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AN AUTOCORRELATION APPROACH TO THE BROWN-TWISS INTERFEROMETER

by Frank C. Jones

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Greenbelt, Md.

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ABSTRACT

A question that often arises about the principles underlying the Brown-Twiss stellar interferometer is "How can interference phenomena occur with light produced by incoherent sources?" Starting from the familiar interference pattern produced by two coherent sources, one can proceed in simple steps to a picture of two incoherent sources producing an interference pattern that moves about at random. This randomly moving pattern leaves behind a "footprint" in the form of the intensity autocorrelation function. This report describes how the autocorrelation function for an extended, incoherent source may be constructed. It is this function that is measured by the Brown-Twiss stellar interferometer.

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INTRODUCTION

The physical principles underlying interference phenomena, such as the pattern produced by a diffraction grating or a double-slit interferometer, are understood by practically everyone who has studied physical optics. However, 10 or more years after the basic papers of Brown and Twiss (References 1 and 2) describing the principles and operation of their post-detection stellar interferometers, much confusion still exists in the minds of many physicists concerning the fundamental concepts employed. This condition persists in spite of the considerable number of published papers (Reference 3) that have covered the phenomenon in great detail.

There appear to be two main areas of confusion. The first concerns the correlation of arrival times of two separate photons at two separate detectors. This situation appears to contradict a basic principle of quantum electrodynamics: as stated by Dirac, (Reference 4) "Each photon then interferes only with itself. Interference between two different photons never occurs."

This question has been answered quite clearly by Purcell (Reference 5) and Mandel (Reference 6) to the effect that, while two photons cannot interfere with each other, they may be correlated by virtue of their being bosons and their tendency to clump together in phase space. It would be difficult to improve on the clarity of the treatment of the quantum problem found in Reference 5 and 6. Therefore, only a brief discussion of this matter is given in Appendix A for background. In fact, this treatment leans heavily on that of Mandel (Reference 6). The primary result of this discussion is that the operation of the stellar interferometer can be understood entirely from the point of view of classical electromagnetic theory—a point that has been emphasized by Brown and Twiss (Reference 2). In fact, the validity of the semiclassical approach to a wide variety of electromagnetic phenomena has been demonstrated recently by Mandel and Wolf (Reference 7).

The second question is purely classical in nature: "How is it possible to have any kind of interference phenomena associated with light from incoherent sources?" Although this question has been discussed extensively, it still remains somewhat of a puzzle to those not familiar with the literature on the coherence properties of radiation from incoherent sources. A thorough review of this subject and extensive references have been given by L. Mandel and E. Wolf (Reference 8). It is to this question that the present paper is addressed. The following paragraphs describe the spatial dependence of intensity correlations, starting with the simple, first-order interference pattern produced by two coherent sources and proceeding the physical principles of the post-detection stellar interferometer of Brown and Twiss.

SPATIAL DEPENDENCE OF INTENSITY FLUCTUATION CORRELATION

The essential element in the post-detection interferometer is an electromagnetic radiation detector whose response is proportional to the intensity I of the radiation. This can be a radio receiver with a "square-law" detector circuit or, in the optical region of the spectrum, a photoelectric cell. In Appendix A it is shown that, although the photoelectric cell responds to individual photons, this fact can be ignored for the purpose of studying the statistical correlation between the output of two separate detectors, and its output can be assumed to be directly proportional to the classical intensity.

In reality, however, detectors do not have zero resolution time and also do not have an output $O(t)$ that is directly proportional to the instantaneous intensity $I(t)$; the appropriate quantity will be $u(t)$, where

$$u(t) \propto \int_{t-T}^{t} I(t)\, dt \quad . \tag{1}$$

In this expression, T represents the resolution time of the detector. If the coherence time Δt of the radiation is defined as the characteristic time during which the intensity varies only slightly, and if $T \ll \Delta t$, we have

$$u(t) \propto T\, I(t) \quad . \tag{2}$$

This assumption is quite reasonable for the radio-interferometer (Reference 1) but not in the optical case (Reference 2). The complications involved in the case where $T \geq \Delta t$ in no way change the basic principles to be considered, but they do make the treatment more difficult. For this reason, we shall assume that the output of any detector under discussion is directly proportional to the instantaneous intensity of radiation falling upon it.

In the operation of the post-detection interferometer, two detectors, A and B, are separated by a distance δ. Their outputs are passed through filters, which remove the steady component and allow only the fluctuations to pass. The filter output is proportional to $I(t) - \langle I \rangle$, where the brackets $\langle \; \rangle$ represent a time average. The outputs from the two filters are multiplied together and time-averaged in a circuit called a correlator; the output of the correlator C_{AB} is therefore

$$C_{AB} \propto \Big\langle \big[I_A(t) - \langle I_A(t) \rangle\big] \; \big[I_B(t) - \langle I_B(t) \rangle\big] \Big\rangle$$

$$= \langle I_A I_B \rangle - \langle I_A \rangle \; \langle I_B \rangle \quad . \tag{3}$$

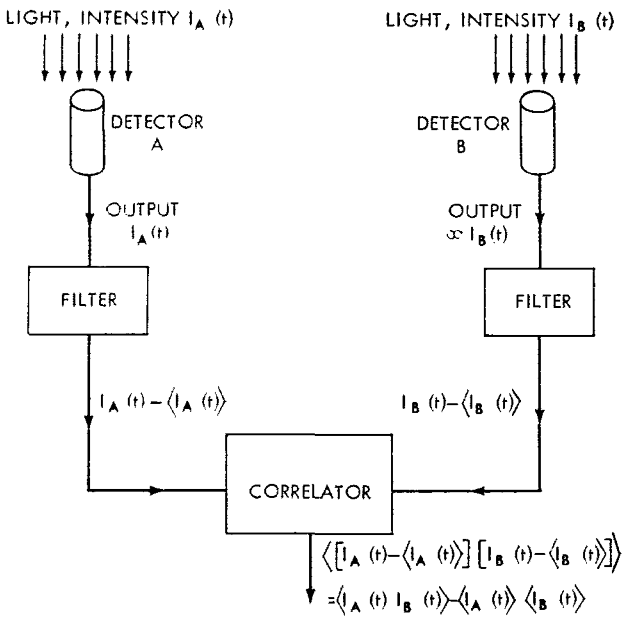

Figure 1—Post detection interferometer block diagram.

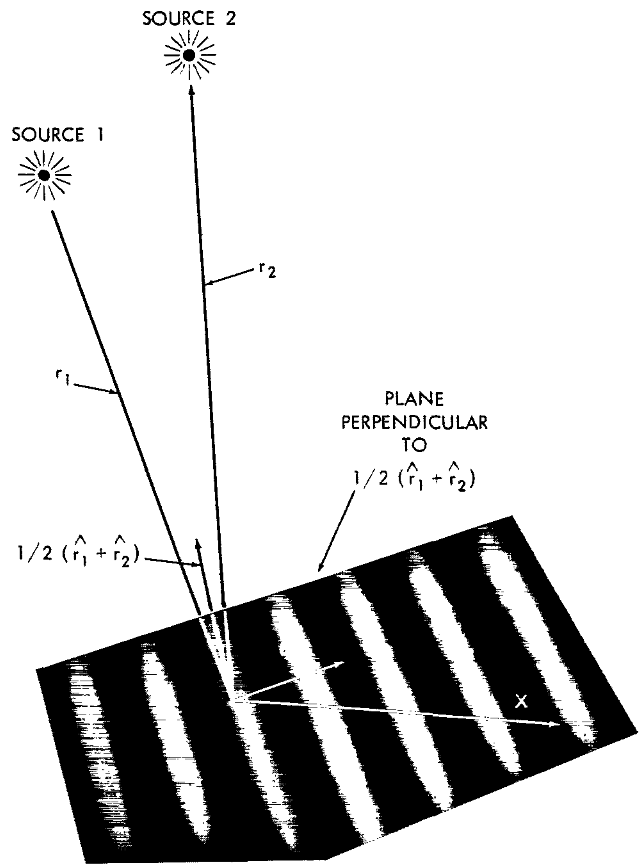

Figure 2—Instantaneous interference pattern produced by two sources.

Figure 1 is a schematic representation of this arrangement.

It is desirable now to examine the relationship between the correlation of intensity fluctuations in two separated detectors and the structure of the radiation source. We start with the simplest possible picture of two sources, 1 and 2, located at $\mathbf{r}_1$ and $\mathbf{r}_2$ (Figure 2), each radiating a monochromatic wave of frequency ω and amplitudes A_1 and A_2. At first, consider their amplitudes to be constant. However, each source will have a phase, $\phi_1(t)$ and $\phi_2(t)$, that will change with time in a random way, but with $d\phi/dt \approx 1/\Delta t << \omega$. The important point to note is that, although the two sources have no fixed relative phase, at any given time a certain instantaneous phase difference, $\phi_1(t) - \phi_2(t)$, does exist. Neglect of this point causes much of the confusion concerning the possible existence of interference phenomena for incoherent sources.

At the origin of the coordinate system, the amplitude from both sources is given by

$$A = A_1 \exp i\left[r_1 + \omega t + \phi_1\left(t - r_1/c\right)\right] + A_2 \exp i\left[r_2 + \omega t + \phi_2\left(t - r_2/c\right)\right] . \qquad (4)$$

In this expression, the amplitudes A_1 and A_2 are the amplitudes at the origin from source 1 and source 2, respectively, and r_1 and r_2 are measured in wavelengths c/ω for simplicity.

The corresponding intensity will be given by

$$I(t) = A_1^2 + A_2^2 + 2A_1A_2\cos\left[r_1 - r_2 + \phi_1(t - r_1/c) - \phi_2(t - r_2/c)\right] . \quad (5)$$

If we move to a point at a distance $\mathbf{x}$ away from the origin, keeping t fixed, and assume that $x << r_1, r_2$ (this is extremely valid for earth observations of astronomical objects), the distances from the sources to the new position are given by

$$r_1' \approx r_1 - \hat{\mathbf{r}}_1 \cdot \mathbf{x} ; \qquad r_2' \approx r_2 - \hat{\mathbf{r}}_2 \cdot \mathbf{x} ,$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$, a unit vector.

The intensity at the point $\mathbf{x}$ (since we may safely take $A_1(\mathbf{x}) = A_1$ etc.) is

$$I(\mathbf{x}, t) = A_1^2 + A_2^2 + 2A_1A_2\cos\left[r_1 - r_2 - \mathbf{x}\cdot\boldsymbol{\theta}_{12} + \Phi_{12}(\mathbf{x}, t)\right] , \quad (6)$$

where $\boldsymbol{\theta}_{12} = \hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2$ and, for small values, is just the angular separation between sources 1 and 2; $\Phi_{12}(\mathbf{x}, t)$ is the new relative phase angle given by

$$\Phi_{12}(\mathbf{x}, t) = \phi_1\left(t - r_1/c + \hat{\mathbf{r}}_1 \cdot \mathbf{x}/c\right) - \phi_2\left(t - r_2/c + \hat{\mathbf{r}}_2 \cdot \mathbf{x}/c\right) .$$

We may now ask to what extent Equation 6 represents a well defined interference pattern. Note first of all that the overall phase of the cosine function is unknown, since $r_1 - r_2$ (to within a wavelength) and the phase factor Φ_{12} are not known. However, the behavior of Equation 6 as a function of $\mathbf{x}$ is well defined if the phase angles do not change appreciably (if they do, they change in a random way). To assure this, the condition $\hat{\mathbf{r}}_1 \cdot \mathbf{x} \approx \hat{\mathbf{r}}_2 \cdot \mathbf{x} << c\Delta t$ must exist since $\phi(t)$ is essentially constant for times that are small compared to Δt. This can be achieved by requiring $\hat{\mathbf{r}}_1 \cdot \mathbf{x} = \hat{\mathbf{r}}_2 \cdot \mathbf{x} = 0$; however, this would make $\mathbf{x} \cdot \boldsymbol{\theta}_{12} = 0$, and we would have no dependence on $\mathbf{x}$. The best compromise, therefore, is to set $(\mathbf{x}/2)\cdot(\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2) = 0$, or in other words to stay on a plane perpendicular to the average direction of the sources (see Figure 2). In practice (References 1 and 2) this is achieved by inserting a time delay in the circuitry of the detector at $\mathbf{x}$ equal to $-\delta t = (1/2)\left(\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2\right) \cdot \mathbf{x}/c$, which has the same effect on the arguments of ϕ_1 and ϕ_2.

Using either approach, the arguments of ϕ_1 and ϕ_2 may be written as

$$t - r_1/c + \hat{\mathbf{r}}_1 \cdot \mathbf{x}/c - (1/2)\left(\hat{\mathbf{r}}_1 + \mathbf{r}_2\right) \cdot \mathbf{x}/c = t - r_1/c + \boldsymbol{\theta}_{12} \cdot \mathbf{x}/2c$$

and

$$t - r_2/c - \boldsymbol{\theta}_{12} \cdot \mathbf{x}/2c$$

respectively. We now require that $\boldsymbol{\theta}_{12} \cdot \mathbf{x} << 2c\Delta t$. But we have required $d\phi/dt \approx 1/\Delta t << \omega$, and, since $\omega = c$ in our units, $2c\Delta t >> 1$. Therefore, $\boldsymbol{\theta}_{12} \cdot \mathbf{x}$ may vary through many times 2π without appreciably changing ϕ_1 or ϕ_2.

With these considerations Equation 6 can be written

$$I(\mathbf{x}, t) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left[\boldsymbol{\theta}_{12} \cdot \mathbf{x} + \Phi_{12}(t)\right] , \tag{7}$$

where we have dropped the constant (and unknown) factor $r_1 - r_2$ and have written I_1 and I_2 for A^2 and A_2^2, respectively. Thus we have a standard coherent interference pattern—but one that is not stationary. It jumps from place to place, staying in one spot only for times less than Δt. It would be extremely difficult to determine its structure (and hence $\boldsymbol{\theta}_{12}$, which would be of astronomical interest) by taking its photograph—although this has been done for the interference fringes produced by the light from two lasers (Reference 9)—or by scanning it with a single phototube. On the other hand, it is important to realize that the spatial pattern represented by Equation 7 has a certain internal structure that is independent of its overall position. The internal structure of a function can be described in terms of its autocorrelation function (Reference 10), which is a measure of how the values of a function at two points separated by a fixed amount are related.

For a given function $F(x)$ the autocorrelation function $\psi(\delta)$ is defined by

$$\psi(\delta) = \lim_{x\to\infty} \frac{1}{x} \int_{-x/2}^{x/2} F(x')F(x' + \delta)\, dx' . \tag{8}$$

It is easy to see that if $F(x)$ is a periodic function with a period X,

$$\psi(\delta) = \frac{1}{X} \int_{x}^{x+X} F(x')F(x' + \delta)\, dx' , \tag{9}$$

which is independent of x. Forming this quantity for the intensity pattern (Equation 7) gives

$$\psi(\boldsymbol{\delta}) = \frac{1}{|X|} \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{X}} I(\mathbf{x}', t)\, I(\mathbf{x}' + \boldsymbol{\delta}, t)\, d\mathbf{x}' = \left(I_1 + I_2\right)^2 + 2 I_1 I_2 \cos\left(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}\right) , \tag{10}$$

where

$$\mathbf{X} = 2\pi\boldsymbol{\theta}_{12}/|\theta_{12}|^2 ,$$

Equation 10 is independent of $\mathbf{x}$ and the phase $\Phi_{12}(t)$. This quantity, viewed as a function of $\boldsymbol{\delta}$, allows one to extract the physically interesting parameter $\boldsymbol{\theta}_{12}$, even though there is no information about the position of the interference pattern.

From our point of view, the significance of the quantity $\psi(\delta)$ lies in the fact that changing the phase angle Φ_{12} is equivalent to changing $(\theta_{12} \cdot \mathbf{x})$. Therefore, the integral over $\mathbf{x}$ in Equation 10 can be replaced by an integral over Φ_{12}: Thus,

$$\psi(\delta) = \frac{1}{2\pi} \int_0^{2\pi} I(\mathbf{x}', t)\, I(\mathbf{x}' + \delta, t)\, d\Phi_{12} \quad . \tag{11}$$

If we take a time-average of the integrand $I(\mathbf{x}, t)\, I(\mathbf{x} + \delta, t)$, the phase angle Φ_{12} will vary randomly and uniformly over the range 0 to 2π, and we will have performed a Monte Carlo integration of Equation 11. Therefore,

$$\langle I(\mathbf{x}, t)\, I(\mathbf{x} + \delta, t)\rangle = \psi(\delta) = (I_1 + I_2)^2 + 2\, I_1 I_2 \cos(\theta_{12} \cdot \delta) \quad . \tag{12}$$

Equation 12 demonstrates the underlying principle of the stellar interferometer of Brown and Twiss. It shows that two sources of radiation that have no fixed phase relationship can cause the intensity at two different points to be correlated. Furthermore, the correlation depends on the separation of the detectors δ and the separation of the sources θ_{12} in a way that is very reminiscent of the interference pattern that would have been obtained had the sources been coherent. This is not surprising since the correlation pattern is closely related to the underlying interference pattern that is moving about too rapidly to be seen. The picture of the sources considered so far is very simple and not very realistic. In the following paragraphs, the picture will be refined to resemble a real astronomical object; the basic principle will remain the one expressed in Equation 12.

REFINEMENTS OF THE MODEL

The foregoing paragraphs described sources that radiated a scalar amplitude A. The description therefore corresponded to a completely polarized source or to measurements made with aligned polarizing filters placed over the detectors. The treatment of partially polarized sources is rather complicated and will not be attempted here. However, the extension to completely unpolarized sources is almost trivial if one assumes that waves of opposite polarization are completely uncorrelated and also remembers that they do not interfere with each other. Thus, for completely unpolarized sources, $I_{\parallel} = I_{\perp} = I/2$, and

$$I(\mathbf{x}, t) = I_1 + I_2 + \sqrt{I_1 I_2}\left\{\cos\left[\theta_{12} \cdot \mathbf{x} + \Phi_{12(\parallel)}(t)\right] + \cos\left[\theta_{12} \cdot \mathbf{x} + \Phi_{12(\perp)}\right]\right\} . \tag{7'}$$

From this,

$$\langle I(\mathbf{x}, t)\, I(\mathbf{x} + \delta, t)\rangle = (I_1 + I_2)^2 + I_1 I_2 \cos(\theta_{12} \cdot \delta) \quad . \tag{12'}$$

Combining Equations 12 and 12' into one equation gives

$$\langle I(\mathbf{x}, t) I(\mathbf{x} + \boldsymbol{\delta}, t)\rangle = (I_1 + I_2)^2 + \gamma I_1 I_2 \cos(\boldsymbol{\theta}_{12} \cdot \boldsymbol{\delta}) \quad , \tag{12''}$$

where $\gamma = 1$ for completely unpolarized light, and $\gamma = 2$ for completely polarized light.

It is a straightforward matter to generalize Equation 12'' for the case of N sources with direction vectors $\hat{\mathbf{r}}_i$; thus

$$\langle I(\mathbf{x}, t) I(\mathbf{x} + \boldsymbol{\delta}, t)\rangle = \left(\sum_{i=1}^{N} I_i\right)^2 + \frac{\gamma}{2} \sum_{i \neq j = 1}^{N} \left[I_i I_j \cos(\boldsymbol{\theta}_{ij} \cdot \boldsymbol{\delta})\right] \quad , \tag{13}$$

where

$$\frac{1}{N}\left[\sum_{i}^{N} \hat{\mathbf{r}}_i\right] \cdot \boldsymbol{\delta} = 0 \quad .$$

At this point, a rather peculiar fact becomes apparent. Until now the point oscillators have been described as radiating a constant-amplitude wave with only the phase varying at random. Therefore for a single source, I_i = const., and $\langle I_i^2\rangle = \langle I_i\rangle^2$. However, allowing N of the point sources in Equation 13 to coincide at a point to form a single source (setting $\theta_{ij} = 0$), gives

$$\begin{aligned} \langle I(\mathbf{x})\, I(\mathbf{x} + \boldsymbol{\delta})\rangle &= \langle I^2\rangle \\ &= \left(\sum_{i}^{N} I_i\right)^2 + \frac{\gamma}{2} \sum_{i \neq j = 1}^{N} I_i I_j \\ &= \langle I\rangle^2 + \frac{\gamma}{2} \sum_{i \neq j = 1}^{N} I_i I_j \quad . \end{aligned} \tag{14}$$

This situation indicates that there is no longer a simple, point source with constant amplitude, but rather one whose intensity fluctuates in time so that $\langle I^2\rangle > \langle I\rangle^2$. Realizing that any macroscopic radiation source (where macroscopic implies a dimension that is very large compared to the wavelength) is in fact composed of many microscopic oscillators, we see that our "point" sources should be considered as being composed of many, small identical oscillators.

To answer the question of how many oscillators should contribute to one source, consider Equation 14 for N equal oscillators where $I_i = \langle I \rangle / N$. Thus,

$$\langle I^2 \rangle = \langle I \rangle^2 + \frac{\gamma}{2} \sum_{i \neq j = 1}^{N} I_i I_j$$

$$= \langle I \rangle^2 \left[1 + \frac{N(N-1)}{N^2} \frac{\gamma}{2} \right]$$

$$= \langle I \rangle^2 \left(1 + \frac{\gamma}{2} - \frac{\gamma}{2N} \right) \tag{15}$$

Choosing $N = \infty$ gives for any point oscillator the relationship

$$\langle I_i^2 \rangle = \left(1 + \frac{\gamma}{2} \right) \langle I_i \rangle^2 \quad . \tag{16}$$

It may be stated that the picture of a point source composed of a very large number of oscillators, all oscillating with a steady amplitude but constantly changing phase, is not very realistic. This is a valid objection. However, in Appendix B, it is shown that a far more realistic model of a radiation source also yields Equation 16. This fact is not too surprising; it is merely one more example of the statistical law of large numbers. This law asserts that the probability distributions of a quantity that is a sum of N random quantities approaches a gaussian as N becomes infinite. In our case, this means that the statistical properties of the output from N oscillators become independent of the properties of the individual oscillators as N becomes very large.

Thus in deriving the expression for the autocorrelation of the intensity (Equation 12) we should have taken time-averages of the individual terms, taking into account the fluctuations of I_1 and I_2. Equation 12'' becomes

$$\langle I(\mathbf{x}, t)\, I(\mathbf{x} + \delta, t) \rangle = \langle (I_1 + I_2)^2 \rangle + \gamma \langle I_1 I_2 \rangle \cos(\theta_{12} \cdot \delta) \quad , \tag{17}$$

and instead of Equation 13 we have

$$\left\langle I(\mathbf{x}, t)\, I(\mathbf{x} + \delta, t) \right\rangle = \left\langle \left(\sum_{i=1}^{N} I_i \right)^2 \right\rangle + \frac{\gamma}{2} \sum_{i \neq j = 1}^{N} \langle I_i I_j \rangle \cos(\theta_{ij} \cdot \delta)$$

$$= \left\langle \sum_{i=1}^{N} I_i^2 \right\rangle + \sum_{i \neq j = 1}^{N} \langle I_i I_j \rangle \left(1 + \frac{\gamma}{2} \cos(\theta_{ij} \cdot \delta) \right) \quad . \tag{18}$$

Applying Equation 16 and noting that $\langle I_i I_j \rangle = \langle I_i \rangle \langle I_j \rangle$, since the intensity fluctuations of two different sources are independent, gives

$$\langle I(\mathbf{x}, t), I(\mathbf{x} + \boldsymbol{\delta}, t) \rangle = \left(1 + \frac{\gamma}{2}\right) \sum_{i=1}^{N} \langle I_i \rangle^2 + \sum_{i \neq j = 1}^{N} \langle I_i \rangle \langle I_j \rangle \left[1 + \frac{\gamma}{2} \cos\left(\boldsymbol{\theta}_{ij} \cdot \boldsymbol{\delta}\right)\right] . \qquad (19)$$

Allowing these sources to coincide to form one source will give

$$\langle I^2 \rangle = \left(1 + \frac{\gamma}{2}\right) \sum_{i=1}^{N} \langle I_i \rangle^2 + \left(1 + \frac{\gamma}{2}\right) \sum_{i \neq j = 1}^{N} \langle I_i \rangle \langle I_j \rangle$$

$$= \left(1 + \frac{\gamma}{2}\right) \sum_{i,j=1}^{N} \langle I_i \rangle \langle I_j \rangle = \left(1 + \frac{\gamma}{2}\right) \left(\sum_{i=1}^{N} \langle I_i \rangle\right)^2$$

$$= \left(1 + \frac{\gamma}{2}\right) (\langle I \rangle)^2 , \qquad (20)$$

which is the same as Equation 16 for a point source. This expression shows that an infinity of oscillators plus several other infinities of oscillators behaves like an infinity of oscillators.

Equation 19 can be cast into a form that is quite suggestive; recalling that $\boldsymbol{\theta}_{ij} = \hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j$,

$$\langle I(\mathbf{x})\ I(\mathbf{x} + \boldsymbol{\delta}) \rangle = \sum_{i,j=1}^{N} \langle I_i \rangle \langle I_j \rangle \left[1 + \frac{\gamma}{2} \cos\left(\hat{\mathbf{r}}_i \cdot \boldsymbol{\delta} - \hat{\mathbf{r}}_j \cdot \boldsymbol{\delta}\right)\right]$$

$$= \left(\sum_{i}^{N} \langle I_i \rangle\right)^2 + \frac{\gamma}{2} \sum_{i,j=1}^{N} \langle I_i \rangle \langle I_j \rangle \exp\left(i\hat{\mathbf{r}}_i \cdot \boldsymbol{\delta} - i\hat{\mathbf{r}}_j \cdot \boldsymbol{\delta}\right)$$

$$= \langle I \rangle^2 + \frac{\gamma}{2} \left| \sum_{i=1}^{N} \langle I_i \rangle \exp\left(i\hat{\mathbf{r}}_i \cdot \boldsymbol{\delta}\right) \right|^2 . \qquad (21)$$

The generalization from N discrete sources to a continuous distribution of sources is immediate. With the substitutions $\langle I_i \rangle \rightarrow I(\hat{r})d\Omega$, $\sum_i \rightarrow \int$, Equation 21 becomes

$$\langle I(x)\ I(x+\delta)\rangle = \langle I\rangle^2 + \frac{\gamma}{2}\left|\int F(\hat{r})\ \exp\ (i\hat{r}\cdot\delta)\ d\Omega\right| , \tag{22}$$

where $F(\hat{r})$ is the source brightness per solid angle in the direction $\hat{r}$.

Returning to Equation 3, we may identify I_A with $I(x)$, and I_B with $I(x+\delta)$. It is easy to verify that $\langle I(x)\rangle = \langle I(x+\delta)\rangle = \langle I\rangle$; therefore Equation 22 can be substituted into Equation 3 to obtain for the output of the correlator

$$C_{AB} \propto \left[\langle I(x)\ I(x+\delta)\rangle - \langle I(x)\rangle\ \langle I(x+\delta)\rangle\right] = \left[\frac{\gamma}{2}\left|\int F(\hat{r})\ \exp\ (i\hat{r}\cdot\delta)\ d\Omega\right|^2\right] . \tag{23}$$

The quantity $\tilde{F}(\delta) = \int F(\hat{r})\ \exp\ (i\hat{r}\cdot\delta)d\Omega$ is the quantity determined with the Michelson interferometer. By determining this quantity as a function of δ, the complete brightness distribution over the face of a star $F(\hat{r})$ may be determined, using the inverse Fourier transform

$$F(\hat{r}) = (2\pi)^{-2}\int \tilde{F}(\delta)\ \exp\ (-i\hat{r}\cdot\delta) . \tag{24}$$

With the post-detection interferometer, only $|\tilde{F}(\delta)|^2$ can be determined, and there is no information about the phase of $\tilde{F}(\delta)$. We are, therefore, unable to reconstruct $F(\hat{r})$ uniquely without making additional assumptions.

The foregoing result has been obtained under the condition that $d\phi/dt$ be small compared to the frequency ω. This is equivalent to assuming that the source is quasi-monochromatic, since any change in ϕ introduces additional frequencies to the wave, and that the condition $d\phi/dt << \omega$ is the same as $\Delta\omega << \omega$; in other words, we are dealing with a narrow band of frequencies. If this condition is relaxed, Equation 7 is no longer valid since the phase difference Φ_{12} cannot be considered as independent of x. Because of this, Equation 12 should be replaced by

$$\langle I(x,\ t)\ I(x+\delta,\ t)\rangle = (I_1 + I_2)^2 + 2I_1 I_2 C(\theta_{12}\cdot\delta) , \tag{25}$$

where

$$C(\theta_{12}\cdot\delta) = 2\left\langle\cos\left[\theta_{12}\cdot x + \Phi_{12}(x,\ t)\right]\cos\left[\theta_{12}\cdot x + \theta_{12}\cdot\delta + \Phi_{12}(x+\delta,\ t)\right]\right\rangle . \tag{26}$$

If we write $\Phi_{12}(x+\delta, t) = \Phi_{12}(x, t) + \Delta\Phi_{12}(x+\delta, t)$, we can expand the cosine functions to obtain

$$
\begin{aligned}
C(\theta_{12}\cdot\delta) = \Big\langle \cos(\theta_{12}\cdot\delta)\Big[&\cos^2(\theta_{12}\cdot\mathbf{x})\left(\cos^2\Phi_{12}\cos\Delta\Phi_{12}\right.\\
&\left.-\cos\Phi_{12}\sin\Phi_{12}\sin\Delta\Phi_{12}\right) + \sin^2(\theta_{12}\cdot\mathbf{x})\left(\sin^2\Phi_{12}\cos\Delta\Phi_{12}\right.\\
&\left.+\sin\Phi_{12}\cos\Phi_{12}\sin\Delta\Phi_{12}\right) - \cos(\theta_{12}\cdot\mathbf{x})\sin(\theta_{12}\cdot\mathbf{x})\left(\cos^2\Phi_{12}\sin\Delta\Phi_{12}\right.\\
&\left.-\sin^2\Phi_{12}\sin\Delta\Phi_{12} + 2\sin\Phi_{12}\cos\Phi_{12}\cos\Delta\Phi_{12}\right)\Big]\\
&-\sin(\theta_{12}\cdot\delta)\Big[\cos^2(\theta_{12}\cdot\mathbf{x})\left(\cos^2\Phi_{12}\sin\Delta\Phi_{12}\right.\\
&\left.+\cos\Phi_{12}\sin\Phi_{12}\cos\Delta\Phi_{12}\right) + \sin^2(\theta_{12}\cdot\mathbf{x})\left(\sin^2\Phi_{12}\sin\Delta\Phi_{12}\right.\\
&\left.-\sin\Phi_{12}\cos\Phi_{12}\cos\Delta\Phi_{12}\right) + \cos(\theta_{12}\cdot\mathbf{x})\sin(\theta_{12}\cdot\mathbf{x})\left(\cos^2\Phi_{12}\cos\Delta\Phi_{12}\right.\\
&\left.-\sin^2\Phi_{12}\cos\Delta\Phi_{12} - 2\sin\Phi_{12}\cos\Phi_{12}\sin\Delta\Phi_{12}\right)\Big]\Big\rangle ,
\end{aligned}
\tag{27}
$$

where the arguments of Φ_{12} and $\Delta\Phi_{12}$ are understood. We shall now assume that Φ_{12} performs a random walk so that the distribution of $\Delta\Phi_{12}$ does not depend on Φ_{12}. The primary justification for such an assumption is that any natural model of a light source would have this property.

Recall that a uniform distribution of Φ_{12} implies

$$\left\langle\cos^2\Phi_{12}\right\rangle = \left\langle\sin^2\Phi_{12}\right\rangle = \frac{1}{2}$$

and

$$\left\langle\sin\Phi_{12}\cos\Phi_{12}\right\rangle = 0 \quad ;$$

thus Equation 27 becomes

$$
\begin{aligned}
C(\theta_{12}\cdot\delta) &= \cos(\theta_{12}\cdot\delta)\left\langle\cos\Delta\Phi_{12}\right\rangle - \sin(\theta_{12}\cdot\delta)\left\langle\sin\Delta\Phi_{12}\right\rangle\\
&= R_e\left[\exp(i\theta_{12}\cdot\delta)\left\langle\exp(i\Delta\Phi_{12})\right\rangle\right] ,
\end{aligned}
\tag{28}
$$

where R_e means the real part, and $\langle \exp(i\Delta\Phi_{12}) \rangle$ can depend only on $\theta_{12} \cdot \delta$ (not on **x**) since we have assumed throughout the discussion that the statistical properties of the source do not depend on time and the arguments of Φ_{12} are just retarded time.

The development from this point on proceeds exactly as before with the simple substitution of the function $C(\theta_{12} \cdot \delta)$ for $\cos(\theta_{12} \cdot \delta)$. Recall that $\Delta\Phi_{12} = \Delta\phi_2 - \Delta\phi_1$; thus Equation 21 becomes

$$\langle I(\mathbf{x})\, I(\mathbf{x}+\delta) \rangle = R_e \left\{ \sum_{i,j=1}^{N} \langle I_i \rangle \langle I_j \rangle \left[1 + \frac{\gamma}{2} \exp\left(i\hat{\mathbf{r}}_i \cdot \delta - i\hat{\mathbf{r}}_j \cdot \delta\right) \left\langle \exp\left(i\Delta\phi_i - i\Delta\phi_j\right) \right\rangle \right] \right\}$$

$$= \langle I \rangle^2 + \frac{\gamma}{2} \sum_{i,j=1}^{N} \langle I_i \rangle \exp\left(i\hat{\mathbf{r}}_i \cdot \delta\right) \left\langle \exp\left(i\Delta\phi_i\right) \right\rangle \times \langle I_j \rangle \exp\left(-i\hat{\mathbf{r}}_j \cdot \delta\right) \left\langle \exp\left(-i\Delta\phi_j\right) \right\rangle$$

$$= \langle I \rangle^2 + \frac{\gamma}{2} \left| \sum_{i=1}^{N} \langle I_i \rangle \left\langle \exp\left(i\Delta\phi_i\right) \right\rangle \exp\left(i\hat{\mathbf{r}}_i \cdot \delta\right) \right|^2 . \tag{29}$$

Before proceeding, pause and consider the function $\langle \exp(i\Delta\phi_i) \rangle$. Forming the autocorrelation function of the amplitude $A_i(t)$ gives

$$\psi(\tau) \equiv \left\langle A_i^*(t)\, A(t+\tau) \right\rangle \Big/ |A|^2$$

$$= \exp(i\omega\tau) \left\langle \exp\left[i\Delta\phi(\tau)\right] \right\rangle = \exp(i\omega\tau)\, \psi'(\tau) . \tag{30}$$

The Fourier transform of $\psi(\tau)$ is

$$\tilde{\psi}(\omega') = \int_{-\infty}^{\infty} \psi(\tau) \exp(i\omega'\tau)\, d\tau = \tilde{\psi}'(\omega' + \omega) , \tag{31}$$

where $\tilde{\psi}'(\omega')$ is the transform of $\psi'(\tau)$. The function $\tilde{\psi}(\omega')$ is called the normalized power spectrum (Reference 10) of the amplitude $A_i(t)$. It has this name because it represents the relative amount of power radiated in the frequency interval ω' and $\omega' + d\omega'$ (Reference 10). Therefore writing

$$\left\langle \exp\left[i\Delta\phi_i(\tau)\right] \right\rangle = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{\psi}'(\omega') \exp(-i\omega'\tau)\, d\omega' ,$$

and substituting this expression in Equation 29 gives

$$\langle I(\mathbf{x})\, I(\mathbf{x}+\delta)\rangle - \langle I\rangle^2 = \frac{\gamma}{4\pi}\left|\sum_{i=1}^{N}\langle I_i\rangle\int_{-\infty}^{\infty}\exp\left(i\hat{\mathbf{r}}_i\cdot\boldsymbol{\delta} - i\omega'\tau_i\right)\tilde{\psi}_i'(\omega')\,d\omega'\right|^2 . \tag{32}$$

Abandoning the practice of writing lengths in units of the wavelength gives $\boldsymbol{\delta} \rightarrow \boldsymbol{\delta}/\lambda = \boldsymbol{\delta}\omega/c$. We also have $\tau_i = \hat{\mathbf{r}}_i \cdot \boldsymbol{\delta}/c$ and therefore can write

$$\langle I(\mathbf{x})\; I(\mathbf{x}+\delta)\rangle - \langle I\rangle^2 = \frac{\gamma}{4\pi}\left|\sum_{i=1}^{N}\langle I_i\rangle\int_{-\infty}^{\infty}\exp\left[i\hat{\mathbf{r}}_i\cdot\boldsymbol{\delta}\,(\omega-\omega')/c\right]\tilde{\psi}_i'(\omega')\,d\omega'\right|^2 . \tag{33}$$

Assuming that all of the sources have the same spectral distribution, i.e., $\tilde{\psi}_i'(\omega') = \psi'(\omega')$ for all i, we may go to the limit of a continuously distributed source and write as the equivalent of Equation 23

$$C_{AB} \propto \left\{\frac{\gamma}{4\pi}\left|\iiint_{-\infty}^{\infty} F(\hat{\mathbf{r}})\,\exp\left[i\hat{\mathbf{r}}\cdot\boldsymbol{\delta}(\omega-\omega')/c\right]\tilde{\psi}'(\omega')\,d\omega'\,d\Omega\right|^2\right\}$$

$$= \left\{\frac{\gamma}{4\pi}\left|\iiint_{-\infty}^{\infty} F(\hat{\mathbf{r}})\,\exp\left[i\hat{\mathbf{r}}\cdot\boldsymbol{\delta}\omega'/c\right]\tilde{\psi}(\omega')\,d\omega'\,d\Omega\right|^2\right\}$$

$$= \left\{\frac{\gamma}{4\pi}\left|\int_{-\infty}^{\infty}\tilde{F}(\boldsymbol{\delta}\omega'/c)\,\tilde{\psi}(\omega')\,d\omega'\right|^2\right\} \tag{34}$$

This result is valid for any spectral distribution, assuming that the detector itself is not frequency-dependent. In any real situation, $\tilde{\psi}(\omega')$ should be multiplied by the frequency-response function of the detector since any real detector is equivalent to an ideal detector with a frequency filter placed in front of it.

From Equation 34 it can be seen that if $\tilde{\psi}(\omega')$ is sharply peaked about a particular frequency (quasi-monochromatic), we obtain the previous result, Equation 23. The finite bandwidth of the input radiation causes a loss of information in the sense that the function $\tilde{F}(\boldsymbol{\delta}\omega/c)$ gets "smeared out" over the frequency distribution. For example, if the characteristic size of the source is Θ, $\tilde{F}$ will have structure over values of $\boldsymbol{\delta}$ of order $c/\omega\,\Theta$ or larger. If the spread in frequency $\Delta\,\omega$ is such that $\Delta(c/\omega\,\Theta) = (c/\omega\,\Theta)\,\Delta\omega/\omega \approx c/\omega\,\Theta$ or $\Delta\,\omega/\omega \approx 1$, the structure will be lost and even the overall size of the source cannot be determined with great accuracy. Therefore, even though the initial requirement $d\phi/dt << \omega$ is not strictly required, it certainly indicates the best operating conditions, and, if it is violated such that $d\phi/dt \approx \omega$, it is very difficult to obtain any information about the source.

CONCLUSIONS

Beginning with the simple notion of an interference pattern that moves randomly about, one can proceed through a series of simple steps to the spatial correlation pattern of light from an extended source. This correlation pattern exists because so-called incoherent sources are not totally incoherent. There are short periods of time during which some phase-relations exist between the various parts of the source. During this same short period, an interference pattern exists at the point of observation. Since these phase-relations between the various parts of the source do not persist, this interference pattern is constantly shifting and changing. However, during this change, certain internal structures of the interference pattern are constant because of the overall structure of the source. The correlation pattern observed with the Brown-Twiss stellar interferometer is a measure of this constant, internal structure.

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Appendix A

THE CONNECTION BETWEEN QUANTUM AND CLASSICAL DESCRIPTION

Photons in the optical range are detected primarily by means of a detector that employs the photoelectric effect (e.g., a photoelectric tube). In fact, the detection of a photon in a light beam can be defined as the ejection of an electron from the photocathode of some suitable detector. In this context, the concepts of photon and classical electromagnetic field are related through a single consideration; that is, for a photocathode illuminated by a light beam, the instantaneous probability per unit time of emission of a photoelectron (detection of a photon) is proportional to the instantaneous intensity of the electromagnetic wave falling on the photocathode. Thus,

$$d\mathbb{P}(1) = \alpha I(t)\, dt \quad , \tag{A1}$$

where $d\mathbb{P}(1)$ is the probability for one, and only one, photon to be detected in a time dt; α is the proportionality coefficient. The probability for two photons to be detected is of second order in dt and is considered negligible. From probability theory, we know that for a finite interval of time between t_1 and t_2, the probability for the detection of n photons is given by the Poisson distribution

$$\mathbb{P}_u(n) = \frac{\left[u\left(t_1, t_2\right)\right]^n \exp\left[-u\left(t_1, t_2\right)\right]}{n!} \quad , \tag{A2}$$

where

$$u\left(t_1, t_2\right) \equiv \int_{t_1}^{t_2} \alpha I(t)\, dt \quad .$$

Unfortunately, unless the intensity $I(t)$ is a constant, the distribution $\mathbb{P}_u(n)$ is not observable, for if one took statistics on successive intervals $\left(t_1, t_2\right)$, $\left(t_2, t_3\right) \cdots \left(t_n, t_{n+1}\right)$, one would find that in general the value of $u\left(t_n, t_{n+1}\right)$ would vary from interval to interval. Equation A2 would describe the distribution for only the collection of intervals that had a common value (e.g., α_1) for the variable u (see Figure A1).

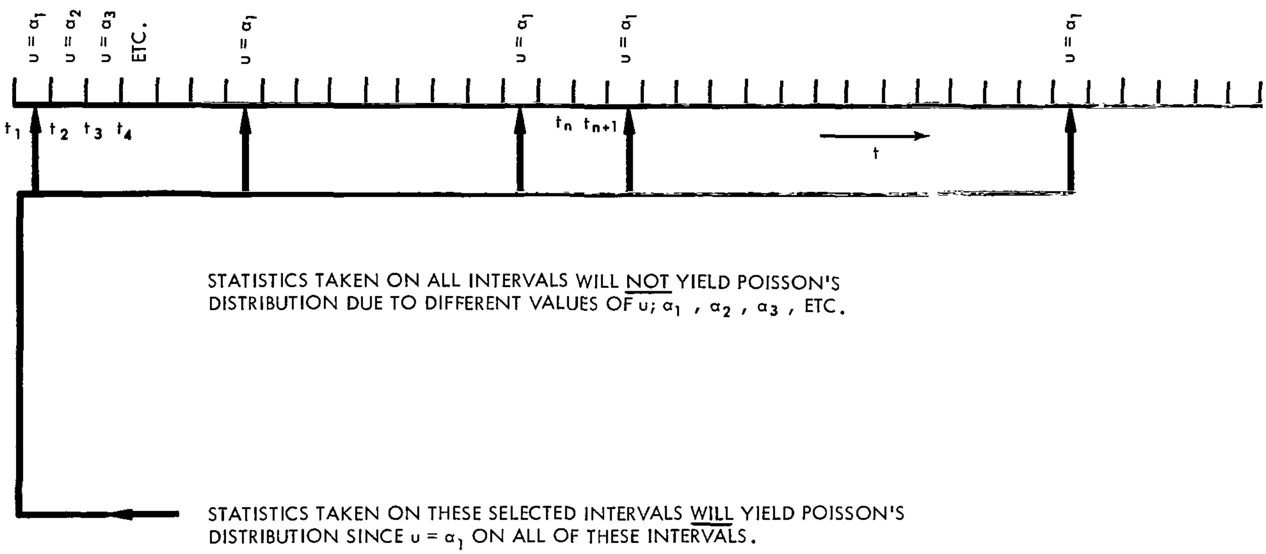

Figure A1—Sequence of time intervals having different values of $u = \int_{t_i}^{t_{i+1}} I(t)\,dt$.

If the variable u varies from interval to interval such that the probability of any interval having a particular value of u is given by $\mathbb{P}(u)$, the probability of detecting n photons in any interval selected at random will be given by the Poisson distribution averaged over all possible values of u; that is,

$$\mathbb{P}(n) = \int_0^\infty \mathbb{P}(u) \frac{u^n \exp(-u)\,du}{n!} . \tag{A3}$$

It would be well at this point to ask what possible meaning can be given to the instantaneous intensity $I(t)$ or its integral over an interval $u(t_n, t_{n+1})$ since there seems to be no operational way to determine their values. The number of photons detected in any given interval has no unique relationship to the value of u for that interval. Therefore, I and u are calculational devices used to determine the statistical properties of photoelectrons from a photocathode. If one calculates the probability per unit time for ejection of a photoelectron caused by a source of electromagnetic radiation some distance away and does so strictly from the theory of quantum electrodynamics, one finds that this probability is proportional to the absolute value squared of a vector quantity **A**. Furthermore, if the source is composed of a large number of elementary quantum systems and is some distance away, the quantity **A** may be calculated to a very good degree of approximation by using Maxwell's equations and considering **A** to be the classical vector potential and hence $|\mathbf{A}|^2$ to be the classical intensity $I(t)$. Therefore, although the quantities $I(u)$ and $u(t_n, t_{n+1})$ are, strictly speaking, not observable (although their average values may be defined operationally), they are conceptually and calculationally useful quantities.

Consider now a large number of nonoverlapping intervals of length T and the mean value and rms fluctuation of the number n of detected photons in each interval. Employing the definition of a mean value

$$\langle f(n) \rangle = \sum_{n=0}^{\infty} f(n) \, \mathbb{P}(n) \qquad \text{(A4)}$$

and making use of the relationship

$$\sum_{n=0}^{\infty} \frac{n!}{(n-p)!} \frac{u^n}{n!} \exp(-u) = u^p \quad , \qquad \text{(A5)}$$

we obtain

$$\langle n \rangle = \sum_{n=0}^{\infty} \int_0^{\infty} \mathbb{P}(u) \frac{n\, u^n}{n!} \exp(-u)\, du$$

$$= \int_0^{\infty} \mathbb{P}(u)\, u \, du = \langle u \rangle \quad . \qquad \text{(A6)}$$

This may be taken to be the operational definition of $\langle u \rangle$, the average value of u.

The mean square fluctuation of n about its mean value will be given by

$$\langle \Delta n^2 \rangle = \langle (n - \langle n \rangle)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2$$

$$= \langle u^2 \rangle + \langle u \rangle - \langle u \rangle^2$$

$$= \langle n \rangle + \langle (u - \langle u \rangle)^2 \rangle$$

$$= \langle n \rangle + \langle \Delta u^2 \rangle \quad . \qquad \text{(A7)}$$

The first term on the right-hand side of Equation A7 is the result that would be obtained if light were a stream of classical particles; the statistics of the photons then would be Poissonian. The second term $\langle \Delta u^2 \rangle$, which is a measure of the fluctuations of the wave field caused by some sort of interference phenomena, shows that the deviation of photon statistics from Poissonian because of their boson nature can be determined solely from an investigation of the associated classical electromagnetic field. In fact it can be shown (Reference 11) that the very existence of the classical electromagnetic field is intimately connected with the boson character of the photon.

Consider now the situation where two different detectors A and B are situated at different points of space. For fixed values of u_A and u_B, the emission of electrons from the two cathodes is completely independent, giving

$$\mathbb{P}_{u_A u_B}(n_A, n_B) = \frac{(u_A)^{n_A} (u_B)^{n_B}}{n_A!\, n_B!} \exp(-u_A - u_B) = \mathbb{P}_{u_A}(n_A)\, \mathbb{P}_{u_B}(n_B) \tag{A8}$$

and

$$\mathbb{P}(n_A, n_B) = \iint \mathbb{P}(u_A, u_B)\, \mathbb{P}_{u_A, u_B}(n_A, n_B)\, du_A\, du_B \quad . \tag{A9}$$

Equation A8 indicates the complete independence of the distributions of n_A and n_B for given values of u_A and u_B. However, if u_A and u_B are not distributed independently, i.e., $\mathbb{P}(u_A, u_B) \neq \mathbb{P}(u_A)\, \mathbb{P}(u_B)$, then $\mathbb{P}(n_A, n_B) \neq \mathbb{P}(n_A)\, \mathbb{P}(n_B)$ in general, and n_A and n_B are not distributed independently. In other words a correlation between u_A and u_B imposes a correlation between n_A and n_B.

In the operation of the stellar interferometer of Brown and Twiss, the fluctuations in the "output" of two different detectors are multiplied together in a circuit called a correlator, and this product is time-averaged. The "output" of a photoelectric detector will be a current or voltage that is proportional to the number of photoelectrons ejected from its cathode during some interval of time τ, which is essentially the resolution time of the detector.

The averaged output of the correlator therefore is proportional to

$$\left\langle \left(n_A - \langle n_A \rangle\right) \left(n_B - \langle n_B \rangle\right) \right\rangle = \langle n_A n_B \rangle - \langle n_A \rangle \langle n_B \rangle = \langle u_A u_B \rangle - \langle u_A \rangle \langle u_B \rangle \quad ; \tag{A10}$$

this output is directly expressible in terms of the fluctuations of the classical electromagnetic intensity. If the fluctuations of u_A and u_B are independent, then $\langle u_A u_B \rangle = \langle u_A \rangle \langle u_B \rangle$, and there is no averaged output from the correlator.

Appendix B

A DETAILED SOURCE MODEL

Consider a model of a radiation source that is considerably more realistic than the one employed in the section entitled "Refinements of the Model." Instead of constructing the source from a large number of oscillators with constant amplitudes and phases that change randomly over a period of time Δt, we shall construct the source from a collection of oscillators that turn on at random times with random starting phases and then after a time Δt turn off again. This model is suggested by the picture of a hot gas whose atoms are constantly being excited through collisions, resulting in the radiation of their extra energy during a short time Δt. The method to be used was employed by Rice (Reference 12) in his calculations of the "shot effect;" however, for a different treatment of the same problem the reader is referred to a paper by Janossy (Reference 13).

Consider a collection of oscillators situated at a point S. If the i^{th} oscillator at S turns on at $t = 0$, it will produce a field at R given by

$$A_i(t) = \hat{e}_i \exp\left(i\,\phi_i\right) F(t - |SR|/c) \quad , \tag{B1}$$

where ϕ_i is an overall phase angle, and $\hat{e}_i$ is a unit polarization vector perpendicular to the line joining S and R. $F(t)$ describes the characteristic output of an oscillator and has the properties $F(t) = 0$ for $t < 0$ and $F(t) \approx 0$ for $T >> \Delta t$. Since the retardation factor $|SR|/c$ will apply equally to all times at R, we shall drop it in subsequent calculations, remembering that any event at time t at S corresponds to an effect at R at time $t + |SR|/c$.

The total field at R caused by many oscillators turning on at times t_i with phase angle ϕ_i and polarization $\hat{e}_i$ is given by

$$A(t) = \sum_i \hat{e}_i \exp\left(i\phi_i\right) F\left(t - t_i\right) \quad . \tag{B2}$$

The field produced by N oscillators will be considered to be a random function of the $3N$ random variables ϕ_i, $\hat{e}_i$, and t_i. If the random variables have probability distribution $\mathbb{P}(\phi_i)$, $\mathbb{P}(\hat{e}_i)$, and $\mathbb{P}(t_i)$, the average of a random function can be defined as the weighted integral of the function over all values of the random variables,

$$\left\langle F(\phi_i,\ \hat{e}_i,\ t_i)\right\rangle = \prod_i \int \mathbb{P}(\phi_i)\, d\phi_i \int \mathbb{P}(\hat{e}_i)\, d\hat{e}_i \int \mathbb{P}(t_i)\, dt_i\, F(\phi_i,\ \hat{e}_i,\ t_i) \quad . \tag{B3}$$

The normalized autocorrelation function of the function $F(t)$ is defined as

$$\Phi(\tau) = \frac{\int_{-\infty}^{\infty} F^*(t)F(t+\tau)\,dt}{\int_{-\infty}^{\infty} |F(t)|^2\,dt} , \tag{B4}$$

where it is assumed that the integrals exist. It is easy to verify that $\Phi(0) = 1$ and $\Phi(-\tau) = \Phi^*(\tau)$. Since $F(t)$ is essentially zero everywhere except for $0 \le t \le \Delta t$, $\Phi(\tau)$ is essentially zero everywhere except for $-\Delta t \le \tau \le \Delta t$.

Consider again the amplitude function, Equation B2. For a process that is stationary in time, the sum should be taken over an infinite number of oscillators over all times. Since it would be difficult to calculate averages with this quantity directly, we shall calculate with the subsidiary quantity $A_T(t)$, which is the amplitude produced by all of those oscillators with turn-on times t_i such that $-T/2 \le t_i \le T/2$. At the end of the calculation, we shall always take the limit $T \to \infty$.

It is assumed that the t_i are independently and uniformly distributed with a probability per unit time η, where ηT is the average number turning on in time T. The phase angles ϕ_i are independently and uniformly distributed between 0 and 2π. The distribution of $\hat{e}_i$ will be considered later.

The averages can be calculated in three steps; first, $\langle F \rangle_{TN}$ is calculated for the case where exactly N oscillators turn on in time $-T/2 \le t \le T/2$. We will then calculate

$$\langle F \rangle_T = \sum_{N=0}^{\infty} \mathbb{P}_T(N) \langle F \rangle_{TN} ,$$

where $\mathbb{P}_T(N)$ is the Poisson distribution

$$\frac{(\eta T)^N}{N!} \exp(-\eta T) .$$

Finally,

$$\langle F \rangle = \lim_{T \to \infty} \langle F \rangle_T .$$

Now consider the intensity

$$I(t) = \mathbf{A}^*(t) \cdot \mathbf{A}(t) = \sum_{ij} (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) \exp(i\phi_j - i\phi_i) F^*(t - t_i) F(t - t_j) \tag{B5}$$

$$\langle I(t) \rangle_{NT} = \left[\prod_{i=1}^{N} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \int \mathcal{P}(\hat{\mathbf{e}}_i)\, d\hat{\mathbf{e}}_i \int_{-T/2}^{T/2} \frac{dt_i}{T} \right]$$

$$\times \left[\sum_{i,j=1}^{N} (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) \exp(i\phi_j - i\phi_i) F^*(t - t_i) F(t - t_j) \right] . \tag{B6}$$

Only those terms in the sum where $i = j$ give a non-zero contribution after the phase angle averaging; therefore (recalling $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = 1$),

$$\langle I(t) \rangle_{NT} = \frac{N}{T} \int_{-T/2}^{T/2} dt_i \, |F(t - t_i)|^2 . \tag{B7}$$

Since the time interval T can be as large as desired, as long as t is not too close to $-T/2$ or $T/2$, we may replace the integral over $|F|^2$ by the infinite integral to obtain

$$\langle I(t) \rangle_{NT} = \frac{N}{T} \int_{-\infty}^{\infty} |F(t)|^2 \, dt . \tag{B8}$$

Also,

$$\langle I(t) \rangle_T = \sum_{N=0}^{\infty} \langle I(t) \rangle_{NT} \, \mathcal{P}_T(N) = \frac{\eta T}{T} \int_{-\infty}^{\infty} |F(t)|^2 \, dt = \eta \int_{-\infty}^{\infty} |F(t)|^2 \, dt . \tag{B9}$$

The passage to the limit $T \to \infty$ is now trivial, and thus

$$\langle I(t) \rangle = \eta \int_{-\infty}^{\infty} |F(t)|^2 \, dt . \tag{B10}$$

The average intensity is just the intensity produced by a single oscillator multiplied by the average number of oscillators turned on per unit time—a rather intuitive result. The result is independent of the state of polarization, i.e., independent of $\mathbb{P}(\hat{e}_i)$.

Consider now the autocorrelation function of the intensity $\langle\psi(\tau)\rangle$ defined as $\psi(\tau) = \langle I(t)\, I\,(t+\tau)\rangle$:

$$(\psi(\tau))_{NT} = \left[\prod_{i=1}^{N} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \int \mathbb{P}(\hat{e}_i)\, d\hat{e}_i \int_{-T/2}^{T/2} \frac{dt_i}{T}\right]$$

$$\times \left[\sum_{i,j,k,\ell=1}^{N} (\hat{e}_i \cdot \hat{e}_j)(e_k \cdot e_\ell) \exp i(\phi_j - \phi_i + \phi_\ell - \phi_k)\right.$$

$$\left. \times\, F^*(t-t_i) F(t-t_j) F^*(t+\tau-t_k) F(t+\tau-t_\ell)\right] . \tag{B11}$$

Again phase angle averaging gives a zero result unless $i = j$ and $k = \ell$, or $i = \ell$ and $j = k$, or $i = j = k = \ell$. This gives

$$(\psi(\tau))_{NT} = \sum_{i\neq k=1}^{N} \frac{1}{T}\left[\int_{-T/2}^{T/2} |F(t-t_i)|^2\, dt_i\right]\left[\frac{1}{T}\int_{-T/2}^{T/2} |F(t+\tau-t_k)|^2\, dt_k\right]$$

$$+ \sum_{i\neq k=1}^{N} \iint \mathbb{P}(\hat{e}_i)\, \mathbb{P}(\hat{e}_k)(\hat{e}_i \cdot \hat{e}_k)^2\, d\hat{e}_i\, d\hat{e}_k$$

$$\times \left[\frac{1}{T}\int_{-T/2}^{T/2} F^*(t-t_i) F(t+\tau-t_i)\, dt_i\right]\left[\frac{1}{T}\int_{-T/2}^{T/2} F^*(t+\tau-t_k)\right.$$

$$\left. \times\, F(t-t_k)\, dt_k\right]$$

$$+ \sum_{i}^{N} \frac{1}{T}\int_{-T/2}^{T/2} |F(t-t_i)|^2\, |F(t+\tau-t_i)|^2\, dt_i \tag{B12}$$

Converting the integrals to infinite integrals under the assumption that both t and $t + \tau$ are far removed from $-T/2$ and $T/2$, gives

$$[\psi(\tau)]_{NT} = \frac{N(N-1)}{T^2} \left(\int_{-\infty}^{\infty} |F(t)|^2 \, dt \right)^2 + \frac{N(N-1)}{T^2} \left\langle (\mathbf{e}_i \cdot \mathbf{e}_k)^2 \right\rangle \left| \int_{-\infty}^{\infty} F^*(t) F(t+\tau) \, dt \right|^2 + \frac{N}{T} \int_{-\infty}^{\infty} |F(t)|^2 \, |F(t+\tau)|^2 \, dt \quad , \tag{B13}$$

where

$$\left\langle (\hat{e}_i \cdot \hat{e}_k)^2 \right\rangle = \iint P(\hat{e}_i) \, P(\hat{e}_k) (\hat{e}_i \cdot \hat{e}_k)^2 \, d\hat{e}_i \, d\hat{e}_n \quad .$$

Averaging over N and taking the limit $T \to \infty$ gives

$$\psi(\tau) = \eta^2 \left(\int_{-\infty}^{\infty} |F(t)|^2 \, dt \right)^2 + \eta^2 \left| \int_{-\infty}^{\infty} F^*(t) \, F(t+\tau) \, dt \right|^2 \left\langle (\hat{e}_i \cdot \hat{e}_k)^2 \right\rangle + \eta \int_{-\infty}^{\infty} |F(t)|^2 \, |F(t+\tau)|^2 \, dt \quad . \tag{B14}$$

Turning now to the quantity $\left\langle (\hat{e}_i \cdot \hat{e}_k)^2 \right\rangle$ we define two states of polarization, completely polarized by $P(\hat{e}_i) = \delta(\hat{e}_i - \hat{e}')$ and completely unpolarized by $P(\hat{e}_i) = (2\pi)^{-1}$. The quantity $\left\langle (\hat{e}_i \cdot \hat{e}_k)^2 \right\rangle = 1$ for completely polarized and $\left\langle (\hat{e}_i \cdot \hat{e}_k)^2 \right\rangle = 1/2$ for completely unpolarized. Equation B14 can be simplified to be

$$\psi(\tau) = \langle I \rangle^2 \left[1 + \frac{\gamma}{2} |\Phi(\tau)|^2 \right] + \eta \int_{-\infty}^{\infty} |F(t)|^2 \, |F(t+\tau)|^2 \, dt \quad , \tag{B15}$$

where $\gamma = 1$ for unpolarized and $= 2$ for polarized.

In the limit of a very large η (equivalent to a large number of oscillators being on at any given time), we may neglect the term linear in η compared to $\langle I \rangle^2$, which is quadratic in η, giving

$$\psi(\tau) = \langle I \rangle^2 \left[1 + \frac{\gamma}{2} |\Phi(\tau)|^2\right] , \tag{B16}$$

a familiar result (Reference 14). For $\tau = 0$, $\Phi(t) = 1$; this gives

$$\psi(0) = \langle I^2(t) \rangle = \left(1 + \frac{\gamma}{2}\right) \langle I(t) \rangle^2 , \tag{B17}$$

our desired result.

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