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METHOD FOR THE SOLUTION OF THE GENERAL BOUNDARY VALUE PROBLEM OF PROPAGATION OF LONG AND ULTRALONG RADIOWAVES AROUND THE EARTH

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SUMMARY

The method consists in the consideration of these waves in a boundless medium subdivided into three regions, the Earth, atmosphere and ionosphere. The solution of the boundary value problem is sought by using the method of connected lines. In this work the author refers to a number of his previous works on the subject of long and ultralong radiowaves, alongside with some earlier foreign works.

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1. The basic traits of the waves considered are contained in the following boundary value problem. Let us consider a boundless medium, described by Maxwellian equations in spherical coordinates r, θ, ϕ with dielectric constant tensor $\vec{\varepsilon}((\theta, \phi, r)$ and subdivided into three regions:

1) The Earth, $0 < r < a(\theta, \phi)$, where ε degenerates into scalar $\varepsilon(\theta, \phi, r) = \varepsilon' + i\varepsilon''$, $\varepsilon'' \neq 0$;

2) the atmosphere, $a(\theta, \phi) < r < c(\theta, \phi)$, where $\vec{\varepsilon} = 1$ and

3) the ionosphere, $c(\theta, \phi) < r < \infty$, where $\vec{\varepsilon}$ is arbitrary, but as $r \to \infty$ $\varepsilon \to \bar{\varepsilon} = $ const.

We seek in this boundless region the amplitudes $\vec{E}(M)$ and $\vec{H}(M)$ of electromagnetic fields induced by Hertz dipole $P\delta(\theta, r - b)\exp(-i\omega t)$ (δ being a delta-function). For the same of simplicity of the expose we consider $\vec{\varepsilon}$ as independent of ϕ and equal to $\vec{\varepsilon}(\theta, \phi = \vec{\phi}, \vec{r})$, where $\phi = \vec{\phi} =$ const is a plane

(*) METOD RESHENIYA OBSHCHEY KRAYEVOY ZADACHI RASPROSTRANENİYA DLINNYKH I SEVRKHDLINNYKH RADIOVOLN VOKRUG ZEMLI passing through the polar axis and observation point M. Then, introducing the potentials A(θ, r) and B(θ, r), linked with $H_\phi = \partial B/r\,\partial\theta$ and $E_\phi = \partial A/\,r\,\partial\theta$ and their derivatives with respect to θ, $C = \partial B/\partial\theta$ and $D = \partial A\,/\partial\theta$, we shall write the boundary value problem in the form

$$\frac{\partial}{\partial\theta}\cdot\vec{B} - L_{r\theta}\cdot\vec{B} = \vec{P}\delta(\theta, r-b), \tag{1}$$

where $\vec{B}(\theta, r)$ is a single-column matrix with elements B, A, C, D, by which all the components are expressed of fields' $\vec{E}$ and $\vec{H}$ amplitudes, designated below as vector-function; $L_{r\theta}$ is the differentiel operator, represented by a 4 × 4-matrix $\|l_{ji}\|$, where

$$\begin{gathered}
l_{11} = l_{12} = l_{14} = l_{21} = l_{22} = l_{23} = 0; \quad l_{13} = l_{24} = 1; \\
l_{31} = -\frac{r^2\Delta}{\varepsilon_{\theta\theta}}\left[\frac{\partial}{\partial r}\left(\frac{\varepsilon_{rr}}{\Delta}\frac{\partial}{\partial r}\cdot\right) + k^2\cdot\right]; \quad l_{32} = -\frac{ikr^2\Delta}{\varepsilon_{\theta\theta}}\frac{\partial}{\partial r}\left[\frac{\varepsilon^*}{\Delta}\cdot\right]; \\
l_{33} = -\frac{r^2\Delta}{\varepsilon_{\theta\theta}}\left[\frac{\partial}{\partial r}\left(\frac{\varepsilon_{\theta r}}{r\theta}\cdot\right) + \frac{\varepsilon_{r\theta}}{r\Delta}\frac{\partial}{\partial r}\cdot + \operatorname{ctg}\theta\cdot\right]; \quad l_{34} = \frac{ikr\varepsilon^{**}}{\varepsilon_{\theta\theta}}; \\
l_{41} = \frac{ikr^2\bar{\varepsilon}^*}{\Delta}\frac{\partial}{\partial r}\cdot; \quad l_{42} = -r^2\left[\frac{\partial^2}{\partial r^2}\cdot + k^2\tilde{\varepsilon}\cdot\right]; \\
l_{43} = \frac{ikr\bar{\varepsilon}^{**}}{\Delta}\cdot; \quad l_{44} = -\operatorname{ctg}\theta\cdot;
\end{gathered} \tag{2}$$

$$\begin{gathered}
\Delta = \varepsilon_{\theta\theta}\varepsilon_{rr} - \varepsilon_{\theta r}\varepsilon_{r\theta}; \quad \tilde{\varepsilon} = \varepsilon_{\varphi\varphi} + (\varepsilon_{\varphi\theta}\varepsilon^* + \varepsilon_{\varphi r}\varepsilon^{**})/\Delta; \\
\varepsilon^* = \varepsilon_{r\varphi}\varepsilon_{\theta r} - \varepsilon_{\theta\varphi}\varepsilon_{rr}; \quad \varepsilon^{**} = \varepsilon_{r\theta}\varepsilon_{\theta\varphi} - \varepsilon_{\theta\theta}\varepsilon_{r\varphi}; \\
\bar{\varepsilon}^* = -\varepsilon_{\varphi r}\varepsilon_{r\theta} + \varepsilon_{\varphi\theta}\varepsilon_{rr}; \quad \bar{\varepsilon}^{**} = \varepsilon_{\theta r}\varepsilon_{\varphi\theta} - \varepsilon_{\theta\theta}\varepsilon_{\varphi r}
\end{gathered} \tag{3}$$

at continuity conditions of $\vec{B}$ on the interfaces of regions r = a and r = c and and limitation as $r \to 0$, $r \to \infty$, $\theta \to 0$, $\theta \to \pi$.

Eq.(1) is approximate, for in it the integral is dropped, under the sign of which stand the products of elements $\vec{B}$ by the derivatives of components of $\vec{\varepsilon}$ with respect to Θ.

2. <u>Method of Connected Lines</u>. We seek the solution of (1) in the form

$$B = \sum Y_k(r)\cdot b_k(\theta), \tag{4}$$

where $\vec{b}_k$ is a vector-function with elements b_k, a_k, c_k, d_k, and $\vec{Y}_k$ is a vector-function with elements Y_k, Z_k, U_k, V_k, which is the eigenfunction of operator $L_{r\theta}$, obtained from $L_{r\theta}$, by "freezing" with respect to θ of coefficients in $\|l_{ji}\|$; θ enters into $L_{r\bar{\theta}}$ as a parameter. $\vec{Y}_k$ are determined from the equation

$$(Y_h, Y_r^*) = \delta_{hr}; \quad \text{under conditions} \quad L_{r\bar{\theta}}\cdot Y = \lambda Y \tag{5}$$

λ(θ) are eigenvalues of $L_{r\bar{\theta}}$; they lie in the 2nd and 4th quadrants of plane λ. We shall number them in order of modulus increase, respectively k = 1, 2,... and k =-1, -2, ... Parentheses in (5) denote the scalar products, while the star denotes the eigenfunctions of the conjugate operator. Let us introduce $L_{r\bar{\theta}}$ into (1):

$$\frac{\partial}{\partial\theta}\cdot B - L_{r\bar{\theta}}\cdot B + [L_{r\bar{\theta}} - L_{r\theta}]\cdot B = P\delta(\theta, r-b). \tag{6}$$

Substituting (4) into (6), multiplying scalarly by $\vec{Y}_r{}^*$, and taking into account (5), we shall obtain a system of equations of connected lines [1], integrated by means of a computer:

$$\frac{d}{d\theta}\cdot \mathbf{b}_k - i\nu_k(\theta)\,\mathbf{b}_k + \sum_{r=-\infty}^{\infty}\left\| S_{ji}^{(k,r)}(\theta)\right\|\cdot \mathbf{b}_r = \mathbf{P}_k\delta(\theta);\quad \lambda_k = i\nu_k. \tag{7}$$

S_{ji} are 4 × 4-matrices of complex numbers, formed from scalar products $[L_{r\bar\theta} - L_{r0}]\cdot\vec{Y}_k$ by $\vec{Y}_k{}^*$. If the spectrum of $L_{r\bar\theta}$ has a continuous part, then still another integral over ν enters into (7). For small S_{ji} and $\nu_k = \nu_r$, i. e. at range from spatial resonance cones [12], neglecting S_{ji}, we shall obtain a solution in the form of a sum of modulated normal waves (*)

$$\mathbf{B}\exp(-i\omega t) = \sum_{k=-\infty}^{+\infty} C_k \mathbf{Y}_k(r,\theta)\exp[-i(\omega t - \textstyle\int \nu_k\, d\theta)]. \tag{4'}$$

In (7), S_{ji} account for the interaction of normal waves over portions of the course where $\vec{\varepsilon}$ depends on θ, for example in the sunrise and sunset band, and also on account of inhomogeneity of space metrics in spherical coordinates. It should be noted that when the Earth's magnetic field $\vec{H}_0$ is not vertical, when $\varepsilon_{\theta r}$ and $\varepsilon_{r\phi}$ are not zero, $\nu_k \neq -\nu_{-k}$, and this is why the principle of interaction is disrupted as waves propagate along θ.(refer to [11]).

-:-:-

3. In order to find the wave numbers ν_k, it is convenient to exclude C and D from (1) and pass to equations of connected lines containing derivatives of 2nd order with respect to θ. Then (5) passes into the following boundary value problem on parameter ν eigenvalues:

$$\begin{aligned} &Y_{rr}'' + \mathfrak{a}Y_r' + \mathfrak{b}Y + \mathfrak{c}Z_r' + \mathfrak{d}Z = 0;\\ &Z_{rr}'' + \mathfrak{e}Z + \mathfrak{f}Y_r' + \mathfrak{g}Y = 0, \end{aligned} \tag{5'}$$

where the prime denotes the derivative with respect to <u>r</u>;

$$\mathfrak{a} = \frac{\Delta}{\varepsilon_{rr}}\left[\left(\frac{\varepsilon_{rr}}{\Delta}\right)_r' \pm ik\frac{(\varepsilon_{\theta r}+\varepsilon_{r\theta})}{\Delta}S\right] \qquad \mathfrak{b} = \frac{k^2\Delta}{\varepsilon_{rr}}\left[1 - \frac{\varepsilon_{\theta\theta}}{\Delta}S^2 \pm \frac{ir}{k}\left(\frac{\varepsilon_{\theta\theta}}{r\Delta}\right)_r' S\right];$$

$$\mathfrak{c} = \frac{ik\varepsilon^*}{\varepsilon_{rr}};\qquad \mathfrak{d} = ik\left(\frac{\varepsilon^*}{\Delta}\right)_r' \frac{\Delta}{\varepsilon_{rr}} \pm \frac{k^2\varepsilon^{**}}{\varepsilon_{rr}}S;\ \mathfrak{e} = k^2(\tilde{\varepsilon} - S^2);$$

$$\mathfrak{f} = -\frac{\widetilde{ik\varepsilon^*}}{\Delta};\qquad \mathfrak{g} = \pm\frac{\widetilde{k^2\varepsilon^{**}}}{\Delta}S;\qquad S = \frac{\nu}{kr}.$$

(*) The normal waves [1 - 4, 11] are also called free, proper (eigen), and in english they are called "residue waves and modes [5 - 7], which is not properly translated as "modes"

It is necessary to find ν, assuring a nonzero solution of (5') at conditions of limitation $|Y|$ and $|Z|$ as $r \to 0$ and $r \to \infty$. It is practical to resolve (5') on a computer by the "drive through" method of conditions as $r \to 0$ and $r \to \infty$ toward a certain point $r = \bar{r}$, where Y, Z, Y_r' and Z_r' must be continuous. Such a method is equivalent to the more economical "drive through" and joining of impedances $|z|$ of the surface $\bar{r}$ = const for regions $(\bar{r},\infty)$ and $(\bar{r},0)$, denoted by $|z(\bar{r}+0)|$ and $|z(\bar{r}-0)|$ and determinable by the expressions

$$E_\theta = z_{11}^\nu(\bar{r}\pm 0)H_\theta + z_{12}^\nu(\bar{r}\pm 0)H_\varphi; \quad E_\varphi = z_{21}^\nu(\bar{r}\pm 0)H_\theta + z_{22}^\nu(\bar{r}\pm 0)H_\varphi;$$

$|z|$ are expressed by Y, Y_r', Z and Z_r'. The joining equation of $|z|$ will be:

$$\det\begin{vmatrix} z_{11}^\nu(\bar{r}+0) - z_{11}^\nu(\bar{r}-0) & z_{12}^\nu(\bar{r}+0) - z_{12}^\nu(\bar{r}-0) \\ z_{21}^\nu(\bar{r}+0) - z_{21}^\nu(\bar{r}-0) & z_{22}^\nu(\bar{r}+0) - z_{22}^\nu(\bar{r}-0) \end{vmatrix} = 0. \tag{8}$$

Its roots will be ν_k. Let us choose for $\bar{r}$ the surface $\bar{r} = c$. In order to find $|z|$ from (5') let us introduce the impedance functions $\underline{u}$, κ; $Y = \exp\int u\,dr$; $Z = \kappa Y$; substituting them into (5), we obtain

$$\begin{aligned} &u_r' + u^2 + \mathfrak{a}u + \mathfrak{b} + \mathfrak{c}(\varkappa_r' + u\varkappa) + \mathfrak{d}\varkappa = 0, \\ &\varkappa_{rr}'' + 2u\varkappa_r' + u^2\varkappa + u_r'\varkappa + \mathfrak{e}\varkappa + \mathfrak{f}u + \mathfrak{g} = 0. \end{aligned} \tag{9}$$

For the computation of $|z^\nu(c+0)|$ we shall integrate (9) on the computer from r_∞ to $r = c$, where $r_\infty \gg c$ is chosen in a region in which $\vec{\varepsilon}$ is practically constant. The initial values of $u(r_\infty)$ will be obtained from Eq. (9) at $u_r' = \kappa_r' = \kappa_r'' = 0$:

$$u^4 + \bar{\mathfrak{a}}u^3 + (\bar{\mathfrak{e}} + \bar{\mathfrak{b}} - \overline{\mathfrak{c}\mathfrak{f}})u^2 + (\overline{\mathfrak{e}\mathfrak{a}} - \overline{\mathfrak{c}\mathfrak{g}} - \overline{\mathfrak{d}\mathfrak{f}})u + (\overline{\mathfrak{e}\mathfrak{b}} - \overline{\mathfrak{d}\mathfrak{g}}) = 0, \tag{10}$$

passing to the Booker equation [8] as $r \to \infty$, provided the link of $\vec{\varepsilon}$ with plasma parameters is determined from the Lorentz equation [formula (1) from [11]]. From the four roots of (10) we shall select $u^o(r_\infty)$ and $u^e(r_\infty)$, corresponding to ordinary and extraordinary waves escaping to $+\infty$. The initial $\kappa(r_\infty)$ for these $\underline{u}$ will be found from the formula $\varkappa = -(\bar{\mathfrak{g}} + \bar{\mathfrak{f}}\bar{u}) / (\bar{\mathfrak{e}} + u^2)$. Integrating (9) from $r = r_\infty$ toward the side of $\underline{r}$ decrease, i. e. toward traveling waves, implies a continuous transformation of impedances of type-o and -e waves

$$\begin{aligned} &Z_y^{eo}(r) = \frac{E_\theta}{H_\varphi} = \frac{1}{\Delta}\left[\frac{\varepsilon_{rr}}{ik}u^{e,o}(r) + \varepsilon^*\varkappa^{e,o}(r) \pm \varepsilon_{0r}S\right]; \\ &Z_z^{e,o}(r) = \frac{H_\theta}{E_\varphi} = -\frac{1}{ik}\left[u^{e,o}(r) + \frac{\varkappa_r'^{e,o}}{\varkappa^{e,o}}\right]; \quad X^{e,o}(r) = \frac{E_\varphi}{H_\varphi} = \varkappa^{e,o}(r) \end{aligned} \tag{11}$$

from adiabatic values at $r = r_\infty$ to values at the surface $r = \bar{r}$. These impedances are linked with the earlier introduced impedances z_{jk} of the surface $r = \bar{r}$ by the expressions

$$\begin{aligned} &z_{11} = (Z_y^e - Z_y^o)/\delta; \quad z_{12} = (X^eZ_z^eZ_y^o - X^oZ_z^oZ_y^e)/\delta; \\ &z_{21} = (X^e - X^o)/\delta, \quad z_{22} = (X^e - X^o)Z_z^eZ_z^o/\delta, \end{aligned} \tag{12}$$

where $\delta = X^eZ_z^e - X^oZ_z^o$. Having carried integration to the point $r = c$, we shall obtain the impedances of $z_{jk}(c+0)$ according to formulas (11) and (12).

The impedances z_{jk} (c + 0) are found in two stages. At first for two values $u^{o,e}(r_0)$, $\varkappa^{o,e}(r_0)$, where $r_0 \ll a$, corresponding to waves traveling toward the center of the Earth, we integrate (9) from r = r_0 to the surface of the Earth, r = a, and we find analogously to the case of the ionosphere

$$z_{12}(a-0) = Z_y^e = Z_y^o \text{ and } z_{21}^{-1}(a-0) = Z_z^e = Z_z^o.$$

Because of Earth's isotropy $z_{11}(a-0) = z_{22}(a-0) = 0$. The transformation of impendances on the following segment (a, c) is performed analytically by formulas

$$\begin{aligned} iz_{12}^{\nu}(c-0) &= \frac{D_\nu(a',c') - iz_{12}^{\nu}(a-0)D_\nu(a,c')}{D_\nu(a',c) - iz_{12}^{\nu}(a-0)D_\nu(a,c)} = \frac{\overline{D_\nu(a',c')}}{\overline{D_\nu(a',c)}}, \\ iz_{21}^{\nu}(c-0) &= \frac{D_\nu(a,c) - iz_{21}^{\nu}(a-0)D_\nu(a',c)}{D_\nu(a,c') - iz_{21}^{\nu}(a-0)D_\nu(a',c')} = \frac{\overline{D_\nu(a,c)}}{\overline{D_\nu(a,c')}}, \end{aligned} \tag{13}$$

where $D_\nu(a,c)$ are two-argument functions from from products of modified Hankel functions $h_\nu^{(1,2)}(ka)$ & $h_\nu^{(1,2)\prime}(kc)$ [2 - 4]; $z_{11}{}^\nu$(c - 0) and $z_{22}\nu$(c - 0) are zero. Substituting (11), (12), (13) into (8), we obtain the final equation for the determination of wave numbers of normal waves

$$\begin{aligned} &[\overline{D_\nu(a',c')} - iZ_y^e \overline{D_\nu(a',c)}][\overline{D_\nu(a,c')} + iZ_z^0 \overline{D_\nu(a,c)}] - \\ &- (X^e/X^o)[\overline{D_\nu(a',c')} - iZ_y^0 \overline{D_\nu(a',c)}][\overline{D_\nu(a,c')} + iZ_z^e \overline{D_\nu(a,c)}] = 0. \end{aligned} \tag{14}$$

Concomittantly with (9) and (11), Eq.(14) was utilized in [2, 3] for the case of vertical $\vec{H}_0$, when (10) becomes biquadratic. The values of $\bar{u}$ and $\bar{\kappa}$ for such case are given by formulas (2.40) - (2.41) of [3]. If the ionosphere is uniform, that is, if N_e = 0, for r < c and N_e = N_e,r > c, there is no necessity to integrate (9), for the impedances are obtainable directly from (11)

$$Z_y^{e,o}(c) = \frac{1}{\Delta}\left[\frac{\varepsilon_{rr}}{ik}\bar{u}^{e,o} + \varepsilon^* \bar{\varkappa}^{e,o} \pm \varepsilon_{\theta r} S\right]; \quad Z_z^{e,o} = -\frac{1}{ik}\bar{u}^{e,o}; \quad X^{e,o} = \bar{\varkappa}^{e,o}. \tag{15}$$

With these impedances Eq.(14) was utilized in the work [4] dor the case of vertical magnetic field of the Earth, when $\varepsilon_{\theta r}$ = 0, $\varepsilon^* = -\varepsilon_{\theta\phi}\varepsilon_{rr}$, $\Delta = \varepsilon_{\theta\theta}\varepsilon_{rr}$. $\bar{u}^{e,o}$ and $\bar{\kappa}^{e,o}$ are determined in this case from the biquadratic Eq.(10) according to formulas (2.40-2.41) of the work [3]. In the vicinity of ν = ka and ν = kc functions $h_\nu^{(1)}$ and $h_\nu^{(2)}$ in (14) were approximated by Eyri functions:

$$h_\nu^{(1,2)}(z) = z^{1/6} h_{1,2}(\zeta); \quad \zeta = -\eta(\nu - z); \quad \eta = \sqrt[3]{z/2} \tag{16}$$

by the "comparison method" of [9]. In the first of works [4], wher manual calculation was applied, we utilized the ables of functions $h_{1,2}$ (ζ) [10] with 8 significant numerals for the complex argument ζ in the circle $|\zeta| \leqslant 6$. In [2, 3] for the computations of $h_{1,2}$ (ζ) series were applied, that were available in preface to [10].

Introducing ν_k and Y_k into (4'), we obtain I (one) approximation of the solution of the boundary value problem. The solution of (7), taking into account the terms S_{ji} will give II approximation. To make more precise the interaction of normal waves, we should take into account the integral terms, dropped in (7).

**** THE END ****

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