X-641-68-188

PREPRINT

GUIDING CENTER DRIFTS IN TIME-DEPENDENT MERIDIONAL MAGNETIC FIELDS

Thomas J. Birmingham

Laboratory for Theoretical Studies

May 1968

GODDARD SPACE FLIGHT CENTER

Greenbelt, Maryland

GUIDING CENTER DRIFTS IN TIME-DEPENDENT MERIDIONAL MAGNETIC FIELDS

Thomas J. Birmingham

Laboratory for Theoretical Studies

ABSTRACT

The problem of a particle trapped in a time dependent meridional magnetic mirror field and at the same time subjected to a time dependent perpendicular electric field ($\mathbf{E} \cdot \mathbf{B} = 0$) is considered. Bounce averaged guiding center theory is used to derive expressions for the drift velocity components. These drift expressions depend in a straightforward fashion on the structures of the magnetic and electric fields and on the particle charge, kinetic energy, longitudinal invariant, and bounce period. In the special case of a static magnetic field, no electric field, and $J = 0$ particles, the drift equations are integrated and the bounce averaged guiding center trajectory obtained.

GUIDING CENTER DRIFTS IN TIME-DEPENDENT MERIDIONAL MAGNETIC FIELDS

INTRODUCTION

We are concerned in this paper with the guiding center drifts of particles trapped in mirror magnetic fields whose field lines are totally contained in constant longitude—i.e., constant ϕ—planes (in a spherical coordinate system). We call magnetic fields of this type meridional magnetic fields, the field of a magnetic dipole being an example. However, no requirement of axi-symmetry is imposed on the field. Time variations which preserve the meridional character of the field are also allowed.

Electric fields accompany such magnetic time variations. In addition there may exist electrostatic electric fields ($\nabla \times \mathbf{E} = 0$). (A source of such electrostatic fields might be a low-β plasma confined by the magnetic geometry.) We here assume that the total electric field lies perpendicular to the magnetic field ($\mathbf{E} \cdot \mathbf{B} = 0$).

For the field of a static magnetic dipole and in the absence of electric fields, Northrop [1966], using guiding center theory, has derived a rather simple expression for $\langle \dot{\phi} \rangle$, the longitudinal drift rate of a particle averaged over its bounce motion (Bounce averages are denoted by $\langle \ \rangle$.),

$$\langle \dot{\phi} \rangle = - \frac{2 c r_0 W}{e \mu} \left(\frac{3}{2} - \frac{J}{4WT} \right). \tag{1}$$

Here c is the speed of light; μ is the magnetic moment of the dipole; r_0 is the equatorial crossing radius of the guiding center drift shell; and e, W, J, and T are respectively the charge, kinetic energy, longitudinal invariant, and bounce period of the drifting particle,

$$J = m \oint v_{\|} \, ds \,, \tag{2}$$

$$T = \oint \frac{ds}{v_{\|}} \,, \tag{3}$$

m being the particle mass and the s-integrals being extended over a complete guiding center bounce period. Equation (1) for the dipole field drift rate can be shown (with some amount of algebra) to be exactly equivalent to the expression derived by Hamlin, Karplus, Vik, and Watson [1961]. Northrop's result is aesthetically pleasing in that only the familiar particle parameters e, W, J, and T appear in it; our results for the drifts in more general fields share this property.

THEORY

We have found that for other meridional (but not necessarily either axi-symmetric or time independent) magnetic fields, the drift rate is simply expressed in terms of the particle parameters e, W, J, and T; the quantities α and β identifying the field line upon which the guiding center is instantaneously bouncing; and derivatives of the electric potential Φ. A derivation of these results is communicated here.

A guiding center drifts across magnetic field lines at a rate which depends on its bounce phase. We here assume that the size of the electric field and the space and time variations of both the magnetic and electric fields are sufficiently small that over a complete bounce period a guiding center drifts only a small amount. The bounce path is then nearly periodic, and it is significant to consider the rate of guiding center drift averaged over the bounce motion. We are here concerned with this average drift of guiding centers.

The bounce averaged drift rate is resolved into components $\langle\dot{\alpha}\rangle$ and $\langle\dot{\beta}\rangle$, the Euler potentials α and β being defined in the usual fashion

$$\mathbf{B}(\mathbf{r}, t) = \nabla\alpha(\mathbf{r}, t) \times \nabla\beta(\mathbf{r}, t) . \tag{4}$$

The drifts $\langle\dot{\alpha}\rangle$ and $\langle\dot{\beta}\rangle$ determine a trajectory in a cartesian two-dimensional α, β space, each point of which represents a magnetic field line. The relations $\alpha = \alpha(\mathbf{r}, t)$, $\beta = \beta(\mathbf{r}, t)$ may subsequently be used to translate the α, β trajectory into one—from field line to field line—in real space.

Northrop [1963] provides us with the dynamical equations for averaged guiding centers,

$$\langle\dot{\beta}\rangle = -\frac{c}{eT}\frac{\partial J}{\partial \alpha}(\alpha, \beta, K, M, t) , \tag{5a}$$

$$\langle\dot{\alpha}\rangle = \frac{c}{eT}\frac{\partial J}{\partial \beta} , \tag{5b}$$

correct to leading order in the adiabatic (m/e) expansion. The quantity K is given for each particle by the expression

$$K = \frac{mv_{\|}^2}{2} + MB + e\left[\Phi + \frac{\alpha}{c}\frac{\partial\beta}{\partial t}(\mathbf{r}, t)\right], \tag{6}$$

with M the particle magnetic moment $(M = W_{\perp}/B)$. In the absence of time variations K is thus the total energy of a particle moving in the magnetic field **B** through the electrostatic potential Φ.

For meridional magnetic fields we can and shall choose β as longitude ϕ. With this choice $\nabla\beta = (r \sin\vartheta)^{-1}\,\hat{e}_{\phi}$ is totally azimuthal (hence perpendicular to the magnetic field) and $\partial\beta/\partial t\,(\mathbf{r}, t) = 0$. The Euler potential $\alpha(r, \vartheta, \phi, t)$ is then from Equation (4) any well behaved solution to

$$\frac{\partial\alpha}{\partial\vartheta} = r^2 \sin\vartheta\, B_r\,, \tag{7a}$$

$$\frac{\partial\alpha}{\partial r} = -r\sin\vartheta\, B_{\vartheta}\,. \tag{7b}$$

On the time scale of many bounces K is not a conserved quantity in this time dependent situation.

We now solve Equation (6) for $v_{\|}$ and substitute the result into the defining Equation (2) for J. We do so, noting that the infinitesimal arc length ds is (from the field line equation) $rB\,d\vartheta/B_{\vartheta}$, $d\vartheta$ being the increment in co-latitude subtended by ds. Substituting the ensuing expression for J into Equations (5), we obtain

$$\langle\dot{\phi}\rangle = -(2m)^{1/2}\,\frac{2c}{eT}\int_{\vartheta_\ell}^{\vartheta_\mu} d\vartheta \left\{ \left[K - MB - e\Phi\right]^{1/2} \frac{\partial}{\partial\alpha}\left(\frac{rB}{B_\vartheta}\right) - \frac{rB}{2B_\vartheta}\,\frac{\frac{\partial}{\partial\alpha}(MB + e\Phi)}{\left[K - MB - e\Phi\right]^{1/2}} \right\}, \quad (8a)$$

$$\langle\dot{\alpha}\rangle = (2m)^{1/2}\,\frac{2c}{eT}\int_{\vartheta_\ell}^{\vartheta_\mu} d\vartheta \left\{ \left[K - MB - e\Phi\right]^{1/2} \frac{\partial}{\partial\phi}\left(\frac{rB}{B_\vartheta}\right) - \frac{rB}{2B_\vartheta}\,\frac{\frac{\partial}{\partial\phi}(MB + e\Phi)}{\left[K - MB - e\Phi\right]^{1/2}} \right\}. \quad (8b)$$

The integrals in Equations (8) are carried out at constant α and ϕ between the mirror co-latitudes, ϑ_ℓ and ϑ_μ, of the bouncing particle. The multiplicative factor of 2 in each appears to compensate for the fact that the ϑ-integral represents only half the bounce motion. No contribution results from differentiation of the limits of integration since $v_\parallel$ vanishes at both ϑ_ℓ and ϑ_μ.

We now rearrange the integrands of Equations (8)

$$\langle\dot{\phi}\rangle = -(2m)^{1/2}\,\frac{2c}{eT}\int_{\vartheta_\ell}^{\vartheta_\mu} d\vartheta\,\frac{rB}{B_\vartheta} \left\{ \left[K - MB - e\Phi\right]^{1/2} \left[\frac{B_\vartheta}{rB}\frac{\partial}{\partial\alpha}\left(\frac{rB}{B_\vartheta}\right) + \frac{1}{2B}\frac{\partial B}{\partial\alpha}\right] \right.$$

$$\left. - \frac{1}{2}\left[K - MB - e\Phi\right]^{-1/2} \left[\frac{(K - e\Phi)}{B}\frac{\partial B}{\partial\alpha} + e\,\frac{\partial\Phi}{\partial\alpha}\right] \right\}, \quad (9a)$$

$$\langle\dot{\alpha}\rangle = (2m)^{1/2} \frac{2c}{eT} \int_{\vartheta_\ell}^{\vartheta_\mu} d\vartheta \frac{rB}{B_\vartheta} \left\{ \left[K - MB - e\Phi\right]^{1/2} \left[\frac{B_\vartheta}{rB} \frac{\partial}{\partial\phi}\left(\frac{rB}{B_\vartheta}\right) + \frac{1}{2B}\frac{\partial B}{\partial\phi}\right] \right.$$

$$\left. - \frac{1}{2}\left[K - MB - e\Phi\right]^{-1/2} \left[\frac{K - e\Phi}{B}\frac{\partial B}{\partial\phi} + e\frac{\partial\Phi}{\partial\phi}\right] \right\} \cdot \qquad (9b)$$

The general expression for the electric field [Northrop, 1963]

$$\mathbf{E} = \frac{1}{c}\left(\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right) - \nabla\left(\Phi + \frac{\alpha}{c}\frac{\partial\beta}{\partial t}\right) \qquad (10)$$

reduces for the choice $\beta = \phi$ to

$$\mathbf{E} = -\frac{\hat{e}_\phi}{c r \sin\vartheta}\frac{\partial\alpha}{\partial t} - \nabla\Phi \; . \qquad (11)$$

In our model $\mathbf{E} \cdot \mathbf{B} = 0$ and it follows that magnetic field lines are equipotentials, so that Φ, $\partial\Phi/\partial\alpha$, and $\partial\Phi/\partial\phi$ remain constant for the integrations occurring in Equations (9). Furthermore, to the order in m/e which we are working the variation in K over the bounce motion is negligible [Northrop, 1963].

Consider now mirror-type, meridional magnetic fields having the properties

$$\frac{B_\vartheta}{rB}\frac{\partial}{\partial\alpha}\left(\frac{rB}{B_\vartheta}\right) + \frac{1}{2B}\frac{\partial B}{\partial\alpha} = a_0(\alpha, \phi, t)\,, \qquad (12a)$$

$$\frac{1}{B}\frac{\partial B}{\partial\alpha} = a_1(\alpha, \phi, t)\,, \qquad (12b)$$

a_0 and a_1 being functions independent of distance along the field line α, ϕ. Fields with the properties Equations (12) [and similarly fields satisfying Equations (14) following] are selected for consideration because of the simplicity of the resultant expressions for the drifts. For fields of the form Equation (12), $\langle\dot{\phi}\rangle$ becomes

$$\langle\dot{\phi}\rangle = -\frac{c}{eT}\left[Ja_0 - T\left(Wa_1 + e\frac{\partial\Phi}{\partial\alpha}\right)\right] . \tag{13}$$

In obtaining Equation (13) from Equation (9a) the definitions of J and T were used and the particle kinetic energy $W = K - e\Phi$ was identified.

Similarly, for magnetic fields with the properties

$$\frac{B_\vartheta}{rB}\frac{\partial}{\partial\phi}\left(\frac{rB}{B_\vartheta}\right) + \frac{1}{2B}\frac{\partial B}{\partial\phi} = b_0\,(\alpha, \phi, t)\,, \tag{14a}$$

$$\frac{1}{B}\frac{\partial B}{\partial\phi} = b_1\,(\alpha, \phi, t)\,, \tag{14b}$$

the drift $\langle\dot{\alpha}\rangle$ assumes the form

$$\langle\dot{\alpha}\rangle = \frac{c}{eT}\left[Jb_0 - T\left(Wb_1 + e\frac{\partial\Phi}{\partial\phi}\right)\right] . \tag{15}$$

When both the meridional magnetic field and the potential Φ are axi-symmetric, $\langle\dot{\alpha}\rangle = 0$.

Mirror-type meridional magnetic fields of the form

$$B_r = \frac{1}{r^p}\frac{1}{(p-2)}\frac{g(\phi, t)}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\, f(\vartheta, t)\right], \tag{16a}$$

$$B_\vartheta = \frac{1}{r^p} f(\vartheta, t)\, g(\phi, t), \tag{16b}$$

$$B_\phi = 0, \tag{16c}$$

separable in the coordinates r, ϑ, and ϕ, have the properties indicated in Equations (12) and (14). In Equations (16) p is an arbitrary number greater than 2, and f and g are subject to the restriction that the magnetic field be well defined for all $r > 0$. The axi-symmetric multipole fields are special, curl-free cases of Equations (16) and correspond to integral $p(\geq 3)$, g independent of ϕ, and $f(\vartheta) = -dP_{p-2}(\cos\vartheta)/d\vartheta$, P_{p-2} being the Legendre polynomial of order $p-2$.

Equations (16) describe a mirror geometry in a region of space provided that at least one magnetic minimum occurs on field lines, i.e., $\partial B/\partial\vartheta|_{\alpha,\phi,t} = 0$, $\partial^2 B/\partial\vartheta^2|_{\alpha,\phi,t} > 0$ for at least one point on B-lines in that spatial region.

For the field, Equation (16), we choose

$$\alpha = \frac{r^{2-p}}{p-2}\sin\vartheta\, f(\vartheta, t)\, g(\phi, t). \tag{17}$$

With this choice all gauge freedom in this problem is eliminated, for $\partial\Phi/\partial\alpha$ and $\partial\Phi/\partial\phi$ must now be such as to yield the correct electric field through Equation (11).

It may now be directly verified that

$$a_0 = \frac{1}{2\alpha}, \qquad (18a)$$

$$a_1 = \frac{p}{p-2}\frac{1}{\alpha}, \qquad (18b)$$

$$b_0 = 0, \qquad (18c)$$

and

$$b_1 = \frac{2}{2-p}\frac{1}{g}\frac{\partial g}{\partial \phi}. \qquad (18d)$$

Substituting into Equations (13) and (15), we find for the drift components

$$\langle\dot{\phi}\rangle = -\frac{c}{eT\alpha}\left[\frac{J}{2} - T\left(\frac{\bar{W}p}{p-2} + e\alpha\frac{\partial\Phi}{\partial\alpha}\right)\right], \qquad (19)$$

$$\langle\dot{\alpha}\rangle = -\frac{c}{e}\left(\frac{2W}{2-p}\frac{1}{g}\frac{\partial g}{\partial\phi} + e\frac{\partial\Phi}{\partial\phi}\right). \qquad (20)$$

Northrop's dipole field result, Equation (1), is recovered from Equation (19) by the identification $\Phi = 0, p = 3, g = -\mu, f = \sin\vartheta$, and (from Equation 17) $\alpha = -\mu\sin^2\vartheta/r = -\mu/r_0$.

In the general, time dependent case, the complicated α, ϕ, t dependence of the right sides of Equations (19) and (20) prohibits analytic determination of the guiding center trajectory. In simplification we therefore focus attention on the situation where the magnetic field is static and there are no electric fields. Now the particle kinetic energy W is constant, Φ is zero, and we obtain

$$\frac{d\alpha}{d\phi} = \frac{\langle\dot{\alpha}\rangle}{\langle\dot{\phi}\rangle} = \frac{2WT}{2-p}\,\frac{\alpha}{g}\,\frac{dg}{d\phi}\left[\frac{J}{2} - \frac{pWT}{p-2}\right]^{-1} \tag{21}$$

by dividing Equation (20) by Equation (19).

Even in the form Equation (21), the α, ϕ dependence of the bounce period T inhibits further progress. The case of $J = 0$ particles is, however, one limiting situation which can be handled. These particles have no component of motion along the magnetic field and drift so as to always remain at a local magnetic minimum, $\partial B/\partial \vartheta|_{\alpha,\phi} = 0$. The bounce period T for such particles defined in the $J \to 0$ limit is non-vanishing. For $J = 0$ particles, Equation (21) reduces to

$$\frac{d\alpha}{d\phi} = \frac{2\alpha}{pg}\,\frac{dg}{d\phi}\,. \tag{22}$$

Equation (22) can be integrated to yield

$$\alpha = Ag^{2/p}\,, \tag{23}$$

A being a constant determined for each trajectory. In order that the magnetic field be well defined for all $r > 0$, g must be a periodic function of ϕ. It follows then from Equation (23) that a particle executes closed loops in α, β space, indicative of the fact that after each real space excursion of 2π in longitude a particle returns to the same field line for this static situation.

The time development of the trajectory may now be traced using Equation (23) coupled with Equation (19) simplified for static **B**, $\Phi = 0$, and $J = 0$,

$$\frac{d\phi}{dt} = \frac{c}{e\alpha}\frac{p}{p-2}W = \frac{c}{eA}\frac{p}{p-2}Wg^{-2/p}. \tag{24}$$

Integration of Equation (24) depends, of course, on a choice of g, and for most g's the ϕ-integral cannot be explicitly evaluated. For the interesting case of small azimuthal modulations of the form $g = 1 + \epsilon \cos^n \phi$, $\epsilon << 1$, $n > 0$, it is possible, however, to develop from Equation (24) the time history of ϕ as an asymptotic series in ϵ.

ACKNOWLEDGMENT

Dr. T. Northrop's suggestion of this problem and discussions with him, Dr. J. Burns, Dr. F. Jones, and Dr. D. Stern are gratefully acknowledged.

REFERENCES

Hamlin, D. A., R. Karplus, R. C. Vik, and K. M. Watson, Mirror and Azimuthal Drift Frequencies for Geomagnetically Trapped Particles, J. Geophys. Res., 66, 1, 1961.

Northrop, T. G., Adiabatic Theory of Charged Particle Motion, in Radiation Trapped in the Earth's Magnetic Field, edited by B. M. McCormac, p. 26, D. Reidel Publishing Company, Derdrecht Holland, 1966.

Northrop, T. G., The Adiabatic Motion of Charged Particles, Interscience Publishers, New York, 1963.