~~Schardt~~ SG

National Aeronautics and Space Administration
Goddard Space Flight Center
Contract NAS-5-12487

ST-OA-CM-10712

N.L.

DETERMINATION OF THE PHOTOGRAPHIC POSITION OF AN OBJECT WITH THE AID OF TWO AND THREE REFERENCE STARS

by

A. N. Deych

22 May 1968

ST-OA-CM-10712

DETERMINATION OF THE PHOTOGRAPHIC POSITION OF AN OBJECT WITH THE AID OF TWO AND THREE REFERENCE STARS

Astronomicheskiy Zhurnal
Tom 25, No.2, pp 44-58
Moskva, 1948.

By

A.N. Deych **

SUMMARY

In this article formulas (3) are derived for determining the position of a celestial object on the basis of two reference stars given by equatorial coordinates. In this connection, second-order terms are taken into account which are derived for a 5° x 5° field with an error not greater than 1 sec. Analysis of third-order terms shows that such terms can be discarded for the same limitations and for declinations not greater than 60°. If the object being determined is located close to the optical center the precision of formulas (3) is increased tenfold.

The method described here can be easily extended to the case of three reference stars [formulas (4)] whereupon in this case all terms with the coefficient b_1 vanish, which may significantly increase the precision of determination of object's position. Tables are given at the end of the paper for a rapid accounting of second-order terms.

*

* *

Numerous methods are known for determining the position of a celestial object on a photographic plate with the aid of two reference stars, for example, the methods proposed by Wolf, Reger, Kaiser, Blazhko, Fick, Arend, etc., These methods are based on isolated particular considerations which are not connected with the general theory of photographic determination of celestial coordinates. At the same time, the conditions limiting to a greater or lesser extent the practical application of the method are usually not clearly defined. The simplicity of the formulas and computation approaches also leaves much to be desired. At the same time, a tying to two or three stars is important in determining the position of small planets and comets when high

** Transliteration of "Deutsch".

precision is often not required but when it is important to curtail the time needed for measurements and plate processing. In this connection, we must remember that a rational use of even two reference stars can often allow us to determine the position of an object just as effectively as with classical methods utilizing a large number of stars.

In the well-known Schlesinger method [1] the position of the celestial object is determined with the aid of three reference stars. It may be shown (see, for example, our note [2]) that Schlesinger's method stems from Turner's method of six constants. In 1933, Arend [3] developed the Schlesinger's method of dependences in its application to two reference stars. We will show that Arend's formulas result from the classical method of four constants. We shall further derive more convenient and exact formulas for obtaining α and δ of the object directly from equatorial coordinates of reference stars. We shall calculate the effect exerted by third-order terms in the cases of remote position of the object from the optical center. Finally, we shall extend our method to the case of three reference stars and we shall present a simple method for determining the "dependences".

As is well known, in the four-constant method the ideal coordinates are linked with the measured coordinates by a linear dependence but, at the same time, the coordinates are rectangular and the scale is the same in all directions. Therefore, we must first consider the nonorthogonal terms of differential refraction.

Let us assume that there are two reference stars and that object to be determined is located between these stars. Then, on the basis of the four-constant method we may write:

$$\begin{aligned} ax_1+by_1+c&=X_1, & ay_1-bx_1+d&=Y_1,\\ ax_2+by_2+c&=X_2, & ay_2-bx_2+d&=Y_2,\\ ax_0+by_0+c&=X_0, & ay_0-bx_0+d&=Y_0, \end{aligned} \tag{1}$$

where x_i, y_i are the measured coordinates, X_i, Y_i are the ideal coordinates, and a, b, c, d are the four unknown plate constants. The third line refers to the object to be determined. Let us subtract the first lines from the second lines:

$$\begin{aligned} a(x_2-x_1)+b(y_2-y_1)&=X_2-X_1,\\ a(y_2-y_1)-b(x_2-x_1)&=Y_2-Y_1. \end{aligned} \quad (2$$

After introducing evident denotations, we have :

$$a\Delta x + b\Delta y = \Delta X,$$
$$a\Delta y - b\Delta x = \Delta Y,$$

from which we find

$$a = \frac{\Delta X\,\Delta x + \Delta Y\,\Delta y}{\Delta x^2 + \Delta y^2} \quad \text{и} \quad b = \frac{\Delta X\,\Delta y - \Delta Y\,\Delta x}{\Delta x^2 + \Delta y^2}.$$

Let us now subtract the first line from the third line of Eqs.(1):

$$X_0 - X_1 = a\,(x_0 - x_1) + b\,(y_0 - y_1) = a\Delta x_1 + b\Delta y_1,$$
$$Y_0 - Y_1 = a\,(y_0 - y_1) - b\,(x_0 - x_1) = a\Delta y_1 - b\Delta x_1.$$

Let us substitute into these expressions the found values of a and b and let us factor out ΔX and ΔY:

$$X_0 - X_1 = a_1\,\Delta X + b_1\,\Delta Y,$$
$$Y_0 - Y_1 = a_1\,\Delta Y - b_1\,\Delta X,$$

where

$$a_1 = \frac{\Delta x\,\Delta x_1 + \Delta y\,\Delta y_1}{\Delta x^2 + \Delta y^2}, \tag{2}$$

and

$$b_1 = \frac{\Delta y\,\Delta x_1 - \Delta x\,\Delta y_1}{\Delta x^2 + \Delta y^2},$$

According to Schlesinger coefficients a and b can be called "dependence ". They depend only on the measured coordinates.

With respect to the second reference star we can write analogously

$$X_2 - X_0 = a_2\Delta X + b_2\Delta Y.$$
$$Y_2 - Y_0 = a_2\Delta Y - b_2\Delta X,$$

where

$$a_2 = \frac{\Delta x\,\Delta x_2 + \Delta y\,\Delta y_2}{\Delta x^2 + \Delta y^2},$$

and

$$b_2 = \frac{\Delta y\,\Delta x_2 - \Delta x\,\Delta y_2}{\Delta x^2 + \Delta y^2},$$

whereby $\Delta x_2 = x_2 - x_0$ and $\Delta y_2 = y_2 - y_0$. We can establish once and for all the numbering order of the stars so that $x_2 > x_0 > x_1$. In this case, for the other coordinate we will have either $y_2 > y_0 > y_1$ or $y_2 < y_0 < y_1$. Consequently, a_1 and a_2 are always positive. It may be easily seen that

$$a_1 + a_2 = 1 \quad \text{и} \quad b_1 + b_2 = 0.$$

In practice, a_1 is usually close to 1/2 and b_1 is smaller than 0.1. Coeffjcients a_1 and b_1 (a_2 and b_2) have an important geometric significance which will be utilized later. Arend (loc. cit.) has pointed out that $a_1 = \frac{m}{l}$; $b_1 = \frac{p}{l}$. In Fig.1 the segment $S_2S_1 = l = \sqrt{\Delta x^2 + \Delta y^2}$. Points S_1 and S_2 are the reference stars.

Point A represents the object to be determined. Line segment $AB = p \perp S_1S_2$, and line segment $BS_1 = m$.

To provide demonstration let us represent

$$a_1 = \frac{1}{l}\left(\Delta x_1 \frac{\Delta x}{l} + \Delta y_1 \frac{\Delta y}{l}\right) \quad \text{и} \quad b_1 = \frac{1}{l}\left(\Delta x_1 \frac{\Delta y}{l} - \Delta y_1 \frac{\Delta x}{l}\right).$$

The ratios $\frac{\Delta x}{l}$ and $\frac{\Delta y}{l}$ are the cosine and sine of the angle formed by the straight line S_1S_2 with the axis $\underline{x}$. The expressions in parentheses are therefore the projections of Δx_1 and Δy_1 onto the straight line S_1S_2 and onto the line AB perpendicular to the former. The sum of these projections yields respectively the line segments $BS_1 = m$ and $AB = p$. By making use of this geometric interpretation, the calculation of coefficients a_1 and b_1 can be greatly simplified by orienting the plate in the measuring instruments in such a way that the straight line S_1S_2 coincide with the axis $\underline{x}$. Then, after measuring the coordinates of points S_1, A and S_2 we shall obtain the differences $\underline{m}$ and $\underline{l}$ and their ratio a_1. The line segment AB can be measured by means of a perpendicular screw, if one is available, or plate rotation by 90°. In order that the error of the coordinate be smaller than 0.1 min, it is necessary to obtain in the measuring instrument two mutually perpendicular directions with an error not greater than 1 min. Indeed, we have seen that $a = \frac{m}{l}$, whence $da = \frac{dm}{l}$. However, $dm = p \operatorname{tg} i$, where $\underline{i}$ is the slope angle of the axes. If we postulate $l = 1° = 60'$ and $b_1 < 0.1$, we have p = 5 min. Assuming i = 1 min, we obtain $dm \approx 0.0015$ min, hence da = 0.000025. Considering that ΔX or ΔY are not greater than 1° = 3600 sec, we will obtain according to formulas (2) an error in the coordinate smaller than 0.1 sec.

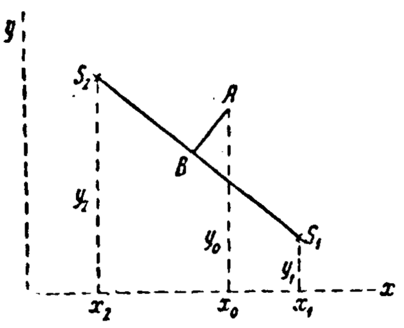

Fig.1.

Formulas (2) are exact to the same extent as formulas (1). If the object is located on the straight line connecting both reference stars, then $b_1 = 0$. The reservation concerning the accounting of nonorthogonal terms of refraction can then be disregarded, and the accuracy of formulas (2) is just as good as the one obtained by the six-constant method. In other words in this case the third star becomes superfluous, of which one can be easily convinced by applying the Schlesinger's method of "dependences".

In the general case, the course of our problem's solution is as follows.

Knowing the optical center of the plate, we calculate by the equatorial coordinates of the reference stars the ideal coordinates of stars X_i and Y_i. According to the measured coordinates of stars and of the object, we calculate the coefficients a_1 and b_1. We obtain the ideal coordinates of the object X_0, Y_0 from formulas (2), and from these coordinates we obtain the equatorial coordinates α_0 and δ_0. When utilizing the photographic catalogs of the "Sky Chart", it is possible to apply formulas (2) directly to the rectangular coordinates of reference stars listed in these catalogs. In this case the choice of stars is considerably increased since the positions of stars up to the 12th visible magnitude are given in the photographic catalogs. We shall thus obtain the rectangular coordinates of the object in the "Sky Chart" plate system, and afterwards we shall calculate the ideal coordinates with the aid of constants of this plate and, finally, the equatorial coordinates of the object sought for. At the same time we should remember that an additional source of errors arises from the non coincidence of the optical centers of our plate and of the "Sky Chart" plate. Schlesinger (loc.cit, p 78) notes that for a difference of 1° between the optical centers of the plates the error in the position of the object will be less than 0.5 sec.

However, the field covered in the sky can be much larger than the area of the plate of a normal astrograph (4 square degrees). For example, the plate of a zonal astrograph covers an area of 25 square degrees. The object may be located far away from the optical center and then the suitable plate of the "Sky Chart" with reference stars will have an inclination to our plate greater than 1°, which may lead to an error in the position of the object greater than 1 sec. It should also be noted that the use of photographic catalogs generally is a rather complex matter in view of diversified forms of publication of such catalogs by different observatories. In addition, so far the system of stellar positions has not been defined in these catalogs.

For this reason, we believe that it is important to derive formulas which will make it possible to obtain α_0 and δ_0 of the object directly from the equatorial coordinates of reference stars. For this purpose expansions in series are used, in which in order that the problem does not lose its practical importance it makes no sense to utilize terms above the second order. Further in this article we shall list those limiting conditions which permit us to discard third-order terms while still retaining the required precision.

Let us use the formulas proposed by A. König [4] for the expansion in series up to and including third-order terms:

$$X = (\alpha - A)\cos D - (\alpha - A)(\delta - D)\sin D + \frac{1}{6}(\alpha - A)^3\cos D\,(3\cos^2 D - 1),$$

$$Y = (\delta - D) + \frac{1}{2}(\alpha - A)^2\sin D\cos D + \frac{1}{2}(\alpha - A)^2(\delta - D)\cos 2D + \frac{1}{3}(\delta - D)^3.$$

Let us substitute these expressions into formulas (2) instead of coordinates X_0, X_1, X_2 and Y_0, Y_1, Y_2:

$$X_0 - X_1 = a_1(X_2 - X_1) + b_1(Y_2 - Y_1),$$
$$Y_0 - Y_1 = a_1(Y_2 - Y_1) - b_1(X_2 - X_1).$$

Then, we obtain

$$\begin{aligned}
\alpha_0 - \alpha_1 = {} & a_1(\alpha_2 - \alpha_1) + b_1(\delta_2 - \delta_1)\sec D + \\
& + [(\alpha_0 - A)(\delta_0 - D) - (\alpha_1 - A)(\delta_1 - D) - a_1(\alpha_2 - A)(\delta_2 - D) + \\
& + a_1(\alpha_1 - A)(\delta_1 - D)]\,\mathrm{tg}\, D + \frac{b_1}{2}[(\alpha_2 - A)^2 - (\alpha_1 - A)^2]\sin D + \\
& + \frac{1}{6}[a_1(\alpha_2 - A)^3 - a_1(\alpha_1 - A)^3 - (\alpha_0 - A)^3 + (\alpha_1 - A)^3](3\cos^2 D - 1) + \\
& + \frac{b_1}{2}[(\alpha_2 - A)^2(\delta_2 - D) - (\alpha_1 - A)^2(\delta_1 - D)]\cos 2D\sec D + \\
& + \frac{b_1}{3}[(\delta_2 - D)^3 - (\delta_1 - D)^3]\sec D;
\end{aligned}$$

$$\begin{aligned}
\delta_0 - \delta_1 = {} & a_1(\delta_2 - \delta_1) - b_1(\alpha_2 - \alpha_1)\cos D + \\
& + \frac{1}{2}[a_1(\alpha_2 - A)^2 - a_1(\alpha_1 - A)^2 - (\alpha_0 - A)^2 + (\alpha_1 - A)^2]\sin D\cos D + \\
& + b_1[(\alpha_2 - A)(\delta_2 - D) - (\alpha_1 - A)(\delta_1 - D)]\sin D + \\
& + \frac{1}{2}[a_1(\alpha_2 - A)^2(\delta_2 - D) - a_1(\alpha_1 - A)^2(\delta_1 - D) - \\
& - (\alpha_0 - A)^2(\delta_0 - D) + (\alpha_1 - A)^2(\delta_1 - D)]\cos 2D + \\
& + \frac{1}{3}[a_1(\delta_2 - D)^3 - a_1(\delta_1 - D)^3 - (\delta_0 - D)^3 + (\delta_1 - D)^3] - \\
& - \frac{b_1}{6}[(\alpha_2 - A)^3 - (\alpha_1 - A)^3]\cos D\,(3\cos^2 D - 1).
\end{aligned}$$

In these formulas, the differences between right ascensions and declinations are expressed in radians.

Let us start with an estimate of third-order terms in the first equation. The bracket in the first such term with the coefficient $\frac{1}{6}$ $(3\cos^2 D - 1)$ can be represented as follows (remembering that $a_1 + a_2 = 1$):

$$\begin{aligned}
& [a_1(\alpha_2 - A)^3 + a_2(\alpha_1 - A)^3 - (\alpha_0 - A)^3] = \\
& = a_1[(\alpha_2 - A)^3 - (\alpha_0 - A)^3] + a_2[(\alpha_1 - A)^3 - (\alpha_0 - A)^3] = \\
& = a_1(\alpha_2 - \alpha_0)[(\alpha_2 - A)^2 + (\alpha_2 - A)(\alpha_0 - A) + (\alpha_0 - A)^2] - \\
& - a_2(\alpha_0 - \alpha_1)[(\alpha_1 - A)^2 + (\alpha_1 - A)(\alpha_0 - A) + (\alpha_0 - A)^2]
\end{aligned}$$

Let us substitute here:

$$\alpha_2-\alpha_0=a_2(\alpha_2-\alpha_1)+b_2(\delta_2-\delta_1)\sec D,$$
$$\alpha_0-\alpha_1=a_1(\alpha_2-\alpha_1)+b_1(\delta_2-\delta_1)\sec D.$$

Then, we obtain:

$$\begin{aligned}&a_1a_2(\alpha_2-\alpha_1)[(\alpha_2-A)^2-(\alpha_1-A)^2+(\alpha_0-A)(\alpha_2-\alpha_1)]-\\&-b_1(\delta_2-\delta_1)\sec D[a_1(\alpha_2-A)^2-a_1(\alpha_1-A)^2+a_1(\alpha_2-A)(\alpha_0-A)+\\&+a_2(\alpha_1-A)(\alpha_0-A)+(\alpha_1-A)^2+(\alpha_0-A)^2]=\\&=a_1a_2(\alpha_2-\alpha_1)[(\alpha_2-\alpha_1)(\alpha_2+\alpha_1-2A)+(\alpha_0-A)(\alpha_2-\alpha_1)]-\\&-b_1(\delta_2-\delta_1)\sec D[a_1(\alpha_2-\alpha_1)(\alpha_2+\alpha_1-2A)+(\alpha_0-A)(a_1\alpha_2+a_2\alpha_1-A)+\\&+(\alpha_1-A)^2+(\alpha_0-A)^2].\end{aligned}$$

Disregarding the term with b_1^2, we may postulate $a_1\alpha_2+a_2\alpha_1=\alpha_0 = \alpha_0$. Then:

$$\begin{aligned}&a_1a_2(\alpha_2-\alpha_1)^2(\alpha_2+\alpha_1+\alpha_0-3A)-b_1(\delta_2-\delta_1)\sec D[a_1(\alpha_2-\alpha_1)\times\\&\times(\alpha_2+\alpha_1-2A)+2(\alpha_0-A)^2+(\alpha_1-A)^2]=\\&=3a_1a_2(\alpha_2-\alpha_1)^2\left(\frac{\alpha_2+\alpha_1+\alpha_0}{3}-A\right)-\\&-b_1(\delta_2-\delta_1)\sec D\left[(\alpha_0-\alpha_1)2\left(\frac{\alpha_2+\alpha_1}{2}-A\right)+2(\alpha_0-A)^2+(\alpha_1-A)^2\right].\end{aligned}$$

Assuming in term b_1 approximately $\frac{\alpha_2+\alpha_1}{2}=\alpha_0$, we obtain:

$$3a_1a_2(\alpha_2-\alpha_1)^2\left(\frac{\alpha_2+\alpha_1+\alpha_0}{3}-A\right)-b_1(\delta_2-\delta_1)\sec D[2(\alpha_0-A)(\alpha_2-A)+$$
$$+(\alpha_1-A)^2].$$

For the numerical tabulation of these terms, taking the coefficient $\frac{1}{6}$ (3 $\cos^2 D$ -1) into account, we can write the following expression:

$$\frac{1}{2}a_1a_2(\alpha_2-\alpha_1)^2(\alpha-A)(3\cos^2 D-1)-$$
$$-\frac{1}{2}b_1(\delta_2-\delta_1)(\alpha-A)^2\sec D(3\cos^2 D-1).$$

By the difference (α-A) we imply the order of the mean value of the differences (α_0 - A), (α_1 - A) and (α_2 - A).

The factor (3 $\cos^2$D -1) varies from 2 to 1 as D varies from 0° to 90°. The difference (α_2 - α_1) is seldom greater than 1° or 3600 sec. If at higher declinations this difference is twice as great, the factor (3 $\cos^2$D -1) then becomes twice as small. The coefficient $\frac{1}{2}$ $a_1a_2 \leqslant \frac{1}{8}$. Thus, if we want the term $\frac{1}{2}a_1a_2(\alpha_2-\alpha_1)^2(\alpha-A)(3\cos^2 D-1)$ to be smaller than 1 sec, it is necessary that

$$\frac{(3600'')^2(\alpha-A)\cdot 2}{8\cdot(206264)^2}<1'',$$

whence (α - A) $\leqslant$ 3°30 min. In case an error < 0.1 sec is allowed, (α - A) must be < 20'.

We shall examine the term with b_1 together with the following third-order term of our expansion. The bracket of this term can be represented in the form:

$$(\alpha_2 - A)^2(\delta_2 - \delta_1 + \delta_1 - D) - (\alpha_1 - A)^2(\delta_1 - D) =$$
$$= [(\alpha_2 - A)^2 - (\alpha_1 - A)^2](\delta_1 - D) + (\alpha_2 - A)^2(\delta_2 - \delta_1) =$$
$$= (\alpha_2 - \alpha_1)(\alpha_2 + \alpha_1 - 2A)(\delta_1 - D) + (\alpha_2 - A)^2(\delta_2 - \delta_1) =$$
$$= 2(\alpha_2 - \alpha_1)\left(\frac{\alpha_2 + \alpha_1}{2} - A\right)(\delta_1 - D) + (\alpha_2 - A)^2(\delta_2 - \delta_1).$$

By substituting we finally obtain:

$$b_1 \cos 2D \sec D (\alpha_2 - \alpha_1)(\alpha_0 - A)(\delta_1 - D) +$$
$$+ \frac{b_1}{2}(2\cos^2 D - 1)\sec D (\alpha_2 - A)^2(\delta_2 - \delta_1).$$

The second term in this expression will be partially reduced with the corresponding term of the previous expansion, so that there remains:

$$-\frac{b_1}{2}(\delta_2 - \delta_1)(\alpha - A)^2 \cos D.$$

The coefficient of the first term $\cos^2 D \sec D$ varies from 1 to 0 as D varies from 0° to 45° and from 0 to - 1 as D varies from 45° to 60°. Assuming $b_1 < 0.1$ we may see that in order that the whole term be < 1 sec $(\alpha_0 - A)$ and $(\delta_1 - D)$ should not be greater than 3°. Declinations > 60° will give a greater error. In regard to the last third-order term in the expansion of $\alpha_0 - \alpha_1$, we can easily see from the tables proposed by Koenig (loc.cit.p 545) for third-order differences between the arc and tangent that for $D \ll 60°$ this term will be < 1 sec when $(\delta - D) < 3°30$ min.

Thus, we may conclude that the influence exerted by third-order term generally will not be greater than 1 sec if the object is not located further than 2°5 from the optical center. The error will not exceed 0.1 sec if the object is close to the optical center. If the reference stars are located close to the object being determined and if, in addition, b_1 is very small, then the influence exerted by third-order terms becomes even lesser.

It may be shown by means of similar transformations of third-order terms in the expansion of $\delta_0 - \delta_1$ that these terms too will not exceed 1 sec on a 5° x 5° plate and 0.1 sec in a 30' x 30' area around the optical center so long as the distance between the reference stars is not greater than 1°, $b_1 < 0.1$ and $D \leq 60°$.

Let us now analyze the second-order terms. Let us rewrite the bracket around the first such term in the expansion of $\alpha_0 - \alpha_1$ as follows:

$$[(\alpha_0-A)(\delta_0-D)-a_1(\alpha_2-A)(\delta_2-D)-a_2(\alpha_1-A)(\delta_1-D)]=$$
$$=[(\alpha_0-A)(\delta_0-D)-a_1(\alpha_2-\alpha_0+\alpha_0-A)(\delta_2-\delta_0+\delta_0-D)-$$
$$-a_2(\alpha_1-\alpha_0+\alpha_0-A)(\delta_1-\delta_0-D)]=a_2[(\alpha_0-\alpha_1)(\delta_1-D)+$$
$$+(\delta_0-\delta_1)(\alpha_0-A)]-a_1[(\alpha_2-\alpha_0)(\delta_2-D)+(\delta_2-\delta_0)(\alpha_0-A)].$$

Let us substitute here:

$$\alpha_0-\alpha_1=a_1(\alpha_2-\alpha_1)+b_1(\delta_2-\delta_1)\sec D,$$
$$\delta_0-\delta_1=a_1(\delta_2-\delta_1)-b_1(\alpha_2-\alpha_1)\cos D,$$
$$\alpha_2-\alpha_0=a_2(\alpha_2-\alpha_1)+b_2(\delta_2-\delta_1)\sec D,$$
$$\delta_2-\delta_0=a_2(\delta_2-\delta_1)-b_2(\alpha_2-\alpha_1)\cos D.$$

Then, we obtain:

$$-a_2a_1(\alpha_2-\alpha_1)(\delta_2-\delta_1)+b_1(\delta_2-\delta_1)\sec D\,(a_2\delta_1+a_1\delta_2-D)-$$
$$-b_1(\alpha_2-\alpha_1)\cos D\,(\alpha_0-A).$$

The second second-order term at sin D can be transformed as follows:

$$\frac{b_1}{2}(\alpha_2-\alpha_1)(\alpha_2+\alpha_1-2A)=b_1(\alpha_2-\alpha_1)\frac{\alpha_2+\alpha_1}{2}-A\Big)\,.$$

If both these second-order terms are combined and the coefficients tgD and sin D are taken into consideration, we obtain:

$$-a_1a_2(\alpha_2-\alpha_1)(\delta_2-\delta_1)\operatorname{tg}D+b_1(\delta_2-\delta_1)[\delta_1+a_1(\delta_2-\delta_1)-D]\sec D\operatorname{tg}D+$$
$$+b_1(\alpha_2-\alpha_1)\left(\frac{\alpha_2+\alpha_1}{2}-\alpha_0\right)\sin D.$$

The first term with b_1 acquires substantial value when the object is located far away from the optical center along the declination and at the same time at large declinations.

The second term with b_1 can be transformed as follows:

$$b_1(\alpha_2-\alpha_1)\left[\frac{(\alpha_2-\alpha_0)-(\alpha_0-\alpha_1)}{2}\right]\sin D=$$
$$=b_1(\alpha_2-\alpha_1)^2\frac{a_2-a_1}{2}\sin D-b_1^2(\alpha_2-\alpha_1)(\delta_2-\delta_1)\operatorname{tg}D.$$

Therefore, this term can be disregarded when a_2 is close to a_1, i.e. when the object is close to the middle point between the reference stars. Let us consider the second-order terms in the expansion of $\delta_0 - \delta_1$. The first such term can be rewritten as follows, omitting for the time being the factor $\frac{1}{2}$ sin D cos D:

$$a_2[(\alpha_1-A)^2-(\alpha_0-A)^2]+a_1[(\alpha_2-A)^2-(\alpha_0-A)^2]=$$
$$=a_2(\alpha_1-\alpha_0)(\alpha_1+\alpha_0-2A)+a_1(\alpha_2-\alpha_0)(\alpha_2+\alpha_0-2A).$$

TABLE I

$$0.25(\alpha_2 - \alpha_1)^s (\delta_2 - \delta_1)'' \sin 1''$$

$\Delta\alpha$ / $\Delta\delta$	20^s	40^s	60^s	80^s	100^s	120^s	140^s	160^s	180^s	200^s	220^s	240^s	260^s	280^s	300^s	320^s	340^s	360^s	$\Delta\alpha$ / $\Delta\delta$
0°2′	$0^s.00$	$0^s.01$	$0^s.01$	$0^s.01$	$0^s.01$	$0^s.02$	$0^s.02$	$0^s.02$	$0^s.03$	$0^s.03$	$0^s.03$	$0^s.04$	$0^s.04$	$0^s.04$	$0^s.04$	$0^s.05$	$0^s.05$	$0^s.05$	0°2′
4	.01	.01	.02	.02	.03	.04	.04	.05	.05	.06	.06	.07	.08	.08	.09	.09	.10	.10	4
6	.01	.02	.03	.04	.04	.05	.06	.07	.08	.09	.10	.10	.11	.12	.13	.14	.15	.16	6
8	.01	.02	.04	.05	.06	.07	.08	.09	.10	.12	.13	.14	.15	.16	.17	.19	.20	.21	8
10	.01	.03	.04	.06	.07	.09	.10	.12	.13	.15	.16	.17	.19	.20	.22	.23	.25	.26	10
12	.02	.04	.05	.07	.09	.10	.12	.14	.16	.17	.19	.21	.23	.24	.26	.28	.30	.31	12
14	.02	.04	.06	.08	.10	.12	.14	.16	.18	.20	.22	.24	.26	.28	.31	.33	.35	.37	14
16	.02	.05	.07	.09	.12	.14	.16	.19	.21	.23	.26	.28	.30	.33	.35	.37	.40	.42	16
18	.03	.05	.08	.10	.13	.16	.18	.21	.24	.26	.29	.31	.34	.37	.39	.42	.44	.47	18
20	.03	.06	.09	.12	.15	.17	.20	.23	.26	.29	.32	.35	.38	.41	.44	.47	.49	.52	20
22	.03	.06	.10	.13	.16	.19	.22	.26	.29	.32	.35	.38	.42	.45	.48	.51	.54	.58	22
24	.04	.07	.10	.14	.17	.21	.24	.28	.31	.35	.38	.42	.45	.49	.52	.56	.59	.63	24
26	.04	.08	.11	.15	.19	.23	.26	.30	.34	.38	.42	.45	.49	.53	.57	.60	.64	.68	26
28	.04	.08	.12	.16	.20	.24	.28	.33	.37	.41	.45	.49	.53	.57	.61	.65	.69	.73	28
30	.04	.09	.13	.17	.22	.26	.31	.35	.39	.44	.48	.52	.57	.61	.65	.70	.74	.79	30
32	.05	.09	.14	.19	.23	.28	.33	.37	.42	.47	.51	.56	.60	.65	.70	.74	.79	.84	32
34	.05	.10	.15	.20	.25	.30	.35	.40	.44	.49	.54	.59	.64	.69	.74	.79	.84	.89	34
36	.05	.10	.16	.21	.26	.31	.37	.42	.47	.52	.58	.63	.68	.73	.79	.84	.89	.94	36
38	.06	.11	.17	.22	.28	.33	.39	.44	.50	.55	.61	.66	.72	.77	.83	.88	.94	0.99	38
40	.06	.12	.17	.23	.29	.35	.41	.47	.52	.58	.64	.70	.76	.81	.87	.93	0.99	1.05	40
42	.06	.12	.18	.24	.31	.37	.43	.49	.55	.61	.67	.73	.79	.86	.92	0.98	1.04	1.10	42
44	.06	.13	.19	.26	.32	.38	.45	.51	.58	.64	.70	.77	.83	.90	0.96	1.02	1.09	1.15	44
46	.07	.13	.20	.27	.33	.40	.47	.54	.60	.67	.74	.80	.87	.94	1.00	1.07	1.14	1.20	46
48	.07	.14	.21	.28	.35	.42	.49	.56	.63	.70	.77	.84	.91	0.98	1.05	1.12	1.19	1.26	48
50	.07	.15	.22	.29	.36	.44	.51	.58	.65	.73	.80	.87	.95	1.02	1.09	1.16	1.24	1.31	50
52	.08	.15	.23	.30	.38	.45	.53	.60	.68	.76	.83	.91	0.98	1.06	1.13	1.21	1.29	1.36	52
54	.08	.36	.24	.31	.40	.47	.55	.63	.71	.79	.86	.94	1.02	1.10	1.18	1.26	1.34	1.41	54
56	.08	.16	.24	.32	.41	.49	.57	.65	.73	.81	.90	0.98	1.06	1.14	1.22	1.30	1.38	1.47	56
58	.08	.17	.25	.33	.42	.51	.59	.67	.76	.84	.93	1.01	1.10	1.18	1.27	1.35	1.43	1.52	58
60	.09	.17	.26	.35	.44	.52	.61	.70	.79	.87	.96	1.05	1.13	1.22	1.31	1.40	1.48	1.57	60

Let us substitute, as earlier:

$$\alpha_0-\alpha_1=a_1(\alpha_2-\alpha_1)+b_1(\delta_2-\delta_1)\sec D,$$
$$\alpha_2-\alpha_0=a_2(\alpha_2-\alpha_1)+b_2(\delta_2-\delta_1)\sec D.$$

Then, we obtain:

$$a_1a_2(\alpha_2-\alpha_1)^2-b_1(\delta_2-\delta_1)\sec D\,(a_1\alpha_2+a_2\alpha_1+\alpha_0-2A).$$

Let us now examine the following second-order term, omitting for the time geing the factor b_1 sin D. The bracket can be represented as follows:

$$[(\alpha_2-\alpha_1+\alpha_1-A)(\delta_2-\delta_1+\delta_1-D)-(\alpha_1-A)(\delta_1-D)]=$$
$$=(\alpha_2-\alpha_1)(\delta_2-D)+(\alpha_1-A)(\delta_2-\delta_1).$$

Taking into account the previously omitted factors and examining together both terms with b_1 we obtain:

$$b_1\sin D\,[(\alpha_2-\alpha_1)(\delta_2-D)+(\alpha_1-A)(\delta_2-\delta_1)-(\delta_2-\delta_1)(\alpha_0-A)]=$$
$$=b_1\sin D\,[(\alpha_2-\alpha_1)(\delta_2-D)+(\delta_2-\delta_1)(\alpha_1-\alpha_0)]=b_1\sin D\,[(\alpha_2-\alpha_1)(\delta_2-D)-$$
$$-(\delta_2-\delta_1)a_1(\alpha_2-\alpha_1)]=b_1\sin D\,(\alpha_2-\alpha_1)[\delta_2-a_1(\delta_2-\delta_1)-D].$$

Thus, the second-order terms for $\delta_2-\delta_1$ will be:

$$\frac{1}{4}a_1a_2(\alpha_2-\alpha_1)^2\sin 2D+b_1(\alpha_2-\alpha_1)[\delta_2-a_1(\delta_2-\delta_1)-D]\sin D.$$

In our transformations we omitted the terms with b_1^2. In a second-order term, when $\alpha_0-\alpha_1$: $b_1(\alpha_2-\alpha_1)^2\frac{a_2-a_1}{2}\sin D$ the factor $\frac{a_2-a_1}{2}$ usually does not exceed 0.15. Therefore, we will also omit this term. Thus, finally, we can write the following formulas, which are suitable for practical application:

$$\alpha_0=\alpha_1+a_1(\alpha_2-\alpha_1)^s+b_1(\delta_2-\delta_1)''\frac{\sec D}{15}-a_1a_2(\alpha_2-\alpha_1)^s(\delta_2-\delta_1)''\operatorname{tg}D\sin 1''+$$
$$+b_1(\delta_2-\delta_1)[\delta_1+a_1(\delta_2-\delta_1)-D]\frac{\sec D}{15}\operatorname{tg}D\sin 1'';$$
$$\delta_0=\delta_1+a_1(\delta_2-\delta_1)''-b_1(\alpha_2-\alpha_1)^s\,15\cos D+ \qquad (3)$$
$$+\frac{1}{4}a_1a_2(\alpha_2-\alpha_1)^{2s}\sin 2D\,15^2\sin 1''+$$
$$+b_1(\alpha_2-\alpha_1)^s[\delta_2-a_1(\delta_2-\delta_1)-D]\,15\cos D\operatorname{tg}D\sin 1''.$$

Tables have been compiled to facilitate the calculation of second-order terms. The first table gives the term:

$$0.25(\alpha_2-\alpha_1)^s(\delta_2-\delta_1)''\sin 1'',$$

The auxiliary table gives the factor $\frac{a_1a_2}{0.250}$. The second table gives the term:

$$\frac{1}{4}0.25(\alpha_2-\alpha_1)^{2s}\sin 2D\,15^2\sin 1''.$$

The third table gives second-order terms with b_1 with respect to the arguments:

$$b_1(\delta_2-\delta_1)\frac{\sec D}{15} \quad \text{and} \quad [\delta_1 + a_1(\delta_2-\delta_1)-D]$$

or

$$b_1(\alpha_2-\alpha_1)\,15\cos D \quad \text{and} \quad [\delta_2 - a_1(\delta_2-\delta_1)-D].$$

The values given in the tables should then be multiplied by tgD. Before we undertake the solution of an example based on formulas (3) we wish to say a few words about the influence exerted by nonorthogonal refraction terms, not accounted for in the original formula (1). According to König's formulas loc. cit. p.530) these terms have the form of corrections made on the measured coordinates, whereupon here one may assume the origin of coordinates to be located not necessarily in the optical center but, for example, in the first reference star:

correction for x = $+2[\beta+2\beta'(1+k_1^2+k_2^2)]k_1k_2\Delta y_i$,

correction for y = $+[\beta+2\beta'(1+k_1^2+k_2^2)](k_1^2-k^2)\Delta y_i$,

AUXILIARY TABLE

a_1		$\frac{a_1 \cdot a_2}{0.250}$	a_1		$\frac{a_1 \cdot a_2}{0.250}$
0.500	0.500	1.000	0.400	0.600	0.960
0.490	0.510	1.000	0.390	0.610	0.952
0.480	0.520	0.998	0.380	0.620	0.942
0.470	0.530	0.996	0.370	0.630	0.932
0.460	0.540	0.994	0.360	0.640	0.922
0.450	0.550	0.990	0.350	0.650	0.910
0.440	0.560	0.986	0.340	0.660	0.898
0.430	0.570	0.980	0.330	0.670	0.884
0.420	0.580	0.974	0.320	0.680	0.870
0.410	0.590	0.968	0.310	0.690	0.856

a_1		$\frac{a_1 \cdot a_2}{0.250}$	a_1		$\frac{a_1 \cdot a_2}{0.250}$	a_1		$\frac{a_1 \cdot a_2}{0.250}$
0.300	0.700	0.875	0.200	0.800	0.640	0.100	0.900	0.260
0.290	0.710	0.824	0.190	0.810	0.616			
0.280	0.720	0.806	0.180	0.820	0.590			
0.270	0.730	0.788	0.170	0.830	0.564			
0.260	0.740	0.770	0.160	0.840	0.538			
0.250	0.750	0.750	0.150	0.850	0.510			
0.240	0.760	0.730	0.140	0.860	0.470			
0.230	0.770	0.708	0.130	0.870	0.452			
0.220	0.780	0.686	0.120	0.880	0.422			
0.210	0.790	0.634	0.110	0.890	0.392			

T A B L E 2

$$\frac{1}{4}\,0.25\,(\alpha_2 - \alpha_1)^{2s} \sin 2\,D15^2 \sin 1''$$

δ \ Δx	20^s	40^s	60^s	80^s	100^s	120^s	140^s	160^s	180^s	200^s	220^s	240^s	260^s	280^s	300^s	320^s	340^s	360^s	Δx / δ
2°	0″.0	0″.0	0″.0	0″.0	0″.0	0″.1	0″.1	0″.1	0″.2	0″.2	0″.2	0″.3	0″.3	0″.4	0″.4	0″.5	0″.6	0″.6	88°
4	.0	.0	.0	.1	.1	.1	.2	.2	.3	.4	.5	.5	.6	.7	.9	1.0	1.1	1.2	86
6	.0	.0	.0	.1	.2	.2	.3	.4	.5	.6	.7	.8	1.0	1.1	1.3	1.5	1.6	1.8	84
8	.0	.0	.1	.1	.2	.3	.4	.5	.6	.8	.9	1.1	1.3	1.5	1.7	1.9	2.2	2.4	82
10	.0	.0	.1	.2	.2	.3	.5	.6	.8	.9	1.1	1.3	1.6	1.8	2.1	2.4	2.7	3.0	80
12	.0	.0	.1	.2	.3	.4	.5	.7	.9	1.1	1.3	1.6	1.9	2.2	2.5	2.8	3.2	3.6	73
14	.0	.0	.1	.2	.3	.5	.6	.8	1.0	1.3	1.6	1.8	2.2	2.5	2.9	3.3	3.7	4.1	76
16	.0	.1	.1	.2	.4	.5	.7	.9	1.2	1.4	1.7	2.1	2.4	2.8	3.3	3.7	4.2	4.7	74
18	.0	.1	.1	.3	.4	.6	.8	1.0	1.3	1.6	1.9	2.3	2.7	3.1	3.6	4.1	4.6	5.2	72
20	.0	.1	.2	.3	.4	.6	.9	1.1	1.4	1.8	2.1	2.5	3.0	3.4	3.9	4.5	5.1	5.7	70
22	.0	.1	.2	.3	.5	.7	.9	1.2	1.5	1.9	2.3	2.7	3.2	3.7	4.3	4.8	5.5	6.1	68
24	.0	.1	.2	.3	.5	.7	1.0	1.3	1.6	2.0	2.5	2.9	3.4	4.0	4.6	5.2	5.9	6.6	66
26	.0	.1	.2	.3	.5	.8	1.1	1.4	1.7	2.2	2.6	3.1	3.6	4.2	4.8	5.5	6.2	7.0	64
28	.0	.1	.2	.4	.6	.8	1.1	1.4	1.8	2.3	2.7	3.3	3.8	4.4	5.1	5.8	6.5	7.3	62
30	.0	.1	.2	.4	.6	.8	1.2	1.5	1.9	2.4	2.9	3.4	4.0	4.6	5.3	6.0	6.8	7.7	60
32	.0	.1	.2	.4	.6	.9	1.2	1.6	2.0	2.4	3.0	3.5	4.1	4.8	5.5	6.3	7.1	7.9	58
34	.0	.1	.2	.4	.6	.9	1.2	1.6	2.0	2.5	3.1	3.6	4.3	5.0	5.7	6.5	7.3	8.3	56
36	.0	.1	.2	.4	.6	.9	1.3	1.7	2.1	2.6	3.1	3.7	4.4	5.1	5.8	6.6	7.5	8.5	54
38	.0	.1	.2	.4	.7	1.0	1.3	1.7	2.1	2.6	3.2	3.8	4.5	5.2	6.0	6.8	7.6	8.6	52
40	.0	.1	.2	.4	.7	1.0	1.3	1.7	2.2	2.7	3.2	3.9	4.5	5.3	6.0	6.9	7.8	8.8	50
42	.0	.1	.2	.4	.7	1.0	1.3	1.7	2.2	2.7	3.3	3.9	4.6	5.3	6.1	6.9	7.8	8.8	48
44	.0	.1	.2	.4	.7	1.0	1.3	1.7	2.2	2.7	3.3	3.9	4.6	5.3	6.1	7.0	7.9	8.8	46

where the quantities k_1 and k_2 are determined from the corresponding parallactic triangle and β and β are constant refractions. Under Δy_i we imply $y_2 - y_1 = \Delta y$ or $y_0 - y_1 = \Delta y_1$. In order to get an idea of the numerical value of these corrections, let us turn to Küstner's tables for the Bonn Observatory printed in [5]. From these tables we can see that corrections in large hour angles and at zenith distances up to 75° amount to as much as 0.0002 of the coordinate. The effect of this error on constants a_1 and b_1 can be found by differentiation of these constants with respect to Δx_i and Δy_i. In fact, if we denote by dx and dy the corrections per coordinate unit (i.e. the coefficients in front of Δy_i, depending on β, β', and k_2), we obtain:

$$da_1 = d\left(\frac{\Delta x\Delta x_1 + \Delta y\Delta y_1}{\Delta x^2+\Delta y^2}\right) = \frac{(\Delta y\Delta x_1 dx + \Delta x\Delta y_1 dx + 2\Delta y\Delta y_1 dy)(\Delta x^2+\Delta y^2)}{(\Delta x^2+\Delta x^2)^2} -$$

$$- \frac{(\Delta x\Delta x_1+\Delta y\Delta y_1)2(\Delta x\Delta y dx + \Delta y^2 dy)}{(\Delta x^2+\Delta y^2)^2} =$$

$$= \frac{2\Delta y\Delta x dy(\Delta x\Delta y_1 - \Delta y\Delta x_1) + \Delta x^2 dx(\Delta x\Delta y_1 - \Delta y\Delta x_1) - \Delta y^2 dx(\Delta x\Delta y_1 - \Delta y\Delta x_1)}{(\Delta x^2+\Delta y^2)^2} =$$

$$= \frac{(\Delta x\Delta y_1 - \Delta y\Delta x_1)[2\Delta y\Delta x\, dy + \Delta x^2 dx - \Delta y^2 dx]}{(\Delta x^2+\Delta y^2)^2} = -b_1\frac{(\Delta x^2-\Delta y^2)dx + 2\Delta x\Delta y dy}{\Delta x^2+\Delta y^2};$$

similarly, we obtain:

$$db_1 = b_1\frac{(\Delta x^2-\Delta y^2)dy - 2\Delta x\Delta y dx}{\Delta x^2+\Delta y^2}.$$

TABLE 3

Second-order terms with b_1

Δα / Δδ	10^s	20^s	30^s	40^s	Δα / Δδ	10^s	20^s	30^s	40^s	Δα / Δδ	10^s	20^s	30^s	40^s
2′	0ʺ.01	0ʺ.01	0ʺ.01	0ʺ.02	1°2′	0ʺ.18	0ʺ.36	0ʺ.54	0ʺ.72	2°2′	0ʺ.36	0ʺ.71	1ʺ.06	1ʺ.42
4	.01	.02	.04	.05	4	.19	.37	.56	.74	4	.36	.72	1.08	1.44
6	.02	.04	.05	.07	6	.19	.38	.58	.77	6	.37	.73	1.10	1.47
8	.02	.05	.07	.09	8	.20	.40	.59	.79	8	.37	.74	1.12	1.49
10	.03	.06	.09	.12	10	.20	.41	.61	.82	10	.38	.76	1.14	1.51
12	.04	.07	.10	.14	12	.21	.42	.63	.84	12	.38	.77	1.15	1.54
14	.04	.08	.12	.16	14	.22	.43	.65	.86	14	.39	.78	1.17	1.56
16	.05	.09	.14	.19	16	.22	.44	.66	.88	16	.40	.79	1.19	1.58
18	.05	.10	.16	.21	18	.23	.45	.68	.91	18	.40	.80	1.20	1.61
20	.06	.12	.18	.23	20	.23	.46	.70	.93	20	.41	.81	1.22	1.63
22	.06	.13	.19	.26	22	.24	.48	.72	.96	22	.41	.83	1.24	1.65
24	.07	.14	.21	.28	24	.24	.49	.73	.98	24	.42	.84	1.26	1.68
26	.08	.15	.23	.30	26	.25	.50	.75	1.00	26	.43	.85	1.28	1.70
28	.08	.16	.24	.33	28	.26	.51	.77	1.02	28	.43	.86	1.29	1.72
30	.09	.18	.26	.35	30	.26	.52	.79	1.05	30	.44	.87	1.31	1.75
32	.09	.19	.28	.37	32	.27	.54	.80	1.07	32	.44	.88	1.33	1.77
34	.10	.20	.30	.40	34	.27	.55	.82	1.10	34	.45	.90	1.34	1.79
36	.10	.21	.31	.42	36	.28	.56	.84	1.12	36	.46	.91	1.36	1.82
38	.11	.22	.33	.44	38	.29	.57	.86	1.14	38	.46	.92	1.38	1.84
40	.12	.23	.35	.46	40	.29	.58	.87	1.16	40	.47	.93	1.40	1.86
42	.12	.24	.37	.49	42	.30	.59	.89	1.19	42	.47	.94	1.42	1.89
44	.13	.26	.38	.51	44	.30	.60	.91	1.21	44	.48	.95	1.43	1.91
46	.13	.27	.40	.54	46	.31	.62	.93	1.23	46	.48	.97	1.45	1.93
48	.14	.28	.42	.56	48	.32	.63	.94	1.26	48	.49	.98	1.47	1.96
50	.15	.29	.44	.58	50	.32	.64	.96	1.28	50	.50	.99	1.48	1.98
52	.15	.30	.45	.60	52	.33	.65	.98	1.30	52	.50	1.00	1.50	2.00
54	.16	.31	.47	.63	54	.33	.66	1.00	1.33	54	.51	1.01	1.52	2.03
56	.16	.33	.49	.65	56	.34	.68	1.01	1.35	56	.51	1.02	1.54	2.05
58	.17	.34	.51	.68	58	.34	.69	1.03	1.37	58	.52	1.04	1.55	2.07
1°00	.18	.35	.52	.70	2°00	.35	.70	1.05	1.40	3°00	.52	1.05	1.57	2.10

Thus, as was to be expected, these corrections vanish completely at $b_1 = 0$. If we assume as an example $\Delta x = \Delta y$, the fractions are respectively transformed into dy and dx. Postulating $b_1 = 0.1$ we obtain a maximum value for da_1 and db_1 of the order 0.0002. When the distance between reference stars is about 1° i.e. 3600 sec, we will obtain an error < 1 sec even at very large zenithal distances.

As an example, we have chosen the determination of the position of a star located between two reference stars whose positions were borrowed by us from the photographic catalog of Helsinki Observatory, Vol. VI, plate No.665, with the optical center $A = 15^h\,45^m\,0^s\!.00$ and $D = +45°00'\!.0$ (1900). According to the catalog the position of the star sought for is:

$$\alpha_0 = 15^h\,45^m\,35^s\!.78,\quad \delta_0 = +45°6'3''\!.0.$$

The solution is:

$$\alpha_1 = 15^h42^m20^s\!.49,\qquad \delta_1 = +44°25'29''\!.5,$$
$$\alpha_2 = 15\;47\;\;18.88.\qquad \delta_2 = +46\;16\;11.9$$
$$\alpha_2-\alpha_1 = +4^m58^s\!.39 \qquad \delta_2-\delta_1 = +50'42''\!.4$$
$$= +298^s\!.39 \qquad = +3042''\!.4$$

$$x_0 = +6^{mm}557,\quad y_0 = +6^{mm}183,\quad a_1 = +0.7248,$$
$$x_1 = -28.\,375,\quad y_1 = -34.\,285,\quad a_2 = +0.2752,$$
$$x_2 = +24.\,731,\quad y_2 = +16.\,402,\quad b_1 = -0.0702.$$
$$x_2 - x_1 = +53.106 = \Delta x,\qquad y_2 - y_1 = +50.687 = \Delta y,$$
$$x_0 - x_1 = +34.932 = \Delta x_1,\qquad y_0 - y_1 = +40.468 = \Delta y_1,$$

α_1	$15^h\,42^m\,20^s\,49,$	δ_1	$+44°25'29''\!.5,$
$a_2(\alpha_2-\alpha_1)$	$+3\quad 36.27,$	$a_1(\delta_2-\delta_1)$	$+36\;45.1,$
$b_1(\delta_2-\delta_1)\frac{\sec D}{15}$	-20.08	$-b_1(\alpha_2-\alpha_1)\,15\cos D$	$+\;3\;42.2$
	15 45 36.68		+45 5 56.8
second-order term a_1a_2.	$-.88$	second-order term a_1a_2.	$+4.8$
" " b_1	$-.01$	" " b_1.	$+1.4$
α_0	$15^h45^m\,35^s\!.79$	δ_0	$+45°6'3''\!.0$

From formulas (3) and also from analysis of third-order terms we can see that terms with b_1 create additional difficulties and errors. Therefore, it is advantageous to have $b_1 = 0$. This can be achieved with the aid of a third reference star in the following manner. Let there be an object sought for which is lo ated inside (and in exceptional cases also outside) the triangle formed by the three reference stars. Then, after connecting one of the stars and the object sought for by a straight line, let us extend this line until it intersects the straight line connecting the other two reference stars.

The point of intersection can be assumed as a fictitious star whose coordinates we can determine from the two reference stars; in this case, it is evident that $b_1 = 0$. After calculating the equatorial coordinates of the fictitious star, let us use these coordinates for determining α_0, and δ_0 of the object sought for, whereby in this case b_1 is also equal to zero. Thus, the process is broken down into using formulas (3) twice, but without terms with b_1.

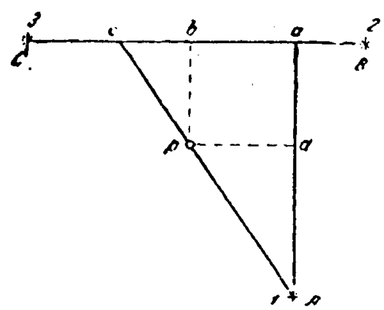

Fig.2

In order to find coefficients a_1 we must determine the coordinates $\underline{x}$ and $\underline{y}$ of the fictitious star. This can be done by means of a calculation. Indeed, since $b_1 =$ $= 0$ in both cases, we obtain the following two conditions for determining the coordinates $\underline{x}$ and $\underline{y}$:

$$(y_3-y_2)(x-x_2)-(x_3-x_2)(y-y_2)=0,$$
$$(y-y_1)(x_0-x_1)-(x-x_1)(y_0-y_2)=0,$$

whence

$$x(y_3-y_2)-y(x_3-x_2)=x_2y_3-y_2x_3,$$
$$x(y_0-y_1)-y(x_0-x_1)=x_1y_0-y_1x_0.$$

From these two equations we can find $\underline{x}$ and $\underline{y}$ of the fictitious star and then we calculate coefficients a_1 with the aid of known formulas. The problem of determining a_1 can be greatly simplified by measuring the plate in a special manner. Namely, after orienting the plate in such a way that the straight line connecting two reference stars coincide with the horizontal filament (line) of the instrument, we measure the coordinates of all reference stars and of the object sought for (Fig.2). The fictitious star is denoted by the letter c.

Then, we rotate (turn) the plate in such a way that the vertical filament (line) coincide with the selected pair of stars, and we again measure all coordinates. From such triangles we can see that

$$a_1=\frac{Ap}{Ac}=\frac{Ad}{Aa}.$$

The segments Ad and Aa are known from measurements. In order to find $a_1'=\frac{Bc}{BC}=\frac{Ba+ac}{BC}$, we find the segment

$$ac=ab\frac{Aa}{Ad}=\frac{ab}{a_1'}.$$

In the case of three stars and with such a method of measurement it is not necessary that the vertical motion be necessarily perpendicular to the horizontal direction in which the measurement scale is disposed, as was the case of two reference stars. Thus, three reference stars give a better precision with lesser limitations.

Therefore, the formulas used for calculating the coordinates of the object sought for with the aid of three reference stars will be as follows:

$$\alpha_0 = \alpha_1 + a_1 (\alpha - \alpha_1)^s - a_1 a_2 (\alpha - \alpha_1)^s (\delta - \delta_1)'' \operatorname{tg} D \sin 1'',$$

where

$$\alpha = \alpha_2 + a_1' (\alpha_3 - \alpha_2)^s - a_1' a_2 (\alpha_3 - \alpha_2)^s (\delta_3 - \delta_2)'' \operatorname{tg} D \sin 1'', \qquad (4)$$

$$\delta_0 = \delta_1 + a_1 (\delta - \delta_1)'' + \frac{1}{4} a_1 a_2 (\alpha - \alpha_1)^{2s} \sin 2D\, 15^2 \sin 1'',$$

where

$$\delta = \delta_2 + a_1' (\delta_3 - \delta_2)'' + \frac{1}{4} a_1' a_2' (\alpha_3 - \alpha_2)^{2s} \sin 2D\, 15^2 \sin 1''.$$

Nonorthogonal refraction terms are also taken into account by the very same formulas.

<u>EXAMPLE</u>. We shall resolve by means of our method the example given in Schlesinger's article (loc. cit. p.84).

Given are the following three reference stars:

$$\alpha_1 = 15^h 31^m 30^s.41, \quad \delta_1 = -17^\circ 42' 18''.1,$$
$$\alpha_2 = 15\ 33\ 0.28, \quad \delta_2 = -17\ 20\ 11.1,$$
$$\alpha_3 = 15\ 34\ 40.24, \quad \delta_3 = -17\ 23\ 75.5.$$

Measurements gave:

$$x_1 = -71.537, \quad y_1 = +18.453,$$
$$x_2 = -37.744, \quad y_2 = +53.408,$$
$$x_3 = 0.000, \quad y_3 = 0.000,$$
$$x_0 = -43.396, \quad y_0 = +30.209.$$

Coordinates of the point of intersection (of the fictitious star) will be found analytically. We obtain:

$$x = -26.374,$$
$$y = +37.319.$$

Hence we find the coefficients:

$$a_1 = 0.62310, \quad a_1' = 0.30124.$$

From formulas (4) we obtain the coordinates of the fictitious star:

$$\alpha = 15^h\,33^m\,30^s.33, \quad \delta = -17^\circ 30' 21''.7,$$

and finally, the coordinates of the object sought for:

$$\alpha_0 = 15^h\,32^m\,45^s.16, \quad \delta_0 = -17^\circ\,34'\,52''.3.$$

These values are in precise agreement with Schlesinger's data.

Pul'kovo Observatory
10 February, 1947.

* * * THE END * * *

REFERENCES

1. Schlesinger, A.J. 875, 1926.
2. Deutsch, A.N. 249, 102, 1933.
3. Arend, A.N. 246, 13, 1922.
4. A. König, Handbuch der Astrophysik 1,516.
5. Küstner, Veroff. Univ. Sternw. Bonn 14, 1926, Anhang.

\- - - - -

CONTRACT No.NAS-5-12487
VOLT TECHNICAL CORPORATION
1145 19th St. N.W.
Washington, D.C. 20036
Telephone: 223-6700 [x-36-37]

Translated by
A. Schidlovsky
May 12, 1968

Revised by
Dr. Andre L. Brichant
May 14, 1968

ALB/ldf