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A USE OF LIGHTHILL'S GENERALIZED FUNCTIONS IN THE SOLUTION OF OPTIMAL CONTROL PROBLEMS

Ernest S. Armstrong

NASA Langley Research Center
Langley Station, Hampton, Va.

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The usual definition of the Dirac "δ- function" as a function which vanishes everywhere except at a single point but whose integral over the entire real line is unity is not a rigorous mathematical definition. No mathematical function can have these properties.

There are, however, several methods of rigorously using such symbolic expressions. In each the concept of a function is generalized to include all continuous functions, all Lebesque integrable functions, and new objects a simple example of which is the "δ- function" . As a result of this generality the theory is referred to as the theory of generalized functions.

Probably the most mathematically sophisticated approach, requiring a knowledge of measure theory and Lebesque integration, is due to L. Schwartz. An introduction to Schwartz's French work (reference 1) has been prepared in English by I. Halperin (reference 2). Fortunately, there is another formulation, presented in a text by M. T. Lighthill (reference 3) and originally due to G. Temple (reference 4), requiring only a knowledge of ordinary calculus. Lighthill's approach, a brief introduction to which is presented in Appendix A, is to construct generalized functions in terms of sequences of continuous functions. For example, a representation

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(not unique) for the Dirac "δ- function" is the sequence $\left\{e^{-nt^2}(n/\pi)^{\frac{1}{2}}\right\}$. In the limit as $n \to \infty$ $\int_{-\infty}^{\infty} e^{-nt^2}(n/\pi)^{\frac{1}{2}}\,dt \to 1$ and $e^{-nt^2}(n/\pi)^{\frac{1}{2}}$ takes on the properties of the "δ- function" . In Lighthill's text it is essentially established that sets of equivalent sequences (see Appendix A) form an additive abelian group over which differentiation is defined. It was found that this theory could be applied to a class of optimal control problems. Appendix A contains the basic definitions and properties of Lighthill's formulation which are needed to present the application.

The general result of applying an optimal control theory, such as Pontryagin's maximum principle (reference 5), to a controllable dynamic system is a two-point boundary value problem. A large class of these boundary value problems take the form.

I. Given the dynamic equation (written in scalar form for simplicity)

$$\dot{x} = f(x, \operatorname{sgn} \rho, \alpha, t), \quad t \in \left[t_o, t_1\right], \; x(t_o) = x^o,$$

determine the parameter α such that at t_1, $e(t_1) = 0$.

The variable x is a state or conjugate variable of the control theory, ρ is a continuous switching function, and e is a function of $x(t_1)$, α, and t_1 representing a terminal condition. The function f is continuous in x, $\operatorname{sgn} \rho$ (see Appendix A for definition), and α, and piecewise continuous in t with points of discontinuity occurring at the zeros of the continuous function $\rho\left\{x(t), \alpha, t\right\}$. The usual approach for

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the solution of such problems is to devise an algorithm whereby an assumed value of α can be corrected so as to drive $e(t_1)$ to zero. This requires the evaluation of the expression $\frac{dx}{d\alpha}(t_1)$ describing the influence of α on $x(t_1)$. It is the purpose of this discussion to indicate how Lighthill's generalized function theory can be used to obtain an equation for $\frac{dx}{d\alpha}(t)$, $t \in \left[t_o, t_1\right]$.

In the sense of ordinary functions we can write

$$\frac{df}{d\alpha}(x, \operatorname{sgn}\rho, \alpha, t) = \frac{\partial f}{\partial(\operatorname{sgn}\rho)} \frac{d(\operatorname{sgn}\rho)}{d\rho} \frac{d\rho}{d\alpha} + \frac{\partial f}{\partial x}\frac{dx}{d\alpha} + \frac{\partial f}{\partial \alpha}$$

Let t^* be a zero of $\rho(x(t), \alpha, t)$ and $S(t^*)$ be the set of zeros of ρ on $\left[t_o, t_1\right]$. If $\left.\frac{d\rho}{dt}\right|_{t=t^*} \neq 0$ then as a generalized function

$$\frac{\partial f}{\partial(\operatorname{sgn}\rho)} \quad \frac{d(\operatorname{sgn}\rho)}{d\rho} \frac{d\rho}{d\alpha}$$ is represented

by $\sum_{S(t^*)} Q(t^*)\, \delta(t - t^*)$ where

$$Q(t^*) = -\,2 \operatorname{sgn}\rho\,(t^{*-})\, \frac{\partial f\,(t^*)}{\partial(\operatorname{sgn}\rho)} \frac{\frac{d\rho}{d\alpha}(t^*)}{\dot{\rho}(t^*)}$$

$\delta(t - t^*)$ = the Dirac "δ- function" of $t - t^*$

and $\sum_{S(t^*)}$ represents the sum over all t^* of $\rho(x(t), \alpha, t)$ in $\left[t_o, t_1\right]$.

Let $P(t)$ be the generalized function corresponding to the ordinary function $\frac{\partial f}{\partial x}\frac{dx}{d\alpha} + \frac{\partial f}{\partial \alpha}$. As generalized functions we can write by property 3 of Appendix A.

$$\frac{d}{dt}\ \frac{dx}{d\alpha} = \frac{d}{d\alpha}\ (\dot{x}) = \sum_{S(t^*)} Q(t^*)\ \delta(t - t^*) + P(t).$$

Applying first property 1 and then 2 of Appendix A the (ordinary function) equation results.

$$\int_{t_o}^{t_1} \left[\frac{dx}{d\alpha}\ (t) - \frac{dx}{d\alpha}\ (t_o) - \sum_{S(t^*)} Q(t^*)\ H(t - t^*) - \int_{t_o}^{t} (\frac{\partial f}{\partial x}\frac{dx}{d\alpha} + \frac{\partial f}{\partial \alpha})ds\right] \frac{dF}{dt}\ dt = 0$$

for all "good" functions $F(t)$.

(see Appendix)

This in turn leads, by the integral form of the fundament lemma of the calculus of variations (reference 6), to the integral equation

$$\text{II.}\quad \frac{dx}{d\alpha}\ (t) = \frac{dx}{d\alpha}\ (t_o) + \sum_{S(t^*)} Q(t^*)\ H(t - t^*) + \int_{t_o}^{t} (\frac{\partial f}{\partial x}\frac{dx}{d\alpha} + \frac{\partial f}{\partial \alpha})ds.$$

By solving this equation in conjunction with the $\dot{x}$ equation, $\frac{dx}{d\alpha}\ (t_1)$ results.

By way of application of generalized functions in a computational algorithm, the following problem was solved:

> Find the optimal thrust magnitude and direction of a space vehicle, launched from the surface of the moon, such that it will rendezvous with a minimum of fuel expenditure with a target vehicle in a circular orbit about the moon. The moon is assumed to have an inverse square law gravitational field. Rendezvous is interpreted to be the nulling of the relative position and velocity of the two vehicles.

Pontryagin's maximum principle (reference 5) was applied to the three dimensional equations describing the relative motion of the two spacecraft yielding an optimal control law and a (vector) two point boundary problem of the form I. An algorithm was formulated requiring the evaluation of $\frac{dx(t_1)}{d\alpha}$ (now a matrix) which was computed by an integral equation of the form II. The equations were programed and solved on an IBM 7094 digital computer. The algorithm was shown to work quite well yielding both planar and non-planar minimum fuel rendezvous trajectories in about 7 minutes of computer time.

Space does not permit a complete discussion of the algorithm and optimization results. It is intended that this discussion will serve to make the reader aware of a particular use of Lighthill's theory of generalized functions in analysis and of a specific application resulting from the use of optimal control theory. More complete details of both the theory and application are found in reference 7.

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APPENDIX A

GENERALIZED FUNCTIONS

Certain definitions and properties of generalized functions as presented in reference 3 are contained in this section for reference in the main text and to give the reader a general idea of the concepts involved.

Definition 1 - A <u>good function</u>, $F(t)$, is one which is everywhere differentiable any number of times such that it and all its derivatives approach zero as its argument approaches $\pm\infty$.

Definition 2 - A sequence $h_n(t)$ of good functions is called <u>regular</u> if, for any good function $F(t)$, the limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n(t)\, F(t) \quad \text{exists.}$$

Definition 3 - A <u>generalized</u> function, $h(t)$, is defined as a regular sequence $h_n(t)$ of good functions, i.e. $h(t) \rightarrow \left[h_n(t)\right]$.

Thus each generalized function is really the class of all regular sequences equivalent, in the sense that the limit in Definition 2 is the same for each sequence, to a given regular sequence.

By the symbol $\int_{-\infty}^{\infty} h(t)\, F(t)\, dt$ one means

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n(t)\, F(t)\, dt.$$

Definition 4 - Two generalized functions, $h(t)$ and $g(t)$, are equal if and only if

$$\int_{-\infty}^{\infty} h(t)\ F(t)\ dt = \int_{-\infty}^{\infty} g(t)\ F(t)\ dt$$

for all good functions $F(t)$.

Definition 5 - The derivative $h'(t)$ is defined by the sequence $h'_n(t)$, i.e. $h'(t) \rightarrow \left[h_n'(t)\right]$

Property 1 $\int_{-\infty}^{\infty} h'(t)\ F(t)\ dt = - \int_{-\infty}^{\infty} h(t)\ F'(t)\ dt$

Property 2 If the ordinary function $f(t)$ is such that $\int_{-\infty}^{\infty} (t^2+1)^{-n}\ f(t)\ dt$ exists (for some natural number N) then a generalized function $h(t)$ exists such that for all $F(t)$

$$\int_{-\infty}^{\infty} h(t)\ F(t)\ dt = \int_{-\infty}^{\infty} f(t)\ F(t)\ dt.$$

The integral on the right is the integral in the ordinary sense. When the generalized function $h(t)$ has been defined, this integral has a meaning also in the theory of generalized functions and the above equation states that these two meanings are the same.

If $\mathrm{sgn}(t) = \begin{cases} 1, & t>0 \\ -1, & t<0 \end{cases}$ Signum function

$H(t) = 1/2\ (1 + \mathrm{sgn}\ t)$ Heaviside function

$$\delta(t) = \begin{cases} 0 & , \quad t \neq 0 \\ \text{at } t=0, \ \delta(t) \text{ is infinite in such} \\ \text{a way that} \\ \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \end{cases}$$

Dirac "δ- function"

then $\dfrac{d}{dt}$ sgn $t = 2 \dfrac{dH}{dt} = 2\delta(t)$

Definition 6 - If $h_\alpha(t)$ is a generalized function of t for each value of a parameter α then

$$\frac{d}{d\alpha} h_\alpha(t) = \lim_{\alpha' \to \alpha} \frac{h_{\alpha'}(t) - h_\alpha(t)}{\alpha' - \alpha}$$

Property 3 If $\dfrac{dh_\alpha(t)}{d\alpha}$ exists, $\dfrac{d}{d\alpha} \dfrac{d}{dt} h_\alpha(t)$

$$= \frac{d}{dt} \frac{d}{d\alpha} h_\alpha(t) .$$