010 A-LOW=.004 A-HIGH=.024

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3809	.9000	.9000	.8998	.9004
1.8290	.9501	.9500	.9498	.9505
2.2554	.9751	.9750	.9748	.9754
2.8099	.9901	.9900	.9899	.9903
4.2662	.9990	.9990	.9990	.9991

A-TRUE=.050 A-LOW=.021 A-HIGH=.114

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3727	.9002	.9000	.8991	.9017
1.8135	.9502	.9500	.9491	.9518
2.2298	.9753	.9750	.9742	.9765
2.7666	.9902	.9900	.9895	.9910
4.1515	.9991	.9990	.9989	.9992

A-TRUE=.100 A-LOW=.044 A-HIGH=.213

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3636	.9002	.9000	.8984	.9025
1.7963	.9503	.9500	.9484	.9526
2.2016	.9754	.9750	.9736	.9772
2.7191	.9903	.9900	.9893	.9915
4.0276	.9991	.9990	.9987	.9993

A-TRUE=.200 A-LOW=.093 A-HIGH=.379

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3489	.9002	.9000	.8975	.9025
1.7685	.9503	.9500	.9474	.9525
2.1565	.9753	.9750	.9728	.9772
2.6438	.9903	.9900	.9884	.9915
3.8360	.9991	.9990	.9986	.9993

A-TRUE=.300 A-LOW=.149 A-HIGH=.511

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3386	.9001	.9000	.8973	.9013
1.7492	.9502	.9500	.9472	.9514
2.1253	.9752	.9750	.9726	.9762
2.5923	.9902	.9900	.9863	.9908
3.7081	.9991	.9990	.9985	.9992

A-TRUE=.400 A-LOW=.215 A-HIGH=.619

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3324	.9000	.9000	.8977	.8999
1.7378	.9500	.9500	.9476	.9499
2.1070	.9750	.9750	.9729	.9749
2.5623	.9900	.9900	.9885	.9899
3.6482	.9990	.9990	.9986	.9990

A-TRUE=.500 A-LOW=.291 A-HIGH=.709

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3304	.9000	.9000	.8986	.8986
1.7341	.9500	.9500	.9485	.9485
2.1009	.9750	.9750	.9737	.9737
2.5524	.9900	.9900	.9891	.9891
3.6107	.9990	.9990	.9988	.9988

N1=15 N2=15

A-TRUE=.010 A-LOW=.005 A-HIGH=.020

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3437	.9000	.9000	.8999	.9002
1.7589	.9500	.9500	.9499	.9502
2.1408	.9750	.9750	.9749	.9752
2.6179	.9900	.9900	.9899	.9901
3.7713	.9990	.9990	.9990	.9990

A-TRUE=.050 A-LOW=.025 A-HIGH=.096

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3387	.9001	.9000	.8995	.9008
1.7496	.9501	.9500	.9495	.9509
2.1268	.9751	.9749	.9745	.9757
2.5955	.9901	.9899	.9896	.9904
3.7316	.9991	.9989	.9989	.9991

A-TRUE=.100 A-LOW=.052 A-HIGH=.183

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3332	.9001	.9000	.8991	.9013
1.7392	.9501	.9500	.9491	.9513
2.1091	.9751	.9750	.9742	.9761
2.5658	.9901	.9900	.9895	.9906
3.6433	.9991	.9990	.9989	.9992

A-TRUE=.200 A-LOW=.110 A-HIGH=.336

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3241	.9001	.9000	.8987	.9013
1.7223	.9501	.9500	.9486	.9514
2.0822	.9751	.9750	.9738	.9762
2.5218	.9901	.9900	.9892	.9908
3.5372	.9991	.9990	.9988	.9992

A-TRUE=.300 A-LOW=.175 A-HIGH=.464

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3176	.9000	.9000	.8986	.9008
1.7105	.9501	.9500	.9486	.9509
2.0641	.9751	.9750	.9738	.9758
2.4932	.9901	.9900	.9891	.9906
3.4824	.9991	.9990	.9983	.9992

A-TRUE=.400 A-LOW=.248 A-HIGH=.574

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3138	.9000	.9000	.8989	.9001
1.7035	.9500	.9500	.9488	.9501
2.0521	.9750	.9750	.9740	.9751
2.4731	.9900	.9900	.9893	.9901
3.4222	.9990	.9990	.9988	.9990

A-TRUE=.500 A-LOW=.331 A-HIGH=.669

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3125	.9000	.9000	.8994	.8994
1.7011	.9500	.9500	.9494	.9494
2.0484	.9750	.9750	.9744	.9744
2.4672	.9900	.9900	.9896	.9896
3.4084	.9990	.9990	.9989	.9989

TABLE XIII. - COMPARISON OF WELCH APPROXIMATION $[G(x),\ G_{LOW}(x),\ G_{HIGH}(x)]$

WITH TRUE VALUE $[F(x)]$ - Continued

N1=20 N2=20

A-TRUE=.010 A-LOW=.006 A-HIGH=.017

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3268	.9000	.9000	.8999	.9001
1.7274	.9500	.9500	.9499	.9501
2.0925	.9751	.9751	.9751	.9752
2.5401	.9901	.9901	.9901	.9902
3.6169	.9991	.9991	.9991	.9991

A-TRUE=.050 A-LOW=.029 A-HIGH=.085

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3232	.9000	.9000	.8997	.9005
1.7208	.9500	.9500	.9497	.9505
2.0797	.9751	.9750	.9747	.9754
2.5177	.9900	.9900	.9898	.9903
3.5275	.9990	.9990	.9990	.9991

A-TRUE=.100 A-LOW=.060 A-HIGH=.163

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3192	.9000	.9000	.8995	.9007
1.7133	.9501	.9500	.9494	.9508
2.0678	.9751	.9749	.9744	.9755
2.4985	.9901	.9899	.9896	.9903
3.4820	.9990	.9989	.9988	.9990

A-TRUE=.200 A-LOW=.125 A-HIGH=.305

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3126	.9000	.9000	.8992	.9008
1.7013	.9500	.9500	.9492	.9508
2.0507	.9752	.9750	.9743	.9757
2.4723	.9902	.9900	.9895	.9905
3.4517	.9991	.9990	.9989	.9991

A-TRUE=.300 A-LOW=.196 A-HIGH=.429

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3079	.9000	.9000	.8992	.9005
1.6927	.9500	.9500	.9491	.9506
2.0372	.9751	.9751	.9744	.9756
2.4504	.9901	.9901	.9896	.9905
3.3996	.9991	.9991	.9990	.9992

A-TRUE=.400 A-LOW=.275 A-HIGH=.539

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3052	.8999	.9000	.8994	.9001
1.6876	.9500	.9500	.9493	.9501
2.0271	.9750	.9750	.9744	.9751
2.4328	.9900	.9900	.9895	.9900
3.3289	.9990	.9990	.9988	.9990

A-TRUE=.500 A-LOW=.363 A-HIGH=.637

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3042	.8999	.9000	.8997	.8997
1.6860	.9500	.9500	.9497	.9497
2.0244	.9750	.9750	.9747	.9747
2.4286	.9900	.9900	.9898	.9898
3.3193	.9990	.9990	.9989	.9989

N1=30 N2=30

A-TRUE=.010 A-LOW=.006 A-HIGH=.016

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3108	.8999	.9000	.9000	.9001
1.6980	.9500	.9500	.9500	.9501
2.0435	.9750	.9750	.9749	.9750
2.4592	.9900	.9900	.9899	.9900
3.3898	.9990	.9990	.9989	.9990

A-TRUE=.050 A-LOW=.032 A-HIGH=.076

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3085	.8999	.9000	.8998	.9002
1.6938	.9500	.9500	.9498	.9502
2.0368	.9750	.9750	.9748	.9752
2.4485	.9900	.9900	.9899	.9902
3.3651	.9990	.9990	.9990	.9990

A-TRUE=.100 A-LOW=.066 A-HIGH=.149

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3060	.8999	.8983	.8980	.8986
1.6891	.9500	.9483	.9480	.9487
2.0302	.9751	.9732	.9729	.9735
2.4384	.9901	.9882	.9880	.9884
3.3534	.9991	.9972	.9972	.9973

A-TRUE=.200 A-LOW=.137 A-HIGH=.282

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3017	.8999	.9000	.8996	.9004
1.6814	.9500	.9500	.9496	.9505
2.0172	.9750	.9750	.9745	.9754
2.4170	.9900	.9900	.9897	.9903
3.2927	.9990	.9990	.9989	.9991

A-TRUE=.300 A-LOW=.214 A-HIGH=.402

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.2987	.8999	.9000	.8996	.9003
1.6759	.9500	.9500	.9496	.9503
2.0106	.9751	.9751	.9747	.9754
2.4078	.9901	.9901	.9898	.9903
3.2992	.9991	.9991	.9990	.9992

A-TRUE=.400 A-LOW=.298 A-HIGH=.512

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.2969	.8999	.9000	.8997	.9001
1.6726	.9500	.9500	.9497	.9501
2.0054	.9751	.9751	.9748	.9752
2.3995	.9901	.9901	.9899	.9901
3.2802	.9991	.9991	.9990	.9991

A-TRUE=.500 A-LOW=.389 A-HIGH=.611

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.2963	.8999	.9000	.8999	.8999
1.6716	.9500	.9500	.9499	.9499
2.0017	.9750	.9750	.9749	.9749
2.3924	.9900	.9900	.9899	.9899
3.2368	.9990	.9990	.9990	.9990

TABLE XIII. - COMPARISON OF WELCH APPROXIMATION $[G(x),\ G_{LOW}(x),\ G_{HIGH}(x)]$

WITH TRUE VALUE [F(x)] - Continued

N1=50 N2=50

A-TRUE=,010 A-LOW=,007 A-HIGH=,014

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,2987	,8999	,9000	,9000	,9000
1,6759	,9499	,9500	,9500	,9500
2,0094	,9750	,9749	,9749	,9750
2,4051	,9900	,9899	,9899	,9899
3,2765	,9990	,9989	,9989	,9989

A-TRUE=,050 A-LOW=,036 A-HIGH=,069

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,2974	,8999	,9000	,8999	,9001
1,6735	,9500	,9500	,9499	,9501
2,0048	,9750	,9750	,9749	,9751
2,3972	,9900	,9900	,9899	,9901
3,2477	,9990	,9990	,9990	,9990

A-TRUE=,100 A-LOW=,073 A-HIGH=,136

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,2959	,8999	,9000	,8999	,9002
1,6708	,9500	,9500	,9499	,9502
2,0005	,9750	,9750	,9749	,9752
2,3904	,9900	,9900	,9899	,9901
3,2322	,9990	,9990	,9990	,9990

A-TRUE=,200 A-LOW=,150 A-HIGH=,261

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,2934	,8999	,9000	,8998	,9002
1,6663	,9500	,9500	,9498	,9502
1,9934	,9750	,9750	,9748	,9752
2,3792	,9900	,9900	,9898	,9901
3,2072	,9990	,9990	,9989	,9990

A-TRUE=,300 A-LOW=,233 A-HIGH=,377

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,2917	,8997	,9000	,8998	,9002
1,6631	,9499	,9500	,9498	,9502
1,9884	,9750	,9750	,9748	,9751
2,3713	,9900	,9900	,9899	,9901
3,1895	,9990	,9990	,9990	,9990

A-TRUE=,400 A-LOW=,321 A-HIGH=,485

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,2906	,8991	,9000	,8999	,9001
1,6633	,9498	,9502	,9501	,9503
1,9893	,9751	,9752	,9751	,9753
2,3750	,9902	,9902	,9901	,9903
3,2479	,9992	,9992	,9992	,9992

A-TRUE=,500 A-LOW=,414 A-HIGH=,586

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1,2903	,8986	,9000	,9000	,9000
1,6606	,9494	,9500	,9500	,9500
1,9845	,9747	,9750	,9750	,9750
2,3650	,9899	,9900	,9900	,9900
3,1756	,9990	,9990	,9990	,9990

N1= 6 N2= 2

A-TRUE=.010 A-LOW=.000 A-HIGH=.366

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
3.0130	.9085	.9000	.8960	.9608
6.1052	.9633	.9500	.9484	.9912
12.1245	.9902	.9750	.9738	.9982
29.8228	.9995	.9900	.9893	.9998
365.6413	1.0000	.9991	.9990	1.0000

A-TRUE=.050 A-LOW=.001 A-HIGH=.751

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.7783	.9216	.9000	.8902	.9831
5.3696	.9773	.9497	.9412	.9988
10.1310	.9967	.9747	.9685	.9998
23.3322	.9999	.9897	.9861	.9998
193.6779	1.0000	.9984	.9977	.9998

A-TRUE=.100 A-LOW=.002 A-HIGH=.864

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.5316	.9266	.9000	.8806	.9776
4.6350	.9805	.9497	.9324	.9980
8.2856	.9973	.9748	.9618	.9997
17.6370	.9999	.9898	.9819	.9998
128.2795	1.0000	.9988	.9973	.9998

A-TRUE=.200 A-LOW=.004 A-HIGH=.935

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.1560	.9257	.9000	.8627	.9612
3.6038	.9781	.9500	.9147	.9936
5.7847	.9958	.9749	.9462	.9992
10.5909	.9998	.9899	.9705	.9999
53.2457	1.0000	.9989	.9938	1.0000

A-TRUE=.300 A-LOW=.007 A-HIGH=.961

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.9287	.9225	.9022	.8492	.9462
3.0302	.9744	.9522	.9000	.9867
4.5840	.9939	.9771	.9330	.9975
7.9220	.9995	.9919	.9611	.9998
49.7630	1.0000	.9998	.9940	1.0000

A-TRUE=.400 A-LOW=.012 A-HIGH=.974

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.7187	.9137	.9000	.8344	.9282
2.5322	.9657	.9497	.8824	.9747
3.5240	.9882	.9747	.9140	.9920
5.2460	.9977	.9897	.9417	.9982
13.3214	1.0000	.9987	.9771	.9997

A-TRUE=.500 A-LOW=.017 A-HIGH=.983

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5945	.9081	.9000	.8248	.9151
2.2605	.9594	.9500	.8708	.9642
3.0097	.9834	.9750	.9013	.9857
4.1968	.9955	.9900	.9288	.9960
8.8375	.9999	.9990	.9666	.9999

TABLE XIII. - COMPARISON OF WELCH APPROXIMATION $[G(x),\ G_{LOW}(x),\ G_{HIGH}(x)]$

WITH TRUE VALUE [F(x)] - Continued

N1= 2 N2= 6

A-TRUE=.010 A-LOW=.002 A-HIGH=.039

Y	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4716	.9001	.9000	.8996	.9014
2.0065	.9502	.9499	.9495	.9513
2.5557	.9752	.9749	.9746	.9761
3.3377	.9902	.9899	.9897	.9907
5.8081	.9991	.9988	.9988	.9990

A-TRUE=.050 A-LOW=.013 A-HIGH=.176

Y	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4573	.9033	.9000	.8983	.9023
1.9778	.9504	.9500	.9483	.9523
2.5099	.9755	.9751	.9737	.9770
3.2587	.9905	.9901	.9892	.9914
5.6672	.9992	.9991	.9989	.9993

A-TRUE=.100 A-LOW=.027 A-HIGH=.311

Y	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4454	.9002	.9000	.8974	.8972
1.9544	.9502	.9500	.9474	.9472
2.4660	.9752	.9750	.9727	.9726
3.1765	.9902	.9900	.9885	.9883
5.3102	.9991	.9990	.9986	.9986

A-TRUE=.200 A-LOW=.058 A-HIGH=.504

Y	F(X)	G(X)	G-LOW(Y)	G-HIGH(X)
1.4412	.9001	.9000	.8982	.8814
1.9460	.9501	.9500	.9481	.9308
2.4517	.9751	.9750	.9734	.9581
3.1510	.9901	.9900	.9890	.9775
5.2334	.9990	.9990	.9987	.9947

A-TRUE=.300 A-LOW=.095 A-HIGH=.635

X	F(X)	G(X)	G-LOW(Y)	G-HIGH(X)
1.4627	.9011	.9000	.9022	.8679
1.9887	.9514	.9500	.9522	.9166
2.5249	.9763	.9750	.9768	.9450
3.2820	.9910	.9900	.9912	.9668
5.6334	.9993	.9990	.9992	.9894

A-TRUE=.400 A-LOW=.141 A-HIGH=.730

Y	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5121	.9038	.9000	.9093	.8596
2.0885	.9545	.9500	.9597	.9080
2.6994	.9792	.9750	.9821	.9372
3.6020	.9938	.9900	.9943	.9605
6.7366	.9996	.9990	.9997	.9865

A-TRUE=.500 A-LOW=.198 A-HIGH=.802

X	F(X)	G(X)	G-LOW(Y)	G-HIGH(X)
1.5945	.9081	.9000	.9189	.8564
2.2653	.9597	.9502	.9678	.9055
3.0207	.9836	.9752	.9882	.9355
4.2321	.9956	.9902	.9972	.9600
9.6056	.9999	.9992	1.0000	.9882

N1=10 N2= 2

A-TRUE=.010 A-LOW=.000 A-HIGH=.377

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
3.0130	.9087	.9000	.8980	.9636
6.1051	.9638	.9500	.9483	.9925
12.1242	.9908	.9750	.9738	.9986
29.8218	.9998	.9900	.9893	.9998
285.0695	1.0000	.9990	.9989	1.0000

A-TRUE=.050 A-LOW=.001 A-HIGH=.759

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.7777	.9225	.9000	.8902	.9883
5.3676	.9788	.9499	.9414	.9996
10.1504	.9977	.9749	.9687	.9999
23.3578	1.0000	.9899	.9863	.9999
197.0376	1.0000	.9989	.9983	.9999

A-TRUE=.100 A-LOW=.002 A-HIGH=.869

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.5294	.9282	.9000	.8805	.9849
4.6381	.9827	.9501	.9327	.9995
8.3036	.9984	.9753	.9621	1.0000
17.8046	1.0000	.9903	.9823	1.0000
143.0244	1.0000	.9993	.9978	1.0000

A-TRUE=.200 A-LOW=.004 A-HIGH=.937

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.1508	.9281	.9000	.8623	.9713
3.5850	.9809	.9500	.9142	.9975
5.7422	.9973	.9750	.9458	.9999
10.4809	.9999	.9900	.9703	1.0000
46.4778	1.0000	.9990	.9934	1.0000

A-TRUE=.300 A-LOW=.007 A-HIGH=.963

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.8856	.9228	.9000	.8461	.9550
2.9200	.9755	.9500	.8964	.9920
4.3027	.9947	.9750	.9287	.9991
6.9636	.9997	.9900	.9557	1.0000
23.9768	1.0000	.9991	.9873	1.0000

A-TRUE=.400 A-LOW=.011 A-HIGH=.976

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.6985	.9163	.9000	.8326	.9389
2.4870	.9687	.9500	.8805	.9832
3.4364	.9907	.9750	.9120	.9965
5.0623	.9988	.9900	.9399	.9997
12.7006	1.0000	.9990	.9763	1.0000

A-TRUE=.500 A-LOW=.016 A-HIGH=.984

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5668	.9104	.9000	.8221	.9247
2.2063	.9624	.9502	.8678	.9730
2.9126	.9860	.9752	.8980	.9916
4.0309	.9971	.9903	.9257	.9986
8.8005	1.0000	.9992	.9663	1.0000

TABLE XIII. - COMPARISON OF WELCH APPROXIMATION $[G(x),\ G_{LOW}(x),\ G_{HIGH}(x)]$

WITH TRUE VALUE $[F(x)]$ - *Continued*

N1= 2 N2=10

A-TRUE=.010 A-LOW=.003 A-HIGH=.033

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3610	.9000	.9000	.8998	.9006
1.8292	.9500	.9500	.9498	.9506
2.2556	.9751	.9750	.9748	.9755
2.8104	.9900	.9900	.9898	.9903
4.3209	.9991	.9990	.9990	.9991

A-TRUE=.050 A-LOW=.015 A-HIGH=.150

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3749	.9000	.8993	.8985	.8993
1.8176	.9501	.9493	.9485	.9493
2.2366	.9751	.9743	.9736	.9743
2.7780	.9901	.9893	.9888	.9893
4.2017	.9991	.9983	.9982	.9983

A-TRUE=.100 A-LOW=.032 A-HIGH=.272

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3722	.9000	.9000	.8993	.8953
1.8125	.9500	.9500	.9492	.9451
2.2281	.9750	.9750	.9743	.9707
2.7638	.9900	.9900	.9895	.9870
4.1936	.9991	.9991	.9990	.9983

A-TRUE=.200 A-LOW=.069 A-HIGH=.457

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3830	.9005	.9000	.9015	.8818
1.8359	.9508	.9502	.9517	.9311
2.2676	.9758	.9752	.9765	.9582
2.8351	.9907	.9902	.9911	.9774
4.5033	.9994	.9993	.9994	.9950

A-TRUE=.300 A-LOW=.113 A-HIGH=.590

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4166	.9022	.9000	.9065	.8699
1.8978	.9527	.9500	.9565	.9184
2.3699	.9776	.9750	.9803	.9463
3.0072	.9920	.9900	.9934	.9676
4.8123	.9995	.9990	.9996	.9893

A-TRUE=.400 A-LOW=.166 A-HIGH=.601

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4759	.9055	.9000	.9139	.8624
2.0187	.9569	.9502	.9638	.9110
2.5782	.9814	.9752	.9858	.9398
3.3864	.9946	.9902	.9963	.9626
6.1996	.9999	.9992	.9999	.9882

A-TRUE=.500 A-LOW=.229 A-HIGH=.771

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5668	.9104	.9000	.9231	.8595
2.2063	.9624	.9501	.9715	.9085
2.9177	.9861	.9753	.9908	.9383
4.0355	.9971	.9903	.9982	.9622
8.8808	1.0000	.9991	.9999	.9891

N1=15 N2= 2

A-TRUE=.010 A-LOW=.000 A-HIGH=.381

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
3.0130	.9088	.9000	.8980	.9648
6.1050	.9640	.9500	.9483	.9930
12.1241	.9912	.9750	.9738	.9987
29.8215	.9998	.9900	.9893	.9999
285.0781	1.0000	.9990	.9989	1.0000

A-TRUE=.050 A-LOW=.001 A-HIGH=.762

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.7774	.9230	.9000	.8902	.9904
5.3986	.9798	.9503	.9419	.9999
10.2685	.9983	.9754	.9692	1.0000
24.0812	1.0000	.9903	.9868	1.0000
277.0766	1.0000	.9989	.9984	1.0000

A-TRUE=.100 A-LOW=.002 A-HIGH=.871

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.5284	.9290	.9000	.8805	.9880
4.6256	.9836	.9500	.9326	.9998
8.2173	.9988	.9750	.9617	1.0000
17.3556	1.0000	.9900	.9819	1.0000
112.4585	1.0000	.9989	.9971	1.0000

A-TRUE=.200 A-LOW=.004 A-HIGH=.938

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
2.1476	.9292	.9000	.8621	.9757
3.5766	.9822	.9500	.9140	.9986
5.7233	.9980	.9750	.9456	1.0000
10.4322	1.0000	.9900	.9701	1.0000
46.0955	1.0000	.9990	.9932	1.0000

A-TRUE=.300 A-LOW=.007 A-HIGH=.963

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.9086	.9265	.9022	.8477	.9620
2.9609	.9789	.9519	.8981	.9950
4.4776	.9967	.9768	.9311	.9995
7.6568	.9999	.9916	.9595	.9998
46.2219	1.0000	.9995	.9932	.9998

A-TRUE=.400 A-LOW=.011 A-HIGH=.976

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.6896	.9176	.9000	.8319	.9438
2.4732	.9705	.9502	.8799	.9869
3.4129	.9920	.9752	.9114	.9980
5.0399	.9992	.9902	.9395	.9999
13.4082	1.0000	.9992	.9775	1.0000

A-TRUE=.500 A-LOW=.016 A-HIGH=.984

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5546	.9115	.9000	.8209	.9291
2.1764	.9635	.9500	.8661	.9767
2.8564	.9871	.9750	.8960	.9938
3.8991	.9975	.9900	.9232	.9992
7.9329	1.0000	.9991	.9624	1.0000

TABLE XIII. - COMPARISON OF WELCH APPROXIMATION [G(x), $G_{LOW}(x)$, $G_{HIGH}(x)$] WITH TRUE VALUE [F(x)] - Continued

N1= 2 N2=15

A-TRUE=.010 A-LOW=.003 A-HIGH=.030

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3438	.9000	.9000	.8999	.9003
1.7590	.9500	.9500	.9499	.9503
2.1411	.9750	.9750	.9749	.9753
2.6184	.9900	.9900	.9899	.9902
3.7724	.9990	.9990	.9990	.9990

A-TRUE=.050 A-LOW=.017 A-HIGH=.140

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3409	.9000	.9000	.8997	.8992
1.7536	.9500	.9500	.9496	.9492
2.1323	.9750	.9750	.9746	.9742
2.6039	.9900	.9900	.9897	.9894
3.7365	.9990	.9990	.9989	.9988

A-TRUE=.100 A-LOW=.035 A-HIGH=.256

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3417	.9000	.9000	.9000	.8946
1.7551	.9500	.9500	.9500	.9443
2.1348	.9750	.9750	.9750	.9700
2.6080	.9900	.9900	.9900	.9864
3.7466	.9990	.9990	.9990	.9979

A-TRUE=.200 A-LOW=.075 A-HIGH=.437

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3585	.9008	.9000	.9028	.8820
1.7865	.9510	.9500	.9529	.9310
2.1857	.9760	.9750	.9774	.9579
2.6926	.9908	.9900	.9916	.9769
3.9594	.9992	.9990	.9994	.9939

A-TRUE=.300 A-LOW=.121 A-HIGH=.571

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3968	.9028	.9000	.9081	.8709
1.8596	.9535	.9500	.9582	.9193
2.3060	.9784	.9750	.9817	.9471
2.8965	.9926	.9900	.9942	.9680
4.5011	.9996	.9990	.9998	.9894

A-TRUE=.400 A-LOW=.177 A-HIGH=.674

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4602	.9064	.9000	.9156	.8637
1.9837	.9578	.9500	.9652	.9120
2.5200	.9824	.9751	.9869	.9408
3.2769	.9952	.9901	.9969	.9632
5.7675	.9999	.9991	1.0000	.9881

A-TRUE=.500 A-LOW=.244 A-HIGH=.756

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5546	.9115	.9000	.9244	.8609
2.1764	.9635	.9500	.9727	.9098
2.8564	.9871	.9750	.9914	.9392
3.8991	.9975	.9900	.9985	.9626
7.9329	1.0000	.9991	1.0000	.9884

N1=10 N2= 5

A-TRUE=.010 A-LOW=.003 A-HIGH=.038

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5273	.9003	.9000	.8995	.9019
2.1198	.9504	.9500	.9495	.9519
2.7549	.9754	.9750	.9746	.9765
3.7060	.9904	.9900	.9897	.9910
7.0318	.9992	.9990	.9989	.9992

A-TRUE=.050 A-LOW=.013 A-HIGH=.172

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.5051	.9011	.9000	.8975	.9073
2.0742	.9514	.9500	.9475	.9572
2.6740	.9765	.9750	.9730	.9807
3.5549	.9914	.9900	.9887	.9935
6.5156	.9995	.9990	.9987	.9996

A-TRUE=.100 A-LOW=.027 A-HIGH=.304

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4799	.9016	.9000	.8954	.9105
2.0231	.9520	.9500	.9453	.9603
2.5846	.9772	.9750	.9711	.9831
3.3903	.9919	.9900	.9874	.9949
5.9759	.9996	.9990	.9984	.9998

A-TRUE=.200 A-LOW=.060 A-HIGH=.496

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4374	.9017	.8993	.8912	.9101
1.9386	.9522	.9493	.9411	.9599
2.4425	.9774	.9744	.9674	.9829
3.1379	.9920	.9894	.9845	.9945
5.2857	.9996	.9984	.9971	.9991

A-TRUE=.300 A-LOW=.098 A-HIGH=.628

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.4094	.9020	.9007	.8905	.9087
1.8837	.9524	.9507	.9402	.9587
2.3495	.9774	.9756	.9665	.9821
2.9876	.9921	.9906	.9842	.9945
5.1975	.9998	.9994	.9980	.9998

A-TRUE=.400 A-LOW=.145 A-HIGH=.724

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3798	.9008	.9000	.8890	.9045
1.8269	.9510	.9500	.9385	.9546
2.2519	.9760	.9750	.9649	.9788
2.8040	.9909	.9900	.9826	.9925
4.2714	.9993	.9990	.9966	.9995

A-TRUE=.500 A-LOW=.202 A-HIGH=.798

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3628	.9003	.9000	.8895	.9014
1.7948	.9504	.9500	.9390	.9514
2.1991	.9755	.9749	.9652	.9761
2.7150	.9904	.9899	.9828	.9907
4.0170	.9991	.9989	.9965	.9991

TABLE XIII. - COMPARISON OF WELCH APPROXIMATION $\left[G(x),\ G_{LOW}(x),\ G_{HIGH}(x)\right]$

WITH TRUE VALUE [F(x)] - Concluded

N1= 5 N2=10

A-TRUE=.010 A-LOW=.004 A-HIGH=.026

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3809	.9000	.8982	.8980	.8987
1.8290	.9501	.9482	.9480	.9487
2.2554	.9751	.9732	.9730	.9736
2.8100	.9901	.9882	.9881	.9885
4.2663	.9990	.9971	.9971	.9972

A-TRUE=.050 A-LOW=.019 A-HIGH=.124

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3731	.9001	.9000	.8991	.9017
1.8141	.9502	.9500	.9491	.9518
2.2309	.9752	.9750	.9742	.9765
2.7684	.9902	.9900	.9894	.9910
4.1562	.9991	.9989	.9988	.9992

A-TRUE=.100 A-LOW=.040 A-HIGH=.230

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3650	.9002	.9000	.8985	.9020
1.7988	.9502	.9500	.9484	.9520
2.2057	.9752	.9750	.9737	.9767
2.7260	.9902	.9900	.9891	.9912
4.0455	.9991	.9990	.9987	.9992

A-TRUE=.200 A-LOW=.085 A-HIGH=.402

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3541	.9001	.9000	.8980	.9001
1.7783	.9501	.9500	.9479	.9501
2.1724	.9751	.9750	.9732	.9751
2.6703	.9901	.9900	.9887	.9900
3.9027	.9990	.9990	.9986	.9990

A-TRUE=.300 A-LOW=.137 A-HIGH=.536

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3502	.9000	.9000	.8985	.8973
1.7710	.9500	.9500	.9484	.9472
2.1604	.9750	.9750	.9736	.9725
2.6504	.9900	.9900	.9890	.9883
3.8525	.9990	.9990	.9987	.9985

A-TRUE=.400 A-LOW=.199 A-HIGH=.642

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3531	.9001	.9000	.8998	.8946
1.7764	.9501	.9500	.9498	.9444
2.1692	.9751	.9750	.9748	.9701
2.6649	.9901	.9900	.9899	.9865
3.8892	.9990	.9990	.9990	.9980

A-TRUE=.500 A-LOW=.271 A-HIGH=.729

X	F(X)	G(X)	G-LOW(X)	G-HIGH(X)
1.3628	.9003	.9000	.9019	.8927
1.7948	.9504	.9500	.9519	.9424
2.1991	.9755	.9750	.9767	.9683
2.7150	.9904	.9900	.9911	.9852
4.0170	.9991	.9990	.9993	.9975

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