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LIE SERIES FOR CELESTIAL MECHANICS, ACCELERATORS, SATELLITE STABILIZATION AND OPTIMIZATION

by F. Cap, F. Ehlotzky, D. Floriani, W. Groebner, H. Knapp, A. Schett, and J. Weil

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Preface

The Lie series method for the solution of differential equations, for the inversion of systems of functions, for the investigation of the zeros of polynomials, etc., is now 10 years old. In a monograph published in August 1966 by NASA under No. NASA CR-552, Solution of Ordinary Differential Epuations by Means of Lie Series, the theoretical background (including numerical computation of e. g. Mathieu and Weber functions) was given.

In this monograph not only practical applications of Lie series are considered, but also a nearly complete bibliography of all work done using Lie series is given.

A short survey on the contents of this monograph is given in the Introduction, page 1. Futhermore, there are short abstracts describing the contents of each chapter. These abstracts can be found at the beginning of each chapter. We treat particle accelerators, the gravity gradient stabilization method of artificial satellites, orbit calculations of celestial mechanics and optimization and nonlinear control problems.

In the US Dr. Wilson, Chief, Applied Mathematics Section of NASA was the first to recognize the theoretical and practical advantages of the new method.

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Introduction

by

A. Schett, J. Weil

This monograph comprises the research work done in the field of Lie series and their physical and technical applications during the second year of the contract. A Lie series is a series of the form

$$\sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} D^{\nu} f(z) = f(z) + tDf(z) + \frac{t^2 D^2 f(z)}{2!} \ldots \qquad (I,1)$$

where $f(z)$ is any function depending on the complex variables z_1, z_2, ... z_n; D is a linear differential operator defined by:

$$D = \delta_1(z) \frac{\partial}{\partial z_1} + \delta_2(z) \frac{\partial}{\partial z_2} + \ldots + \delta_n(z) \frac{\partial}{\partial z_n} \qquad (I,2)$$

where the coefficients $\delta_i(z)$ represent functions of the complex variables z_1, z_2, ... z_n which are holomorphic in a certain neighborhood of z_0. As to the proof of convergence, see Ref.1,2.

Lie series have been used to solve differential equations of various kinds (see Ref.1-16). Other applications are inversions of functional systems (see Ref.17) and parameter representations of algebraic manifolds (see Ref.18). Furthermore, Lie series may be used in algebraic geometry (see Ref.19) and to represent implicitly given functions by means of Lie series (see Ref.20). Generalized Lie series using higher-order operators are treated in Ref.21. A very interesting example of the usefulness of Lie series is its application to the Hamilton-Jacobi theory (see Ref.2, Chapter IV). Their physical applications comprise many problems of technological and theoretical significance. F.CAP and J.MENNIG (see Ref.22)

***Extensive tables of numerical results for Chapter VI are available upon request from Chief, Applied Mathematics, (RRA), Research Division, OART, NASA, Washington, D.C. 20546**

and F.CAP and A.SCHLTT (see Ref.23) have solved initial and boundary value problems, respectively, occuring in reactor theory. The "Three-Body Problem Earth-Moon-Spaceship" was treated by W.GROEBNER and F.CAP (see Ref.25), the "Perturbation Theory of Celestial Mechanics Using Lie Series" by W.GROEBNER and F.CAP (see Ref.26) whereas a paper by W. GROEBNER and I.RAAB (see Ref.27) was concerned with rocket orbits in the field of several gravity centers (see Ref.27). The investigations of H.KNAPP are of fundamental importance for the attempts to improve the convergence (see Ref.28,29). Further boundary value problems were treated by G.WANNER (see Ref.3,38) and J.MENNIG (see Ref.30), who is particularly concerned with neutron flux problems. Many of these references are also cited in a summary volume covering many works on Lie series done by the Department of Mathematics of the University of Innsbruck (W.GROEBNER), edited by W.GROEBNER and H.KNAPP "Contributions to the Method of Lie Series" which appeared recently. As to further physical applications, we refer to the problems stated in the monograph of the previous contract: "Solution of Ordinary Differential Equations by Means of Lie Series", NASA Contractor Report CR-552, 1966.

The present monograph is concerned with the following problem:

Chapter I presents "The Solution of a System of n-th Order Differential Equations Using Lie Series", an extension of the considerations on second-order differential equations of the past year.

Chapter II is concerned with the Laplace equation; in the course of these investigations WEBER, HEINE, WANGERIN etc, functions were formally represented by Lie series.

Chapter III gives a physical application of great significance in high-energy physics. Lie series are used to calculate particle orbits in circular accelerators.

Chapter IV gives another physical application. Covering older work, it gives a survey on the application and the advantages of the Lie series method in celestial mechanics, especially, it deals with the numerical computation of satellite orbits using Lie series and compares them with other current methods.

Chapter V is devoted to an optimization problem. The Euler-Lagrangian equations of a fuel minimization problem connected with soft landing on the moon's surface are solved with the help of Lie series. A discussion of related problems is annexed.

Chapter VI deals with gravity-gradient stabilized satellites whose equations of motions are solved by means of Lie series. In (6.1) the general theory is developed and the equations of motion of a spinning satellite about its center of mass are derived. In (6.2) the equations of motion of a gyroscope are solved. In (6.3) problems of numerical evaluation of these solutions are discussed. In (6.4) some aspects of our numerical calculations are considered. (6.5) deals with some more papers.

Further applications of Lie series in physics and engineering can be found in the papers quoted under the References.

Chapter I

The Solution of a System of n-th-Order Differential Equations Using Lie Series

by F.Cap and D.Floriani

Abstract: In the present work, the solution of a system of n-th order ordinary differential equations which is solved for $y_Q^{(n)} = f_Q(x,y,y', \dots y^{(n-1)})$ is obtained by means of the Lie series as introduced by Groebner. For this purpose, the concept of a "Lie series" is defined initially and some important properties are quoted. In the third part, the system of equations is solved.

(1.1) Definition of Lie Series

We shall introduce a linear differential operator in the following way:

$$D: = \sum_{o}^{n} F_\varrho(z_o, z_1, \dots, z_n) \cdot \frac{\partial}{\partial z_\varrho} \qquad (I,1)$$

The F_Q are assumed to be holomorphic functions of the complex variables $z_o, \dots, z_Q$. If this operator is applied to another holomorphic function $f(z_o, \dots z_n)$, we have

$$g(z_o, \dots, z_n) = Df(z_o, \dots, z_n)$$

which is again holomorphic. The same holds, of course, if D is applied n times (in the same domain of holomorphy).

With the help of this operator, we may formally set up an infinite series

$$\sum_{o}^{\infty} \frac{t^k}{k!} \cdot D^k f(z_o, z_1 \dots, z_n) \qquad (I,2)$$

which will be written symbolically as

$$e^{tD}f(z) = (\exp tD)\, f(z) \qquad (I,2a)$$

in the following. The series defined in this way have some properties which enable them to become valuable tools in several fields of mathematics.

(1.2) Properties of Lie Series

(1.21) Absolute Convergence

It is shown in Ref.1, p.7, theorem 2, that, if G is a finite closed domain of the z space in which $f(z_o,\dots,z_n)$ and D are holomorphic, a number $T > 0$ can be found such that the Lie series (I,2) converge absolutely and uniformly in the whole of G. The function

$$g(t;z_o,\ \dots,\ z_n):= e^{tD}f(z_o,\ \dots,z_n)$$

is, therefore, holomorphic in t, z_o, ..., z_n.

(1.22) Differentiation

By virtue of this convergence property we have

$$\frac{\partial}{\partial t}\, g(t;z) = \frac{\partial}{\partial t} \sum_{o}^{\infty} \frac{t^k}{k!} \cdot D^k f(z) = \sum_{o}^{\infty} \frac{t^k}{k!} \cdot D^{k+1} f(z) \qquad (I,3)$$

since the series (I,2) may be differentiated term by term with respect to t.

Furthermore we have

$$\frac{\partial}{\partial z_\varrho}\, g(t;z) = \frac{\partial}{\partial z_\varrho} \sum_{o}^{\infty} \frac{t^k}{k!} \cdot D^k f(z) = \sum_{o}^{\infty} \frac{t^k}{k!} \cdot \frac{\partial}{\partial z_\varrho}\, D^k f(z) \qquad (I,4)$$

since the series on the right-hand converges uniformly (Proof: Ref.1,p.7).

(1.23) Commutation theorem

The proof of this theorem is here briefly sketched, on account of its significance.

It is easily shown that

$$D\left(\sum_{0}^{n} a_{\varrho}\cdot f_{\varrho}(z_0, \ldots, z_n)\right) = \sum_{0}^{n} a_{\varrho}\cdot Df_{\varrho}(z_0, \ldots, z_n) \qquad (I,5)$$

(with a_{ϱ} being constants) and, generally,

$$D^k\left(\sum_{0}^{n} a_{\varrho}\cdot f_{\varrho}(z_0, \ldots, z_n)\right) = \sum_{0}^{n} a_{\varrho}\cdot D^k f_{\varrho}(z_0, \ldots, z_n) \qquad (I,6)$$

for any natural number n.

Furthermore, the validity of

$$D^k(f_1(z)\cdot f_2(z)) = \sum_{0}^{k} \binom{k}{\varrho}\cdot(D^{\varrho} f_1(z))\cdot(D^{k-\varrho} f_2(z)) \qquad (I,7)$$

follows from the usual rules of differentiation, where again z is written instead of $z_0, z_1, \ldots, z_n$.

With this we have (see theorem 5 in Ref.1):

$$e^{tD}\left(\sum_{0}^{n} a_{\varrho}\cdot f_{\varrho}(z)\right) = \sum_{0}^{n} a_{\varrho}\cdot e^{tD} f_{\varrho}(z) \qquad (I,8)$$

$$e^{tD}(f_1(z)\cdot f_2(z)) = (e^{tD} f_1(z))\cdot(e^{tD} f_2(z)) \qquad (I,9)$$

In particular, it follows from (8) and (9) for a polynomial:

$$e^{tD}\left(\sum a_{\varrho}\cdot z_0^{\alpha}\cdot z_1^{\beta}\cdot \ldots \cdot z_r^{\varrho}\right) = \sum a_{\varrho}\cdot(e^{tD} z_0)^{\alpha}\cdot(e^{tD} z_1)^{\beta}\cdot \ldots \cdot(e^{tD} z_r)^{\varrho} \qquad (I,10)$$

or briefly:

$$e^{tD} P(z_0, z_1, \ldots, z_r) = P(e^{tD} z_0, \ldots, e^{tD} z_r)$$

As is shown in the general commutation theorem for Lie series, this equation holds for any functional relationship.

The funtions

$$Z_\varrho(t;\ z_o,\ z_1,\ \dots,z_n) := e^{tD} z_\varrho \qquad (I,11)$$

that are holomorphic in t and z_o, ..., z_n are introduced. From it follows:

$$Z_\varrho(t=0;z_o,\ \dots,\ z_n) = z_\varrho. \qquad (I,12)$$

We have then (theorem 6 in Ref.1):

If for a holomorphic function F(z) the power series expansion valid at the point z_o, z_1, ...,z_n converges also in Z_o, Z_1, ...,Z_n (which will certainly be the case for sufficiently small $|Z_\varrho - z_\varrho|$, i.e., for sufficiently small t), we have:

$$e^{tD} F(z_o,\ \dots,z_n) = F(e^{tD} z_o,\ \dots e^{tD} z_n) = F(Z_o,\ \dots,Z_n) \qquad (I,13)$$

This follows for polynomials from (I,10). Let $F_n(z)$ be the portion of the power series for F(z) up to the degree n. We then have because of the presupposed holomorphy:

$$\lim F_n(z) = F(z) \qquad \lim F_n(Z) = F(Z)$$

$$\lim \frac{\partial}{\partial z_\varrho} F_n(z) = \frac{\partial F(z)}{\partial z_\varrho} \qquad (I,14)$$

$$\lim D^k F_n(z) = D^k F(z)$$

For $n \to \infty$. Because of (I,10) we have

$$F_n(Z) = \sum^{\infty} \frac{t^k}{k!} \cdot D^k F_n(z) \qquad (I,15)$$

Since a majorant exists for F(z) the right-hand series converge uniformly with respect to n, i.e., we have:

$$\lim \sum_{0}^{\infty} \frac{t^k}{k!} \cdot D^k F_n(z) = \sum_{0}^{\infty} \frac{t^k}{k!} \cdot \lim D^k F_n(z) \tag{I,16}$$

and with (I,14) to (I,16):

$$F(Z) = \lim F_n(Z) = \lim \sum_{0}^{\infty} \frac{t^k}{k!} \cdot D^k F_n(z) = \sum_{0}^{\infty} \frac{t^k}{k!} \cdot \lim D^k F_n(z) =$$

$$= \sum_{0}^{\infty} \frac{t^k}{k!} \cdot D^k F(z) = e^{tD} F(z)$$

for $n \to \infty$.

(1.3) Construction of the Solutions

Let be given a system of differential equations:

$$y_{\varrho}^{(n)} = f_{\varrho}(x, y, y', \ldots, y^{(n-1)}) \qquad (\varrho=1, \ldots, r) \tag{I,17}$$

with holomorphic functions $f_{\varrho} \cdot y, y', \ldots$ is here symbolic for y_{σ}, y'_{σ} with $\sigma=1, \ldots, r$. (I,17) can be represented by an equivalent system of first-order differential equations:

$$z_0 := x \qquad z_{\varrho,\sigma} := y_{\varrho}^{(\sigma-1)} \qquad (\sigma-1, \ldots, n) \tag{I,18}$$

$$\left.\begin{aligned} z'_{\varrho,\sigma} &= z_{\varrho,\sigma+1} \qquad \text{for } \sigma=1, \ldots, n-1 \\ z'_{\varrho,n} &= f_{\varrho}(x, z_{\varrho\sigma}) \end{aligned}\right\} \tag{I,19}$$

This system (I,19) is now solved by the Lie series which is formed by the operator

$$D := \frac{\partial}{\partial z_0} + \sum_{\varrho} \left[\sum_{\sigma} z_{\varrho,\sigma+1} \cdot \frac{\partial}{\partial z_{\varrho,\sigma}} + f_{\varrho} \cdot \frac{\partial}{\partial z_{\varrho,n}} \right] \tag{I,20}$$

with $\sigma=1, \ldots, n-1$ and $\varrho=1, \ldots, r$:

$$\frac{\partial}{\partial t}(e^{tD} z_0) = \sum_{0}^{\infty} \frac{t^k}{k!} \cdot D^k (D z_0) = 1 \tag{I,21}$$

$$\frac{\partial}{\partial t}(e^{tD}z_{\varrho,\sigma}) = \sum_0^\infty \frac{t^k}{k!} \cdot D^k(Dz_{\varrho,\sigma}) = e^{tD}z_{\varrho,\sigma+1} \quad (\sigma=1,\ldots,n-1) \qquad (I,21)$$

$$\frac{\partial}{\partial t}(e^{tD}z_{\varrho,n}) = \sum_0^\infty \frac{t^k}{k!} \cdot D^k(Dz_{\varrho,n}) = e^{tD}f_\varrho(z)$$

(follows from (I,3)). With

$$Z_o: = e^{tD}z_o = z_o + t \qquad Z_{\varrho,\sigma}: = e^{tD}z_{\varrho,\sigma} \qquad (I,22)$$

(I,21) can be written in the form (because of (I,13)):

$$\frac{\partial}{\partial t} Z_o(t;z) = 1$$

$$\frac{\partial}{\partial t} Z_{\varrho,\sigma}(t;z) = Z_{\varrho,\sigma+1}(t;z)$$

$$\frac{\partial}{\partial t} Z_{\varrho,n}(t;,z) = f_\varrho(Z)$$

or

$$\frac{\partial}{\partial t} Z_o(t;z) = 1$$

$$\frac{\partial^n}{\partial t^n} Z_{\varrho,1}(t;z) = f_\varrho(Z_o; Z_{\varrho,\sigma})$$

or, in terms of the original variables:

$$Z_o = x = z_o + t, \quad Z_{\varrho,\sigma} = y_\varrho^{(\sigma-1)}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x} \qquad \frac{\partial^n}{\partial x^n} y_\varrho(x;z) = f_\varrho(x,y,y', \ldots, y^{(n-1)}).$$

(I,21) is therefore identical with the original system (I,17), since $\frac{\partial^n}{\partial x^n}$ may also be written as $\frac{d^n}{dx^n}$, if they are understood to be parameters.

Consequently, the solution of (I,17) reads:

$$x(t) = z_o + t, \quad y_\varrho(t) = e^{tD}z_{\varrho,1} \quad \text{or} \quad y_\varrho(x) = e^{(x-z_o)\cdot D}z_{\varrho,1} \qquad (I,23)$$

where (see (I,12)) the $z_{\varrho,\sigma}$ are the initial values of $y_{\varrho}^{(\sigma-1)}$ for $t = 0$:

$$y_{\varrho}^{(\sigma-1)}(t=0) = y_{\varrho}^{(\sigma-1)}(x=z_0) = z_{\varrho,\sigma} \tag{I,24}$$

Using (I,23) and (I,24) the problem of solving (I,25) with the initial conditions $y_{\varrho}^{(\sigma-1)}(0) = z_{\varrho,\sigma}$ has been accomplished.

Chapter II

On the Solution of the Differential Equations Resulting from the Separation of Laplace Equation in Various Coordinate Systems

by F.Cap and A.Schett

Abstract: First the R and S-separability is discussed, then several functions (Weber functions, Bessel functions, Baer functions, Mathieu functions, Legendre functions, Lamé functions, Wangerin functions and Heine functions) are formally presented by Lie series.

The Helmholtz equation

$$\Delta \varrho + \varkappa^2 \varrho = 0 \tag{II,1}$$

and the Laplace's equation

$$\Delta \varrho = 0 \tag{II,2}$$

have a great significance in physics. There are many equations, important for physical and technical applications, which reduce to Helmholtz equation if time dependence is separated. These equations are i.a.

1) The diffusion equation:

$$\nabla^2 \varrho = \frac{1}{h^2} \frac{\partial \varrho}{\partial t}$$

This type of equations appears, f.e., in heat conduction theory, diffusion theory and circulatory motion theory.

2) The wave equation:

$$\nabla^2 \varrho = \frac{1}{c^2} \frac{\partial^2 \varrho}{\partial t^2}$$

3) The damped wave equation:

$$\nabla^2 \varphi = \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} + R\frac{\partial \varphi}{\partial t}$$

4) The transmission line equation:

$$\nabla^2 \varphi = \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} + R\frac{\partial \varphi}{\partial t} + S\varphi$$

5) The vector wave equation:

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

The equations enumerated under 1) ÷ 5) describe quite generally the propagation of waves.

The <u>Laplace equation</u> occurs, f.e., in elasticity theory (stress problems, tortion problems, distortion problems, thermal elasticity problems a.s.o.), in potential theory and in potential flow problems. Concerning the separability it is well-known that these equations can be separated in special coordinate systems. One distinguishes R and S separability.

S-separability: If the assumption

$$\varphi = U_1(n_1) \cdot U_2(n_2) \cdot U_3(n_3) \tag{II,3}$$

permits the separation of the partial differential equations (II,1) and (II,2), respectively, into three ordinary differential equations, the equation is said to be simply separable or S-separable.

R-separability: If the assumption

$$\varphi = \frac{U_1(n_1).U_2(n_2).U_3(n_3)}{R(n_1,\, n_2,\, n_3)} \tag{II,4}$$

permits the separation of the partial differential equations (II,1) and

(II,2), respectively, into three ordinary differential equations, and if R = const., the equation is said to be R-separable. The quantity R is defined in Ref.31.

No case is known in which the Helmholtz equation is R-separable, so the question that arises is merely whether the Laplace equation is R-separable in some coordinate systems. In the following table we list the R and S separability of the Laplace and Helmholtz equation, respectively in various coordinate systems. We restrict ourselve to the well-known 11 coordinate systems in which the Helmholtz equation is separable and the most important coordinate systems with regard to technical problems in which the Laplace Equation is R-separable.

In Table II,1 S indicates S-separability

R	"	R- "
X	"	non-separability

TABLE II,1

Coordinate System	$\Delta\psi + \varkappa^2\psi = 0$		$\Delta\psi = 0$	
	$\psi(n_1n_2n_3)$	$\psi(n_1n_2)$	$\psi(n_1n_2n_3)$	$\psi(n_1n_2)$
1. Rectangular Coordinates	S	S	S	S
2. Circular-Cylinder Coordinates	S	S	S	S
3. Elliptic-Cylinder Coordinates	S	S	S	S
4. Parabolic-Cylinder Coordinates	S	S	S	S
5. Spherical Coordinates	S	S	S	S
6. Prolate spheroidal Coordinates	S	S	S	S
7. Oblate spheroidal Coordinates	S	S	S	S
8. Parabolic Coordinates	S	S	S	S
9. Conical Coordinates	S	S	S	S
10. Ellipsoidal Coordinates	S	S	S	S
11. Paraboloidal Coordinates	S	S	S	S
12. Tangent-Cylinder Coordinates	X	X	X	S
13. Cardioid-Cylinder Coordinates	X	X	X	S
14. Hyperbolic-Cylinder Coordinates	X	X	X	S
15. Rose Coordinates	X	X	X	S
16. Cassian-Oval Coordinates	X	X	X	S
17. Inverse Cassian-Oval Coordinates	X	X	X	S

18. *Bi-Cylindrical Coordinates	X	X	X	S
19. Maxwell-Cylinder Coordinates	X	X	X	S
20. Logarithmic-Cylinder Coordinates	X	X	X	S
21. ln tang Cylinder Coordinates	X	X	X	S
22. ln cosh Cylinder Coordinates	X	X	X	S
23. sn-Cylinder Coordinates	X	X	X	S
24. cn-Cylinder Coordinates	X	X	X	S
25. Inverse sn-Cylinder Coordinates	X	X	X	S
26. ln- sn-Cylinder Coordinates	X	X	X	S
27. ln cn-Cylinder Coordinates	X	X	X	S
28. Zeta Coordinates	X	X	X	S
29. Tangent Sphere Coordinates	X	X	R	R
30. Cardioid Coordinates	X	X	R	R
31. Bespherical "	X	X	R	R
32. Toroidal "	X	X	R	R
33. Inverse prolate spheroidal Coordinates	X	X	R	R
34. Inverse oblate spheroidal Coordinates	X	X	R	R
35. Bi-Cyclide Coordinates	X	X	R	R
36. Flat-Ring Cyclide Coordinates	X	X	R	R
37. Disk-Cyclide Coordinates	X	X	R	R
38. Cap-Cyclide Coordinates	X	X	R	R

All differential equations which result from a separation of the Helmholtz equation are special cases of the Bôcher equation. Initial value problems of this general equation were solved by Lie Series in Rep.2 and Rep.3 under the Contract NGR 52-046-001. Special cases were treated in Rep.7 and Rep.8 under the same Contract and in Ref.16.

The differential equations which result from a separation of the Laplace equation are also contained in the Bôcher equation. It means that for initial value problems these ordinary differential equations are solved too. Here we enumerate for the sake of completeness the types of the differential equations resulting from a separation of Laplace's equation in various coordinate systems. Concerning the solution of different types we refer to earlier report under the Contract NGR 52-046-001, if the equation is treated already or shall solve the equation for initial value problems, if the equation is not investigated in earlier reports already. We emphasize, Lie series can only be used to representate functions in regular domains.

(2.1) Types of Differential Equations Resulting from a Separation of Laplace Equation in some Important Coordinate Systems $\rho=\rho(n_1 n_2 n_3)$

Type 1: $Z''(t) - cZ(t) = 0$ (II,1)

c being a constant. This type appears in:

rectangular coordinates,
circular cylinder coordinates,
elliptic-cylinder coordinates,
parabolic-cylinder coordinates,
spherical coordinates,
prolate spheroidal coordinates,
cardioid coordinates,
bispherical coordinates,
toroidal coordinates,
inverse prolate spheroidal coord.,
inverse oblate spheroidal coord.,
bi-cyclide coordinates,

oblate spheroidal coordinates,

parabolic coordinates,

tangent-sphere coordinates,

flat-ring cyclide coordinates,

disk-cyclide coordinates,

cap-cyclide coordinates.

This type is already treated in Rep.7.

Type 2: $Z''(t) + \frac{a}{t} Z'(t) - (\frac{b}{t^2} + c) Z(t) = 0$ (II,2)

a, b, c being constants. This equation appears among the equation of

circular cylinder coord.(a = 1),

parabolic coordinates,

tangent-sphere coordinates,

spherical coordinates,

conical coordinates,

cardioid coordinates.

Eq.(II,2) is already treated in Chapt.IV, Ref.16.

Type 3: $Z''(t) + (a + bt^2) Z(t) = 0$ (II,3)

Eq.(II,3) appears among the equations of parabolic cylinder coordinates. For the solution of this type see Chapt.IV, Ref.16.

Type 4: $Z''(t) - (\alpha_2 + \alpha_3 a^2 \cosh^2 t) Z(t) = 0$ (II,4)

α_2, α_3, a being constants. This equation results from a separation of the Laplace equation in elliptic cylinder coordinates. For solving this equation see Chapt.IV, Ref.16.

Type 5: $Z''(t) + \coth t\, Z'(t) + (\varkappa^2 a^2 \sinh^2 t - \alpha_2 - \frac{\alpha_3}{\sinh^2 t}) Z(t) = 0$ (II,5)

$\varkappa$, a, α_2, α_3 being constants.

This equation appears among the equations in

prolate spheroidal (a=0) coord., toroidal (a=0) coordinates,

inverse prolate spheroidal (a=0) coordinates

Type 6:

$$Z''(t) + \cot t\, Z'(t) + (\varkappa^2 a^2 \sin^2 t + \alpha_2 - \frac{\alpha_3}{\sin^2 t})Z(t) = 0 \qquad (II,6)$$

$\varkappa$, a, α_2, α_3 being constants.

This equation results from:

spherical (a = 0) coordinates,

prolate spheroidal (a = 0) coordinates,

oblate spheroidal (a = 0) coordinates,

bispherical (a = 0) coordinates,

inverse prolate spheroidal (a = 0) coordinates,

inverse oblate spheroidal (a = 0) coordinates.

Eq.(II,6) was investigated in Chapt.IV, Ref.16.

Type 7:

$$Z''(t) + \tanh t\, Z'(t) + (\varkappa^2 a^2 \cosh^2 t - \alpha_2 + \frac{\alpha_3}{\cosh^2 t})\, Z(t) = 0 \qquad (II,7)$$

$\varkappa$, a, α_2, α_3 being constants. This equation results from:

oblate spheroidal (a = 0) coordinates,

inverse oblate spheroidal (a = 0) coordinates.

The solution is given in Chapt.IV, Ref.16.

Type 8:

$$Z''(t) + \frac{t(2t^2-(b^2+c^2))}{(t^2-b^2)(t^2-c^2)}\, Z'(t) + \frac{(\varkappa^2 t^4+\alpha_3 t^2+\alpha_2)}{(t^2-b^2)(t^2-c^2)}\, Z(t) = 0 \qquad (II,8)$$

b, c, $\varkappa$, α_2, α_3 being constants.

This equation appears among the equations in

conical coordinates,

ellipsoidal coordinates.

For solving Eq.(II,8) see Chapt.IV, Ref.16.

Type 9:

$$Z''(t) + \frac{1}{2}\frac{(2t-(b+c))}{(t-b)(t-c)} Z'(t) + \frac{\varkappa^2 t^2 + \alpha_3 t - \alpha_2}{(t-b)(t-c)} Z(t) = 0 \qquad \text{(II,9)}$$

b, c, $\varkappa$, α_2, α_3 being constants.

Eq.(II,9) results from a separation of the Laplace equation in paraboloidal coordinates. The solution for initial value problems is given in Chapt.IV, Ref.16.

Type 10:

$$Z''(t) + \left[\frac{1}{2}\frac{1}{t-a_1} + \frac{2}{t-a_2} + \frac{3}{t-a_3}\right] Z'(t) + \frac{1}{4}\left[\frac{b_0+b_1t+b_2t^2+b_3t^3}{(t-a_1)(t-a_2)^2(t-a_3)^2}\right] Z(t) = 0 \qquad \text{(II,10)}$$

where k, α_2, α_3, a_i, b_j being constants ($i = 1,2,3$; $j = 0,1,2,3$)

The solution functions of Eq.(II,10) are Heine functions (Ref.31).

Let $a_1 = 0$, $a_2 = 1$, $a_3 = 1/k^2$,

$$b_0 = -\frac{\alpha_2}{k^2} \qquad\qquad b_2 = (\alpha_2+2) + 2k^2$$

$$b_1 = (\alpha_2+2) + \frac{\alpha_2}{k^2} - \frac{\alpha_3 k'^4}{k^2} \qquad\qquad b_3 = 2k^4$$

$0 < k^2 < 1 \qquad 0 < k'^2 < 1 \qquad$ and $t = \mathrm{sn}^2\xi$

then one obtains the equation

$$Z''(\xi) - \frac{\mathrm{sn}\xi(\mathrm{dn}^2\xi + k^2\mathrm{cn}^2\xi)}{\mathrm{cn}\xi\,\mathrm{dn}\xi} Z'(\xi) + \left[2k^2\mathrm{sn}^2\xi - \alpha_2 - \alpha_3\frac{k'^4\mathrm{sn}^2\xi}{\mathrm{cn}^2\xi\,\mathrm{dn}^2\xi}\right] Z(\xi) = 0 \qquad \text{(II,10a)}$$

where the Jacobi elliptic functions:

sn - sinus amplitudinis

cn - cosinus amplitudinis

dn - delta amplitudinis

Eq.(II,10a) results from a separation of the Laplace equation in <u>bi-cyclide</u> coordinates. Obviously the solution functions of Eq.(II,10a) are <u>Heine functions</u> (see Ref.31).

Let $a_1 = 0$, $a_2 = 1$, $a_3 = k^2$

$b_o = -\alpha_2 k^2$ $\qquad$ $b_2 = -(\alpha_2+2) - 2k^2$

$b_1 = (\alpha_2-\alpha_3) + k^2(\alpha_2+2)$ $\qquad$ $b_3 = 2$

and

$t = dn^2\zeta$

we obtain from Eq.(II,10) the equation

$$Z''(\zeta) + \frac{dn\zeta(cn^2\zeta - sn^2\zeta)}{sn\zeta\, cn\zeta} Z'(\zeta) + \left[-2dn^2\zeta + \alpha_2 + \alpha_3 \frac{dn^2\zeta}{sn^2\zeta\, cn^2\zeta} \right] Z(\zeta) = 0 \qquad (II,10b)$$

which is again solved by <u>Heine functions</u> (see Ref.31).

Eq.(II,10b) appears among the equations which one obtains by separation of Laplace's equation in <u>bi-cyclide</u> coordinates. The general solution of Eq.(II,10) is given by

$$Z(t) = A\mathcal{H}_1(t) + B\mathcal{H}_2(t) \qquad (II,11)$$

where $\mathcal{H}_i$ $(i = 1,2)$ are Heine functions.

For <u>regular</u> domains we can solve Eq.(II,10) by Lie series. As Eqs.(II,10a), (II,10b) are special cases of Eq.(II,10) we have only to treat Eq.(II,10). The solution representation for initial value problems is given in Chapt.II, Ref.16.

The solution reads:

$$Z(t) = \sum_{o}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_1 = \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} \cdot$$

$$\cdot \left[f_1^{(\varrho)}(z_o) D^{\nu-1-\varrho} z_1 + f_2^{(\varrho)}(z_o)\, D^{\nu-2-\varrho} z_1 \right] + z_1 + tz_2 \qquad (II,12)$$

The operator D is given by:

$$D = \frac{\partial}{\partial z_o} + z_2 \frac{\partial}{\partial z_1} - \left(\frac{1}{2}\left[\frac{1}{t-a_1} + \frac{2}{t-a_2} + \frac{2}{t-a_3} \right] z_2 + \right.$$

$$\left. - \frac{1}{4}\left[\frac{b_o+b_1 t+b_2 t^2+b_3 t^3}{(t-a_1)(t-a_2)^2(t-a_3)^2} \right] \right) \frac{\partial}{\partial z_2} \qquad \text{(II,13)}$$

$$t \neq a_1, \quad t \neq a_2, \quad t \neq a_3$$

$$f_1(t) = -\left(\frac{1}{2(t-a_1)} + \frac{1}{t-a_2} + \frac{1}{t-a_3}\right)$$

$$f_2(t) = -\left(\frac{b_o+b_1 t+b_2 t^2+b_3 t^3}{4(t-a_1)(t-a_2)^2(t-a_3)^2}\right) =$$

$$= \frac{A_1}{t-a_1} + \frac{A_2}{(t-a_2)^2} + \frac{A_3}{t-a_2} + \frac{A_4}{(t-a_3)^2} + \frac{A_5}{t-a_3}$$

$$f_1^{(\varrho)}(t) = \frac{(-1)^{\varrho+1}\varrho!}{2(t-a_1)^{\varrho+1}} + \frac{(-1)^{\varrho+1}\varrho!}{(t-a_2)^{\varrho+1}} + \frac{(-1)^{\varrho+1}\varrho!}{(t-a_3)^{\varrho+1}}$$

$$f_2^{(\varrho)}(t) = \frac{A_1(-1)^{\varrho}\varrho!}{(t-a_1)^{\varrho+1}} + \frac{A_2(-1)^{\varrho}(\varrho+1)!}{(t-a_2)^{\varrho+2}} + \frac{A_3(-1)^{\varrho}\varrho!}{(t-a_2)^{\varrho+1}} +$$

$$+ \frac{A_4(-1)^{\varrho}(\varrho+1)!}{(t-a_3)^{\varrho+2}} + \frac{A_5(-1)^{\varrho}(\varrho)!}{(t-a_3)^{\varrho+1}} \qquad \text{(II,14)}$$

Eqs.(II,12), (II,13), (II,14) solve equation (II,10) for regular domains. Another solution representation derived in Chapt.II, Ref.16 reads:

$$\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = (T^{-1})^T \cdot \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} \cdot T^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} +$$

$$+ \sum_{\alpha=0}^{\infty} \int_{0}^{t} \frac{(t-\tau)^\alpha}{\alpha!} \left[D_2 D^\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right]_{\bar{a}} d\tau \qquad \text{(II,15)}$$

The integral can be evaluated by an iterative method according to Ref.29. The symbol $\bar{a}$ added after the bracket is to indicate that after application of the D-operators z_1, z_2 have to be replaced by $e^{tD_1} z_1$ and $e^{tD_1} z_2$, respectively. λ_1, λ_2, T and D_2 in Eq. (II,15) are given by the relations:

$$\lambda_{1,2} = \frac{f_1}{2} \pm \sqrt{\frac{f_1^2}{4} + f_2}$$

$$T = \begin{pmatrix} f_2 & f_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \qquad \text{(II,16)}$$

$$D_2 = \frac{\partial}{\partial z_o}$$

Eqs.(II,15), (II,16) solve Eq.(II,10). If the initial values $Z(t=t_o)$ and $Z'(t=t_o)$ are given, the solution of Eq.(II,10) can be evaluated for regular domains.

The values $Z(t=t_o)$ and $Z'(t=t_o)$ can be looked up in tables.

The question arises how we can compute the <u>Heine functions</u> by Lie series representation Eqs.(II,12), (II,15).

The general solution of Eq.(II,10) is given by

$$Z(t) = A\mathcal{H}_1(t) + B\mathcal{H}_2(t)$$

A and B being arbitrary constants, $\mathcal{H}_1$, $\mathcal{H}_2$ are Heine functions.

The solution and its derivative is given by:

$$Z(t) = A\mathcal{H}_1 + B\mathcal{H}_2 = Z_1(t)$$

$$Z'(t) = A\mathcal{H}'_1 + B\mathcal{H}'_2 = Z_2(t)$$

Without restriction of generality we may choose:

$$Z(t=t_o) = \varkappa_1(t=t_o) = z_1$$

$$Z'(t=t_o) = \varkappa_2(t=t_o) = z_2$$

i.e. we have put $A = 1$ and $B = 0$. Further the equations are valid.

$$Z(t) = \varkappa_1(t) = \sum_0^{\infty} \frac{t^\nu}{\nu!} D^\nu z_1$$

$$Z'(t) = \varkappa_2(t) = \sum_1^{\infty} \frac{t^\nu}{\nu!} D^{\nu+1} z_1 \tag{II,17}$$

For numerical evaluation of $\varkappa_1$ we expand $Z(t)$ in the neighborhood of $t = t_o$ and choose a step size of Δt. As t increases more terms $\frac{t^\nu D^\nu}{\nu!} z_1$ have to be calculated if the accuracy is prescribed. Since the computers have a limited numerical range, only a limited number of terms $\frac{t^\nu D^\nu}{\nu!} z_1$ can be calculated.

Consequently, we expand the functions $Z(t)$ at $t = t_o$ and using a certain step size Δt we calculate the function $\varkappa_1$ in the region (t_o, t_1). At t_1 the function $\varkappa_1$ will be expanded again. Continuing this method, we can compute $\varkappa_1$ and $\varkappa'_1$ for regular domains.

In an analogous way one may calculate the function $\varkappa_2$ by means of Lie series. Concerning the important problem of error estimation of the solution representations Eq.(II,12) and Eq.(II,15) we refer to Ref.14, 29. G.MAESS treats in Ref.14 an error estimation, which may perhaps be used for numerical computation of Eq.(II,12). As we have never used this method, we cannot decide, wether this error estimation is suitable for numerical evaluation of Eq.(II,12).

H.KNAPP discusses in Ref.29 the error estimation of the representation Eq.(II,15). The usefulness of this method was already proved by numerical calculations (see Ref.29).

Whether Eq.(II,12) or Eq.(II,15) is more advantageous for computing the solutions can only be decided by help of a computer.

Type **11:**

$$Z''(t) + \frac{1}{2}\left[\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{2}{t-a_3}\right] Z'(t) + \frac{1}{4}\left[\frac{b_o+b_1t+b_2t^2}{(t-a_1)(t-a_2)(t-a_3)^2}\right] Z(t) = 0 \qquad \text{(II,18)}$$

where a_1, a_2, a_3, b_o, b_1, b_2, b_3 being constants. The solution functions of Eq.(II,18) are Wangerin functions.

If $a_1 = 1$, $a_2 = 1/k^2$, $a_3 = 0$

$$b_o = -\frac{\alpha_3}{k^2}$$

$$b_1 = \frac{-\alpha_2}{k^2} \qquad \text{(II,19)}$$

$$b_2 = 1 - \alpha_3$$

and

$$t = \mathrm{sn}^2\zeta$$

one obtains from Eq.(II,18) the equation

$$Z''(\zeta) + \frac{\mathrm{cn}\zeta \cdot \mathrm{dn}\zeta}{\mathrm{sn}\zeta} Z''(\zeta) + \left[k^2\mathrm{sn}^2\zeta - \alpha_2 - \alpha_3\left(k^2\mathrm{sn}^2\zeta + \frac{1}{\mathrm{sn}^2\zeta}\right)\right] Z(\zeta) = 0 \qquad \text{(II,18a)}$$

k, α_2, α_3 being constants

cn cosinus amplitudinis

sn sinus

dn delta amplitudinis

This equation results from a separation of Laplace's equation in

flat-ring coordinates,

cap-cyclide coordinates.

By the transformation $t = cn^2\zeta$ one obtains with Eqs.(II,19) and Eq.(II,18) the differential equation

$$Z''(\zeta) - \frac{sn\zeta dn\zeta}{cn\zeta} Z'(\zeta) +$$

$$+\left[k^2 sn^2\zeta - \alpha_2 + \alpha_3(k^2 cn^2\zeta - \frac{k'^2}{cn^2\zeta})\right] Z(\zeta) = 0 \qquad \text{(II,18b)}$$

If $a_1 = 1$, $a_2 = -(\frac{k'}{k})^2$, $a_3 = 0$

$$b_o = (k'/k)^2$$

$$b_1 = (\alpha_2 - k^2)/k^2$$

$$b_2 = \alpha_3 - 1$$

This equation appears among the separated Laplace equation of disk-cyclide coordinates.

If $a_1 = 1$, $a_2 = -(k'/k)^2$, $a_3 = 0$

$$b_o = \alpha_3$$

$$b_1 = (k'^2 - \alpha_2)/k^2$$

$$b_2 = 1 - \alpha_3(k'/k)^2$$

and

$$t = cn^2\zeta$$

one obtains from (II,18) the equation

$$Z''(\zeta) - \frac{sn\zeta dn\zeta}{cn\zeta} Z'(\zeta) + \left[- dn^2\zeta + \alpha_2 + \alpha_3(k'^2 cn^2\zeta - \frac{k^2}{cn^2\zeta})\right] Z = 0 \quad \text{(II,18c)}$$

This equation appears among the separated Laplace equations in disk-cyclide coordinates.

If $a_1 = 1$, $a_2 = k^2$, $a_3 = 0$

$$b_o = -\alpha_3 k^2$$

$$b_1 = -a_2$$

$$b_3 = 1-\alpha_3$$

and

$$t = \mathrm{dn}^2\zeta$$

one obtains from the origin equation (II,18) the equation:

$$Z''(\zeta) - \frac{k'^2 \mathrm{sn}\zeta\,\mathrm{cn}\zeta}{\mathrm{dn}\zeta} Z'(\zeta) + \left[-\mathrm{dn}^2\zeta + \alpha_2 + \alpha_3(\mathrm{dn}^2\zeta + \frac{k^2}{\mathrm{dn}^2\zeta})\right] Z(\zeta) = 0 \quad \text{(II,18d)}$$

This equation results from a separation of the Laplace's equation in flat-ring coordinates, cap-cyclide coordinates.

Eqs.(II,18a), (II,18b), (II,18c) and (II,18d) are solved by Wangerin functions.

The above enumerated equations are special cases of Eq.(II,18). Therefore we have only to solve Eq.(II,18).

For regular domains we can representate the solution by Lie series.

For this case the solution is given by (II,12) and (II,15), respectively.

The operator D reads:

$$D = \frac{\partial}{\partial z_o} + z_2 \frac{\partial}{\partial z_1} + \Big(-\frac{1}{2}\left[\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{2}{t-a_3}\right] z_2 - \frac{1}{4}\left[\frac{b_o+b_1 t+b_2 t^2}{(t-a_1)(t-a_2)(t-a_3)^2}\right] z_1\Big) \frac{\partial}{\partial z_2} \quad \text{(II,20)}$$

for $t \neq a_1$, $t \neq a_2$, $t \neq a_3$

$$f_1(t) = -\frac{1}{2}\left(\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{2}{t-a_3}\right)$$

$$f_2(t) = -\frac{1}{4}\left[\frac{b_0+b_1t+b_2t^2}{(t-a_1)(t-a_2)(t-a_3)^2}\right] =$$

$$= \frac{A_1}{t-a_1} + \frac{A_2}{t-a_2} + \frac{A_3}{(t-a_3)^2} + \frac{A_4}{t-a_3} \qquad (II,21)$$

A_1, A_2, A_3, A_4 being constants.

$$f_1^{(\nu)}(t) = \frac{(-1)^{\nu+1}\nu!}{2(t-a_1)^{\nu+1}} + \frac{(-1)^{\nu+1}\nu!}{2(t-a_2)^{\nu+1}} + \frac{(-1)^{\nu+1}\nu!}{(t-a_3)^{\nu+1}}$$

$$f_2^{(\nu)}(t) = \frac{A_1(-1)^{\nu+1}\nu!}{(t-a_1)^{\nu+1}} + \frac{A_2(-1)^{\nu+1}\nu!}{(t-a_2)^{\nu+1}} + \frac{A_4(-1)^{\nu+1}\nu!}{(t-a_3)^{\nu+1}} +$$

$$+ \frac{A_3(-1)^{\nu+1}\nu!}{(t-a_3)^{\nu+2}} \qquad (II,22)$$

With Eqs.(II,12), (II,15), (II,20), (II,21) and (II,22) Eq.(II,18) is solved and the Wangerin functions can be computated by Lie series. The Lie series solution (II,12) and (II,15) of Eq.(II,18) converges within a circle whose center is at $t = t_0$ and whose radius extends to the nearest singularity of the differential equation.

The general solution of Eq.(II,18) is given by

$$Z(t) = AW_1 + BW_2$$

where A and B are arbitrary constants and W_1, W_2 are Wangerin functions. For computing the Wangerin functions by means of Lie series we refer to the treatment under type 10 in this work.

In the monograph Ref.16 Weber functions and Mathieu functions were evaluated numerically. Results of these calculations were published in Ref.16, Chapt.VII and Chapt.VI, respectively.

(2.2) Appendix

Legendre Polynomials

Here we show that for special differential equations the series $\sum_0^\infty \frac{t^\nu}{\nu!} D^\nu z_0$ breaks off, i.e., the solution is represented by polynomials. As a special example we investigate Legendre polynomials.

We consider the differential equation which appears in the separation equations of spherical polar coordinates, after splitting up the singularities.

$$(t^2-1)Z'' + 2tZ' - n(n+1)Z = 0 \tag{II,23}$$

$$Z(t) = \sum_0^\infty \frac{t^\nu}{\nu!} D^\nu z$$

$$Z'(t) = \sum_0^\infty \frac{t^{\nu-1}}{(\nu)!} \nu D^\nu z \tag{II,24}$$

$$Z''(t) = \sum_0^\infty \frac{t^{\nu-2}}{\nu!} \nu(\nu-1) D^\nu z$$

Inserting Eq.(II,24) in Eq.(II,23) one obtains:

$$(t^2-1) \sum_0^\infty \frac{t^{\nu-2}}{\nu!} \nu(\nu-1) D^\nu z + 2t \sum_0^\infty \frac{t^{\nu-1}}{\nu!} \nu D^\nu z -$$

$$- n(n+1) \sum_0^\infty \frac{t^\nu}{\nu!} D^\nu z = 0$$

so that

$$\sum_0^\infty \frac{t^\nu}{\nu!} \nu(\nu-1) D^\nu z - \sum_0^\infty \frac{t^{\nu-2}}{\nu!} \nu(\nu-1) D^\nu z + 2 \sum_0^\infty \frac{t^\nu}{\nu!} \nu D^\nu z -$$

$$- n(n+1) \sum_0^\infty \frac{t^\nu}{\nu!} D^\nu z = 0$$

if $\nu-2 = \mu$ one obtains

$$\sum_{0}^{\infty} \frac{t^{\nu}}{\nu!} \left\{ \nu(\nu-1)\, D^{\nu} z - \frac{(\nu+2)(\nu+1)}{(\nu+2)(\nu+1)}\, D^{\nu+2} z + 2\nu D^{\nu} z - n(n+1) D^{\nu} z \right\} = 0 \quad (II,24)$$

Necessary and sufficient that Eq.(II,24) is valid is the relation

$$D^{\nu} z \left\{ (\nu-1)\nu + 2\nu - n(n+1) \right\} - D^{\nu+2} z = 0 \qquad (II,25)$$

or

$$D^{\nu+2} z = D^{\nu} z \left\{ (\nu-1)\nu + 2\nu - n(n+1) \right\} \qquad (II,26)$$

If $D^{o} z$ and $D^{1} z$ are given one can calculate all $D^{\nu} z$ by Eq.(II,26).

For $\nu = n$ it follows

$$D^{n+2} z = (n^2 - n + 2n - n^2 - n)\, D^{n} z = 0$$

This means, the series $\sum_{0}^{\infty} \frac{t^{\nu}}{\nu!} D^{\nu} z$ breaks off, in other words we have polynomials.

It is well known, that Eq.(II,23) is solved by Legendre polynomials P_n, i.e., the relation is valid:

$$Z(t) = \sum_{0}^{n} \frac{t^{\nu}}{\nu!} D^{\nu} z = P_n$$

For computing P_n by the series $\sum_{0}^{n} \frac{t^{\nu}}{\nu!} D^{\nu} z$ we need the initial values

$Z(t = t_o)$ and $Z'(t = t_o)$ or

$P_n(t = t_o)$ and $P'(t = t_o)$.

In analogy one can obtain other polynomials.

Chapter III

Numerical Computation of Satellite Orbits Using Lie Series. Comparison with other Methods.

by H.Knapp

Abstract: A survey (reprinted) on the application and the advantages of the Lie series method in celestial mechanics is given.

Using the Lie series theory the formal solution of the astronomical n-body problem in a region where no collisions take place, is easy. It could be demonstrated by a special example (J.Kovalevsky chose this example to test the Lie series method for celestial mechanics) that after the transformation given by W.Groebner (see Ref.1) the Lie series converge so rapidly that the method in its present form can be successfully employed for calculating the orbits in celestial mechanics. This method of solution is particularly flexible and very general, and good estimates can be given since the theoretical expansions and estimations can be directly applied to general multi-body problems.

(3.1) Presentation of the problems

(3.11) Preparation

(3.111) Coordinate system: Our calculations are based on the following coordinate system: Let the center of mass of the three celestial bodies be the origin. Due to the vanishingly small mass of the 8th moon of Jupiter, it lies on the connection line Sun-Jupiter. Let the x-axis indicate the direction of the ascending node of Jupiter for the year 1950, let the y-axis be rotated in the direction of Jupiter motion by 90° relative to the x-axis in the Jupiter orbital plane, let the z-axis be directed such that we have an orthogonal right-handed system. This coordinate system is then assumed to be an inertial system since only in such a system Newton's law of gravitation holds in the simple form. This may be regarded as fulfilled within the accuracy of calculation required here (up to and inclusive of the 9th significant figure of each step).

(3.112) Designations: For reasons of simplicity we use vectors, thus e.g.

$\vec{x} = \{x, y, z\}$	is a position vector
$\vec{u} = \{u, v, w\}$	is a velocity vector
$\vec{x}\,\vec{u} = xu + yv + zw$	is a scalar product
$\lvert\vec{x}\rvert = \sqrt{x^2 + y^2 + z^2}$	is the absolute amount
$[\vec{x}\,\vec{u}] = \{yw-zv,\ zu-xw,\ xv-yu\}$	is the vector product
$\frac{\partial}{\partial x} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$	is the gradient symbol

E.g.

$$\vec{u}\,\frac{\partial}{\partial \vec{x}} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

Furthermore, we use the following designations:

	Sun	Jupiter	8th moon
position vectors	$\vec{x}_3$	$\vec{x}_2$	$\vec{x}_1$
velocities	$\vec{u}_3$	$\vec{u}_2$	$\vec{u}_1$
masses	M_3	M_2	M_1
mass numbers	m_3	m_2	m_1

f is the gravitational constant and $m_i = fM_i$ holds.

All quantities occuring in our calculations are assumed to be differentiable. The three celestial bodies, the Sun, Jupiter and its eighth satellite are assumed to be replaced by mass points which are subject to gravitation according to Newton's law.

The positions and velocities

$$\vec{x}_i(t_o) = \vec{x}_i^{(o)} \qquad \text{and} \qquad \vec{u}_i(t_o) = \vec{u}_i^{(o)}$$

of the three celestial bodies are given for the initial moment $t = t_o$. The 18 components of the vectors $\vec{x}_i$ and $\vec{u}_i$ $(i = 1,2,3)$ are to be determined as functions of time such that the mass points move according to the laws of a three-body problem.

(3.113) Units:

Unit length	$1\ L = 1$ astronomical unit $= 1495{,}04200 \cdot 10^{10}$ cm
unit time	$1\ d = 1$ mean solar day
unit velocity	$1\ Ld^{-1}$
unit mass	$1\ \mu =$ mass of the Sun

In these units the gravitational constant f assumes the numerical value:

$$f = 0{,}29591220828559 \cdot 10^{-3}\mu^{-1}L^3d^{-2} \qquad \text{x)}$$

mass numbers:

$$m_3 = 0{,}295912208 \cdot 10^{-3}L^3d^{-2}$$

$$m_2 = 0{,}282532864 \cdot 10^{-6}L^3d^{-2} = m_3 : 1047{,}355$$

$$m_1 = 0 \text{ (vanishingly small as compared to } m_2 \text{ and } m_3)$$

(3.114) Equations of motion of the mechanical system: According to the general theorems of mechanics we obtain the following system of differential equations for the three-body problem (see Ref.1,p.71):

x) This and all other numerical values are taken from a paper by J.Kovalevsky. Since we are concerned with the explanation of the method rather than with the values themselves the problem of their accuracy is of minor importance.

$$\begin{cases} \dot{\vec{x}}_i = u_i \\ \dot{\vec{u}}_i = -\frac{1}{M_i}\frac{\partial U}{\partial x_i} \end{cases} \qquad (i = 1,2,3) \qquad \text{(III,1)}$$

with

$$U = -\sum_{i<k} f\frac{M_i M_k}{r_{ik}} \qquad r_{ik} = \left|\vec{x}_i - \vec{x}_k\right|$$

(the dot denotes differentiation with respect to the time t)

Let the operator belonging to the differential equations (III,1) be designated by D;

Since $m_1 = 0$ it has the following form:

$$D = \vec{u}_1\frac{\partial}{\partial\vec{x}_1} + \vec{u}_2\frac{\partial}{\partial\vec{x}_2} + \vec{u}_3\frac{\partial}{\partial\vec{x}_3} + \left[\frac{m_2(\vec{x}_2-\vec{x}_1)}{r_{12}^3} + \frac{m_3(\vec{x}_3-\vec{x}_1)}{r_{13}^3}\right]\frac{\partial}{\partial\vec{u}_1} +$$

$$+\frac{m_3(\vec{x}_3-\vec{x}_2)}{r_{23}^3}\frac{\partial}{\partial\vec{u}_2} + \frac{m_2(\vec{x}_2-\vec{x}_3)}{r_{23}^3}\frac{\partial}{\partial\vec{u}_3} \qquad \text{(III,2)}$$

(3.115) Known integrals of the system:

Law of conservation of energy

$$E_{kin} + E_{pot} - \frac{1}{2}(M_2\vec{u}_2^2 + M_3\vec{u}_3^{\,2}) + U = \text{const., since } D(E_{kin}+E_{pot}) = 0$$

(III,3)

Law of conservation of angular momentum:

$$\vec{P} = M_2\left[\vec{x}_2.\vec{u}_2\right] + M_3\left[\vec{x}_3.\vec{u}_3\right] = \text{const., since } D\vec{P} = 0 \qquad \text{(III,4)}$$

Conservation of center of gravity:

$$\vec{x}_S = \frac{1}{m}(m_2\vec{x}_2 + m_3\vec{x}_3) \text{ with } m = m_2 + m_3$$

is the position of the center of mass of the three bodies.

Since $D^2\vec{x}_S = 0$ and owing to the special selection of the coordinate system $\vec{x}_S = 0$ is valid for all times (See Ref.1, p.75): the center of gravity rests in the origin of the coordinate system. Hence we have:

$$\vec{x}_S = 0 \quad \text{and} \quad \vec{u}_S = 0 \qquad \text{or:} \qquad \begin{matrix} m_2\vec{x}_2 + m_3\vec{x}_3 = 0 \\ \\ m_2\vec{u}_2 + m_3\vec{u}_3 = 0 \end{matrix} \tag{III,5}$$

The nine components of the vectors $\vec{x}_S$, $\vec{U}_S$, $\vec{P}$ and the constant energy (III,3) are the 10 algebraic integrals of the problem. With these 10 relations between the 18 unknown components of the vectors $\vec{x}_i$ and $\vec{u}_i$ ($i = 1,2,3$) the number of unknown functions could be reduced to eight. In our example the conservation laws for energy and angular momentum refer only to the partial problem Sun-Jupiter and permit its complete integration. With the aid of (III,5) however, the six unknown quantities can be easily eliminated and the motion can then be described by only two position- and two velocity vectors: $\vec{x}_s$ and $\vec{x}_m$, $\vec{u}_s$ and $\vec{u}_m$.

(3.116) Transformation of variables:

$$\begin{cases} \vec{x}_s = \vec{x}_3 - \vec{x}_2 \\ \vec{x}_m = \vec{x}_1 - \vec{x}_2 \end{cases} \qquad \begin{cases} \vec{u}_s = \vec{u}_3 - \vec{u}_2 \\ \vec{u}_m = \vec{u}_1 - \vec{u}_2 \end{cases} \tag{III,6}$$

Due to (III,5) this transformation is always reversible:

$$\begin{cases} \vec{x}_1 = -\frac{m_3}{m}\vec{x}_s + \vec{x}_m \\ \vec{x}_2 = -\frac{m_3}{m}\vec{x}_s \\ \vec{x}_3 = \frac{m_2}{m}\vec{x}_s \end{cases} \qquad \begin{cases} \vec{u}_1 = -\frac{m_3}{m}\vec{u}_s + \vec{u}_m \\ \vec{u}_2 = -\frac{m_3}{m}\vec{u}_s \\ \vec{u}_3 = \frac{m_2}{m}\vec{u}_s \end{cases} \tag{III,7}$$

The converted operator (III,2) has the following form:

$$D = \vec{u}_s \frac{\partial}{\partial x_s} + \vec{u}_m \frac{\partial}{\partial x_m} - \frac{m}{|\vec{x}_s|^3} \vec{x}_s \frac{\partial}{\partial u_s} - \frac{m_2}{|\vec{x}_m|^3} \vec{x}_m \frac{\partial}{\partial u_m} + $$
$$+ m_3 \left[\frac{\vec{x}_s - \vec{x}_m}{|\vec{x}_s - \vec{x}_m|^3} - \frac{\vec{x}_s}{|\vec{x}_s|^3} \right] \frac{\partial}{\partial \vec{u}_m} \qquad \text{(III,8)}$$

(3.12) Formulation of the problem

We now have to integrate the system of differential equations

$$\begin{aligned}
\dot{\vec{x}}_s &= \vec{u}_s \\
\dot{\vec{x}}_m &= \vec{u}_m \\
\dot{\vec{u}}_s &= - \frac{m}{|\vec{x}_s|^3} \vec{x}_s \\
\dot{\vec{u}}_m &= - \frac{m_2}{|\vec{x}_m|^3} \vec{x}_m + m_3 \left[\frac{\vec{x}_s - \vec{x}_m}{|\vec{x}_s - \vec{x}_m|^3} - \frac{\vec{x}_s}{|\vec{x}_s|^3} \right]
\end{aligned} \qquad \text{(III,9)}$$

which belongs to the operator (III,8) under the initial conditions

$$\vec{x}_s(t_o) = \vec{x}_s^{(o)},\ \vec{x}_m(t_o) = \vec{x}_m^{(o)},\ \vec{u}_s(t_o) = \vec{u}_s^{(o)},\ \text{and } \vec{u}_m(t_o) = \vec{u}_m^{(o)}$$

which are to be calculated from the initial conditions $\vec{x}_i(t_o)$ and $\vec{u}_i(t_o)$ for $i = 1,2,3$ according to the formulas (III,6).

The solution can be easily obtained by Lie series:

If $f(t)$ is an arbitrary function holomorphic in the neighborhood of $t = t_o$ of the twelve sought components of the vectors $\vec{x}_s$, $\vec{x}_m$, $\vec{u}_s$, and $\vec{u}_m$, then the Lie series

$$f(t) = \left[e^{(t-t_o)D} f \right]^{(o)} = \sum_{\nu=0}^{\infty} \frac{(t-t_o)^\nu}{\nu!} \left[D^\nu f \right]^{(o)} \qquad \text{(III,10)}$$

holds.

The superscript zero denotes that after application of the operator D instead of the variable components of $\vec{x}_s$, $\vec{x}_m$, $\vec{u}_s$ and $\vec{u}_m$ the components of the constant initial values $\vec{x}_s^{(o)}$, $\vec{x}_m^{(o)}$, $\vec{u}_s^{(o)}$ and $\vec{u}_m^{(o)}$ are substituted. The trajectories are obtained by writing down this formula for the vectors $\vec{x}_s(t)$ and $\vec{x}_m(t)$ and by analytically continuing the series. In this form, the solution can, however, not be used for numerical purposes since the series converge too weakly. (This has been distinctly shown by J.Kovalevsky in a comparison with the Cowel method). Hence a transformation is necessary: First, we determine an approximate orbit which is then corrected by a perturbation calculation.

(3.2) <u>Solution of the problem</u>

<u>Sun-Jupiter as an unperturbed two-body problem</u>

(3.211) Splitting of the operator: We shall now split D into two components:

$$D = D_s + \bar{D} \tag{III,11}$$

where

$$D_s = \vec{u}_s \frac{\partial}{\partial \vec{x}_s} - \frac{m}{|\vec{x}_s|^3} \vec{x}_s \frac{\partial}{\partial \vec{u}_s} \tag{III,12}$$

while the remaining terms of the operator (III,8) are denoted by $\bar{D}$.

(3.212) Calculation of $\vec{x}_s(t)$: The partial operator D_s out of the total operator D will solely act, if in the place of functions depending only on $\vec{x}_s$ and $\vec{u}_s$, but not depending on $\vec{x}_m$ and $\vec{u}_m$, are substituted into the final formula (III,10). Thus, we have, for instance,

$$\vec{x}_s(t) = \left[e^{(t-t_o)D} \vec{x}_s \right]^{(o)} = \left[e^{(t-t_o)D_s} \vec{x}_s \right]^{(o)} \tag{III,13}$$

and the problem visualized by the partial operator D_s can be solved separately. We may say: The variables $\vec{x}_s$ and $\vec{u}_s$ are separated from $\vec{x}_m$ and $\vec{u}_m$ since they do not depend on these. - D_s is, however, the operator of the unpertubed two-body problem Sun-Jupiter. We shall give the solution together with the respective numerical data in (3.3).

(3.22) <u>Construction of the approximative orbit of the eighth satellite of Jupiter</u>

(3.221) Further splitting of the operator: It would be most natural to split up $\bar{D}$ in such a way that its essential part again is the operator of a two-body problem in this case of the fictive two-body problem Jupiter - satellite. Rather voluminous intermediate calculations, which may be a large source of accumulating rounding errors, are required for the determination of the Kepler ellipse as an approximative orbit (particularly in the reversal of Kepler's equation!). In order to avoid these we have decided on calculating with a simpler, although less accurate approximative orbit.

We shall split the operator

$$D = D_2 + D_m + \Delta_m \qquad \text{(III,14)}$$

The abbreviations mean

$$\begin{cases} D_m = \vec{u}_m \dfrac{\partial}{\partial \vec{x}_m} - c^2 \vec{x}_m \dfrac{\partial}{\partial \vec{u}_m} \\ \Delta_m = \vec{\delta}_m \dfrac{\partial}{\partial \vec{u}_m} \end{cases} \qquad \text{(III,15)}$$

where

$$c^2 = \frac{m_2}{|\vec{x}_m(o)|^3} \qquad \text{(III,16)}$$

The perturbation function $\vec{\delta}_m$ has the form

$$\vec{\delta}_m = \vec{\delta}_{m_I} + \vec{\delta}_{m_{II}} \tag{III,17}$$

with

$$\begin{cases} \vec{\delta}_{m_I} = m_3 \left[\dfrac{\vec{x}_s - \vec{x}_m}{|\vec{x}_s - \vec{x}_m|^3} - \dfrac{\vec{x}_s}{|\vec{x}_s|^3} \right] \\ \vec{\delta}_{m_{II}} = \left[c^2 - \dfrac{m_2}{|\vec{x}_m|^3} \right] \vec{x}_m \end{cases} \tag{III,17a}$$

(3.222) Rough estimation of the order of magnitude of $\vec{\delta}_m$:

(3.2221) If $\vec{x}_s$ and $-\vec{x}_m$, respectively, are substituted in the place of $\bar{a}$ and $\bar{b}$ in the formula

$$\frac{\vec{a} + \vec{b}}{|\vec{a}+\vec{b}|^3} = (\vec{a} + \vec{b}) \sum_{\nu=0} \binom{-3/2}{\nu} \frac{(2\vec{a}\vec{b} + \vec{b}^2)^\nu}{|\vec{a}|^{2\nu+3}} \tag{III,18}$$

we obtain for $\vec{\delta}_{m_I}$ an expansion into a series by means of which the order of mangitude can be estimated more easily than by means of the expression (III,17a) for $\vec{\delta}_{m_I}$ which contains differences of approximately equal orders:

$$\vec{\delta}_{m_I} = - \frac{m_3}{|\vec{x}_s|^3} \left[\vec{x}_m - \frac{3\vec{x}_s\vec{x}_m}{|\vec{x}_s|^2} + \frac{3\vec{x}_s\vec{x}_m}{|\vec{x}_s|^2} \vec{x}_m + \ldots\ldots \right] \tag{III,19}$$

If we consider the first two terms of the series jointly and observe that

$$\left| \vec{x}_m - \frac{3\vec{x}_s\vec{x}_m}{|\vec{x}_s|^2} \vec{x}_s \right| \leq 2 \left| \vec{x}_m \right|,$$

$|\vec{x}_s| > 4.95$ L and 0.05 L $< |\vec{x}_m| < 0.25$ L, we will have in the most unfavorable case

$$\left|\vec{\delta}_{m_I}\right|_{max} \approx \frac{2m_3}{\left|\vec{x}_s\right|^3}\left|\vec{x}_m\right| < 1.22\cdot 10^{-6}\ Ld^{-2} \qquad \text{(III,20)}$$

(3.2222) $\vec{\delta}_{m_{II}}$ is less favorable to handle. If we transform $\vec{\delta}_{m_{II}}$ in such a way that the Kepler ellipse relations enter the formula as an approximative orbit we find that

$$\left|\vec{\delta}_{m_{II}}\right|_{max} \approx 4.05\cdot 10^{-6}\ Ld^{-3} \cdot \left|\Delta t\right| \qquad \text{(III,21)}$$

where $\Delta t = t - t_o$ is the length of the concerned step of calculation. However, we shall not go into these details.

(3.223) Relative orbit of the satellite with respect to Jupiter.

We shall first neglect Δ_m in comparison to D_m, since then also the variables $\vec{x}_m$ and $\vec{u}_m$ are separated from $\vec{x}_s$ and $\vec{u}_s$. In this way, the problem represented by the operator D_m may be solved separately. The resulting approximative orbit of course deviates from the true orbit, owing to (III,20) and (III,21). We should note, however, that extremely unfavourable conditions have been assumed in these estimations; the figures in (III,20) and (III,21) will be smaller in general!

The solution of the systems of differential equations

$$\begin{cases} \dot{\vec{x}}_{ma} = \vec{u}_{ma} \\ \dot{\vec{u}}_{ma} = -c^2\,\vec{x}_{ma} \end{cases} \qquad \text{(III,22)}$$

with the operator $D_{ma} = \vec{u}_{ma}\frac{\partial}{\partial \vec{x}_{ma}} - c^2\,\vec{x}_{ma}\frac{\partial}{\partial \vec{u}_{ma}}$ and with the initial values

$$\vec{x}_m^{(o)} = \vec{x}_{ma}^{(o)} \quad \text{and} \quad \vec{u}_m^{(o)} = \vec{u}_{ma}^{(o)}$$

for the moment t_o are obtained in the form of the rather simple approximative orbit (ellipse)

$$\vec{x}_{ma}(t) = \left[e^{(t-t_o)D_{ma}}x_{ma}\right]^{(o)} = \vec{x}_m^{(o)}\cos\left[c(t-t_o)\right] + \vec{u}_m^{(o)}\,\frac{1}{c}\sin\left[c(t-t_o)\right]$$

$$\vec{u}_{ma}(t) = \left[e^{(t-t_o)D_{ma}}u_{ma}\right]^{(o)} = -\vec{x}_m^{(o)}c\,\sin\left[c(t-t_o)\right] + \vec{u}_m^{(o)}\cos\left[c(t-t_o)\right] \qquad \text{(III,23)}$$

(The additional subscript a is to indicate that these approximative functions, in difference from the sought exact solutions $\vec{x}_m$ and $\vec{u}_m$ of the original three-body problem.)

The connection with time t is evident; the reversal of a Kepler equation is superfluous.

(3.23) <u>Solution of the three-body problem by means of the given approximative orbit; perturbation calculus</u>

(3.231) Transformation of the solution (III,10): With the new symbol

$$D_1 = D_s + D_m \qquad \text{(III,24)}$$

we have

$$f(t) = \sum_{\nu=0}^{\infty} \frac{(t-t_o)^{\nu}}{\nu!}\left[(D_1 + \Delta_m)f\right]^{(o)} \qquad \text{(III,25)}$$

Expanding $(D_1 + \Delta_m)^{\nu}$, ordering according to the positions of Δ_m, and applying the exchange theorem to the Lie series one obtains the formula (see Ref.1, p.92, formula (12.3e)):

$$f(t) = f_a(t) + \sum_{\alpha=0}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^{\alpha}}{\alpha!}\left[\Delta_m D^{\alpha} f(t)\right]_a d\tau \qquad \text{(III,26)}$$

which is very important for the subsequent calculations. This formula expresses how the approximative solution $f_a(t)$ has to be modified in order to yield a solution of the original problem. The expression

$$\left[\Delta_m D^{\alpha} f(\tau)\right]_a$$

means that $\Delta_m D^{\alpha} f$ has to be calculated first, and that then the components of $\vec{x}_m$ and $\vec{u}_m$ have to be substituted by the components of the approximative solution $\vec{x}_{ma}(\tau)$ and $\vec{u}_{ma}(\tau)$.

(3.232) Expansion of the essential terms in the series (III,26): We shall now substitute the required special functions $\vec{x}_m(t)$ and $\vec{u}_m(t)$ in the place of the general functions f(t) in formula (III,26). - In the subsequent numerical computation we shall have to break the corresponding series and to confine ourselves to the essential terms. Of course, the accuracy of the result may be increased to any degree if more terms are taken into account. In the present instance, the following approximations may be sufficient:

$$\begin{aligned} \vec{x}_m(t) &= \vec{x}_{ma}(t) + \int_{t_o}^{t} (t-\tau)\, \vec{\delta}_{ma}(\tau)d\tau + \int_{t_o}^{t} \frac{(t-\tau)^3}{3!} \vec{\zeta}_{ma}(\tau)d\tau \\ \vec{u}_m(t) &= \vec{u}_{ma}(t) + \int_{t_o}^{t} \vec{\delta}_{ma}(\tau)d\tau + \int_{t_o}^{t} \frac{(t-\tau)^2}{2!} \vec{\zeta}_{ma}(\tau)d\tau \end{aligned} \qquad \text{(III,27)}$$

with

$$\begin{aligned} \vec{\zeta}_{ma} = &- \frac{m_2}{|\vec{x}_{ma}|^3}\left[\vec{\delta}_{ma} - \frac{3(\vec{x}_{ma}\vec{\delta}_{ma})}{|\vec{x}_{ma}|^2}\vec{x}_{ma}\right] - \\ &- \frac{m_3}{|\vec{x}_s - \vec{x}_{ma}|^3}\left[\vec{\delta}_{ma} - \frac{3([\vec{x}_s - \vec{x}_{ma}]\vec{\delta}_{ma})}{|\vec{x}_s - \vec{x}_{ma}|^2}(\vec{x}_s - \vec{x}_{ma})\right] \end{aligned} \qquad \text{(III,28)}$$

Naturally, the formulas (III,27) are of use only as long as the time space $|t - t_o|$ is chosen so small that the further terms of the series may be neglected according to the required accuracy. (It is obvious that t may never be outside the region of convergence of the series.)

(3.24) <u>Estimation of the error due to breaking off the series</u>

(3.241) Region of validity of the formulas (III,27): We know from formula (III,27) that it is the solution of the problem (III,9) within a certain region of the t-plane. Within this region, the solution functions constructed by means of formula (III,27) have to satisfy the differential equations (III,9). If $\vec{x}_m(t)$ and $\vec{u}_m(t)$ are calculated from (III,27), one obtains

$$\vec{u}_m(t) = - \frac{m_2}{|\vec{x}_{ma}(t)|^3} \vec{x}_{ma}(t) + \vec{\delta}_{ma_I}(t) + \vec{R}(t) \qquad \text{(III,29)}$$

where

$$\vec{R}(t) = \sum_{\alpha=0}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!} \Big[\Delta_m D^{\alpha+2} \vec{x}_m(\tau) \Big]_a d\tau =$$

$$= \int_{t_o}^{t} (t-\tau)\vec{\xi}_{ma}(\tau)d\tau + \sum_{\alpha=2}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!} \Big[\Delta_m D^{\alpha+2} \vec{x}_m(\tau) \Big]_a d\tau \qquad \text{(III,30)}$$

Comparison of (III,29) with (III,9) yields

$$\vec{R}(t) = m_2 \left[\frac{\vec{x}_{ma}(t)}{|\vec{x}_{ma}(t)|^3} - \frac{\vec{x}_m(t)}{|\vec{x}_m(t)|^3} \right] + \vec{\delta}_{m_I}(t) - \vec{\delta}_{ma_I}(t) \qquad \text{(III,31)}$$

We shall make use of this in order to determine the order of magnitude of the expression $\vec{R}(t)$. With the abbreviation

$$\vec{x}_m(t) = \vec{x}_{ma}(t) - \vec{\xi}(t) \qquad \text{(III,32)}$$

where

$$\vec{\varepsilon}(t) = \int_{t_o}^{t} (t-\tau)\vec{\delta}_{ma}(\tau)d\tau + \int_{t_o}^{t} \left[\int_{t_o}^{\tau} \vec{R}(\xi)d\xi \right] d\tau \qquad (III,33)$$

and with the aid of formula (III,18) x) we obtain

$$-m_2 \left[\frac{\vec{x}_{ma} + \vec{\varepsilon}}{|\vec{x}_{ma} + \vec{\varepsilon}|^3} - \frac{\vec{x}_{ma}}{|\vec{x}_{ma}|^3} \right] = - \frac{m_2}{|\vec{x}_{ma}|^3} \left[\vec{\varepsilon} - \frac{3(\vec{x}_{ma}\vec{\varepsilon})}{|\vec{x}_{ma}|^2} \vec{x}_{ma} + \qquad (III,34)$$

$$+ \left(\begin{matrix} \text{terms of higher} \\ \text{order of } |\vec{\varepsilon}| \end{matrix} \right) \Bigg] \text{xx)}$$

Substitution of (III,32) in (III,19) yields

$$\vec{\delta}_{m_I} - \vec{\delta}_{ma_I} = - \frac{m_3}{|\vec{x}_s|^3} \left[\vec{\varepsilon} - \frac{3(\vec{x}_s \vec{\varepsilon})}{|\vec{x}_s|^2} \vec{x}_s + \left(\begin{matrix} \text{terms of higher} \\ \text{order of } \vec{\varepsilon} \end{matrix} \right) \right] \qquad (III,35)$$

so that

$$|\vec{R}(t)|_{max} \approx 2|\vec{\varepsilon}(t)| \left[\frac{m_2}{|\vec{x}_{ma}(t)|^3} + \frac{m_3}{|\vec{x}_s(t)|^3} \right] = 2|\vec{\varepsilon}(t)| K(t) \qquad (III,36)$$

K(t) varies between

2.10^{-5} d^{-2} (for large $|\vec{x}_m|$) and $2.2 \cdot 10^{-3}$ d^{-2} (for small $|\vec{x}_m|$).

By virtue of

$$|\vec{\varepsilon}(t)| < \frac{(t-t_o)^2}{2} \left[|\vec{\delta}_{ma}(t)|_{max} + |\vec{R}(t)|_{max} \right] \qquad (III,37)$$

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x) The series converges for $\frac{2\vec{x}_{ma}\vec{\varepsilon} + \vec{\varepsilon}^2}{|\vec{x}_{ma}|^2} < 1$, which is certainly fulfilled in a region where formula (III,20) represents the solution, when $|t-t_o| = |\Delta t|$ is chosen sufficiently small.

xx) The terms linear in $|\vec{\varepsilon}|$ are sufficient in estimating the order of magnitude.

and with (III,36) we obtain in the most unfavorable case the following estimate for the order of magnitude of $|\vec{R}(t)|$:

$$\left|\vec{R}(t)\right|_{max} \approx \frac{(t-t_o)^2 K(t)}{1-(t-t_o)^2 K(t)} \left| \delta_{ma}(t)\right|_{max} \tag{III,38}$$

This estimate is critical for $1 - (t - t_o)^2 K(t) = 0$, which means near the perijove for $|t - t_o| \approx 21$ d

near the apojove for $|t - t_o| \approx 220$ d

so that, as it was to be expected, the magnitude of the region of convergence of formula (III,26) depends strongly on the distance between the two celestial bodies. Formula (III,26) is valid in any case for a time space of at least 20 days.

In numerically evaluating the formula it will be desirable to chose the interval rather long. One has to be careful, however, not to come close to the edge of the region of convergence since then the rapid convergence of the series, which is desired in practice, will no longer be given.

(3.242) Residue of the series after the second perturbation integral; choice of proper step length Δt: The comprehensive deliberations which have been made to estimate the expression

$$\vec{R}_{\varrho}(t) = \vec{R}(t) - \int_{t_o}^{t} (t-\tau)\vec{\zeta}_{ma}(\tau)d\tau = \sum_{\alpha=2}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^{\alpha}}{\alpha!} \left[\Delta_m D^{\alpha+2}\vec{x}_m(\tau)\right]_a d\tau \tag{III,39}$$

have shown that the step length needs never be shorter than 0.3 d if the error due to the breaking-off of the series is postulated in one step of calculation to amount to not more than 5.10^{-11} L in the case of $\left|\vec{x}_m\right|$ and to not more than 5.10^{-13} Ld^{-1} in the case of $\left|\vec{u}_m\right|$. Moreover, one may conclude that the breaking-off error after the second perturbation integral in first approximation amounts to

$$\int_{t_o}^{t} \vec{R}_\varrho(\tau)d\tau \approx \frac{t}{4}\,\vec{R}_\varrho(t) \tag{III,40}$$

in the case of $\vec{u}_m$, and to

$$\int_{t_o}^{t}\left[\int_{t_o}^{\tau} \vec{R}_\zeta(\zeta)\,d\zeta\right]d\tau \approx \frac{(\Delta t)^2}{20}\,\vec{R}_\zeta(t) \tag{III,41}$$

in the case of $\vec{x}_m$. Therefore, these quantities may be calculated at the end of each step x). After this one may determine the step length permissible at the prescribed accuracy.

In practice one will always stay somewhat below the accuracy limit, but will calculate several steps of equal length. Only when approaching

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x) The program-controlled SIE 2002 computer at the computing center of Aachen Technical University usually calculates with 10 decimal places only. In this way one can obtain only the order of magnitude of $\vec{R}_\zeta(t)$. However, if the solution series are broken off after the first perturbation integral and if the corresponding calculations are carried out for $\vec{R}(t)$, one will obtain 2 or 3 figures of the components of $\vec{R}(t)$. If in analogy to (III,40) and (III,41) the expressions

$$\int_{t_o}^{t} \vec{R}(\tau)d\tau \approx \frac{\Delta t}{3}\,\vec{R}(t) \tag{III,40a}$$

$$\int_{t_o}^{t}\left[\int_{t_o}^{\tau} \vec{R}(\zeta)d\zeta\right]d\tau = \frac{(\Delta t)^2}{12}\,\vec{R}(t) \tag{III,41a}$$

are formed, and if these quantities are added as corrections to $\vec{x}_m$ and $\vec{u}_m$, respectively, one will obtain improved solutions. A checking calculation, also to ten digits, has shown that after 30 steps the result for $\vec{x}_m$ is exactly the same as that obtained when two perturbation integrals were taken into account. The result for $\vec{u}_m$ differed but insignificantly (rounding errors), but the time required for computation was only half as long! The same procedure can be made with $\vec{R}(t)$ if the computation covers more than 10 digits.

this limit one will reduce the step length a little (or increase it if the absolute amounts of the expressions (III,40) and (III,41) have dropped below some certain value). If this is sensibly done by the computer one has nothing to do but to adjust the length of the first step. Obviously, this is of particular significance for calculation of rocket trajectories (when their approximate course is known, and when estimations according to the above pattern can be made only for short sections of the trajectory).

(3.243) Propagation of the breaking off error in the analytical continuation of the solutions: The exact result of the analytical continuation of (III,26) after n steps will be denoted by $f^{(n)}$ throughout this paragraph. The result involving the breaking-off errors (we shall not be concerned with rounding errors) of the previous calculating steps (breaking off after the second perturbation integral) will be termed $\vec{f}^{(n)}$. For the error quantities

$$\begin{aligned} p_n &= \left| \vec{x}_m^{(n)} - \vec{x}_m^{(n)} \right| \\ q_n &= \left| \vec{u}_m^{(n)} - \vec{u}_m^{(n)} \right| \end{aligned} \qquad \text{(III,42)}$$

we obtain the recurrence formulas

$$\begin{aligned} p_n &< (1+P_n)p_{n-1} + (1+P_n)\left|\Delta t\right|_n q_{n-1} + \bar{p}_n \\ q_n &< (1+P_n)7c^2\left|\Delta t\right|_n p_{n-1} + (1+P_n)q_{n-1} + \bar{q}_n \end{aligned} \qquad \text{(III,43)}$$

in which $\bar{p}_n$ denotes the amount of the error in $\vec{x}_m$ at the n-th step, due to breaking-off the series, $\bar{q}_n$ the amount of the breaking-off error in the series for $\vec{u}_m$ after the n-th step.

$$p_n < \frac{3}{2}\left[\frac{m_3}{|\vec{x}_s-\vec{x}_{ma}|^3}+\frac{m_2}{|\vec{x}_{ma}|^3}\right]_{max}|\Delta t|_n^2$$

(i.e. the maximum of this expression in the time interval of the n-th step calculation).

The solution of the recurrence formulas may be written straightforward, if a good part of the path is computed with the same step length $|\Delta t|$, if the breaking off errors $\bar{p}_i$ and $\bar{q}_i$ in the formulas (III,43) are replaced by their maximum values $\bar{p}$ and $\bar{q}$, and if P_n is replaced by the maximum P. Thus,

$$\left.\begin{aligned} p_n &< \beta_1 e^{\alpha_1 n} + \beta_2 e^{\alpha_2 n} - p \\ q_n &< \gamma_1 e^{\alpha_1 n} + \gamma_2 e^{\alpha_2 n} - q \end{aligned}\right\} \qquad \text{(III,44)}$$

where

$$\begin{aligned} e^{\alpha_1} &= (1+P)\,(1+\sqrt{7}\,c|\Delta t|) \\ e^{\alpha_2} &= (1+P)\,(1-\sqrt{7}\,c|\Delta t|) \end{aligned} \qquad \text{(III,45)}$$

$$\left.\begin{aligned} p &= \left[\bar{p}P - \bar{q}\,(1+P)|\Delta t|\right]k_1 \\ q &= \left[\bar{q}P - \bar{p}\,(1+P)\;\;7c^2|\Delta t|\right]k_1 \end{aligned}\right\} \quad k_1 = \frac{1}{P^2-(1+P)^2 7c^2|\Delta t|^2} \qquad \text{(III,46)}$$

β_i and γ_i are the constants of the general solution of the recurrence formulas which make the adaption to the initial conditions possible. With p^* being the error of the initial data of our calculation in $|\vec{x}_m|$ and q^* the error of the initial data in $|\vec{u}_m|$ we have the relations

$$\beta_1 = k_2\left[(p+p^*)e^{\alpha_2} - a\right]$$

$$\gamma_1 = k_2 \left[(q + q^*)e^{\alpha_2} - b \right]$$

$$\beta_2 = k_2 \left[-(p + p^*)e^{\alpha_1} + a \right] \qquad (III,47)$$

$$\gamma_2 = k_2 \left[-(q + q^*)e^{\alpha_1} + b \right]$$

$$k_2 = - \frac{1}{2(1+P)\sqrt{7c}|\Delta t|}$$

$$a = (1+P)\left[p^* + |\Delta t| q^* \right] + \vec{p} + p$$

$$b = (1+P)\left[7c^2 |\Delta t| p^* + q^* \right] + \vec{q} + q$$

(3.25) <u>Calculation of the perturbation integrals</u>

It would be an awful lot of work to evaluate generally the integrals

$$\int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!} \left[\Delta_m D^\alpha f(\tau) \right]_a d\tau \qquad (III,48)$$

occurring in (III,26). We rather go another way which yields the integrals in question with sufficient accuracy. We label the wellknown functions

$$\left[\Delta_m D^\alpha f(\tau) \right]_a = g_\alpha(\tau) \qquad (III,49)$$

for the 4 equidistant instants of time x)

$$t_o,\ t_o+h,\ t_o+2h,\ t_o+3h \qquad (III,50)$$

where

x) This is arbitrary! The functions could as well be labeled more finely (in the case of large step lengths this might be necessary; naturally, the integral formula (III,52) would then have to be changed). But since the step length has to be chosen short anyhow in order to keep the breaking-off errors, low, and since it is evident that few but finely graded steps involve just as much work as more steps with a coarser grading, there is no reason to label the functions more finely since the errors due to the chosen interpolation do not reach the amount of the breaking-off errors. This can be demonstrated the most rapidly by calculating forth and back with different step lengths.

$$h = \frac{\Delta t}{3} = \frac{t-t_o}{3} \tag{III,51}$$

and with the aid of the differentiating scheme of the table

τ	$g_\alpha(\tau)$	$\Delta g_\alpha(\tau)$	$\Delta^2 g_\alpha(\tau)$	$\Delta^3 g_\alpha(\tau)$
t_o	$g_\alpha(t_o)$			
		$\Delta g_\alpha(t_o)$		
t_o+h	$g_\alpha(t_o+h)$		$\Delta^2 g_\alpha(t_o)$	
		$\Delta g_\alpha(t_o+h)$		$\Delta^3 g_\alpha(t_o)$
t_o+2h	$g_\alpha(t_o+2h)$		$\Delta^2 g_\alpha(t_o+h)$	
		$\Delta g_\alpha(t_o+2h)$		
t_o+3h	$g_\alpha(t_o+3h)$			

we replace the function $g_\alpha(\tau)$ by the Newton interpolation polynomial. The differences $\Delta^\nu g_\alpha(t_o)$ are defined as

$$\Delta^\nu g_\alpha(t_o) = \Delta^{\nu-1} g_\alpha(t_o+h) - \Delta^{\nu-1} g_\alpha(t_o) \tag{III,52}$$

We have then

$$\int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!} g_\alpha(\tau)d\tau = \frac{(\Delta t)^{\alpha+1}}{(\alpha+1)!}\left\{g_\alpha(t_o) + \frac{3}{\alpha+2}\Delta g_\alpha(t_o) - \right.$$
$$\left. - \frac{3}{2}\frac{\alpha-3}{(\alpha+2)(\alpha+3)}\Delta^2 g_\alpha(t_o) + \frac{\alpha^2-2\alpha+3}{(\alpha+2)(\alpha+3)(\alpha+4)}\Delta^3 g_\alpha(t_o)\right\} \tag{III,53}$$

When calculating back, Δt (and also h) has to be taken negative. The differences $\Delta^\nu g_\alpha(t_o)$ are calculated from their definition (III,59) also in this case.

(3.3) Numerical computations

(3.31) Compilation of the special initial values and of the formulas for the solution of one operation

(3.311) Initial instant: Timing begins from Oct.29, 1958 - the Julian day 2429200.5 - and continues in days.

(3.312) Relative motion of the sun and Jupiter: tabulation of $\vec{x}_s(t)$: for the instants

$$t_\nu = t_o + \nu h \qquad (\nu = 0,1,2,3) \quad ^{x)} \tag{III,54}$$

the corresponding values of E_ν are to be determined by inversion of the Kepler equation

$$E_\nu - \varepsilon \sin E_\nu = \mu t_\nu + M \tag{III,55}$$

Numerical values:

$$\begin{aligned} \varepsilon &= 0.0484011000 && \text{(eccentricity)} \\ \mu &= 0.001450215293 \text{ d}^{-1} && \text{(mean motion)} \\ M &= 5.645944315 && \text{(mean anomaly)} \\ t_o &= 0 && \text{(calendar day)} \end{aligned} \tag{III,56}$$

The solution of (III,55) with respect to E_ν is most easily achieved by iteration of Newton's approximate formula for solving equations:

$$E_{\nu II} = E_{\nu I} - \frac{E_{\nu I} - \varepsilon \sin E_{\nu I} - \mu t_\nu - M}{1 - \varepsilon \cos E_I} \tag{III,57}$$

where $E_{\nu I}$ $^{xx)}$ is a value which approximately satisfies Eq.(III,55) and $E_{\nu II}$ is an improved approximate value. Formula (III,57) has to be iterated until $E_\nu = E_{\nu N}$ satisfies Eq.(III,55) with a given accuracy.

---

x) The step $\Delta t = 3h$ can be chosen arbitrarely

xx) The value of $E_{\nu-1}$ corresponding to the preceding instant $t_{\nu-1}$ is best taken as the initial value of $E_{\nu I}$ (starting from $E_{oI} = 5.615994607$).

Then, $x_s(t_\nu)$ can be calculated from the resulting values of E_ν:

$$x_s(t_\nu) = \begin{Bmatrix} 0.015676901-4.186636655 \sin E_\nu-0.323895551 \cos E_\nu \\ 0.251333487-0.323515939 \sin E_\nu+5.192722630 \cos E_\nu \\ 0 \end{Bmatrix} L \tag{III,58}$$

(3.313) Initial data for the orbit of the moon: Computation is to be carried out with the mass numbers of page 32

$$\left.\begin{aligned} m_2 &= 0.2825328640 \cdot 10^{-6} L^3 d^{-2} \\ m_3 &= 0.2959122080 \cdot 10^{-3} L^3 d^{-2} \end{aligned}\right\} \tag{III,59}$$

and with the values for the relative position and the relative velocity of the moon, corresponding to the instant t_o:

$$\vec{x}_m^{(o)} = \begin{Bmatrix} -0.1859213874 \\ 0.0071237637 \\ 0.0775628307 \end{Bmatrix} L \qquad \vec{u}_m^{(o)} = \begin{Bmatrix} 0.0002062301590 \\ 0.0008942872800 \\ -0.0003356104520 \end{Bmatrix} L\, d^{-1} \tag{III,60}$$

(3.314) Approximate orbit for Jupiter's moon: We first calculate

$$c = \sqrt{\frac{m_2}{|\vec{x}_m^{(o)}|^3}} \tag{III,61}$$

Then, the position of the moon on its approximate orbit at the instants (III,54) is found from the formula:

$$\vec{x}_{ma}(t_\nu) = \vec{x}_m^{(o)} \cos\left[c(t_\nu - t_o)\right] + \vec{u}_m^{(o)} \frac{1}{c} \sin\left[c(t_\nu - t_o)\right] \tag{III,62}$$

The velocity of the moon on its approximate orbit must be taken only for the end point $t_3 = t_o + \Delta t$ of the interval:

$$\vec{u}_{ma}(t_3) = -x_m^{(o)} c \sin\left[c(t_3-t_o)\right] + \vec{u}_m^{(o)} \cos\left[c(t_3-t_o)\right] \qquad \text{(III,63)}$$

(3.315) Computation of the perturbation integrals: Now the functions $\vec{\delta}_{ma}(t)$ and $\vec{\zeta}_{ma}(t)$ must be tabulated for the instants (III,54) from the formulas x).

$$\vec{\delta}_{ma}(t\ ) = m_3\left[\frac{\vec{x}_s(t_\nu)-\vec{x}_{ma}(t_\nu)}{\left|\vec{x}_s(t_\nu)-\vec{x}_{ma}(t_\nu)\right|^3} - \frac{x_s(t_\nu)}{\left|x_s(t_\nu)\right|^3}\right] + \left[c^2 - \frac{m_2}{\left|\vec{x}_{ma}(t_\nu)\right|^3}\right]\vec{x}_{ma}(t_\nu)$$

$$\vec{\zeta}_{ma}(t_\nu) = -\frac{m_2}{\left|\vec{x}_{ma}(t_\nu)\right|^3}\left[\vec{\delta}_{ma}(t_\nu) - \frac{3(\vec{x}_{ma}(t_\nu)\vec{\delta}_{ma}(t_\nu))}{\left|\vec{x}_{ma}(t_\nu)\right|^2}\ \vec{x}_{ma}(t_\nu)\right] -$$

$$-\frac{m_3}{\left|\vec{x}_s(t\ )-\vec{x}_{ma}(t_\nu)\right|^3}\vec{\delta}_{ma}(t_\nu) - \frac{3(\left[\vec{x}_s(t_\nu)-\vec{x}_{ma}(t_\nu)\right]\vec{\delta}_{ma}(t_\nu))}{\left|\vec{x}_s(t_\nu)-\vec{x}_{ma}(t_\nu)\right|^2}\ .$$

$$.\ (\vec{x}_s(t_\nu) - \vec{x}_{ma}(t_\nu)) \qquad \text{(III,64)}$$

With the aid of the differences between these tables, obtained from (III,52) we are able to calculate the perturbation integrals:

$$\int_{t_o}^{t_o+\ t} \vec{\delta}_{ma}(\tau)d\tau=(\Delta t)\left\{\vec{\delta}_{ma}(t_o)+ \frac{3}{2}\Delta\vec{\delta}_{ma}(t_o)+ \frac{3}{4}\Delta^2\vec{\delta}_{ma}(t_o)+ \frac{1}{8}\Delta^3\vec{\delta}_{ma}(t_o)\right\}$$

$$\int_{t_o}^{t_o+\ t} (t_o+\Delta t-\tau)\vec{\delta}_{ma}(\tau)d\tau=(\Delta t)^2\left\{\frac{1}{2}\ \vec{\delta}_{ma}(t_o)+ \frac{1}{8}\Delta^2\vec{\delta}_{ma}(t_o)+ \frac{1}{60}\Delta^3\vec{\delta}_{ma}(t_o)\right\}$$

$$\int_{t_o}^{t_o+\ t} \frac{(t_o+\Delta t-\tau)^2}{2!}\ \vec{\zeta}_{ma}(\tau)d\tau=(\Delta t)^3\left\{\frac{1}{6}\ \vec{\zeta}_{ma}(t_o)+ \frac{1}{8}\Delta\vec{\zeta}_{ma}(t_o)+ \frac{1}{80}\Delta^2\vec{\zeta}_{ma}(t_o)+\frac{1}{240}\Delta^3\vec{\zeta}_{ma}(t_o)\right\}$$

$$\int_{t_o}^{t_o+\ t} \frac{(t_o+\ t-\tau)^3}{3!}\ \vec{\zeta}_{ma}(\tau)d\tau=(\Delta t)^4\left\{\frac{1}{24}\ \vec{\zeta}_{ma}(t_o)+ \frac{1}{40}\ \Delta\vec{\zeta}_{ma}(t_o)+ \frac{1}{840}\Delta^3\vec{\zeta}_{ma}(t_o)\right\} \ \text{xx)}$$

---

x) The second constituent of $\vec{\zeta}_{ma}$ hardly influences the result. The delay of the computer is, however, very small if this part is included in the calculation, since all the quantities appearing in it had already to be prepared for the calculation of $\vec{\delta}_{ma}$.

xx) The contribution of this integral manifests itself only with great steps, but the situation is about the same as in the foregoing footnote.

(3.316) Formulas of solution: The perturbation integrals (III,65) are used to correct the approximate solutions (III,62) and (III,63):

$$\vec{x}_m(t_o+\Delta t)=\vec{x}_{ma}(t_o+\Delta t)+\int_{t_o}^{t_o+\Delta t}(t_o+\Delta t-\tau)\vec{\delta}_{ma}(\tau)d\tau+\int_{t_o}^{t_o+\Delta t}\frac{(t_o+\Delta t-\tau)^3}{3!}\vec{\zeta}_{ma}(\tau)d\tau$$

$$\vec{u}_m(t_o+\Delta t)=\vec{u}_{ma}(t_o+\Delta t)+\int_{t_o}^{t_o+\Delta t}\vec{\delta}_{ma}(\tau)d\tau+\int_{t_o}^{t_o+\Delta t}\frac{(t_o+\Delta t-\tau)^2}{2!}\vec{\zeta}_{ma}(\tau)d\tau \qquad \text{(III,66)}$$

Now, we replace t_o by $t_o+(\Delta t)$ in all the formulas of (3.31) and proceed to another operation, using the values of (III,66) instead of those of (III,60). Again, Δt can be newly chosen.

(3.317) Precautionary measures taken to avoid unnecessary rounding errors: Since the SIE 2002 computer of the TH Aachen, with which our numerical computations were made, usually calculates with no more than 10 digits some precautionary measures had to be taken to eliminate rounding errors:

(3.3171) Prior to our computations we reduced the quantity M (and E_o) by a factor of 2π in order to maintain the anomaly $|E|<1$ for some hundred days. Thus, the 10th digit cannot be lost during the inversion of the Kepler equation;

(3.3172) Instead of t_o+3h we always calculated $t_o+\Delta t$ since h is equal to $\frac{\Delta t}{3}$ only within rounding errors so that a noticeable error might appear in the time counting;

(3.3173) When calculating solutions from (III,66), we first determined the sum of the perturbation integrals and then added the approximate solution. In this way, the rounding error of the additions enters the result only once.

(3.32) Results

(3.321) Trial computations made so far and experience gathered from them: The following trial calculations were made:

(3.3211) The first informative computations with different steps (one step forward and one step backward) have shown that formula (III,53) is sufficiently accurate and that the step constitent with the considerations in (3.242) is approximately 1d.

(3.3212) 100 steps were calculated forward and backward with $\Delta t = 1d$ x) This was the most important part of our calculations since they could be compared with other results.

J.Kovalevsky pointed out that his 12-digit computations, carried out by Cowell's method with an IBM 650 computer, took 10 sec. for each operation and that the deviations in the coordinates and velocities, obtained when calculating with $(\Delta t) = 5d$ 100 days forward and backward (i.e., in 40 operations) were less than $50.10^{-10}L$ and 100.10^{-10}, respectively (unit not given).

We obtained the following results by this method:

10-digit computation with an SIE 2002 computer took 2 sec. for each operation (the printing of four lines of data after each operation, which was necessary for informative purposes but could be omitted later, took 1.6 sec). When calculating with the step $(\Delta t) = 1d$ 100 days forward and backward (i.e., in 200 operations), the deviations in the coordinates and velocities were less than $15.10^{-10}L$ and $1.2 \cdot 10^{-11} L\, d^{-1}$, respectively. On the basis of this result and with the aid of the (still very rough) estimate it could be shown in (2.243) that the errors in

x) The relevant section of the table may be seen from the enclosed table of data.

the analytic continuation at $\Delta t = 1d$ accounted for no more than 50% of the values indicated, whereas the remaining deviations were due to the rounding errors. The same computation with $\Delta t = 2d$ yielded deviations in the coordinates and velocities of less than $28 \cdot 10^{-10} L$ and $4 \cdot 10^{-11} L d^{-1}$, respectively. The remaining test time was used for informative computations with greater steps (3d, 5d, 10d). Here, the break-off errors were already noticeable. As a result of these computations, we came to the conclusion that the expressions $\vec{R}_{\zeta}(t)$ and $\vec{R}(t)$ might be used for a correction (cf.(2.24)).

(3.3213) Integration was performed from $\Delta t = 1d$ (then 0.8d, 0.6d, 0.4d) beyond the nearest distance between Jupiter and the moon, and the time left was used for backward calculation. The values obtained again agreed very well. In order to save time, only two lines of values were printed.

(3.3214) The modification mentioned in the footnote p.45 was calculated. At the same time, the printing commands were distributed more conveniently in order to stop the computer for a shorter time. Calculation and printing took about 2 sec. for one operation so that the printing process was hardly interrupted.

(3.322) Influence of errors: The results can be falsified in four ways:

(3.3221) by calculating with an insufficient number of protective places. Rounding errors may cause serious errors unless they are smaller than the break-off errors from the very outset;

(3.3222) by using too great steps. If a definite number of terms is used, the required rapid convergence of series can be achieved only if the step Δt is reduced;

(3.3223) by successively performing many, sufficiently accurate operations (if Δt is definitely chosen, the excessively strong propagation of the break-off error can be eliminated only by allowing for further terms of (III,26). This means, however, that the break-off error is reduced simultaneously. Reduction of the step alone is not very advantageous since the required number of operations increases simultaneously, cf.(III,44).

(3.3224) by inexact tabulation of the functions appearing in the perturbation integrals, which can be avoided either by a more exact tabulation or by reducing the step.

The rounding errors show a random character, whereas the other three error sources reside in the method; however, they can all be controlled:

in(3.3222) by observing the increase of (III,39) and by reducing the step in time;

in (3.3223) with the aid of the estimate (III,44) which can be improved since we have always taken the maxima of the absolute values of the quantities involved;

in (3.3224) by calculating forward and backward (random sampling) and, if necessary, by reducing the step.

When choosing the step Δt, it is necessary that conflicting requirements be compensated:

Results of given accuracy are to be obtained with the greatest possible step and the least possible number of operations. The modification mentioned in (3.24) is very helpful in this respect, since it makes it possible to allow for the essential part of the rests of series without determining the required perturbation integrals. Finally, it should be stressed that we have dealt only with a special example and that our method can also be used for the numerical solution of general manybody problems. The elaboration of our method is still under way, and we hope that we shall soon be able to achieve even better results.

Notes on the table of data:

Since the data were originally printed only for the purpose of obtaining information on the efficiency of our method, we expressed the numbers in the way they were stored in the computer. The comma was omitted. The last two figures of each number are the so-called characteristics of the values represented as floating-point numbers (characteristic = exponent + 50; the point of the computer is put behind the sign). The decimal number +0.7, for example, corresponds to the floating-point number + 700 000 000 050. Another disadvantage of the tables is that the printed numerical values are not clearly arranged. After each operation the values were printed in the following four-line arrangement (dimensions are given in brackets):

time t $[d]$, step Δt $[d]$, components of $\vec{x}_m(t)$ $[L]$, $\vec{x}_m(t)$ $[L]$
components of $\vec{u}_m(t)$ $[L\ d^{-1}]$
components of $\vec{R}_\zeta(t)$ $[L\ d^{-2}]$, $\vec{R}_\zeta(t)$ $[L\ d^{-2}]$
components of $\vec{x}_s(t)$ $[L]$; $|\vec{x}_s(t)|$ $[L]$

The numbers in the third line give information only on the order of magnitude of the expression $\vec{R}_\zeta(t)$ (we calculated only with ten digits and several digits were lost in the course of calculation, especially during the determination of the difference between two approximately equal numbers from formula (III,39): The first two figures and the characteristic are valid at most, while the other digits are insignificant.

Table of data:

We do not reproduce the full table which covers 24 pages. Anyone who is interested to have a copy should write to the author.

A short summary reads

time [d]	step [d]	$(\vec{x}_m)_x$	$\lvert\vec{x}_m\rvert$
0.00000	+ 01.00000	- 185921387450	+ 201577536050
1.00000	+ 01.00000	- 185711957150	+ 201288442250
⋮	⋮	⋮	⋮
99.00000	+ 01.00000	- 129514535750	+ 158151320350
100.00000	+ 01.00000	- 128523006850	+ 157550010150
99.00000	- 01.00000	- 129514535750	+ 158151320350
⋮	⋮	⋮	⋮
0.00000	- 01.00000	- 185921386050	+ 201577534650

(3.4) Appendix: The Lie Groebner Method

The solution sought

$$x(t) \equiv (x_1(t),\ x_2(t),\ \ldots,\ x_n(t)) \qquad \text{(III,67)}$$

of a system of n differential equations

$$\frac{d}{dt}\, x(t) = f(x(t),\ t) \tag{III,68}$$

with

$$f(x(t),\ t) \equiv (f_1(x_1(t),\ \ldots\ x_n(t),\ t),\ \ldots\ f_n(x_1(t),\ \ldots\ t) \tag{III,69}$$

satisfying the initial conditions

$$x(t_o) = a \equiv (a_1,\ a_2,\ \ldots\ a_n) \tag{III,70}$$

is given by the following formula (see Ref.1,32)

$$x(t) = \hat{x}(t) + \sum_{\alpha=o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^{\alpha}}{\alpha!} \left[D_2 D^{\alpha} x \right]_{x(\tau),\tau} d\tau \tag{III,71}$$

where

$$\hat{x}(t) \equiv (\hat{x}_1(t),\ \hat{x}_2(t),\ \ldots\ \hat{x}_n(t)) \tag{III,72}$$

are given functions satisfying the system of differential equations

$$\frac{d}{dt}\, \hat{x}(t) = g(\hat{x}(t),\ t) \tag{III,73}$$

with

$$g(\hat{x}(t),\ t) \equiv (g_1(\hat{x}_1(t),\ \ldots\ \hat{x}_n(t),\ t),\ldots,g_n(\hat{x}_1(t),\ldots t)) \tag{III,74}$$

and fulfilling the initial conditions (III,70), i.e., $\hat{x}(t) = a$, whereas the Lie operator D is defined by

$$D = f_K(x(t),\ t)\, \frac{\partial}{\partial x_K} + \frac{\partial}{\partial t} \tag{III,74'}$$

where $D^oF = F$, $D^1F = DF$, $D^{\alpha}F = D(D^{\alpha-1}F)$, if $F(x(t),\ t)$ is differentiable an appropriate number of times, and

$$D_2 = (f_K(x(t),\ t) - g_K(x(t),\ t))\, \frac{\partial}{\partial x_K} = D - D_1 \tag{III,75}$$

Furthermore, to calculate the expressions

$$\left[D_2 D^{\alpha} x_i\right]_{\hat{x}(\tau),\tau} \tag{III,76}$$

the variables $x(t) = (x_1(t), \ldots, x_n(t))$, t are to be considered independent in order to obtain $D^{\alpha}x$ and $D_2 D^{\alpha}x$. Then $x(t)$ is to be replaced by the given functions (III,72), $\hat{x}(\tau)$ and t by τ. Consequently, the sought solutions (III,67) of the system (III,68) may be calculated by formula (III,71) from the given functions (III,72).

In numerical calculations (III,71) has, of course, to be broken off, and only an approximation of the sought solution is obtained. This approximation may, however, be used to define a new decomposition of D into D_1 and D_2 (eqs.(III,74), (III,75)) in order to compute a further approximation; it can be shown that under rather general conditions the repetition of this procedure yields a sequence of approximate solutions having the solution sought as its limit (see Ref.2). Starting from the initial functions

$${}_o\hat{x}(t) \equiv ({}_o\hat{x}_1(t), \ldots, {}_o\hat{x}_n(t)) \tag{III,77}$$

the expression

$${}_{\nu}\hat{x}(t) \equiv ({}_{\nu}\hat{x}_1(t), \ldots, {}_{\nu}x_n(t)) \tag{III,78}$$

is assumed to be the ν-th approximation to the solution (m+1 times differentiable functions), satisfying the initial conditions (III,70), i.e., ${}_{\nu}\hat{x}(t_o) = a$, and let

$${}_{\nu}g(t) = \frac{d}{dt}\,{}_{\nu}x(t) \tag{III,79}$$

by their derivative; in this case, the decomposition of the operator $D = {}_{\nu}D_1 + {}_{\nu}D_2$ with

$$ {}_{\nu}D_1 = {}_{\nu}g_x(t)\,\frac{\partial}{\partial x_K} + \frac{\dot{\partial}}{\partial t} \tag{III,80} $$

yields the ν+1-st approximation in the form

$$ {}_{\nu+1}\hat{x}(t) = {}_{\nu}\hat{x}(t) + \sum_{\alpha=0}^{m} \int_{\infty}^{t} \frac{(t-\tau)^{\alpha}}{\alpha!} \left[{}_{\nu}D_2 D_x^{\alpha}\right]_{\nu}\hat{x}(\tau)_{\tau} d\tau \tag{III,81} $$

resulting from Knapp's general method of iteration (see Ref.2). As mentioned, the sequence ${}_{\nu}\hat{x}(t)$ converges toward the solution x(t) for $\nu \to \infty$, for a certain fixed m and a suitable interval $|t - t_o|$. In order to reduce the computational effort m-3 or 4 should not be exceeded in practical applications.

Since in practice all functions are usually given numerically the iteration rule (III,81) should be reformulated in order to avoid numerical differentiations (III,69). For this purpose, we start from the functions

$$ {}_{o}g(t) \equiv ({}_{o}g_1(t),\ {}_{o}g_2(t),\ \dots\ {}_{o}g_n(t)) \tag{III,82} $$

instead of (III,77) and obtains the ν-th approximation

$$ {}_{\nu}x(t) \equiv ({}_{\nu}g_1(t),\ {}_{\nu}g_2(t)\ \dots.\ {}_{\nu}g_n(t)) \tag{III,83} $$

by iteration; using the initial conditions (III,70) the approximate solution

$$ {}_{\nu}\hat{x}(t) = a + \int_{t_o}^{t} {}_{\nu}g(\tau)d\tau \tag{III,84} $$

is found. With the help of the same decomposition of the D-operator we obtain the iteration formula (see Ref.2)

$$ {}_{\nu+1}g(t) = f({}_{\nu}\hat{x}(t),t) + \sum_{\alpha=1}^{m} \int_{t_o}^{t} \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} \left[{}_{\nu}D_2 D_x^{\alpha}\right]_{\nu}\hat{x}(\tau),\tau d\tau \qquad (III,85) $$

Since the expressions ${}_{\nu}D_2 D^{\alpha} x_i (i=1,2, \ldots n)$ can be calculated once for all it is not necessary to differentiate given functions numerically.

If the constant initial values (III,70) are taken as the initial solution an improved solution can be found by applying the iteration procedure once (as far as there is no better initial solution determined by the problem itself), i.e. (see Ref.2)

$$ {}_{o}\hat{x}(t) = \sum_{\alpha=o}^{\bar{m}} \frac{(t-t_o)^{\alpha}}{\alpha!} \left[D^{\alpha}x\right]_{a,t_o} \qquad (III,86) $$

or

$$ {}_{o}g(t) = \sum_{\alpha=1}^{m} \frac{(t-t_o)^{\alpha-1}}{(\alpha-1)!} \left[D^{\alpha}x\right]_{a,t_o} \qquad (III,87) $$

where $o \leqslant \bar{m} \leqslant m$. In doing so we obtain the operator

$$ {}_{o}D_2 = (f_K(x,t) - \sum_{\alpha-1}^{m} \frac{(t-t_o)^{\alpha-1}}{(\alpha-1)!} \left[D^{\alpha}x_K\right]_{a,t_o} \frac{\partial}{\partial x_K} \qquad (III,88) $$

representing a suitable decomposition for numerical calculations. The functions

$$ {}_{1}\hat{x}(t) \equiv ({}_{1}\hat{x}_1(t), {}_{1}\hat{x}_2(t), \ldots\ldots, {}_{1}\hat{x}_n(t)) \qquad (III,89) $$

found with the help of (III,84) and (III, 88) by applying once more the iteration formula (III,85) are in the most cases a sufficiently exact approximation of the sought solution (III,67) in the interval (t_o, t). If higher accuracy is required it is more expedient to carry out some further iterations instead of including greater values of $\bar{m}$, i.e., to calculate more complex terms of the series (III,71); the numerical integrations needed are, however, apt to aggrevate the iteration process. Fortunately, the method converges rapidly such that there is seldom need of more than two iterations (see Ref.2).

Theoretically, the following three possibilities of constructing the solution result from the aforementioned facts:

a) $m \to \infty$

b) $|t \to t_o| \to 0$

c) $\nu \to \infty$

Practically all three possibilities can be combined, which increases the adaptibility of the method to various problems and to the different domains of their solutions.

The fast convergence of the iteration procedure by Knapp becomes obvious from the error estimate (see Ref.2)

$$|x_i(t) - {}_\nu\hat{x}_i(t)| \leqslant b\,\frac{(K_n|t-t_o|^{m+1})^\nu}{((m+1)\nu)!};\quad (i=1,2,\ \ldots,n) \qquad \text{(III,90)}$$

where b and K are constants and m is the number of equations of the system. Consequently, the differences

$${}_oS(t) = f({}_1\hat{x}(t),\ t) - \frac{d_1\hat{x}(t)}{dt} = f({}_1\hat{x}(t),t) - f({}_o\hat{x}(t),t) -$$

$$- \sum_{\alpha=1}^{m} \int_{t_o}^{t} \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!}\Big[{}_oD_2D^\alpha x\Big]_{{}_o\hat{x}(\tau),\tau} d\tau \qquad \text{(III,91)}$$

are integrated to yield the values

$${}_oR_m(t) = x(t) - {}_1\hat{x}(t) \simeq \int_{t_o}^{t} {}_oS(\tau)d\tau \simeq \frac{(t-t_o)}{m+\bar{m}+2}\ {}_oS(t) \qquad \text{(III,92)}$$

which are a good estimate of the remaining terms; in case of small step length $(t - t_o)$ they can be used to correct the results (III,89) of the first approximation, since in this case they approximately correspond to the improvement due to the second approximation. The expressions (III,92) may also be used to automatically adapt the integration step lengths to the problem to be solved in its various domains of solution. As to the treatment of error propagation, see Ref.2.

Chapter IV

Application of the Method of Lie Series to a Calculation of Particle Orbits in Accelerators

by F.Ehlotzky

Abstract: In this chapter it will be shown how Lie Series can be used for a numerical treatment of differential equations of the isochronous AVF cyclotron, particularly in order to study stability problems.

It is of interest to test the usefulness of the Lie series method by a problem which is, at present, very pressing in high energy physics, i.e., the calculation of particle orbits in accelerators (see Ref.33), viz. an AVF cyclotron (see Ref.34-37).

Generally, the motion of charged particles in a cylindrically symmetric magnetic field $\vec{B}$ (r, θ, z) which is constant in time is described by the following canonical system of equations (see Ref.34,35). (We chose $m_o = c = e = 1$; m_o = rest of mass of the particle; c = speed of light; e = charge of particle.)

$$
\begin{aligned}
p_r' &= q - rB_z + (r/q)\, p_z B_e \\
r' &= (r/q) p_r \\
p_z' &= rB_r - (r/q)\, p_r B_e, \quad q = (p^2 - p_r^2 - p_z^2)^{1/2} \qquad \text{(IV,1)} \\
z' &= (r/q)\, p_z \\
t' &= E(r/q)
\end{aligned}
$$

It proved to be expedient to introduce the azimuth Θ as the independent variable instead of time t; therefore, $r' = dr/d\Theta$, etc. Furthermore, already many important informations are obtained in calculating an accelerator if, to start with, the field accelerating the particles is neglected, i.e., in the case of a cyclotron the h-f electric field in the gaps between the D-s (see Ref.34, 35). This is always justified if it can be proved that the phase integral over a closed path of revolution of a particle is adiabatically invariant (see Ref.33). On this assumption, the system of equations (IV,1) has been derived, and since the Lorentz force is normal to the momentum $\vec{p}$ of the particle, the energy $E = (p+1)^{1/2}$ is constant. The acceleration procedure is not taken into account with the considerations made here.

In a cyclotron with azimuthally varying field (briefly, AVF cyclotron) to which the further investigations refer the azimuthal periodic variability of the magnetic field gives rise to the necessary axial focusing of the particle beam. The latter may also be achieved by purely radial dependence of the field but the condition to be fulfilled in this case $(\partial B_z/\partial r)$ is not consistent with that of isochrony $(\partial B_z/\partial r > 0)$. In the ideal case, the period of azimuthal variation is $2\pi/N$, where N is the number of completely equal magnetic configurations causing the desired field variability in the Θ-direction. The azimuthal variation of the field is also a function of r, such that the field has "swirl" structure. In axial direction the magnetic field is, of course, symmetric with respect to the central plane of the accelerator which may be identified with the plane of the particle equilibrium orbits. Since in the present case rot $\vec{B}(r, \Theta, z) = 0$ the magnetic field may derive from a scalar potential, i.e., $B = \text{grad}\ \Phi$, where Φ may be expanded in a power series of z:

$$\Phi (r, \theta, z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\Delta^n B(r,\theta))z^{2n+1} \qquad (IV,2)$$

which usually may be broken off already at n=1.

$\Delta = (1/r)(\partial/\partial r)r(\partial/\partial r) + (1/r^2)(\partial^2/\partial r^2)$ and $B(r/\theta) = B_z(r, \theta, 0)$.

An analytic expression in the form of a power series in r and a Fourier series in θ may be given for the funtion $B(r, \theta)$; it is, however, more expedient in practice to give $B(r, \theta)$ point by point on a r, θ-net, and, if necessary, to interpolate between these values (see Ref.35-37). At any rate, the system (IV,1) cannot be integrated in an elementary manner, but must be treated numerically.

It is, to start with, of particular interest to determine the equilibrium orbits of a particle and the frequencies ν_r and ν_z (focusing frequencies) of the radial and axial oscillations (see Ref.35-37) belonging to several given particle energies for the given magnetic field configurations $B(r, \theta)$. (The particles of the beam carry out these oscillations about the equilibrium orbits if as is always the case they do not exactly fulfill the initial conditions of the equilibrium orbits, a fact that is also necessary to achieve beam intensity). It is usually sufficient to allow for the linear deviations, as being remarkably great, which excludes a coupling of the radial and axial oscillations. The equilibrium orbit is defined to be that possible particle orbit for which the orbit radius r and the radial momentum p_r are identical (see Ref.35) at the beginning and at the end of one of the N identical magnetic field sectors of an AVF cyclotron. (Since the equilibrium orbits lie in the central plane of the cyclotron we have always $z = p_z = 0$).

The periodically occurring initial values

$$r_o = r(o) = r(\theta_o), \qquad \theta_o = 2\pi/N \qquad \text{(IV,3)}$$
$$p_{ro} = p_r(o) = p_r(\theta_o),$$

defined in this way must be found by successive approximation. For this purpose the system (see Ref.35)

$$p_r' = q - rB(r, \theta), \qquad q = (p^2 - p_r^2)^{1/2} \qquad \text{(IV,1a)}$$
$$r' = (r/q)p_r,$$

found from the system (IV,1) by specializing for the central plane is integrated numerically with approximate initial conditions for the equilibrium orbit. If no better values are available $r_o = p/\bar{B}$, $p_{r_o} = 0$ are chosen as such conditions, where $\bar{B}$ is the value of the magnetic field averaged over θ (see Ref.35). The integration mentioned yields the values $r(\theta_o)$ and $p_r(\theta_o)$ at the end of the sector differing by

$$\varepsilon_1 = r(\theta_o) - r_o, \quad \varepsilon_2 = p_r(\theta_o) - p_{r_o} \qquad \text{(IV,4)}$$

from the initial values. We repeat the integration with two modified initial conditions

$$\begin{aligned} r_1 &= r_o + \delta r & r_2 &= r_o \\ p_{r_1} &= p_{r_o} & p_{r_2} &= p_{r_o} + \delta p_r \end{aligned} \qquad \text{(IV,5)}$$

where the choice of δr and δp_r is suggested by the deviations (IV,4). In doing so, the new values $r_1(\theta_o)$, $r_2(\theta_o)$, $p_1(\theta_o)$, $p_2(\theta_o)$ result at the end of the sector; by means of the transformation matrix (J) they may be represented as linear functions of the initial values (IV,5) (see Ref.35)

$$r_1(\theta_o) = r_o + I_{11}\delta r, \quad p_{r_1}(\theta_o) = p_{r_o} + I_{12}\delta p_r \tag{IV,6}$$

$$r_2(\theta_2) = r_o + I_{21}\delta r, \quad p_{r_2}(\theta_o) = p_{r_o} + I_{22}\delta p_r$$

The unknown a_1 and a_2 to be calculated from the system of equations

$$\begin{aligned} (I_{11} - 1)a_1\delta r + I_{12}a_2\delta p_r &= \varepsilon_1 \\ I_{21}a_1\delta r + (I_{22} - 1)a_2\delta p_r &= \varepsilon_2 \end{aligned} \tag{IV,7}$$

permit the calculation of new improved initial values for the equilibrium orbit, viz.

$$\bar{r}_o = r_o - a_1\delta r, \quad \bar{p}_{r_o} = p_{r_o} - a_2\delta p_r \tag{IV,8}$$

This procedure is repeated until the conditions of the equilibrium orbit (IV,3) are fulfilled with sufficient accuracy. If $\varepsilon = \varepsilon_1 + \varepsilon_2$ is the deviation of the first approximation, then the deviations of the following approximations are ε^2, ε^4, ...; therefore, the method converges very rapidly (see Ref.35). At the same time also the matrix (J) is determined more and more exactly; according to the Floquet theorem it may be used to calculate the focusing frequency from the equation (see Ref.35)

$$2\cos(\nu_r\theta_o) = I_{11} + I_{22} \tag{IV,9}$$

The determination of the axial focusing frequency is carried out in a similar manner, but we shall not treat this problem any longer.

The L.-G.-method will now be used to a stepwise integration of system (IV,1a), in particular, in Knapp's formulation as an iteration method (see Ref.2). The interval of integration $(0, \theta_o)$ will now be subdivided into n equal steps, and, in order to illustrate the scheme of calculation, we pick out the first subinterval $(0, \theta_o/n)$. For = 0 the initial conditions are p_{r_o}, r_o corresponding to the conditions (III,70). First, the Lie operator D defined by formula (III,74) is needed for the calculation. Comparing the general system (III,67), (III,68), (III,69) with the system (IV,1a) of the present problem (θ is to be identified with t) we find

$$D = (q - rB)(\partial/\partial p_r) + (r/q)\ p_r\ (\partial/\partial r) - (\partial/\partial\theta) \qquad \text{(IV,10)}$$

With its help we can immediately write down an initial solution according to (III,86) ($\bar{m} = 2$ was chosen)

$$_o p_r(\theta) = p_{r_o} + \theta(q_o - r_o B(r_{o1} 0)) - (\theta^2/2) p_{r_o} (1 - r_o^2/q_o) \cdot$$
$$\cdot\ ((\partial B/\partial r)_o - r_o(\partial B/\partial\theta)_o)$$

$$_o r(\theta) = r_o + \theta(r_o/q_o) p_{r_o} - (\theta^2/2)(r_o/q_o) \cdot$$
$$\cdot\ ((q_o - r_o B(r_o,\ 0))(1 + p^2_{r_o}/q_o^2) + p^2_{r_o}/q_o \qquad \text{(IV,11)}$$

where $q_o = (p^2 - p^2_{r_o})^{1/2}$, $(\partial B/\partial r)_o = (\partial B/\partial r)_{r-r_o}$, $\theta = \theta_o'$ etc.

The derivative of the solution (IV,11) with respect to θ will be designated by $_o g_p(\ )$, $_o g_r(\ )$. On account of definition (III,88) the operator $_o D_2$ is given in the zeroth approximation

$$_o D_2 = ((q - rB) - {_o g_p})\ \frac{\partial}{\partial p_r} + ((r/q)p_r - {_o g_r})\ \frac{\partial}{\partial r} \qquad \text{(IV,12)}$$

Thus the following first approximation $_1 g_p$, $_1 g_r$ may be obtained with the help of (IV,1a), (IV,10), (IV,11), (IV,12), according to Knapp's iteration formula (III,85). (Let $\bar{m} = m = 2$).

$$
\begin{aligned}
{}_1g_p(\theta) &= ((p^2 - {}_o\hat{p}_r^2(\theta))^{1/2} - {}_o\hat{r}(\theta)B({}_o\hat{r}(\theta), \theta)) + \\
&+ \int_0^\theta ({}_oD_2Dp_r)_o\hat{p}_r(\theta'), {}_o\hat{r}(\theta'), \theta'd\theta' + \\
&+ \int_0^\theta (0 - \theta')({}_oD_2D^2p_r) \\
{}_1g_r(\theta) &= ({}_o\hat{r}(\theta)/(p^2 - {}_o\hat{p}_r^2(\theta))^{1/2}{}_o\hat{p}_r(\theta) + \int_0^\theta ({}_oD_2D_r)_o\hat{p}_r(\theta'), \\
&{}_o\hat{r}(\theta'), \theta'd\theta' + \int_0^\theta (\theta - \theta')({}_oD_2D^2r)_o\hat{p}_r(\theta'), {}_o\hat{r}(\theta')\theta'd\theta
\end{aligned}
\tag{IV,13}
$$

In doing so, the following rule must be taken into account in calculating the integrands (according to (III,76)):

$$
\begin{aligned}
&({}_oD_2Dp_r)_o\hat{p}_r(\theta'), {}_o\hat{r}(\theta'), \theta' = ({}_oD_2(q-rB)_o\hat{p}_r(\theta'), {}_o\hat{r}(\theta'), 0) = \\
&= -((p^2 - {}_o\hat{p}_r^2(\theta'))^{1/2} - {}_o\hat{r}(\theta')B({}_o\hat{r}(\theta'), \theta') - {}_og_p(\theta') \cdot \\
&\cdot ({}_o\hat{p}_r(\theta')/(p^2 - {}_o\hat{p}_r^2(\theta'))^{1/2} - (({}_o\hat{r}(\theta')/(p^2 - {}_o\hat{p}_r^2(\theta))^{1/2}) \cdot \\
&\cdot ({}_o\hat{p}_r(\theta') - {}_og_r(\theta')) \cdot (B({}_o\hat{r}(\theta'), \theta') - {}_o\hat{r}(\theta')(\partial B(r,\theta)/\partial r)
\end{aligned}
\tag{IV,14}
$$

Owing to the general formula (III,84) the first approximation to the solution of the system (IV,1a) is given by

$$
\begin{aligned}
{}_1\hat{p}_r(\theta) &= p_{r_o} + \int_0^\theta {}_1g_p(\theta')d\theta' \\
{}_1r(\theta) &= r_o + \int_0^\theta {}_1g_r(\theta')d\theta'
\end{aligned}
\tag{IV,15}
$$

If this solution does not yet correspond to the accuracy requirements an improved operator ${}_1D_2$ must be determined with the help of (IV,13) and the general definitions (III,75), (III,80). With its help an additional application of the iteration formula (III,84), (III,85) yields the second improved approximation ${}_2p_r$, ${}_2r$ where, of course, the first approximation (IV,15) has to be substituted on the right-hand-side of (III,85). The final values $p_r(\theta_o/n)$, $r(\theta_o/n)$ obtained in this way are then the initial values for the next step of integration etc.

According to the facts indicated at the end of Section 2 the functions

$$
\begin{aligned}
{}_oR_p(\theta) &= p_r(\theta) - {}_1\hat{p}_r(\theta) \simeq \int_0^\theta {}_oS_p(\theta')\,d\theta' \\
{}_oR_r(\theta) &= r(\theta) - {}_1r(\theta) \simeq \int_0^\theta {}_oS_r(\theta')\,d\theta'
\end{aligned}
\qquad \text{(IV,16)}
$$

are a reasonable correction of the first approximation for sufficiently small step length θ_o/n; in this formula, ${}_oS_p(\theta')$, ${}_oS_r(\theta')$ are to be obtained from the general definition (III,91) specializing to the system (IV,1a) and from the approximation (IV,13), (IV,15) given for it above.

This completes the theoretical discussion of finding the equilibrium orbits in an AVF cyclotron with the help of the L.-G.-method.

Chapter V

Optimization Problems Solved by Lie-Series: Soft Landing on the Moon with Fuel Minimization

by F.Cap, W.Groebner and J.Weil

Abstract: The problem of soft landing on the moon with the additional requirement that fuel consumption during the deceleration of the rocket should be minimized is solved formally with the help of Lie series. A corresponding one-dimensional problem having no solution is briefly reviewed.

The enduring effort to improve technology - generally speaking as well as, in our case, the technology of space craft - has given rise to the concept of optimization; optimum control systems are, therefore, gaining more and more importance in space flight. Optimization may be carried out with respect to various parameters, as, e.g., time of flight or consumption of propellant. In this chapter, we shall consider the problem of soft landing on the surface of the moon under the additional condition of minimizing the fuel needed to operate the decelerating rockets.

Optimization problems were for the first time solved with the help of Lie series by Groebner (see Ref.39) and Dotzauer (see Ref.40); the present chapter is closely related to their considerations. - The equations used by us show an intimate resemblance to those used by other authors (see Ref.41). - Before stating and formally solving our specific problem we shall present the general formalism based on considerations of Ref.39,40.

(5.1) General Formalism

Let p functions $x_i(t)$ $(i=1,\ldots,p)$ specifying, e.g., the positions and momenta of a spacecraft and q functions $y_j(t)$ $(j=1,\ldots,q)$ representing control forces be given. The equations to be solved are of the form

$$x_i = G_i(x, y) \qquad (i = 1, \ldots, p) \tag{V,1}$$

They serve as the constraint conditions supplementing the equations stemming from a minimization of the integral T

$$I(y) = \int_0^T \left\{F(x,y,\dot{y}) + \lambda_i G_i(x,y)\right\} dt \tag{V,2}$$

To secure uniqueness, p initial or final conditions of the form

$$x_i(0) = a_i \tag{V,3a}$$

or

$$x_i(T) = c_i \tag{V,3b}$$

respectively, must be given. As will be shown below, Lie series formalism provides a convenient method of transforming final conditions to initial ones.

Briefly, we shall have to find the 2p + q functions $x_i(t)$, $y_j(t)$ and $\lambda_K(t)$ - the Lagrange multipliers - from (V,1) and the p equations:

$$\dot{\lambda}_K = -\sum \lambda_i \frac{\partial G_i}{\partial x_K} + \frac{\partial F}{\partial x_K} \tag{V,4}$$

and the q equations:

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{y}_j} = -\sum \lambda_i \frac{\partial G_i}{\partial y_j} + \frac{\partial F}{\partial y_j} \qquad (V,5)$$

together with boundary conditions.

We shall now pass to the calculation of the corresponding new initial conditions from the final conditions, making use of Lie series formalism. Redefining our variables in a straightforward way appropriate to obtaining Lie solutions our system reads (see Ref.1):

$$\dot{Z}_i = \mathfrak{I}_i(Z) \qquad (i = 1, \ldots, 2p+q) \qquad (V,6)$$

Let, e.g., $2p + q - 1$ initial conditions

$$(Z_i)_{t=0} = a_i \qquad (i = 1, \ldots, 2p + q - 1) \qquad (V,7)$$

and one final condition

$$F(Z_1, \ldots, Z_n)_{t=T} = 0 \qquad (V,8)$$

be given. The solution of this system is given by:

$$Z_i = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu a_i \equiv e^{tD} a_i \qquad (V,9)$$

where the D-operator is composed of the right-hand sides of (V,6) in the well-known manner (see, e.g., Ref.1). Using the well-known commutation theorem (see Ref.1) we have:

$$F(Z_1, \ldots, Z_{2p+q}) = e^{tD}.F(a_i) \equiv \Phi\,(t;a_i) \qquad (V,10)$$

The function Φ defined in this way may be used to reexpress the final condition:

$$\Phi\,(T;\, a_1, \ldots, a_{n-1}, \zeta) = 0 \qquad (V,11)$$

where ζ is considered variable such that

$$\zeta = b \equiv z_n(o) \tag{V,12}$$

The initial condition representing the final condition reads:

$$\Phi(0;\ a_1,\ \ldots,\ a_{n-1},\zeta) = F(a_1,\ \ldots,\ a_{n-1};\zeta) = 0 \tag{V,13}$$

where the value of $\zeta_{t=T}$ is connected with $b \equiv \zeta_{t=o}$ by

$$\zeta_{t=T} = (e^{TD_1}b)_{\tau=o} \tag{V,14}$$

with

$$D_1 = \frac{\partial}{\partial\tau} - \frac{\Phi_\tau(\tau;a_1,\ \ldots,a_{n-1},\zeta)}{\Phi_\zeta(\tau;a_1,\ \ldots,a_{n-1},\zeta)}\frac{\partial}{\partial\zeta} \tag{V,15}$$

$$(\Phi_\tau = \frac{\partial\Phi}{\partial\tau},\ \ \Phi_\zeta = \frac{\partial\Phi}{\partial\zeta})$$

This statement (V,13) is easily proved as follows:

With

$$D_1\tau = 1,\ \ D_1^\nu\ \tau = 0 \tag{V,16}$$

and

$$D_1^\nu\ \Phi\ (\tau;\ a_1,\ \ldots,a_{n-1},\zeta) = 0 \tag{V,17}$$

as well as

$$T = (e^{TD_1}\tau)_{\zeta=b,\tau=o} \tag{V,18}$$

we obtain, using again the commutation theorem (see Ref.1):

$$\Phi(T,\zeta;\ a_1,\ \ldots,n_{n-1}) = (e^{TD_1}\ \Phi(\tau,\zeta;a_1,\ldots,a_{n-1})_{\substack{\zeta=b\\ \tau=o}} =$$

$$= \Phi\ (a,\ a_1,\ \ldots,\ a_{n-1},b) = 0 \tag{V,19}$$

q.e.d.

(5.2) The Problem of Soft Landing on the Surface of the Moon with Fuel Optimization

Our specific problem to be solved by Lie series formalism is a two-body problem, i.e., the decelerated motion of a spacecraft in the neighborhood of the moon subject to the conditions of soft landing ($v_{t=T}$ = 2-3m/sec) as well as of minimum propellant consumption during the action of the decelerating rockets. The equations of motion to be employed read:

$$\dot{\vec{x}}_v = \vec{v}_v \tag{V,20a}$$

$$m_v(t)\dot{\vec{v}}_v(t) = \frac{\gamma m_m m_v \vec{x}_v}{r_v^3} + \vec{y}(t) \tag{V,20b}$$

where $m_v(t)$ is the mass of the vehicle, $\vec{x}_v(t)$ the position of the vehicle in the moon's coordinate system, $\vec{v}_v$ its velocity, r_v its distance from the center of the moon and γ the gravitational constant. $\vec{y}(t)$ is defined by:

$$-c\,\frac{dm_v}{dt} = \sqrt{\vec{y}^{\,2}} \tag{V,21}$$

with the optimization integral

$$\int_0^T \frac{dm_v}{dt}\,dt = -\frac{1}{c}\int_0^T \sqrt{\vec{y}^{\,2}}\,dt = \text{extr.} \tag{V,22}$$

where c is the constant exhaust velocity of the vehicle. Additionally, the following boundary conditions are to be satisfied:

$$\begin{aligned} &\dot{\vec{x}}_v(0) = \vec{v}_v(0),\ \dot{\vec{x}}_v(T) = \vec{v}_v(T) = 2\text{-}3\text{m sec}^{-1} \\ &\vec{x}_v(0) = \vec{x}_v^{\,o},\ \vec{x}_v(T) = \vec{x}_{vT} \end{aligned} \tag{V,23}$$

if we assume the initial and final points of the vehicle's path to be given.

Rewriting our equations to be solved we obtain:

$$
\begin{aligned}
& i = 1,\ (\dot{\vec{x}}_v)_x \equiv \dot{x}_1 = G_1 \equiv x_4; \\
& i = 2,\ (\dot{\vec{x}}_v)_y \equiv \dot{x}_2 = G_2 \equiv x_5; \\
& i = 3,\ (\dot{\vec{x}}_v)_z \equiv \dot{x}_3 = G_3 \equiv x_6; \\
& i = 4,\ (\dot{\vec{v}}_v)_x \equiv x_4, \\
& i = 5,\ (\dot{\vec{v}}_v)_y \equiv x_5, \\
& i = 6,\ (\dot{\vec{v}}_v)_z \equiv x_6
\end{aligned}
\tag{V,20c}
$$

and

$$
\left.\begin{matrix} G_4 \\ G_5 \\ G_6 \end{matrix}\right\} \equiv \frac{\gamma m_m}{r^3_v} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \frac{1}{y_4} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \tag{V,24}
$$

with

$$\vec{x}_v = (x_1,\ x_2,\ x_3) \tag{V,25a}$$

and

$$\vec{y} = (y_1,\ y_2,\ y_3) \tag{V,25b}$$

as well as

$$m_v(t) \equiv y_4 \tag{V,25c}$$

such that we have to solve six equations of the type:

$$\dot{x}_i = G_i(x_1,x_2,x_3,x_4,x_5,x_6,y_1,y_2,y_3,y_4) \tag{V,20d}$$

which are supplemented by the Eulerian equations stemming from the optimization integral:

$$\dot{\lambda}_K = -\sum_i \lambda_i \frac{\partial G_i}{\partial x_K} + \frac{\partial F}{\partial x_K} \qquad (V,4)$$

$$\frac{d}{dt}\frac{\partial F}{\partial y_j} = -\sum_i \lambda_i \frac{\partial G_i}{\partial y_j} + \frac{\partial F}{\partial y_j} \qquad (V,5)$$

where in our case:

$$F = -\frac{1}{c}\sqrt{y_1^2 + y_2^2 + y_3^2} \qquad (V,26)$$

such that (V,4) becomes

$$\dot{\lambda}_K = -\lambda_4 \frac{\partial G_4}{\partial x_K} - \lambda_5 \frac{\partial G_5}{\partial x_K} - \lambda_6 \frac{\partial G_6}{\partial x_K} \quad (K = 1,2,3) \qquad (V,4a)$$

and

$$\dot{\lambda}_4 = -\lambda_1,\ \dot{\lambda}_5 = -\lambda_2,\ \dot{\lambda}_6 = -\lambda_3 \qquad (V,4b)$$

since the $G_i (i = 1,2,3)$ are of the form $G_1 \equiv x_4$, $G_2 \equiv x_5$, $G_3 \equiv x_6$ and the F are independent of x_i. The equations (V,5) yield:

$$\frac{d}{dt}\frac{\partial F}{\partial y_j} = 0 = \lambda_4 \frac{\partial G_4}{\partial y_j} + \lambda_5 \frac{\partial G_5}{\partial y_j} + \lambda_6 \frac{\partial G_6}{\partial y_j} + \frac{\partial F}{\partial y_j} \qquad (V,5a)$$

$$(j = 1,2,3,4)$$

such that we have 16 equations for 16 functions from which 12 equations for 12 functions (i.e., corresponding to the number of boundary conditions) can be deduced by elimination. The final conditions $\dot{\vec{x}}_v(T) = \vec{v}(T)$ and $\vec{x}_v(T) = \vec{x}_{vT}$ are transformed to initial ones by the process given above.

It was not possible to carry out detailed or numerical calculations since NASA stopped its financial contribution.

The one-dimensional problem of soft landing is treated in many papers (see Ref.42-47).

Chapter VI

On the Equations of Motion of Satellites

by

F. Cap, D. Floriani, A. Schett and J. Weil

Abstract: After discussing the forces and torques acting on a a satellite we turn to the expansion of $\vec{g}(\vec{r})$ in a Taylor series $(\vec{F} = \vec{g}(\vec{r})m)$. Explicit expressions of the first four terms of this series and the complete equations of motion of a satellite. In section (6.2) we present the solution of the equations describing the heavy asymmetric gyroscope in the forms:

a) Solution = Solution (heavy, symmetric) + contributions from asymmetry.

b) Solution = Solution (symmetric, forcefree) + contributions from asymmetry and forces.

c) Solution = Solution (asymmetric forcefree) + contributions from forces.

Insection (6.3) the above mentioned equations of motion are solved using Lie series. In section (6.4) some aspects of our numerical calculations concerning the motion of a satellite about its mass center are discussed.

(6.1) The Dynamical Equations Describing the Motion of a Satellite about its Mass Center by A. Schett, J. Weil and D. Floriani

(6.11) Introduction

Many problems posed in satellite research work require the fixation of the satellite's position with respect to the surface of the earth. For this purpose, a number of active methods has been developed since the beginning of space flight whose success depends on the precision with which each element of the regulating mechanism is operating; further a continuous input of energy is necessary. Recently, great attention was paid to the idea of utilizing the earth's gravitational and magnetic fields for the purpose of stabilization. This aim would be achieved if for certain initial positions torques became effective in the further movement of the satellite along its path such that it is, so to speak, rotated back into its original position relative to the earth's surface. Consequently, the influence of very small effects, as, e.g., radiation pressure and air friction (treated in (6.12)) must be regarded in the theoretical treatment of the problem. For this purpose also higher terms of the Taylor series expansion of $\vec{g}(\vec{r})$ were considered here in the equations of motion since the first two terms are certainly insufficient (see Sect. (6.13)).

(6.12) A Survey of Forces and Torques Acting on Satellites and a Survey of Papers Dealing with the Attitude of a Satellite and with Gravity Gradient Stabilization

If we consider a satellite orbiting around the earth, six second-order differential equations are required to describe its motion (see Ref. 52). Three of these equations serve to describe the motion of the center of mass whereas the three remaining ones specify the orientation of the satellite. Several perturbational torques and forces act on the satellite. It seems to be desirable to obtain a solution in which the perturbational torques and forces are represented in separate form.

In Sect. (6.2), we shall show that Lie series representation of the solution to differential equations make it possible to split the solution into several terms each of which is responsible for a definite physical effect. This splitting off procedure can be used to represent the solution describing the motion of the satellite.

In order to find a favorable representation in numerical evaluations, it is, however, necessary to know the order of magnitude of the individual forces and torques. Our interest will, therefore, be focused on the perturbational forces $\vec{F}_\rho$ and the torques $\vec{M}_\rho$. Besides some analytic expressions for these forces and torques we shall present some numerical data for $\vec{F}_\rho$ and $\vec{M}_\rho$ in order to compare the respective orders of magnitude.

(6.121) Survey of Forces and Torques Acting on a Satellite:

(6.1211) The Forces $\vec{F}_\rho$ (Ref. 53):

(6.12111) Analytical expressions for $\vec{F}_\rho$:

With respect to a frame of reference whose origin lies at the center of mass of the earth, we have the following principal forces acting on a satellite:

$\vec{F}_o$: the gravitational forces which are caused by a spherically symmetric potential field.

$\vec{F}_1$: perturbational forces corresponding to an asymmetry of the gravitational field caused by oblateness and inhomogeneity of the earth.

$\vec{F}_2$: drag force, i.e., forces due to the fact that the vehicle moves in a rerified gas rather than in vacuum.

$\vec{F}_3$: force due to radiation pressure.

$\vec{F}_4$: gravitational forces due to the action of the sun and the moon.

$\vec{F}_5$: is the magnetic force and

$\vec{F}_6$: is the force due to casual effects.

We shall now enter a more detailed discussion of the individual forces:

a) $\vec{F}_o$: The gravitational force is given by

$$\vec{F}_o = -\frac{\mu\vec{r}}{r^3} \tag{VI,1}$$

with

$$\mu = h^2 M \cdot m$$

h^2 is here the gravitational constant, M the mass of the earth, m that of the satellite, and $\vec{r}$ is the vector pointing in the direction of the satellite.

b) $\vec{F}_1$: The potential taking account of the asymmetry of the gravitational field is given by (Ref. 55, 56)

$$U_{total} = U_{symm} + U_{asymm}$$

where

$$U_{asymm} = \frac{K^2\mu}{r}\sum_{n=2}^{\infty}\left\{J_n\left(\frac{R}{r}\right)^n P_n(\sin\beta) - \left(\frac{R}{r}\right)^n P_n^m(\sin\beta)(C_{n,m}\cos m\varphi + S_{n,m}\sin m\varphi)\right\} \tag{VI,2}$$

R is the equatorial radius of the earth, the P_n are the n-th order Legendre polynomials, the J_n are coefficients and β is the angle between the equatorial plane and the plane of the orbital motion. The coefficients J_n can be determined by measurement; their values up to the neinth order are (Ref. 57, p. 71).

$$\begin{aligned}
J_2 &= (1082.48 \pm 0.04)\cdot 10^{-6}\\
J_3 &= (-2.566 \pm 0.012)\cdot 10^{-6}\\
J_4 &= (-1.84 \pm 0.09).10^{-6}\\
J_5 &= (-0.063 \pm 0.019)\cdot 10^{-6}\\
J_6 &= (0.39 \pm 0.09)\cdot 10^{-6}\\
J_7 &= (-0.469 \pm 0.021)\cdot 10^{-6}
\end{aligned} \tag{VI,3}$$

$$J_8 = (-0.02 \pm 0.07) \cdot 10^{-6}$$
$$J_9 = (0.114 \pm 0.025) \cdot 10^{-6} \qquad \text{(VI,3)}$$

Using (VI,2), we obtain for the force due to gravitational asymmetry:

$$\vec{F}_1 = \nabla U \qquad \text{(VI,4)}$$

with
$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

x, y, z is a frame of reference fixed with respect to the earth and having its origin at the center of the earth. (Remark: The coefficients $C_{n,m}$ and $S_{n,m}$ are much more difficult to determine than the J_n (Ref. 57, p. 72)).

c) $\vec{F}_2$: The drag force $\vec{F}_2$ acting on a satellite is given by (Ref. 57, p. 242)

$$\vec{F}_2 = -\frac{1}{2} C_D \cdot \rho \cdot S \cdot |\vec{v}| \vec{v} \qquad \text{(VI,5)}$$

where C_D is the drag coefficient usually taken to be 2.0; ρ is the atmospheric density; it is plotted in a diagram in (Ref. 59, p. 88). S is the area occupied by the satellite and projected normally to the velocity vector $\vec{v}$. The following table (VI,I) shows the atmospheric density ρ as a function of the altitude h (Ref. 59, p. 88):

height above the surface of the earth	ρ(density)/$\frac{g}{cm^3}$
400 km	10^{-14}
600 km	10^{-15}
800 km	10^{-16}
1000 km	10^{-17}

Table (VI,I): Atmospheric density as a function of the altitude above the surface of the earth.

In a first approximation the density can be described by the following relation (Ref. 59 , p. 124):

$$\log \rho(h) = \log_{stand}\rho(h) - \left\{ i(220,h)\frac{220-F}{120} + a(h)\cdot g(a) + \right.$$
$$\left. + \Theta(h)\cdot f(\Theta) + K(150,h)\cdot\frac{A_p}{150} \right\} \qquad (VI,6)$$

The functions i(220,h), a(h), g(a), $\Theta(h)$, $f(\Theta)$, K(150,h) and A_p are plotted in diagrams in the paper by Paetzold (Ref. 58). With the help of this diagram $\rho(h)$ can easily be computed. In Ref. 59, p. 86, a more complex formula for $\vec{F}_2$ is given:

$$F_2 = \frac{\rho s U^2}{2s}\exp\left[-(s')^2\right]\left\{\frac{2-\sigma'}{\sqrt{\pi}}(s') + \sqrt{\frac{\sigma'}{2}\frac{Tw}{T}}\right\} +$$
$$+ \left[1+\mathrm{erf}(s')\right]\cdot\left\{(2-\sigma')\left[\frac{1}{2}+(s')^2\right] + \sqrt{\frac{\sigma}{2}\frac{\pi Tw}{T}}\cdot s'\right\} \qquad (VI,7)$$

where S is the projection of the satellite's cross section normal to the velocity vector, ρ is the density, U is the relative velocity of the free stream, s is the speed ratio, $s' = s\cdot\sin\eta$, η is the local incidence measured from the surface. σ' is the surface reflection for normal momentum, T_w is the surface temperature, T the absolute temperature and

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int \exp(-x'^2)dx' \qquad \text{with } 0 \leq \mathrm{erf}(x) < 1.$$

d) $\vec{F}_3$: The force $\vec{F}_3$ is caused by the radiation pressure (Ref. 60, p. 119):

$$F_3 = S_{rad}\cdot\frac{1,4\cdot 10^6}{c} \qquad (VI,8)$$

where c is the velocity of light, and S_{rad} is the area projected normal to the vector pointing from the space vehicle to the sun. The equation holds for an intercepting surface; if the rays are totally reflected, (VI,8) must be multiplied by a factor 2. A more sophisticated formula for $\bar{F}_3$ is given in Ref. 59, p. 89. The solar radiation reflected from the earth is negligible.

e) $\vec{F}_4$: The analytic expression for the perturbational forces caused by the sun and the moon reads:

$$\vec{F}_4 = -\nabla \sum_{i<j} f^2 \cdot \frac{m_j m_i}{|\vec{r}_i - \vec{r}_j|} \qquad i=0,1;\; j=1,2 \qquad \text{(VI,9)}$$

where m_o is the mass of the satellite, m_1 the mass of the sun, m_2 the mass of the moon, f^2 the gravitational constant and $\vec{r}_i$ are the respective position vectors.

(6.12112) Order of Magnitude of the Individual $\vec{F}_\rho$:

The analytic expressions for the perturbational forces $\vec{F}_\rho$ are rather complex and depend on several parameters. The following table (VI,II) presents numerical data from Explorer XI for the purpose of comparing the magnitudes of the different forces.

Table (VI,II): Data of Explorer XI

F_1	F_2	F_3	F_4	F_o
$<10^{-1} m_s$	$\sim 3.10^{-5} \cdot S$ (see Ref. 59, p. 199)	$\sim 6 \cdot 10^{-5} \cdot S$ (see Ref. 59, p. 203)	m_s **) (see Ref. 60, p. 126) ($F_{moon} \sim 10^{-5}$) ($F_{sun} \sim m_s$)	$10^2 m_s$ *)

(The value of F_ρ is given in dyn)

Satellite Explorer XI: r = 7512 km (the half major axis on the average), its velocity is about 8 km/sec

*) m_s is the mass of the satellite (given in gr)

**) this seems to be in contradiction to the effect;

The contradiction mentioned is resolved by realizing that the gravity the earth experiences under the influence of the sun is, on its time average, equal to that felt by the satellite, i.e., that the satellite's path about the earth is only perturbed by the sun's gravity insofar as its distance from the sun is sometimes smaller and sometimes greater than that of the earth. These differences are, however, comparatively small (Compare 7000 km with 150 millions of km!).

The acceleration due to the radiation pressure of the sun is, however, vanishingly small because of the great mass of the earth ($m_E = 6 \cdot 10^{27}$ gr), it is only $3 \cdot 10^{-15}$ cm/sec^2, i. e., in this case the effects do not cancel.

The following table (VI,III) shows the relative values of the drag forces as a function of the altitude:

F_2	Altitude of Explorer XI
100 dyn	400 km
10 dyn	500 km
5 dyn	600 km
1 dyn	800 km
0,1 dyn	1000 km
0,01 dyn	1200 km

Table (VI,III): Relative values of the drag forces as a function of the altitude

The forces F_o, F_1, F_3, F_4 vary only slightly with the altitude.

The following table (VI,IV) shows the corresponding values for Echo I.

F_o	F_1	F_2	F_3	F_4	Altitude
		$\sim 0{,}2\cdot 10^{-2}S$			400 km
		$\sim 0{,}2\cdot 10^{-3}S$			600 km
		$\sim 0{,}2\cdot 10^{-4}S$			800 km
$\sim 10^2\ m_s$	$<10^{-1}\ m_s$	$\sim 0{,}2\cdot 10^{-5}S$	$5\cdot 10^{-5}S_{rad}$	$\sim m_s$	1000 km (v=7,3 km/sec)
		$\sim 0{,}2\cdot 10^{-6}S$			1200 km

The F_ρ are given in dyn; m_s and S_{rad} means the numerical value of this quantities.

Table (VI,IV): Data of Echo I

(6.1212) The Torques $\vec{M}_\rho$ Acting on a Satellite:

(6.12121) Analytic Expressions for the Torques:

The relevant torques will be designated in the following manner:

$\vec{M}_o$ is the gravitational torque, caused by a spherically symmetric potential field

$\vec{M}_1$ is the torque due to gravitational asymmetry

$\vec{M}_2$ is the drag torque

$\vec{M}_3$ is the torque caused by radiation

$\vec{M}_4$ is the torque due to the gravitation of sun and moon

$\vec{M}_5$ is the magnetic torque

$\vec{M}_6$ is the torque due to casual effects, as collisions with meteorites, etc.

Let us now discuss these torques in some detail:

a) $\vec{M}_o$: $\vec{M}_o$ is given by

$$\vec{M}_o = \int \vec{f} x \vec{r} \, dm \tag{VI,10}$$

where the integral is taken over all elements of the extended body, and $\vec{f}$ is the force density (i. e., per unit of mass). $|\vec{r}|$ is the distance from the center of rotation (i. e., the center of the earth) to the mass point.

b) $\vec{M}_2$: $\vec{M}_2$ is given by the following expression (Ref. 57, p. 242):

$$\vec{M}_2 = \vec{a} x \vec{F}_2 \tag{VI,11}$$

where $|\vec{a}|$ is the distance of the center of pressure from the center of mass. $\vec{F}_2$ is defined by Eqs. (VI,5) and (VI,7), resp.

c) $\vec{M}_3$: $\vec{M}_3$ is specified by an equation that is completely analogous to (VI,11):

$$\vec{M}_3 = \vec{b} x \vec{F}_3 \tag{VI,12}$$

where $|\vec{b}|$ is the distance of the center of pressure from the center of mass. $\vec{F}_3$ is defined by Eq. (VI,8).

d) $\vec{M}_4$: In analogy to (VI,10), the torques due to sun and moon gravity are given by:

$$\vec{M}_4 = \int \vec{f} x \vec{r} \, dm$$

where $\vec{f}$ again is a force density and $\vec{r}$ a position vector.

e) $\vec{M}_5$: The magnetic torques may be due to different reasons:

α) The induced magnetic torque ${}_i\vec{M}_5$ (Ref. 57, p. 241 and Ref. 59, p. 203):

$${}_i\vec{M}_5 = \vec{M} x \vec{B}$$

where

$$\vec{M} = \frac{(\mu_r - 1)}{\mu_o} \cdot V \cdot (\vec{B} \cdot \vec{A}) \vec{A} \tag{VI,13}$$

with μ_r for the relative permeability, μ_o for the permeability of free space, V for the volume of the material in the walls and $\vec{A}$ for the unit vector along the longitudinal axis.

β) If the satellite has a permanent magnetic moment, as, e. g., caused by magnetic coils or rods, additional torques $_p\vec{M}_5$ will occur. Equations for these torques can be found in the papers by F. Mesch, et al. (Ref. 54, p. 3, Ref. 57, p. 245-246).

γ) Torques Due to Charge Separation (Ref. 59, p. 203):

If the satellite is constructed by conducting materials it can be considered as a conducting cylinder moving through a magnetic field, and an electric field of approximately 0.4 volts per meter can be induced across the longitudinal axis of the cylinder. This produces a charge separation that may influence the impact parameters of the incident ions. Thus the negative end will appear to have a larger drag cross section than the positive end to the positively charged atmospheric ions. The interesting feature of this mechanism is that each end alternates his sign during a tumble cycle so that the effect does not cancel owing to rotational symmetry. In the case of Explorer XI a net torque of 0,06 dyn·cm due to charge separation is obtained.

δ) Torque Due to Eddy Currents (Ref. 59, p. 201):

A conducting surface rotating in a magnetic field experiences induced eddy currents producing torques tending to oppose the rotation and to decrease the angular momentum of the body. This torque is smaller than 1,8 dyn·cm.

The following table (VI,V) compares the most significant moments in the case of Explorer XI.

M_o	M_2	M_3	M_4
max.: 113 dyn·cm av.: 57 dyn·cm	max.: 168 dyn·cm av.: 0 *)	max.: 3 dyn·cm	max.: 11 dyn·cm av.: 6 dyn·cm

Remark: $(J_1 - J_3) = 1{,}587 \cdot 10^8$ gr·cm^2; R = 7512 km; v = 8 km/sec;
$m \cdot h^2 = 3{,}986 \cdot 10^5$ km^3/sec^2

max.: maximum

av.: average (over a tumble cycle)

Table (VI,V): Data of moments of Explorer XI

(6.122) Survey of Papers Dealing with the Attitude of a Satellite and with Gravity Gradient Stabilization:

In the following a short summary of the most important results concerning attitude and stabilization problems is given. The survey will be arranged with respect to three items:

Derivation of the dynamical equations,

Expressions for gravitational torques acting on a satellite,

Stabilization of satellites.

(6.1221) The Dynamical Equations for a Satellite's Attitude in the Individual Papers:

Using the same frame of reference as in our paper F. W. Raymond

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*) The tumbling motion results in an equal and opposite torque a half tumble cycle; hence, the first-order torques average to zero owing to the rotational symmetry. Second-order effects are negligible.

(Ref. 62) obtains the following components N_ξ, N_η, N_ζ:

$$N_\xi = \left(|\vec{u}|^2 + \frac{3\gamma}{R^3}\right) m_i \eta_i \zeta_i + |\vec{u}| m_i \eta_i \dot{\xi}_i - K\dot{u}_1$$

$$N_\eta = \frac{-3\gamma}{R^3} m_i \zeta_i \xi_i + |\vec{u}| m_i \eta_i \dot{\eta}_i - K\dot{u}_2$$

$$N_\zeta = -\vec{u}^2 m_i \xi_i \eta_i + |\vec{u}| m_i \eta_i \dot{\zeta}_i - K\dot{u}_3$$

where again summation over i is implied (magnetic forces, perturbations of gravity due to sun and moon, radiation pressure and drag are neglected). The terms $K\dot{u}_i$ are damping terms due to the satellite's tumbling. The equations are identical with ours for the spherically symmetric case; Raymond does not solve them.

David L. Mott derives in Ref. 63 the same equations agreeing with ours and those of Raymond.

Linearized dynamical equations of coupled orbital and attitude motion are discussed in a paper by B. Lange (Ref. 64), who uses a reference frame fixed with respect to the satellite.

In his paper Irving Michelson (Ref 67) uses linearized vibrational equations for small angular displacements α, β, γ from a gravity-gradient stabilized equilibrium (the principal inertia axes lying at equilibrium in the directions of orbital angular velocity, the instantaneous earth vertical and the opposite direction of the instantaneous linear orbital velocity, respectively). The equations read:

$$A\ddot{\alpha} + 3\vec{u}^2(B-C)\alpha = 0$$

$$B\ddot{\beta} + 4\vec{u}^2(A-C)\beta + |\vec{u}|(A-B-C)\dot{\gamma} = 0$$

$$C\ddot{\gamma} + \vec{u}^2(A-B)\gamma - |\vec{u}|(A-B-C)\dot{\beta} = 0$$

where A, B, C are the principal moments of inertia, $\vec{u}$ is the orbital angular velocity for a circular orbit. The equations are characterized by the fact that the components of the gravitational torque are included and that they describe a motion about a point not fixed in space. They are solved in a manner indicating that there is an infinity of equilibrium satellite orientations in which, therefore, no attitude control is needed.

In his paper (Ref. 68), Robert R. Newton considers the problem of damping the librations of a prolate axially symmetric and gravitationally stabilized satellite by coupling it to the longitudinal oscillations of a mass-spring system connected to the satellite. Three equations of motions (for the mass of the satellite, the mass connected to the other end of the spring and for the libration component in the plane of orbit) are given. The coupling for librations in the plane of orbit is linear to the libration amplitude, and hence is effective for all amplitudes. Coupling for librations normal to the orbital plane is quadratic in amplitude and has low effectiveness for small amplitudes.

In his paper, B. Etkin (Ref. 69) presents a theoretical framework for analyzing the motion of a multibody satellite in a gravity-stabilized orbiting reference frame. It consists essentially of expressions for the forces and moments of the forcefield on arbitrary bodies and of their utilization in Lagrange's equations to find the equations of motion. It is applied to the analysis of a specific system designed for attitude stabilization. The equations are linearized and separated into two groups (longitudinal, i. e., parallel to the orbit plane, and lateral, i.e., transverse to the orbit plane).

In his paper, E. E. Zajac (Ref. 70) considers the damping of a two-body, viscously coupled, gravitationally oriented satellite that is in a circular orbit. A graphical method for determining the damping rate as a function of the damping coefficient is presented. The following system of equations is quoted for a two-body satellite in a circular orbit linked at the two-bodie's centers of mass by means of a linear spring:

$$A_1\frac{d^2\alpha}{dt^2} + 3\vec{u}^2(B_1-C_1)\alpha + K(\alpha-\beta) + \zeta\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right) = 0$$

$$A_2\frac{d^2\beta}{dt^2} + 3\vec{u}^2(B_2-C_2)\beta - K(\alpha-\beta) - \zeta\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right) = 0$$

where α, β are small deviations from the local vertical, A_i, B_i, C_i are the principal moments of inertia of the two bodies, $|\vec{u}|$ is the orbital frequency, K is the spring constant of the coupling spring, and ζ is the damping coefficient.

In his paper, Paul F. Hultquist (Ref. 71) computes the angular momentum in pitch and roll imparted to a totally stabilized, solar oriented satellite by gravitational torque over a year's time for a jet controlled satellite with one axis normal to the ecliptic and a transverse axis along the solar vector. Both circular and elliptic orbits are considered.

In his paper, Robert E. Roberson (Ref. 72) examines the foundations of methods for determining the vertical by differential gravity measurements.

In their paper R. D. Cole, M. E. Ekstrand and M. R. O'Neill (Ref. 73) consider the problem of what torques are necessary to orient a body in a given manner (in the paper, the satellite is assumed to be a symmetric rotating rigid body). The differential equations of motion of the symmetric body are given in a body-rotating coordinate system X_1, X_2, X_3 by Eulerian equations of the form:

$$
\begin{aligned}
A\dot{u}_x + (C-A)u_y u_z &= L_x \\
A\dot{u}_y + (C-A)u_x u_z &= L_y \\
C\dot{u}_z &= L_z
\end{aligned}
$$

where u_x, u_y, u_z are the components of the angular velocity; L_x, L_y, L_z are the components of the torque, A and C the principal moments of inertia.

In his paper, T. R. Kane (Ref. 74) is concerned with the investigation of the stability of a certain type of motion of an unsymmetric rigid body in the gravitational field of a fixed particle: the mass center of the body describes a circular path centered at the particle, while one of the body's principal axes of inertia remains normal to the orbit and the second one oscillates about the line joining the particle to the mass center of the rigid body. The paper shows that not only the inertia properties of the body, but also the amplitude of the motions must be taken into account, and that the problem is essentially three-dimensional, i. e., incorrect results are obtained when only planar motions are considered.

The equations derived by Kane are essentially the same as ours.

In the ESROTM-27 report (Ref. 93) rigid body kinematics and dynamics are discussed in relation to satellite attitude control problems. Attention has been given to ways of expressing axis transformations in forms suitable for computation. Equations of motion are developed for systems of pivoted rigid bodies, with discussion of the simplifications which can often be used. The free motion of a rigid body is treated and equations are presented for small displacements of a rigid body relative to an earth-pointing axis system.

(6.1222) Expressions of Gravitational Torques Acting on a Satellite:

In their paper, R. E. Roberson (Ref. 84), D. Tatistcheff and Doolin (Ref. 85) derive the gravitational torque on a satellite by expanding the potential energy about the center of mass in a Taylor series and differentiate with respect to the generalized angles expressing the orientation of the body-fixed axes.

In his paper, R. A. Nidey (Ref. 75) shows that the gravitational torque on a rigid unsymmetrical body is normal to the local vertical. The component of the torque in a given horizontal direction is shown to be essentially proportional to the product of inertia relative to the vertical and horizontal planes intersecting in the direction of interest. Since the average gravitational torque on the system is given by

$$M_{avg} = \frac{3}{4} \cdot \vec{u}^{2} \cdot \Delta I \cdot 2\beta$$

where $\vec{u}$ is the angular velocity of the satellite, ΔI the longitudinal principal moment of inertia decreased by the transverse and β the inclination of the longitudinal axis of the satellite to the orbital plane - continual acquisition of angular momentum can only be prevented if the satellite has equal principal moments of inertia or internal weights must be appropriately manipulated such that the gravitational torque vanishes.

In his paper, P. S. Carroll (Ref. 76) gives expressions for the gravitational and the centrifugal torques, respectively:

$$T_g = \frac{3}{2}\vec{u}_o^2(I_s - I_t)\cdot\sin 2\beta'$$

and

$$T_c = \frac{1}{2}\vec{u}_o^2(I_s - I_t)\cdot\sin 2\beta''$$

where

T_g, T_c are the magnitudes of the instantaneous gravity-gradient (centrifugal) torque vectors, $\vec{u}_o$ is the orbital angular velocity, I_s, I_t are the moment of inertia and the transverse moment of inertia, respectively (the satellite is assumed to be symmetric about one axis). β', β'' are the angles between the symmetry axis and the local vertical (for $I_s < I_t$) or the horizontal plane (for $I_s > I_t$).

Moreover, an expression for the total torque on a spinning satellite due to gravity-gradient and centrifugal force is given in term of Eulerian angles:

$$T = \frac{5}{2}\vec{u}_o^2(I_s - I_t)\cdot\left\{-\Psi\cdot\cos\gamma + (\Theta+\rho)\cdot\sin\gamma\right\}$$

where the anges Θ and Ψ are the Eulerian angles representing the deviation of the spin axis from its initial orientation, and γ and ρ represent the orbit-plane orientation.

In his paper, P. F. Hultquist (Ref. 71) also gives an expression for the gravitational torque acting on the satellite:

$$\vec{L} = -R_e^2\int\frac{g}{R^3}\left[\vec{r}\times\vec{R}\right]dV$$

where $\vec{r}$ is the vector from the satellite's center of mass to dV, g is the acceleration of gravity at Earth's surface, R_e is the earth's radius, $\vec{R}$ is the vector from earth's center to dV, and the integration is performed over the volume occupied by the satellite. Using this relation both in elliptic and circular orbit cases, the angular momentum imparted to the totally stabilized satellite over a year's time is computed.

In his paper, C. D. Pengelley (Ref. 78) derives expressions for the torque on a small rigid body due to an arbitrary gravitational field. It is shown that the body can always be placed in an attitude for which the resultant torque will be zero. The torque is expressed explicitly in terms of direction cosines relative to the zero torque attitude and of second partial derivatives of the gravity potential with respect to suitable specified axes. As an example, the general expression is reduced for the case of a radially symmetric field.

In his paper, J. W. Diesel (Ref. 79) derives expressions for the gravity-gradient torque which are extremely simple and involve only the eigenvalues of the gravitational gradient tensor and the eigenvalues of the body inertia tensor $\vec{J}$ with respect to a reference point O. This general theory is presented in order to bypass some difficulties (as, e. g., of extraneous vehicle motion) which were initially connected with the use of gravity-gradient phenomenon.

In his paper (Ref. 87) W. G. Hughes discusses the following torques acting on a satellite: Drag torque, gravitational torque, magnetic torque, torque caused by solar radiation pressure and meteoroid impact. From the discussion one can see that for the majority of satellites, only the gravitational torque is amenable to accurate calculation and expression in a reasonable simple analytical form, suitable for use in overall control system studies.

(6.1223) Stabilization of Satellites:

In his paper (Ref. 80) W. T. Thomson examines the stability of single-axis gyroscopes mounted on a vehicle in circular motion about a central force field for several orientations of the spin vector and output axis. Stability is investigated when the orientation of the output axis is in the radial or tangential direction of the orbit and if it is fixed in inertial space. In the first case, stability depends on the ratio of the spin angular velocity to the vehicle angular velocity around the orbit and the ratio of the moments of inertia of the gyro wheel. In the second case, stability depends on the moment-of-inertia ratio of the wheel, the angular velocity of the vehicle around the orbit, and the desired orientation of the spin velocity vector.

The paper by T. R. Kane and D. Sobala (Ref. 81) deals with motion, in a circular orbit, of a satellite consisting of a rigid body, which possesses an axis of rotational symmetry and carries, on this axis, two particles that performs prescribed oscillations while the axis remains nearly normal to the plane of the orbit. Stability conditions are obtained by using a generalized kind of Floquet theory to study the boundedness of the solutions of the differential equations governing attitude angles.

In their paper, W. T. Thomson and Y. C. Fung (Ref. 82) consider the stability of a spinning space station due to periodic motions of the crew. Several modes of crew motion giving rise to instabilities are studied.

In the paper by T. R. Kane and C. F. Wang (Ref. 83) a detailed exploration of a single-degree-of-freedom gyroscope fixed in rotating satellite (with gimbal ring connected to the satellite by means of a spring and damper) is carried out. The relationship of the motion of the satellite, the physical characteristics of the gyroscope and the spin rate of the rotor is discussed in detail.

In his paper E. E. Zajac (Ref. 86) uses the fact that for a satellite in a circular orbit there exists an energy integral for motion relative to an Earth pointing rotating reference frame. This integral is used to obtain a set of orientation conditions.

In his paper (Ref. 88), W. G. Hughes discusses the use and advantages of momentum exchange control actuators. Furthermore equations of motion for wheel and gyro actuators have been derived. Finally he discusses the mean feature of reaction wheel characteristics and constructions.

In another paper (Ref. 89), W. G. Hugher discusses the stability of a spinning body, nutation damping, and the choice of suitable moment of inertia ratios. Further he discusses the spin rate reduction and spin axis precession, due to external torques, and the possibilities of active control of spin-axis attitude.

Various ways of providing passive damping are discussed in the paper by N. E. Ives (Ref. 90). Furthermore an alternative method of providing the damping, the semi-passive gyro damper, is discussed.

In his paper (Ref. 91), I. K. Abbott proposes first to treat attitude control of communication satellites in general terms and then to refer in more detail to particular systems either of existing satellites or of proposed satellites.

The ESRO reports TR-1 and TR-2 (Ref. 92) are dealt with analytical expressions, which describe the attitude drift of a spin-stabilized satellite controlled magnetically through a coil whose moment is parallel to the spin axis.

(6.13) Expansion of $\vec{g}(\vec{r})$

(6.131) Derivation and Rewriting of the Individual Terms:

To facilitate the integrations $\int \vec{g}(\vec{r})dm$ and $\int (\vec{r}-\vec{r}_s) x \vec{g}(\vec{r})dm$ occuring in sections (6.14) and (6.15) an expansion of $\vec{g}(\vec{r})$ in a Taylor series is recommendable. For this purpose instead of the complete formula for the gravitational potential (Eq. (VI,2)) the simplified form

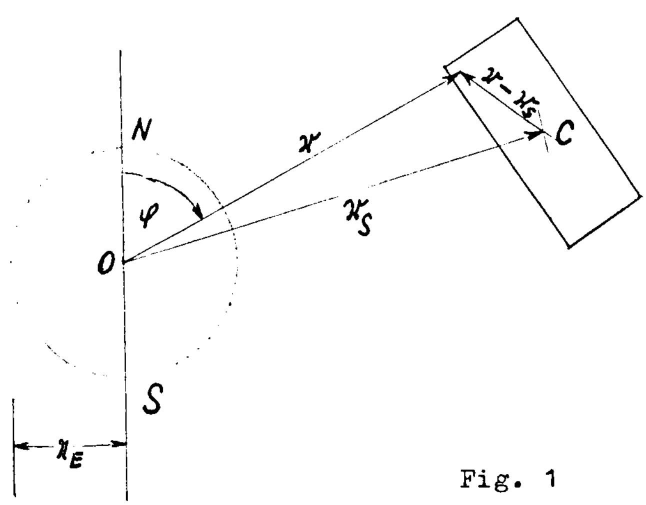
$$U(\vec{r}) = +\frac{\alpha}{r} + \frac{\beta-\gamma\cdot\cos^2\varphi}{r^3} \qquad (VI,14)$$

with

$$\alpha: = -\Gamma\cdot m_E \qquad \beta: = +\frac{\alpha\cdot r_E\cdot J_2}{2} \qquad \gamma: = +3\beta$$

was used as the starting point. φ is the complementary angle of the geographical latitude, viz.

$$\varphi: = 90^{\circ} - \text{geogr. latitude} \qquad (VI,15)$$



C: Center of mass
O: Center of the Earth
N, S: Poles

Fig. 1

From (VI,14) it follows immediately for $\vec{g}(\vec{r})$

$$\vec{g}(\vec{r}) = -\nabla U = +\frac{1}{r^2}(\alpha+3\cdot\frac{\beta-\gamma\cdot\cos^2\varphi}{r^2})\vec{e}^{r} - \frac{2\gamma}{r^3}(\sin\varphi\cdot\cos\varphi)\vec{e}^{\varphi} \tag{VI,16}$$

(The symbols used and the coordinate system are discussed in (6.132)).
As is well-known, $\vec{g}(\vec{r})$ may now be expanded in the following way, under the usual conditions:

$$\begin{aligned}\vec{g}(\vec{r}_s+\vec{s}) = \vec{g}(\vec{r}_s) + \vec{s}\cdot(\nabla o\vec{g})_s + \frac{1}{2!}(\vec{s}o\vec{s})\cdot\cdot(\nabla o\nabla o\vec{g})_s + \\ + \frac{1}{3!}(\vec{s}o\vec{s}o\vec{s})\cdot\cdot\cdot(\nabla o\nabla o\nabla o\vec{g})_s + \cdots\end{aligned} \tag{VI,17}$$

with

$$\vec{s} := \vec{r} - \vec{r}_s \tag{VI,18}$$

Taking account of only the first two terms in (VI,17) would yield the certainly incorrect result that the gravitational field does not exert any torque on a rotationally symmetric satellite if its axis of symmetry is normal to the radius vector. For this reason further approximations are considered here, in contrast to Rep. 13 of this contract.

In the following we shall use the following abbreviations:

$$\begin{aligned}&\vec{S}_2 := (\vec{s}o\vec{s}) && \vec{S}_3 := (\vec{s}o\vec{s}o\vec{s}) && \vec{S}_4 := (\vec{s}o\vec{s}o\vec{s}o\vec{s}) \\ &\vec{A}_{1,s} := (\nabla o\vec{g})_s && \vec{A}_{2,s} := (\nabla o\nabla o\vec{g})_s && \vec{A}_{3,s} := (\nabla o\nabla o\nabla o\vec{g})_s\end{aligned} \tag{VI,19}$$

and for their scalar products:

$$\vec{a}_{(1)} := \vec{s}\cdot\vec{A}_{1,s} \qquad \vec{a}_{(2)} := \vec{S}_2\cdot\cdot\vec{A}_{2,s} \qquad \vec{a}_{(3)} := \vec{S}_3\cdot\cdot\cdot\vec{A}_{3,s} \tag{VI,20}$$

Thus (VI,17) reads:

$$\vec{g}(\vec{r}_s+\vec{s}) = \vec{g}(\vec{r}_s) + \sum_{k=1} \frac{1}{k!}\vec{a}_{(k)} \tag{VI,17 a}$$

The first three $\vec{A}_{k,s}$ tensors read $(R:=r_s)$:

$$\vec{A}_{1,s} = +\frac{1}{R^3}\left(\alpha+\frac{3\beta-5\gamma\cdot\cos^2\varphi}{R^2}\right)(\vec{I}-3\vec{e}^{rr}) + \frac{2}{R^5}(\gamma\cdot\cos^2\varphi-3\beta)\vec{e}^{rr} + \tag{VI,21 a}$$
$$+\frac{8\gamma\cdot\sin\varphi\cdot\cos\varphi}{R^4}(\vec{e}^{r\varphi}+\vec{e}^{\varphi r}) + \frac{2\gamma\cdot\sin^2\varphi}{R^3}\vec{e}^{\varphi\varphi}$$

$$\vec{A}_{2,s} = +\frac{1}{R^4}\left(3\alpha+\frac{15\beta-17\gamma\cdot\cos^2\varphi}{R^2}\right)(7\vec{e}^{rrr}-\vec{I}o\vec{e}^{r}-\vec{e}^{r}o\vec{I}-\vec{e}^{\mu r}{}_{\mu}) +$$
$$+\frac{8\gamma\cdot\cos^2\varphi}{R^6}\vec{e}^{r}o\vec{I} + \frac{2\gamma\cdot\sin\varphi\cdot\cos\varphi}{R^5}(4\vec{I}o\vec{e}^{\varphi}+5\vec{e}^{\varphi}o\vec{I}+4\vec{e}^{\mu\varphi}{}_{\mu}) -$$
$$-\frac{6\alpha}{R^4}\vec{e}^{rrr} - \frac{2\gamma\cdot\sin\varphi\cdot\cos\varphi}{R^5}(24\vec{e}^{rr\varphi}+24\vec{e}^{r\varphi r}+25\vec{e}^{\varphi rr}) + \tag{VI,21 b}$$
$$+\frac{2\gamma(4\cdot\cos^2\varphi-5\cdot\sin^2\varphi)}{R^4}(\vec{e}^{\varphi r\varphi}+\vec{e}^{\varphi\varphi r}) - \frac{10\gamma\cdot\sin^2\varphi}{R^4}\vec{e}^{r\varphi\varphi} +$$
$$+\frac{8\gamma\cdot\sin^2\varphi\cdot\cos^2\varphi}{R^4}(\vec{e}^{\lambda r\lambda}+\vec{e}^{\lambda\lambda r}) + \frac{4\gamma\cdot\sin\varphi\cdot\cos\varphi}{R^3}\vec{e}^{\varphi\varphi\varphi} +$$
$$+\frac{2\gamma\cdot\sin^3\varphi\cdot\cos\varphi}{R^3}(\vec{e}^{\lambda\varphi\lambda}+\vec{e}^{\lambda\lambda\varphi})$$

The calculation of $\vec{A}_{3,s} = (\nabla o\vec{A}_2)_s$ would already be somewhat cumbersome. The terms with β and γ which are by a factor 10^2 smaller than the α-terms were, therefore, neglected in the next step. This simplified tensor is designated by $\vec{A}_{30,s}$:

$$\vec{A}_{30,s} = +\frac{3\alpha}{R^5}\Big\{5\vec{e}^{rr}o\vec{I}+5\vec{e}^{r}o\vec{I}o\vec{e}^{r}+4\vec{e}^{r\mu}{}_{r\mu}-\vec{I}o\vec{I}-\vec{e}^{\mu}o\vec{I}o\vec{e}_{\mu}-\vec{e}^{\varphi\mu}{}_{\varphi\mu}-\vec{e}^{\lambda\mu}{}_{\lambda\mu} - \tag{VI,21 c}$$
$$-35\vec{e}^{rrrr}+5\vec{e}^{\mu r}{}_{r\mu}+5\vec{e}^{\mu r}{}_{\mu r}+5\vec{I}o\vec{e}^{rr}\Big\}$$

After the k-fold scalar multiplications with the $\vec{S}_k$ expressions of the following kind are formed for the $\vec{a}_{(k)}$:

$$\vec{a}_{(1)} = h^{o}_{(1)}\vec{s} + (h^{(\mu)}_{(1)\rho}s^{\rho})\vec{e}_{\mu}$$

$$\vec{a}_{(2)} = (h^{o}_{(2)\rho}s^{\rho})\vec{s} + (h^{(\mu)}_{(2)o}\vec{s}^{2} + h^{(\mu)}_{(2)\rho\sigma}s^{\rho}s^{\sigma})\vec{e}_{\mu} \qquad \text{(VI,22)}$$

$$\vec{a}_{(3)} = (h^{o}_{(3)o}\vec{s}^{2} + h^{o}_{(3)\rho\sigma}s^{\rho}s^{\sigma})\vec{s} + (h^{(\mu)}_{(3)o\rho}s^{\rho}\vec{s}^{2} + h^{(\mu)}_{(3)\rho\sigma\varkappa}s^{\rho}s^{\sigma}s^{\varkappa})\vec{e}_{\mu}$$

where as usually s^{ρ} is understood to be $(\vec{s}\cdot\vec{e}^{\rho})$. The introduction of the following tensors is straightforward:

$$\vec{h}^{(\mu)}_{(1)} := h^{(\mu)}_{(1)\rho}\vec{e}^{\rho} \qquad \vec{h}^{o}_{(2)} := h^{o}_{(2)\rho}\vec{e}^{\rho} \qquad \vec{h}^{(\mu)}_{(3)o} := h^{(\mu)}_{(3)o\rho}\vec{e}^{\rho}$$

$$\vec{H}^{(\mu)}_{(2)} := h^{(\mu)}_{(2)\rho\sigma}\vec{e}^{\rho\sigma} \qquad \vec{H}^{o}_{(3)} := h^{o}_{(3)\rho\sigma}\vec{e}^{\rho\sigma} \qquad \text{(VI,23)}$$

$$\vec{H}^{(\mu)}_{(3)} := h^{(\mu)}_{(3)\rho\sigma\varkappa}\vec{e}^{\rho\sigma\varkappa}$$

Because of

$$\vec{s}^{2} = \vec{S}_{2}\cdot\cdot\vec{I}, \qquad s^{\rho} = \vec{s}.\vec{e}^{\rho}, \qquad s^{\rho}s^{\sigma} = \vec{S}_{2}\cdot\cdot\vec{e}^{\sigma\rho}, \qquad s^{\rho}s^{\sigma}s^{\varkappa} = \vec{S}_{3}\cdot\cdot\cdot\vec{e}^{\varkappa\sigma\rho}$$

(VI,24)

the following expressions result for the $\vec{a}_{(k)}$:

$$\vec{a}_{(1)} = \vec{s}\cdot\left\{h^{o}_{(1)}\vec{I} + \vec{h}^{(\mu)}_{(1)}o\vec{e}_{\mu}\right\}$$

$$\vec{a}_{(2)} = \vec{S}_{2}\cdot\cdot\left\{\vec{h}^{o}_{(2)}o\vec{I} + (h^{(\mu)}_{(2)o}\vec{I}+\vec{H}^{(\mu)}_{(2)})o\vec{e}_{\mu}\right\} \qquad \text{(VI,25)}$$

$$\vec{a}_{(3)} = \vec{S}_{3}\cdot\cdot\cdot\left\{(h^{o}_{(3)o}\vec{I}+\vec{H}^{o}_{(3)})o\vec{I} + (\vec{h}^{(\mu)}_{(3)o}o\vec{I}+\vec{H}^{(\mu)}_{(3)})o\vec{e}_{\mu}\right\}$$

and

$$\vec{a}_{(1)} = \vec{s}\cdot\left\{\vec{T}_{(1)} + \vec{R}^{(\mu)}_{(1)}\circ\vec{e}_{\mu}\right\} = \vec{s}\cdot\vec{B}_{(1)}$$

$$\vec{a}_{(2)} = \vec{S}_2\cdot\cdot\left\{\vec{T}_{(2)} + \vec{R}^{(\mu)}_{(2)}\circ\vec{e}_{\mu}\right\} = \vec{S}_2\cdot\cdot\vec{B}_{(2)} \qquad \text{(VI,25 a)}$$

$$\vec{a}_{(3)} = \vec{S}_3\cdot\cdot\cdot\left\{\vec{T}_{(3)} + \vec{R}^{(\mu)}_{(3)}\circ\vec{e}_{\mu}\right\} = \vec{S}_3\cdot\cdot\cdot\vec{B}_{(3)}$$

respectively, with the abbreviations:

$$\vec{R}^{(\mu)}_{(1)} := \vec{h}^{(\mu)}_{(1)} \qquad \vec{T}_{(1)} := h^{o}_{(1)}\vec{I}$$

$$\vec{R}^{(\mu)}_{(2)} := h^{(\mu)}_{(2)o}\vec{I} + \vec{H}^{(\mu)}_{(2)} \qquad \vec{T}_{(2)} := \vec{h}^{o}_{(2)}\circ\vec{I} \qquad \text{(VI,26)}$$

$$\vec{R}^{(\mu)}_{(3)} := \vec{h}^{(\mu)}_{(3)o}\circ\vec{I} + \vec{H}^{(\mu)}_{(3)} \qquad \vec{T}_{(3)} := (h^{o}_{(3)o}\vec{I} + \vec{H}^{o}_{(3)})\circ\vec{I}$$

The quantities

$$\vec{B}_{(k)} := \vec{T}_{(k)} + \vec{R}^{(\mu)}_{(k)}\circ\vec{e}_{\mu} \qquad \text{(VI,27)}$$

need, of course, not to agree with the $\vec{A}_{k,s}$ and are, generally speaking, simpler than these.

$$\vec{T}_{(1)} = +\frac{1}{R^3}\left(\alpha + \frac{3\beta-5\gamma\cdot\cos^2\varphi}{R^2}\right)\vec{I}$$

$$\vec{R}^{(r)}_{(1)} = -\frac{1}{R^3}\left(3\alpha + \frac{15\beta-17\gamma\cdot\cos^2\varphi}{R^2}\right)\vec{e}^{r} + \frac{8\gamma}{R^4}(\sin\varphi\cdot\cos\varphi)\vec{e}^{\varphi} \qquad \text{(VI,28)}$$

$$\vec{R}^{(\varphi)}_{(1)} = +\frac{2\gamma}{R^5}\left(\frac{4}{R}\vec{e}^{r}\cos\varphi + \vec{e}^{\varphi}\sin\varphi\right)\cdot\sin\varphi$$

and

$$\vec{T}_{(2)} = \left\{-\frac{6}{R^4}\left(\alpha + \frac{5\beta-7\gamma\cdot\cos^2\varphi}{R^2}\right)\vec{e}^{r} + \frac{18\gamma}{R^5}\vec{e}^{\varphi}\sin\varphi\cdot\cos\varphi\right\}\circ\vec{I}$$

$$\vec{R}^{(r)}_{(2)} = -\frac{1}{R^4}\left(3\alpha + \frac{15\beta-17\gamma\cdot\cos^2\varphi}{R^2}\right)\vec{I} + \frac{1}{R^4}\left(15\alpha + 7\frac{15\beta-17\gamma\cdot\cos^2\varphi}{R^2}\right)\vec{e}^{rr} - \qquad \text{(VI,29)}$$

$$-\frac{98\gamma}{R^5}\vec{e}^{r\varphi}\sin\varphi\cdot\cos\varphi + \frac{8\gamma}{R^4}\vec{e}^{\lambda\lambda}\sin^2\varphi\cdot\cos^2\varphi + \frac{2\gamma}{R^4}(4\cdot\cos^2\varphi - 5\cdot\sin^2\varphi)\vec{e}^{\varphi\varphi}$$

$$\vec{R}^{(\varphi)}_{(2)} = + \frac{8\gamma}{R^7}(\sin\varphi\cdot\cos\varphi)\vec{I} - \frac{48\gamma}{R^7}\vec{e}^{rr}\sin\varphi\cdot\cos\varphi + \frac{4\gamma}{R^6}(2\cdot\cos^2\varphi - 5\cdot\sin^2\varphi)\vec{e}^{r\varphi} +$$
$$+ \frac{4\gamma}{R^5}\vec{e}^{\varphi\varphi}\sin\varphi\cdot\cos\varphi + \frac{2\gamma}{R^5}\vec{e}^{\lambda\lambda}\sin^3\varphi\cdot\cos\varphi \qquad \text{(VI,29)}$$
$$\vec{R}^{(\lambda)}_{(2)} = + \frac{8\gamma}{R^6}\vec{e}^{r\lambda}\cos^2\varphi + \frac{2\gamma}{R^5}\vec{e}^{\varphi\lambda}\sin\varphi\cdot\cos\varphi$$

and

$$\vec{T}_{(3)} = - \frac{9\alpha}{R^5}\left\{\vec{I} - 5\vec{e}^{rr}\right\}\circ\vec{I}$$
$$\vec{R}^{(r)}_{(3)} = + \frac{15\alpha}{R^5}(3\vec{I}\circ\vec{e}^{r} - 7\vec{e}^{rrr}) \qquad \text{(VI,30)}$$

In calculating the torques expressions of the form

$$\int \vec{s}\times\vec{a}_{(k)}dm$$

occur in the sections (6.14) and (6.15), which can be rewritten in the following way in order to facilitate integration:

$\vec{R}_{(k)}$ and $\vec{S}_k$ are tensors of the k-th rank; the k-fold scalar products are, therefore, scalar quantities:

$$\alpha^{(\mu)}_{(1)} := \vec{s}\cdot\vec{R}^{(\mu)}_{(1)}, \qquad \alpha^{(\mu)}_{(2)} := \vec{S}_2\cdot\cdot\vec{R}^{(\mu)}_{(2)}, \qquad \alpha^{(\mu)}_{(3)} := \vec{S}_3\cdot\cdot\cdot\vec{R}^{(\mu)}_{(3)} \qquad \text{(VI,31)}$$

The (k+1)-th-rank tensors $\vec{T}_{(k)}$ are of the form

$$\vec{T}_{(k)} = \vec{D}_{(k-1)}\circ\vec{I}$$

$\vec{D}_{(k-1)}$ being (k-1)-th-rank tensors. Putting:

$$\vec{S}_2\cdot\cdot\vec{D}_{(3-1)} =: \beta_2, \qquad \vec{S}_3\cdot\cdot\cdot\vec{D}_{(4-1)} =: \beta_3 \qquad \text{(VI,32)}$$

we have, because of

$$\vec{S}_k = \vec{s}\circ\vec{S}_{k-1} \qquad \text{(VI,33)}$$

by way of example:

$$\vec{S}_3 \cdot\cdot\cdot \vec{T}_{(3)} = (\vec{s}o\vec{S}_2)\cdot\cdot\cdot(\vec{D}_2 o\vec{I}) = \vec{s}\cdot(\vec{S}_2\cdot\cdot\vec{D}_2)o\vec{I} = \beta_2\vec{s}\cdot\vec{I} = \beta_2\vec{s} \qquad (VI,34)$$

Since analogous relations hold for all k, because of $\vec{s}x\vec{s} = \vec{o}$ only the remaining terms with $\vec{R}\binom{\mu}{k}$ must be taken into account in $\vec{s}x\vec{a}_{(k)}$:

$$\vec{s}x\vec{a}_{(k)} = \vec{s}x(\alpha\binom{\mu}{k}\vec{e}_\mu) = -\vec{e}_\mu x\vec{s}\alpha\binom{\mu}{k} = -\vec{e}_\mu x(\vec{s}\alpha\binom{\mu}{k}) = (\vec{s}\alpha\binom{\mu}{k})x\vec{e}_\mu \qquad (VI,35)$$

As in (VI,34) it is now shown that

$$\vec{s}\alpha\binom{\mu}{1} = \vec{S}_2\cdot\vec{R}\binom{\mu}{1}, \qquad \vec{s}\alpha\binom{\mu}{2} = \vec{S}_3\cdot\cdot\vec{R}\binom{\mu}{2}, \qquad \vec{s}\alpha\binom{\mu}{3} = \vec{S}_4\cdot\cdot\cdot\vec{R}\binom{\mu}{3} \qquad (VI,36)$$

holds. Thus, we obtain

$$\begin{aligned} \vec{s}x\vec{a}_{(1)} &= (\vec{S}_2\cdot\vec{R}\binom{\mu}{1})x\vec{e}_\mu \\ \vec{s}x\vec{a}_{(2)} &= (\vec{S}_3\cdot\cdot\vec{R}\binom{\mu}{2})x\vec{e}_\mu \\ \vec{s}x\vec{a}_{(3)} &= (\vec{S}_4\cdot\cdot\cdot\vec{R}\binom{\mu}{3})x\vec{e}_\mu \end{aligned} \qquad (VI,37)$$

Since the tensors $\vec{B}_{(k)}$ in (VI,25 a) and $\vec{R}\binom{\mu}{k}$ in (VI,37) are constant quantities they can be placed in front of the $\int\ldots$dm-integrals. Since a symbolic way of denotation was used throughout the paper all equations written down up to now are valid in all coordinate systems, as, e. g., in the system of the main satellite axès. Thus all integrations $\int\ldots$dm can be reduced to integrations $\int\vec{S}_k$dm in, e. g., the system of main axes.

(6.132) Hints to the Symbols Used:

(6.1321) The coordinate system:

Throughout (6.131) spherical coordinates having the earth's center as origin are used:

r: along a radius from O: $0 < r < +\infty$

φ: along a meridian from N to S: $0^\circ \leqslant \varphi \leqslant 180^\circ$ (VI,38)

λ: along a circle of latitude counted from any fixed point: $0^\circ \leqslant \lambda \leqslant 360^\circ$.

Accordingly, the basis vectors used have the following directions:

$\vec{e}_r$, $\vec{e}^r$: direction of radius vector from O

$\vec{e}_\varphi$, $\vec{e}^\varphi$: direction of meridian to south (VI,39)

$\vec{e}_\lambda$, $\vec{e}^\lambda$: direction of circle of latitude to east or west, depending on the manner of counting.

Since the indices r, φ, λ denote the spherical coordinates it is, of course, not allowed to sum over them. Instead of this, summation is, for example, carried out over ρ, σ, $\varkappa$, μ according to the Einstein convention.

The basis vectors $\vec{e}_\rho$, $\vec{e}^\sigma$ are generally not normalized; their scalar products

$$(\vec{e}_\rho \cdot \vec{e}_\sigma) = : g_{\rho\sigma} \qquad (\vec{e}^\rho \cdot \vec{e}^\sigma) = : g^{\rho\sigma} \qquad (\vec{e}^\rho \cdot \vec{e}_\sigma) = : g^\rho{}_\sigma \ (= \delta_{\rho\sigma}) \tag{VI,40}$$

are, as is well-known, the components of the unit tensor $\bar{I}$. For (VI,39):

$$\begin{aligned} (\vec{e}_r \cdot \vec{e}_r) &= (\vec{e}^r \cdot \vec{e}^r) = 1 \\ (\vec{e}_\varphi \cdot \vec{e}_\varphi) &= r^2 \qquad (\vec{e}^\varphi \cdot \vec{e}^\varphi) = r^{-2} \\ (\vec{e}_\lambda \cdot \vec{e}_\lambda) &= r^2 \cdot \sin^2\varphi \qquad (\vec{e}^\lambda \cdot \vec{e}^\lambda) = r^{-2} \cdot \sin^{-2}\varphi \end{aligned} \tag{VI,40 a}$$

Hence the following unit vectors are obtained from (VI,39):

$$\vec{e}_r = \vec{e}^r, \qquad \frac{1}{r}\vec{e}_\varphi = r\vec{e}^\varphi, \qquad \frac{1}{r \cdot \sin\varphi}\vec{e}_\lambda = r\vec{e}^\lambda \sin\varphi \tag{VI,39 a}$$

The transformation to other coordinate systems $\{x'^{\rho}\}$ is performed by:

$$\vec{e}^{\rho} = \frac{\partial x^{\rho}}{\partial x'^{\sigma}}\,\vec{e}'^{\sigma} \qquad \vec{e}_{\rho} = \frac{\partial x'^{\sigma}}{\partial x^{\rho}}\,\vec{e}'_{\sigma} \tag{VI,41}$$

(6.1322) Symbolics:

The "∇" sign introduced in (VI,16) designates, of course,

$$\nabla := \vec{e}^{\mu}\frac{\partial}{\partial x^{\mu}} \tag{VI,42}$$

In (VI,17) the "o" sign is used for the first time to denote a tensorial product: from $\vec{e}^{\rho}$ and $\vec{e}^{\sigma}$ the basis tensor

$$\vec{e}^{\rho}\circ\vec{e}^{\sigma} =: \vec{e}^{\rho\sigma} \tag{VI,43}$$

is formed, for example. Thus we have, e. g.,(product rule!)

$$(\nabla\circ\nabla\circ\vec{g}) = \vec{e}^{\sigma}\circ\frac{\partial}{\partial x^{\sigma}}(\vec{e}^{\rho}\circ\frac{\partial}{\partial x^{\rho}}\,\vec{g}) = \vec{e}^{\sigma}\circ(\vec{e}^{\rho}\circ\vec{g}_{,\rho})_{,\sigma} = \vec{e}^{\sigma}\circ\left\{\vec{e}^{\rho}{}_{,\sigma}\circ\vec{g}_{,\rho}+\vec{e}^{\rho}\circ\vec{g}_{,\rho,\sigma}\right\} =$$

$$= \vec{e}^{\sigma}\circ\left\{(-{}^{\rho}\Gamma_{\sigma\varkappa}\vec{e}^{\varkappa})\circ(g^{\mu}{}_{,\rho}\vec{e}_{\mu}+\vec{e}_{\varkappa}{}^{\varkappa}\Gamma_{\mu\rho}g^{\mu}) + \vec{e}^{\rho}\circ(g^{\mu}{}_{,\rho,\sigma}\vec{e}_{\mu}+g^{\mu}{}_{,\rho}{}^{\alpha}\Gamma_{\mu\sigma}\vec{e}_{\alpha}+\right.$$

$$\left. g^{\mu}{}_{,\sigma}{}^{\varkappa}\Gamma_{\mu\rho}\vec{e}_{\varkappa}+{}^{\varkappa}\Gamma_{\mu\rho,\sigma}g^{\mu}\vec{e}_{\varkappa}+{}^{\varkappa}\Gamma_{\mu\rho}{}^{\alpha}\Gamma_{\varkappa\sigma}g^{\mu}\vec{e}_{\alpha})\right\} = \tag{VI,44 a}$$

$$= \vec{e}^{\sigma\rho}{}_{\varkappa}\left\{g^{\mu}(-{}^{\alpha}\Gamma_{\sigma\rho}{}^{\varkappa}\Gamma_{\mu\alpha}+{}^{\varkappa}\Gamma_{\mu\rho,\sigma}+{}^{\varkappa}\Gamma_{\alpha\sigma}{}^{\alpha}\Gamma_{\mu\rho}) - g^{\varkappa}{}_{,\alpha}{}^{\alpha}\Gamma_{\sigma\rho} +\right.$$

$$\left. + g^{\mu}{}_{,\rho}{}^{\varkappa}\Gamma_{\mu\sigma}+ g^{\mu}{}_{,\sigma}{}^{\varkappa}\Gamma_{\mu\rho} + g^{\varkappa}{}_{,\rho,\sigma}\right\}$$

Of course, nobody will calculate in this way, but one will first calculate

$$(\nabla\circ\vec{g}) = \vec{e}^{\rho}\circ(g^{\mu}\vec{e}_{\mu})_{,\rho} = \vec{e}^{\rho}\circ(g^{\mu}{}_{,\rho}\vec{e}_{\mu} + g^{\mu}\vec{e}_{\mu,\rho}) = A_{\rho}{}^{\sigma}\vec{e}^{\rho}{}_{\sigma} \tag{VI,44 b}$$

and then

$$(\nabla\circ\vec{A}) = \vec{e}^{\mu}\circ(A_{\rho}{}^{\sigma}\vec{e}^{\rho}{}_{\sigma})_{,\mu} = \vec{e}^{\mu}\circ\left\{A_{\rho}{}^{\sigma}{}_{,\mu}\vec{e}^{\rho}{}_{\sigma} + A_{\rho}{}^{\sigma}(\vec{e}^{\rho}{}_{,\mu}\circ\vec{e}_{\sigma} + \vec{e}^{\rho}\circ\vec{e}_{\sigma,\mu})\right\} \tag{VI,44 c}$$

with

$$\vec{e}_{\rho,\sigma} := {}^{\varkappa}\Gamma_{\rho\sigma}\vec{e}_{\varkappa} \qquad \vec{e}^{\rho}{}_{,\sigma} := -{}^{\rho}\Gamma_{\varkappa\sigma}\vec{e}^{\varkappa} \tag{VI,45}$$

and

$${}^{\varkappa}\Gamma_{\rho\sigma} := g^{\varkappa\mu}\cdot\frac{1}{2}(g_{\mu\rho,\sigma}+g_{\mu\sigma,\rho}-g_{\rho\sigma,\mu}) \tag{VI,45 a}$$

In (VI,17) also the n-fold scalar product appears for the first time:

$$\begin{aligned}(\vec{a}\circ\vec{b}\circ\vec{c}\circ\vec{d})\cdots(\vec{e}\circ\vec{f}\circ\vec{g}) &:= (\vec{a}\circ\vec{b}\circ\vec{c})\cdot\cdot(\vec{f}\circ\vec{g})(\vec{d}\cdot\vec{e}) = (\vec{a}\circ\vec{b})\cdot(\vec{g})(\vec{c}\cdot\vec{f})(\vec{d}\cdot\vec{e}) = \\ &= (\vec{b}\cdot\vec{g})(\vec{c}\cdot\vec{f})(\vec{d}\cdot\vec{e})\vec{a}\end{aligned} \tag{VI,46}$$

as, e. g.,

$$\begin{aligned}\vec{S}_3\cdots\vec{T}_{(3)} &= (S_{\alpha\beta\gamma}\vec{e}^{\alpha\beta\gamma})\cdots(T^{\rho\sigma\varkappa\mu}\vec{e}_{\rho\sigma\varkappa\mu}) = S_{\alpha\beta\rho}T^{\rho\sigma\varkappa\mu}\vec{e}^{\alpha\beta}\cdot\cdot\vec{e}_{\sigma\varkappa\mu} = \\ &= S_{\alpha\sigma\rho}T^{\rho\sigma\varkappa\mu}\vec{e}^{\alpha}\cdot\vec{e}_{\varkappa\mu} = S_{\varkappa\sigma\rho}T^{\rho\sigma\varkappa\mu}\vec{e}_{\mu}\end{aligned} \tag{VI,46 a}$$

(where the sequence of the indices with S is relevant!) If a tensor equation is needed in the components of an arbitrary coordinate system one has to multiply simply with the corresponding basis tensor, e. g.:

$$\begin{aligned}R^{1}{}_{13}{}^{2} &= \vec{R}\cdots\cdot(\vec{e}^{2}{}_{31}{}^{1}) = (((\vec{R}\cdot\vec{e}^{2})\cdot\vec{e}_3)\cdot\vec{e}_1)\cdot\vec{e}^{1} = (\vec{e}^{2}{}_{31}{}^{1})\cdots\cdot\vec{R} = \\ &= \vec{e}^{2}\cdot(\vec{e}_3\cdot(\vec{e}_1\cdot(\vec{e}^{1}\cdot\vec{R})))\end{aligned} \tag{VI,47}$$

a fact which enables us to calculate without reference to a special coordinate system.

(6.14) The Equations of Motion in the Earth' System

The "earth' system" is in the following understood to be the system of refernce of the unrotating earth; one may, therefore, consider the system to be an inertial system, in very good approximation. The equations of motion read

$$m\vec{b}_s = \vec{K} := \int d\vec{K} \tag{VI,48 a}$$

$$\vec{J}_s \cdot \dot{\vec{u}}_t + \vec{u}_t x \vec{J}_s \cdot \vec{u}_t = \vec{M}_s := \int (\vec{r}-\vec{r}_s) x d\vec{K} \tag{VI,48 b}$$

with

$$\vec{b}_s := \frac{d^2\vec{r}_s}{dt^2} \qquad \vec{J}_s := \int \left\{ \vec{s}^2 \vec{I} - \vec{s}o\vec{s} \right\} dm \tag{VI,49}$$

We consider here only gravitational forces. Thus we have:

$$d\vec{K} = \vec{g}(\vec{r})dm \tag{VI,50 a}$$

$$d\vec{M}_s = (\vec{r}-\vec{r}_s) x \vec{g}(\vec{r})dm \tag{VI,50 b}$$

or, because of (VI,17 a) and (VI,18):

$$d\vec{K} = \vec{g}(\vec{r}_s+\vec{s})dm = \left\{ \vec{g}(\vec{r}_s) + \sum_1 \frac{1}{k!}\vec{a}_{(k)} \right\} dm \tag{VI,51 a}$$

$$d\vec{M}_s = \vec{s} x \vec{g}(\vec{r}_s)dm + \sum_1 \frac{1}{k!} \vec{s} x \vec{a}_{(k)} dm \tag{VI,51 b}$$

Using (VI,25 a) we obtain:

$$d\vec{K} = \vec{g}(\vec{r}_s)dm + \vec{s}dm \cdot \vec{B}_{(1)} + \frac{1}{2}\vec{S}_2 dm \cdot\cdot \vec{B}_{(2)} + \frac{1}{6}\vec{S}_3 dm \cdot\cdot\cdot \vec{B}_{(3)} + \cdots \tag{VI,52 a}$$

$$\begin{aligned} d\vec{M}_s = {} & \vec{s}dm x \vec{g}(\vec{r}_s) + (\vec{S}_2 dm \cdot \vec{R}\binom{\mu}{1}) x \vec{e}_\mu + \frac{1}{2}(\vec{S}_3 dm \cdot\cdot \vec{R}\binom{\mu}{2}) x \vec{e}_\mu + \\ & + \frac{1}{6}(\vec{S}_4 dm \cdot\cdot\cdot \vec{R}\binom{\mu}{3}) x \vec{e}_\mu + \cdots \end{aligned} \tag{VI,52 b}$$

Because of $\int \vec{r}dm = \vec{r}_s \int dm$ in both equations (VI,52) the term $\int \vec{s}dm$ cancels. Therefore, the final form of the equations of motion is:

$$\vec{b}_s = \vec{g}(\vec{r}_s) + \frac{1}{2m}(\vec{S}_2 \cdot\cdot \vec{B}_{(2)}) + \frac{1}{6m}(\vec{S}_3 \cdot\cdot\cdot \vec{B}_{(3)}) + \cdots \tag{VI,53 a}$$

$$\vec{J}_s\cdot\dot{\vec{u}}_t + \vec{u}_t x\vec{J}_s\cdot\vec{u}_t = (\vec{S}_2\cdot\vec{R}\binom{\mu}{1})x\vec{e}_\mu + \frac{1}{2}(\vec{S}_3\cdot\cdot\vec{R}\binom{\mu}{2})x\vec{e}_\mu + + \frac{1}{6}(\vec{S}_4\cdot\cdot\cdot\vec{R}\binom{\mu}{3})x\vec{e}_\mu + \cdots \qquad \text{(VI,53 b)}$$

where $\vec{S}_2$, etc., are to designate $\int(\vec{s}o\vec{s})dm$ from now onwards.

The equations (VI,53) are valid in all coordinate systems; the system of reference is, of course, the earth' system defined above. As usually, the transition to components is brought about by scalar multiplication with the corresponding basis vectors.

The rotation vector $\vec{u}_t$ contains the total rotational motion of the satellite with respect to the earth' system; its direction is such that for a point at rest in the satellite we have

$$\vec{v} = \vec{v}_s + \vec{u}_t x(\vec{r}-\vec{r}_s) \qquad \text{(VI,54)}$$

where $\vec{v}$ is the velocity in the earth' system.

(6.15) The Equations of Motion in Orbit Systems

Some people prefer the representation of the equations of motion in a system of reference whose origin is the satellite's center of mass and whose rotational motion $\vec{u}_B$ with respect to the earth' system is fixed in a certain manner, e. g. such that the first of the three Cartesian unit vectors shows into the direction of $\vec{r}_s$ (i. e., from O to C, in Fig. 1) whereas the second one represents the normal to the instantaneous orbital plane of the center of mass. Such a system of reference will be termed "orbit system", in the following. Let $\vec{a}$ be an arbitrary vector, $\dot{\vec{a}}$ its time derivative in the earth' system and $\overset{\shortmid}{\vec{a}}$ in the orbit system. Then we have:

$$\dot{\vec{a}} = \overset{\shortmid}{\vec{a}} + \vec{u}_B x\vec{a} \qquad \text{(VI,55)}$$

and, particularly, in analogy to (VI,54):

$$\vec{v} = \vec{v}_s + \vec{u}_B x \vec{s} \tag{VI,56}$$

for a point at rest in the orbit system. The rotation vector

$$\vec{u}_s := \vec{u}_t - \vec{u}_B \tag{VI,57}$$

then describes the rotation of the satellite with respect to the orbit system.

The equations of motion now read:

$$m\vec{b}_s' = \vec{K}' := \int d\vec{K}' \tag{VI,58 a}$$

$$\vec{J}_s \cdot \ddot{\vec{u}}_s + \vec{u}_s x \vec{J}_s \cdot \vec{u}_s = \vec{M}_s' := \int \vec{s} x d\vec{K}' \tag{VI,58 b}$$

where $d\vec{K}'$ is the force acting on dm in the orbit system and $\bar{b}_s'$ is the acceleration of the center of mass, again the orbit system. On account of the special choice of this system of reference we have

$$\vec{b}_s' = \vec{o} \tag{VI,59}$$

The derivation of $\vec{M}_s'$ can be performed from (VI,48 b) via (VI,57) and (VI,55)

$$\begin{aligned} \vec{J}_s \cdot (\dot{\vec{u}}_s + \dot{\vec{u}}_B - \vec{u}_B x \vec{u}_s + \vec{u}_B x \vec{u}_s) + (\vec{u}_s + \vec{u}_b) x \vec{J}_s \cdot (\vec{u}_s + \vec{u}_B) = \vec{M}_s = \\ = \vec{M}_s' + \vec{J}_s \cdot (\dot{\vec{u}}_B + \vec{u}_B x \vec{u}_s) + \vec{u}_B x \vec{J}_s \cdot (\vec{u}_s + \vec{u}_B) + \vec{u}_s x \vec{J}_s \cdot \vec{u}_B \end{aligned} \tag{VI,60 a}$$

and

$$\vec{M}_s' + \vec{J}_s \cdot \left[\vec{u}_B x \vec{u}_s\right] + \vec{u}_B x \vec{J}_s \cdot \vec{u}_s + \vec{u}_s x \vec{J}_s \cdot \vec{u}_B = \vec{M}_s - \vec{J}_s \cdot \dot{\vec{u}}_B - \vec{u}_B x \vec{J}_s \cdot \vec{u}_B \tag{VI,60 b}$$

respectively. Because of

$$\left\{\overset{\bullet}{\vec{I}}\vec{s}^2 - \vec{s}o\vec{s}\right\}\cdot\left[\vec{u}_B x \vec{u}_s\right] = -\vec{s}x\left[\vec{s}x\left[\vec{u}_B x\vec{u}_s\right]\right] = +\vec{s}x\left\{\vec{u}_s(\vec{s}\cdot\vec{u}_B)-\vec{u}_B(\vec{s}\cdot\vec{u}_s)\right\} \qquad \text{(VI,60 c)}$$

and

$$\vec{u}_B x\left\{\vec{I}\vec{s}^2-\vec{s}o\vec{s}\right\}\cdot\vec{u}_s + \vec{u}_s x\left\{\vec{I}\vec{s}^2-\vec{s}o\vec{s}\right\}\cdot\vec{u}_B = -\vec{u}_B x(\vec{s}o\vec{s})\cdot\vec{u}_s - \vec{u}_s x(\vec{s}o\vec{s})\cdot\vec{u}_B =$$
$$= +\vec{s}x\left\{\vec{u}_B o\vec{s}\cdot\vec{u}_s+\vec{u}_s o\vec{s}\cdot\vec{u}_B\right\} \qquad \text{(VI,60 d)}$$

the left side may be transformed to:

$$\vec{M}'_s + \int \vec{s}x2(\vec{u}_s o\vec{s})\cdot\vec{u}_B dm = \vec{M}'_s - 2\vec{u}_s x\vec{S}_2\cdot\vec{u}_B \qquad \text{(VI, 60 e)}$$

such that:

$$\vec{M}'_s = \vec{M}_s - \vec{J}_s\cdot\dot{\vec{u}}_B - \vec{u}_B x\vec{J}_s\cdot\vec{u}_B + 2\vec{u}_s x\vec{S}_2\cdot\vec{u}_B \qquad \text{(VI,61)}$$

Thanks to (VI,58), (VI,59), (VI,60) and (VI,61) the equations of motion read, therefore:

$$\vec{K}' = \vec{o} \qquad \text{(VI,62 a)}$$

$$\vec{J}_s\cdot\dot{\vec{u}}_s + \vec{u}_s x\vec{J}_s\cdot\vec{u}_s - 2\vec{u}_s x\vec{S}_2\cdot\vec{u}_B = \vec{M}_s - \vec{J}_s\cdot\dot{\vec{u}}_B - \vec{u}_B x\vec{J}_s\cdot\vec{u}_B \qquad \text{(VI,62 b)}$$

with

$$\vec{M}_s = (\vec{S}_2\cdot\vec{R}^{(\mu)}_{(1)})x\vec{e}_\mu + \frac{1}{2}(\vec{S}_3\cdot\cdot\vec{R}^{(\mu)}_{(2)})x\vec{e}_\mu + \frac{1}{6}(\vec{S}_4\cdot\cdot\cdot\vec{R}^{(\mu)}_{(3)})x\vec{e}_\mu + \cdots \qquad \text{(VI,53 b)}$$

$\vec{u}_B$ is given or is yielded by integrating (VI,53 a), i. e., from a knowledge of $\vec{r}_s(t)$, $\dot{\vec{r}}_s(t)$, $\vec{b}_s(t)$. (VI,62 b) can, therefore, be used to determine $\vec{u}_s(t)$. The calculation of $\vec{u}_t(t)$ from (VI,53 b) and of $\vec{u}_s(t)$ from

$$\vec{u}_s = \vec{u}_t - \vec{u}_B \qquad \text{(VI,57)}$$

seems to be simpler.

(6.2) On the solution of the heavy asymmetric gyroscope, using the properties of the Lie operator

by F.Cap and A.Schett

We consider an asymmetric gyroscope with several torque producing forces. A spinning satellite is essentially such a gyroscope; the torque may result in a change in satellite orientation that affects the thermal balance, the operation of solar cells and various scientific measurements.

By means of an operator we can represent the solution of the heavy asymmetric gyroscope such, that the contributions of the different torques appear separately. In other words, using a splitting up procedure of the afore-mentioned operator we can represent the solution in the form, e.g.,:

$$u = \hat{u}_{\substack{\text{global}\\ \text{symmetric}\\ \text{forcefree}}} + \sum_{\alpha=0}^{\infty} \int_{t_0}^{t} \sum_{i} M_i f_\alpha(\tau, M_i, u)\, d\tau + \sum_{\alpha=0}^{\infty} \int_{t_0}^{t} A\, f_{1\alpha}(\tau, M_i, u)\, d\tau$$

where $\hat{u}$ indicates that this function can sometimes be represented in a global form.

The torques M_i $(i = 1,2,3...)$ appearing in the integral terms usually differ by their order of magnitude. For a given degree of accuracy, therefore, the number of summation terms α to be computed depends on the integral considered. The afore-mentioned solution representation renders it possible to compute the single integral terms irrespective of the other integral terms.

Especially we shall present the solution of the equation describing the heavy asymmetric gyroscope in the forms:

(6.211) Solution = Solution (heavy, symmetric) + contributions from asymmetry.

(6.212) Solution = Solution (symmetric, forcefree) + contributions from asymmetry and forces.

The 1-st term is exactly known. The 2-nd term can be split up into several additive integral terms:

a term containing the contributions of asymmetry; this term vanishes if the satellite (gyroscope) is symmetric.

additive integral terms containing the torques $M_i (i=1,2,...)$

$$\sum_{\alpha} \int \sum_{i} M_i f_{\alpha}(\tau, M_i, u) d\tau$$

i.e., these terms vanish if $M_i = 0$

(6.213) Solution = Solution (asymmetric, forcefree) + contributions from forces.

As to the effectiveness of the aforementioned solution representations one can generally say, that it is advantageous to put the main contribution of the solution in the 1-st term and perturbations in the remaining terms.

(6.21) <u>Solution of the equation describing the heavy asymmetric gyroscope</u>.

Using a reference frame (X_1, X_2, X_3) fixed with respect to the body the equation of the heavy asymmetric gyroscope reads

$$\begin{aligned} M_1 &= I_1 \dot{u}_1 + (I_3 - I_2) u_2 u_3 \\ M_2 &= I_2 \dot{u}_2 + (I_1 - I_3) u_1 u_3 \\ M_3 &= I_3 \dot{u}_3 + (I_2 - I_1) u_2 u_1 \end{aligned} \tag{VI,61}$$

where I_1, I_2, I_3 are the moments of inertia, u_1, u_2, u_3 are the angular velocities. $\dot{u}_1$, $\dot{u}_2$, $\dot{u}_3$ are the angular accelerations and M_1, M_2, M_3 are the torques in the reference frame (1,2,3). In Eq.(VI,61) we have substituted X_1, X_2, X_3 by 1, 2, 3.

Using the well-known relations

$$
\begin{aligned}
u_1 &= \dot{\alpha} \sin \gamma \sin \beta + \dot{\beta} \cos \gamma \\
u_2 &= \dot{\alpha} \cos \gamma \sin \beta - \dot{\beta} \sin \gamma \\
u_3 &= \dot{\alpha} \cos \beta + \dot{\gamma}
\end{aligned}
\tag{VI,62}
$$

where α, β, γ are the Eulerian angles defined as follows

$$
\sphericalangle \begin{matrix} z \text{ axis} \\ 3 \text{ axis} \end{matrix} : \beta ; \qquad \sphericalangle \begin{matrix} x' \text{ axis} \\ x \text{ axis} \end{matrix} : \alpha ; \qquad \sphericalangle \begin{matrix} x' \text{ axis} \\ 1 \text{ axis} \end{matrix} : \gamma
$$

where x, y, z are the axis fixed with respect to the space, x' indicates the nodal line of the two planes (xy) and (12).

Differentiating u_i (i=1,2,3) in Eq.(VI,62) with respect to t (time) and substituting the se quantities into Eq.(VI,61) we obtain the following equations

$$
\begin{aligned}
&\ddot{\alpha} + \dot{\alpha}\dot{\gamma}h_{11} + \dot{\alpha}\dot{\beta}h_{12} + \ddot{\beta}h_{13} + \dot{\beta}\dot{\gamma}h_{14} + \dot{\alpha}^2 h_{15} + \dot{\beta}\dot{\alpha}h_{16} + \dot{\alpha}\dot{\gamma}h_{17} + \\
&\qquad + \dot{\beta}\dot{\gamma}h_{18} = S_1(\beta,\gamma,\alpha)
\end{aligned}
\tag{VI,63}
$$

$$
\begin{aligned}
&\ddot{\beta} + \ddot{\alpha}h_{21} + \dot{\gamma}\dot{\alpha}h_{22} + \dot{\alpha}\beta h_{23} + \dot{\beta}\dot{\gamma}h_{24} + \dot{\alpha}^2 h_{25} + \dot{\beta}\dot{\alpha}h_{26} + \dot{\alpha}\dot{\gamma}h_{27} + \\
&\qquad + \dot{\beta}\dot{\gamma}h_{28} = S_2(\beta,\gamma,\alpha)
\end{aligned}
\tag{VI,64}
$$

$$
\ddot{\gamma} + \ddot{\alpha}h_{31} + \dot{\alpha}\dot{\beta}h_{32} + \dot{\alpha}^2 h_{33} + \dot{\alpha}\dot{\beta}h_{34} + \dot{\alpha}\dot{\beta}h_{35} + \dot{\beta}^2 h_{26} = S_3(\beta,\gamma,\alpha) \tag{VI,65}
$$

where

$$h_{ij} = h_{ij}(\beta,\alpha,\gamma);\ i,j = 1,2,\ldots$$

$$S_i = S_i(\beta,\alpha,\gamma);\ \ i = 1,2,3$$

$$h_{11} = \frac{\cos\gamma}{\sin\gamma} \qquad h_{15} = \alpha_1 \frac{\cos\gamma\cos\beta}{\sin\gamma}$$

$$h_{12} = \frac{\cos\beta}{\sin\beta} \qquad h_{16} = -\alpha_1 \frac{\cos\beta}{\sin\beta}$$

$$h_{13} = \frac{\cos\gamma}{\sin\gamma\sin\beta} \qquad h_{17} = \alpha_1 \frac{\cos\gamma}{\sin\gamma} \tag{VI,66}$$

$$h_{14} = \frac{-1}{\sin\beta} \qquad h_{18} = -\frac{\alpha_1}{\sin\beta}$$

$$S_1 = \frac{M_1}{I_1}\frac{1}{cb} \qquad \alpha_1 = \frac{I_3-I_2}{I_1} \tag{VI,67}$$

$$h_{21} = -\frac{\cos\gamma\sin\beta}{\sin\gamma} \qquad h_{24} = \frac{\cos\gamma}{\sin\gamma}$$

$$h_{22} = \sin\beta \qquad h_{25} = -\alpha_2\sin\beta\cos\beta \tag{VI,68}$$

$$h_{23} = -\frac{\cos\gamma\cos\beta}{\sin\gamma} \qquad h_{26} = -\frac{\cos\beta\cos\gamma}{\sin\gamma}\alpha_2$$

$$-h_{27} = \alpha_2\sin\beta \qquad h_{28} = -\frac{\cos\gamma}{\sin\gamma}\alpha_2$$

$$-S_2 = \frac{M_2}{I_2}\frac{1}{c} \qquad \alpha_2 = \frac{I_1-I_3}{I_2} \tag{VI,69}$$

$$h_{31} = \cos\beta \qquad h_{32} = -\sin\beta$$

$$h_{33} = \alpha_3\sin\gamma\cos\gamma\sin^2\beta \qquad h_{34} = \alpha_3\cos^2\gamma\sin\beta \tag{VI,70}$$

$$h_{35} = -\alpha_3\sin^2\gamma\sin\beta \qquad h_{36} = -\alpha_3\cos\gamma\sin\gamma$$

$$S_3 = \frac{M_3}{I_3} \qquad \alpha_3 = \frac{I_2-I_1}{I_3} \tag{VI,71}$$

Inserting $\ddot{\beta}$ from Eq.(VI,64) into Eq.(VI,63) we obtain

$$\ddot{\alpha} + \dot{\alpha}\dot{\gamma}q_{11} + \dot{\alpha}\dot{\beta}q_{12} + \dot{\beta}\dot{\gamma}q_{13} + \dot{\alpha}^2 q_{14} + S_2 q_{15} + S_1 q_{16} = 0 \qquad \text{(VI,72)}$$

where

$$q_{11} = \frac{h_{11} + h_{17} - h_{22}h_{13} - h_{13}h_{27}}{1 - h_{21}h_{13}}$$

$$q_{12} = \frac{h_{12} + h_{16} - h_{23}h_{13} - h_{26}h_{13}}{1 - h_{21}h_{13}}$$

$$q_{13} = \frac{h_{14} + h_{18} - h_{24}h_{13} - h_{28}h_{13}}{1 - h_{21}h_{13}} \qquad \text{(VI,73)}$$

$$q_{15} = \frac{h_{13}}{1 - h_{21}h_{13}}; \qquad q_{14} = \frac{h_{15} - h_{25}h_{13}}{1 - h_{21}h_{13}}$$

$$q_{16} = -\frac{1}{1 - h_{21}h_{13}}$$

Inserting $\ddot{\alpha}$ from Eq.(VI,63) into Eq.(VI,64) one obtains

$$\ddot{\beta} + \dot{\gamma}\dot{\alpha}q_{21} + \dot{\alpha}\dot{\beta}q_{22} + \dot{\beta}\dot{\gamma}q_{23} + \dot{\alpha}^2 q_{24} + S_1 q_{25} + S_2 q_{26} = 0 \qquad \text{(VI,74)}$$

where

$$q_{21} = \frac{h_{22} + h_{27} - h_{11}h_{21} - h_{17}h_{21}}{1 - h_{13}h_{21}}$$

$$q_{22} = \frac{h_{23} + h_{26} - h_{12}h_{21} - h_{16}h_{21}}{1 - h_{13}h_{21}}$$

$$q_{23} = \frac{h_{24} + h_{28} - h_{14}h_{21} - h_{18}h_{21}}{1 - h_{13}h_{21}} \qquad \text{(VI,75)}$$

$$q_{24} = \frac{h_{25} - h_{15}h_{21}}{1 - h_{13}h_{21}}; \qquad q_{25} = \frac{h_{21}}{1 - h_{13}h_{21}}$$

$$q_{26} = -\frac{1}{1 - h_{13}h_{21}}$$

Substituting $\ddot{\alpha}$ in Eq.(VI,65) by Eq.(VI,72) we obtain

$$\ddot{\gamma} + \dot{\alpha}\dot{\beta}q_{31} + \dot{\alpha}^2 q_{32} + \dot{\beta}^2 q_{33} + \dot{\alpha}\dot{\gamma}q_{34} + \dot{\beta}\dot{\gamma}q_{35} + S_2 q_{36} + S_1 q_{37} + $$
$$+ S_3 q_{38} = 0 \qquad \text{(VI,76)}$$

$$\begin{aligned} q_{31} &= h_{32} + h_{34} + h_{35} - q_{12}h_{31}; & q_{35} &= -q_{13}h_{31} \\ q_{32} &= h_{33} - q_{14}h_{31} & q_{36} &= -q_{15}h_{31} \\ q_{33} &= h_{36} & q_{37} &= -q_{16}h_{31} \\ q_{34} &= -q_{11}h_{31} & q_{38} &= -1 \end{aligned} \qquad \text{(VI,77)}$$

Eqs.(VI,72), (VI,74), (VI,76) read

$$\ddot{\alpha} + \dot{\alpha}\dot{\gamma}q_{11} + \dot{\alpha}\dot{\beta}q_{12} + \dot{\beta}\dot{\gamma}q_{13} + \dot{\alpha}^2 q_{14} + S_2 q_{15} + S_1 q_{16} = 0 \qquad \text{(VI,78)}$$

$$\ddot{\beta} + \dot{\gamma}\dot{\alpha}q_{21} + \dot{\alpha}\dot{\beta}q_{22} + \dot{\beta}\dot{\gamma}q_{23} + \dot{\alpha}^2 q_{24} + S_1 q_{25} + S_2 q_{26} = 0 \qquad \text{(VI,79)}$$

$$\ddot{\gamma} + \dot{\alpha}\dot{\beta}q_{31} + \dot{\alpha}^2 q_{32} + \dot{\beta}^2 q_{33} + \dot{\alpha}\dot{\gamma}q_{34} + \dot{\beta}\dot{\gamma}q_{35} + S_2 q_{36} + $$
$$+ S_1 q_{37} + S_3 q_{38} = 0 \qquad \text{(VI,80)}$$

$$\begin{aligned} q_{ij} &= q_{ij}(\beta,\alpha,\gamma); \quad & i, j &= 1,2,, \ldots \\ S_i &= S_i(\beta,\alpha,\gamma); \quad & i &= 1,2,3 \end{aligned} \qquad \text{(VI,81)}$$

Eqs.(VI,78), (VI,79), (VI,80) can be written in the form

$$\begin{aligned} \ddot{\alpha} &= f_1(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma) \\ \ddot{\beta} &= f_2(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma) \\ \ddot{\gamma} &= f_3(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma) \end{aligned} \qquad \text{(VI,82)}$$

For the forcefree, symmetric gyroscope, i.e., $\alpha_3 = 0$, $S_i = 0$ $(i = 1,2,3)$ (see Eqs.(VI,67), (VI,69), (VI,71)), Eq.(VI,82) reads

$$\ddot{\alpha} = f_{1ffs}(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma)$$
$$\ddot{\beta} = f_{2ffs}(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma) \qquad \text{(VI,83)}$$
$$\ddot{\gamma} = f_{3ffs}(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma)$$

where f_{iffs} $(i = 1,2,3)$ indicates <u>f</u>orce<u>f</u>ree <u>s</u>ymmetric

For the forcefree asymmetric gyroscope i.e., $S_i = 0$; $(i = 1,2,3)$ (see Eqs.(VI,67), (VI,69), (VI,71)), Eq.(VI,82) reads

$$\ddot{\alpha} = f_{1ffa}(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma)$$
$$\ddot{\beta} = f_{2ffa}(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma) \qquad \text{(VI,84)}$$
$$\ddot{\gamma} = f_{3ffa}(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma)$$

where f_{iffa} indicates the <u>f</u>orce<u>f</u>ree <u>a</u>symmetric case.

For the <u>s</u>ymmetric <u>h</u>eavy gyroscope, i.e., $\alpha_3 = 0$ (see Eq.(VI,71)), Eq.(VI,82) reads

$$\ddot{\alpha} = f_{1sh}(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma)$$
$$\ddot{\beta} = f_{2sh}(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma) \qquad \text{(VI,85)}$$
$$\ddot{\gamma} = f_{3sh}(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \gamma)$$

Eq.(VI,82) can be written in the form

$$\begin{array}{llll} \dot{Z}_1: \dot{\alpha}_1 = \alpha_2 & & \dot{Z}_4: \dot{\beta}_2 = f_2 & \\ \dot{Z}_2: \dot{\alpha}_2 = f_1 & & \dot{Z}_5: \dot{\gamma}_1 = \gamma_2 & \text{(VI,86)} \\ \dot{Z}_3: \dot{\beta}_1 = \beta_2 & & \dot{Z}_6: \dot{\gamma}_2 = f_3 & \end{array}$$

The sign ":" indicates that $\dot{\alpha}_1 \equiv \dot{Z}_1$, e.g.

For domains, where f_i $(i = 1,2,3)$ are holomorphic the formal solution of Eq.(VI,86) reads (see Ref.1,16)

$$Z_i = e^{tD} z_i \qquad \text{(VI,87)}$$

where

$$D = \alpha_2 \cdot \frac{\partial}{\partial z_1} + f_1 \cdot \frac{\partial}{\partial z_2} + \beta_2 \cdot \frac{\partial}{\partial z_3} + f_2 \cdot \frac{\partial}{\partial z_4} + \gamma_2 \cdot \frac{\partial}{\partial z_5} + f_3 \cdot \frac{\partial}{\partial z_6} \qquad (VI,88)$$

(6.211) Representation of the solution S in the form $S = S_{\text{symmetric, heavy}}$ + contributions from asymmetry.

Starting from Eq.(VI,82) we obtain for the operator D (see Ref.1)

$$D = \alpha_2 \cdot \frac{\partial}{\partial z_1} + \beta_2 \cdot \frac{\partial}{\partial z_2} + \gamma_2 \cdot \frac{\partial}{\partial z_3} + f_{1sh} \cdot \frac{\partial}{\partial z_4} + f_{2sh} \cdot \frac{\partial}{\partial z_5} + f_{3sh} \cdot \frac{\partial}{\partial z_6} +$$

$$+ f_{1a} \cdot \frac{\partial}{\partial z_2} + f_{2a} \cdot \frac{\partial}{\partial z_4} + f_{3a} \cdot \frac{\partial}{\partial z_6} \qquad (VI,89)$$

where f_{ia} $(i = 1,2,3)$ indicate the contribution from asymmetry and f_{ish} $(i = 1,2,3)$ indicates the symmetric heavy case.

We write now the operator D in the form

$$D = D_1 + D_2$$

where

$$D_2 = f_{1a} \cdot \frac{\partial}{\partial z_2} + f_{2a} \cdot \frac{\partial}{\partial z_4} + f_{3a} \cdot \frac{\partial}{\partial z_6} \qquad (VI,90)$$

and D_1 is defined by (VI,89). The solution Eq.(VI,87) reads in this case

$$Z_i = e^{tD} z_i = e^{t(D_1+D_2)} z_i = e^{tD_1} z_i + \sum_{o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^{\alpha}}{\alpha!} \left[D_2 D^{\alpha} z_i \right]_{\bar{a}} d\tau \qquad (VI,91)$$

The subscript $\bar{a}$ indicates that after applying D_2D on z_i, z_i has to be replaced by $e^{tD_1} z_i$. In Eq.(VI,91) the operator D_1 is the operator for the symmetric heavy gyroscope.

The solution representation is recommandable, if the deviations from the symmetric gyroscope are small. In this case only few terms of the sum in Eq.(VI,91) have to be taken into account. For the evaluation of the integral appearing in Eq.(VI,91) suitable methods are already developed (see Ref.2);in these works also the problems of error estimation is treated.

Moreover we will use another method for solving Eq.(VI,61) as proposed by GROEBNER (see Ref.94). For that we put

$$\frac{I_2 - I_1}{I_3} = \lambda \qquad \text{(VI,92)}$$

where λ is a parameter. Using this parameter we obtain

$f_{ia} = \lambda f^*_{ia}$ $(i = 1,2,3)$ and the operator D reads

$D = D_1 + D_2 = D_1 + \lambda D_2^*$, where the operator D_2^* reads

$$D_2^* = f^*_{1a}\cdot\frac{\partial}{\partial z_2} + f^*_{2a}\cdot\frac{\partial}{\partial z_4} + f^*_{3a}\cdot\frac{\partial}{\partial z_6} \qquad \text{(VI,93)}$$

where D_1 and D_2 are given by Eq.(VI,89) and Eq.(VI,90), respectively. With (VI,87) the solution reads

$$Z_i = e^{t(D_1+D_2)} z_i = e^{t(D_1+\lambda D_2^*)} z_i = \sum_{j=o}^{\infty} \lambda^j g_j(t,z_i) =$$

$$= g_o(t,z_i) + \sum_{j=1}^{\infty} \lambda^j g_j \qquad \text{(VI,94)}$$

where $g_o(t,z_i) = e^{tD_1} z_i$ and g_{j+1} can be calculated by the following recurrence formula (see Ref.94)

$$g_{j+1}(t,z_i) = \int_{t_o}^{t} \left[D_2^* g_j(\tau,z_i)\right]_{z_i \rightarrow g_o(t-\tau,z_i)} d\tau \qquad \text{(VI,95)}$$

The subscript $z_i \rightarrow g_o(t-\tau,z_i)$ indicates that after applying the operator D_2^* on g_j, z_i has to be replaced by $g_o(t-\tau,z_i)$.

The proof of formula (VI,94) and formula (VI,95) is given in the work by GROEBNER (see Ref.94). D_1 is the operator for the heavy symmetric gyroscope.

Since the quantity λ defined by Eq.(VI,93) is usually small the factor λ^j $(j = 1,2,3...)$ influences the convergence in a favorable way.

(6.212) Representation of the solution S of Eq.(VI,61) in the form $S = S_{\text{symmetric, forcefree}}$ + contributions from asymmetry and forces.

In this case we write the operator D in the form

$$D = \alpha_2\cdot\frac{\partial}{\partial z_1} + \beta_2\cdot\frac{\partial}{\partial z_3} + \gamma_2\cdot\frac{\partial}{\partial z_5} + f_{1sff}\cdot\frac{\partial}{\partial z_2} + f_{2sff}\cdot\frac{\partial}{\partial z_4} + f_{3sff}\cdot\frac{\partial}{\partial z_6} + $$
$$+ f_{1h}\cdot\frac{\partial}{\partial z_2} + f_{2h}\cdot\frac{\partial}{\partial z_4} + f_{3h}\cdot\frac{\partial}{\partial z_6} + f_{1a}\cdot\frac{\partial}{\partial z_2} + f_{2a}\cdot\frac{\partial}{\partial z_4} + $$
$$+ f_{3a}\cdot\frac{\partial}{\partial z_6} \qquad \text{(VI,96)}$$

where f_{ih} $(i = 1,2,3)$ indicate the contribution of the external force (heavy), f_{isff} and f_{ia} $(i = 1,2,3)$ are explained above. We put now

$$D = D_1 + D_{ha}, \text{ where}$$

$$D_{ah} = f_{1h}\cdot\frac{\partial}{\partial z_2} + f_{2h}\cdot\frac{\partial}{\partial z_4} + f_{3h}\cdot\frac{\partial}{\partial z_6} + f_{1a}\cdot\frac{\partial}{\partial z_2} + f_{2a}\cdot\frac{\partial}{\partial z_4} + $$
$$+ f_{3a}\cdot\frac{\partial}{\partial z_6} \qquad \text{(VI,97)}$$

and D_1 is defined by Eq.(VI,96). Solution (VI,87) reads in our case

$$Z_i = e^{tD} z_i = e^{t(D_1+D_{ha})} z_i = e^{tD_1} z_i + $$
$$+ \sum_{o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!} \left[D_{ha} D^\alpha z_i\right]_{\bar{b}} d\tau \qquad \text{(VI,98)}$$

D_1 is the operator for the symmetric forcefree gyroscope, i.e., $e^{tD_1}z_i$ is the solution of the symmetric forcefree gyroscope.

Taking account of the different torque acting on the gyroscope M_1, M_2 and M_3 in Eq.(VI,61) reads

$$M_1 = \sum_i M_{1i}; \quad M_2 = \sum_i M_{2i}; \quad M_3 = \sum_i M_{3i}, \text{ where } i=1,2,\ldots \qquad (VI,99)$$

indicates the different torques.

Considering a satellite considerable torques are, e.g.: the gravitational torque, the drag torque, the torque caused by radiation and the magnetic torques. Splitting off the operator $D_{ah} = D_a + D_h$, where

$$D_a = f_{1a}\cdot\frac{\partial}{\partial z_2} + f_{2a}\cdot\frac{\partial}{\partial z_4} + f_{3a}\cdot\frac{\partial}{\partial z_6} \qquad (VI,100)$$

$$D_h = f_{1h}\cdot\frac{\partial}{\partial z_2} + f_{2h}\cdot\frac{\partial}{\partial z_4} + f_{3h}\cdot\frac{\partial}{\partial z_6} \qquad (VI,101)$$

D_h again can be written in the form $\sum_l D_{lh}$ $(l=1,2,3\ldots) = D_h$, where

$$D_{lh} = f_{1lh}\cdot\frac{\partial}{\partial z_2} + f_{2lh}\cdot\frac{\partial}{\partial z_4} + f_{3lh}\cdot\frac{\partial}{\partial z_6}; \quad l=1,2,,\ldots \qquad (VI,102)$$

and

$$f_{11h} = S_{21}q_{25}; \quad f_{21h} = S_{11}q_{25} + S_{21}q_{26}; \quad f_{31h} = S_{11}q_{37} +$$
$$+ S_{31}q_{38} \qquad (VI,103)$$

Eq.(VI,98) has now the form

$$Z_i = e^{tD_1}z_i + \sum_0^{\infty}\int_{t_0}^{t} \frac{(t-\tau)^\alpha}{\alpha!}\left[D_a D^\alpha z_i\right]_{\bar{b}}\, d\tau +$$
$$+ \sum_0^{\infty}\int_{t_0}^{t} \frac{(t-\tau)^\alpha}{\alpha!}\left[\sum_l D_{hl} D^\alpha z_i\right]_{\bar{b}}\, d\tau \qquad (VI,104)$$

The subscript $\bar{b}$ indicates, that after applying the operator, z_i has to be replaced by $e^{tD_1}z_i$. The last integral term in Eq.(VI,104) vanishes if $M_i (i = 1,2,3)$ is equal to zero. The solution representation Eq.(VI,104) enables us to evaluate the single integral terms numerically independently from the other terms.

(6.213) Representation of the solution S of Eq.(VI,61) in the form $S = S_{\text{asymmetric, forcefree}}$ + contributions from forces.

In this case we write the operator D in the form

$D = D_{1aff} + D_h$, where the operator D_{1aff} reads

$$D_{1aff} = \alpha_2 \cdot \frac{\partial}{\partial z_1} + \beta_2 \cdot \frac{\partial}{\partial z_3} + \gamma_2 \cdot \frac{\partial}{\partial z_5} + f_{1sff} \cdot \frac{\partial}{\partial z_2} + f_{2sff} \cdot \frac{\partial}{\partial z_4} +$$

$$+ f_{3sff} \cdot \frac{\partial}{\partial z_6} + f_{1a} \cdot \frac{\partial}{\partial z_2} + f_{2a} \cdot \frac{\partial}{\partial z_4} + f_{3a} \cdot \frac{\partial}{\partial z_6} \qquad \text{(VI,105)}$$

$$D_h = f_{1h} \cdot \frac{\partial}{\partial z_2} + f_{2h} \cdot \frac{\partial}{\partial z_4} + f_{3h} \cdot \frac{\partial}{\partial z_6} \qquad \text{(VI,106)}$$

The solution (VI,87) reads in this case

$$Z_i = e^{tD} z_i = e^{t(D_{1aff}+D_h)} z_i = e^{tD_{1aff}} z_i +$$

$$+ \sum_{o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!} \left[D_h D z_i \right]_{\bar{a}} d\tau \qquad \text{(VI,107)}$$

D_{1aff} is the operator for the asymmetric forcefree gyroscope. Putting $e^{tD_{1aff}} z_i = Z_{i1}$ and splitting off the operator D_{1aff} in the form $D_{1aff} = D_{1ff} + \lambda D_2^*$, where D_2^* is defined by (VI,93) and D_{1ff} is defined by Eq.(VI,105), we obtain the solution in the form

$$Z_{i1} = e^{tD_{1aff}} z_i = e^{t(D_{1ff}+\lambda D_2^*)} z_i = \sum_{j=o}^{\infty} \lambda^j g_j(t,z_i) = \qquad (VI,108)$$

$g_o(t,z_i) + \sum_{j=1}^{\infty} \lambda^j g_j(t,z_i)$, where g_o is given by the relation

$$g_o(t,z_i) = e^{tD_{1ff}} z_i \text{ and } g_{j+1}(t,z_i) = \int_{t_o}^{t} \left[D_2^* g_j(\tau,z_i) \right]_{z_i \to g_o(t,z_i)} d\tau \qquad (VI,109)$$

With Eq.(VI,108), Eq.(VI,109) reads

$$Z_i = e^{tD_{1ff}} z_i + \sum_{j=1}^{\infty} \lambda^j g_j(t,z_i) + \sum_{o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!} \left[D_h D^\alpha z_i \right]_{\bar{a}} d\tau \qquad (VI,110)$$

where D_{1ff} is the operator for the forcefree symmetric gyroscope. If several external forces are present we obtain in analogy to Eq.(VI,104) the solution Eq.(VI,110) in the form

$$Z_i = e^{tD_{1ff}} + \sum_{j=1}^{\infty} \lambda^j g_j(t,z_i) + \sum_{o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!} \left[\sum_{l} D_{hl} D^\alpha z_i \right]_{\bar{a}} d\tau \qquad (VI,111)$$

This representation is advantageous insofar as it contains several additive integral terms, which can be computed separately. The number of summation terms $\alpha = 0,1,2,...$ depends on the order of magnitude of the torque appearing in the operators D_{hl} ($l = 1,2,.....$). For the numerical evaluation of the integral terms we refer to the work by H.KNAPP (see Ref.2, e.g.).

Concerning the stability of the solution of Eq.(VI,61) we refer to the books by KLEIN F. and SOMMERFELD A. (see Ref.45) and by GRAMMEL R. (see Ref.96), in which this problem is treated in detail.

Appendix

In the book, "The General Problem of the Motion of Coupled Rigid Bodies About a Fixed Point", by E.Leimanis, Springer Tracts in Natural Philosophy, Vol.7, 1965, p.133, the Euler Poisson equations of motion are solved by Lie Series. These equations read

$$
\begin{aligned}
I_1\dot{u}_1 + (I_2 - I_3)u_2u_3 &= mg(\beta z_o - \gamma y_o) \\
I_2\dot{u}_2 + (I_1 - I_3)u_1u_3 &= mg(\gamma x_o - \alpha z_o) \\
I_3\dot{u}_3 + (I_2 - I_1)u_1u_3 &= mg(\alpha y_o - \beta x_o)
\end{aligned}
\qquad (VI,112)
$$

$$
\begin{aligned}
\dot{\alpha} &= \beta u_3 - \gamma u_2 \\
\dot{\beta} &= \gamma u_1 - \alpha u_3 \\
\dot{\gamma} &= \alpha u_2 - \beta u_1
\end{aligned}
\qquad (VI,113)
$$

where I_i are the moments of inertia, u_i are the angular velocities, 1,2,3 indicate the axis fixed with respect to the body, m is the mass of the body, gm is the weight of the body, $\vec{r}_o = (x_o, y_o, z_o)$ indicates the position of the mass center. (x, y, z) denote the reference frame fixed with respect to the body, α, β, γ are the direction cosines of a fixed axis (Z axis of a space fixed reference frame, e.g.) with respect to x, y, z. E.Leimanis represents the solution of Eqs.(VI,112) and (VI,113) in the form $Z_i = e^{tD}z_i$, where $Z_1 = u_1$, $Z_2 = u_2$, $Z_3 = u_3$, $Z_4 = \alpha$, $Z_5 = \beta$, $Z_6 = \gamma$ and the Lie operator D reads

$$
\begin{aligned}
D = &\left[\frac{mg}{I_1}(\beta z_o - \gamma y_o) - \frac{I_2-I_3}{I_1} u_2u_3\right]\frac{\partial}{\partial u_1} + \frac{mg}{I_2}\left[(\gamma x_o - \alpha z_o) - \right. \\
&\left. - \frac{I_1-I_3}{I_2} u_1u_3\right]\frac{\partial}{\partial u_2} + \frac{mg}{I_3}\left[(\alpha y_o - \beta x_o) - \frac{I_2-I_1}{I_3} u_1u_3\right]\frac{\partial}{\partial u_3} + \\
&+ (\beta u_3 - \gamma u_2)\frac{\partial}{\partial \alpha} + (\gamma u_1 - \alpha u_3)\frac{\partial}{\partial \beta} + (\alpha u_2 - \beta u_1)\frac{\partial}{\partial \gamma}
\end{aligned}
$$

As shown in this paper we have the Euler equation containing not only a term for the gravitational torque, but also several other terms corresponding to other torques (drag torque, centrifugal torque, etc.). Furthermore, as far as the solution representation is concerned experience (see Ref.2, e.g.) has shown, that a representation as it was given by E.Leimanis is not recommandable for numerical computation. A rearrangement of the series $e^{tD} z_i$ by splitting off in the form $e^{t(D_1+D_2)} z_i = e^{tD_1} z_i + R$, as it was done in this report, influences the numerical evaluation in a favorable way, if, e.g., $e^{tD_1} z_i \gg R$.

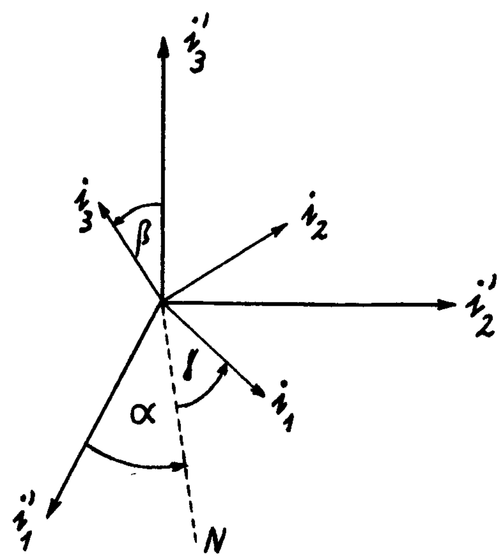
(6.3) The Integration of the Equations of Motions

(6.31) Transition to component representation

(6.311) Derivation of the general formulas:

(6.3111) The Eulerian matrix:

The so-called Eulerian angles are an appropriate means of describing the rotation of two rectangular normalized trihedrals with respect to one another. Since the transformation matrix appearing in this case - the Eulerian matrix - will play a crucial part in the following we shall summarize here the most important formulas:



N: $(\vec{i}'_1, \vec{i}'_2) \,/\, (\vec{i}_1, \vec{i}_2)$

System I: $\vec{i}'_1, \vec{i}'_2, \vec{i}'_3$

System II: $\vec{i}_1, \vec{i}_2, \vec{i}_3$

Fig. 2

Now we have:

$$\vec{i}'_\rho = a_{\rho\sigma}\vec{i}_\sigma \qquad \vec{i}_\rho = a_{\sigma\rho}\vec{i}'_\sigma \tag{VI,114 a}$$

$$a_{\rho\varkappa}a_{\sigma\varkappa} = \delta_{\rho\varkappa} \tag{VI,114 b}$$

with:

$$
\begin{array}{lll}
a_{11} = +\bar{a}\bar{c}-a\bar{b}c & a_{12} = -\bar{a}c-a\bar{b}\bar{c} & a_{13} = +ab \\
a_{21} = +a\bar{c}+\bar{a}\bar{b}c & a_{22} = -ac+\bar{a}\bar{b}\bar{c} & a_{23} = -\bar{a}b \\
a_{31} = +bc & a_{32} = +b\bar{c} & a_{33} = +\bar{b}
\end{array}
\tag{VI,115}
$$

and the following abbreviations:

$$
\begin{array}{lll}
a: = \sin\alpha & b: = \sin\beta & c: = \sin\gamma \\
\bar{a}: = \cos\alpha & \bar{b}: = \cos\beta & \bar{c}: = \cos\gamma
\end{array}
\tag{VI,116}
$$

Since the $a_{\rho\sigma}$ are representing transformations between Cartesian coordinate systems there holds:

$$a_{\rho\sigma} = a^{\rho\sigma} = a_{\rho}{}^{\sigma} = a^{\rho}{}_{\sigma};$$

for such reason in the following there hav been no regard to the position of the indices, in contrast to $b_{\rho\sigma}$ and $c_{\rho\sigma}$ in (VI,121). Where the summation convention it demands, the index concerned is thought to be lifted.

If the vector $\vec{u}$ of the rotation of the two systems with respect to one another is chosen such that the relation

$$\overset{\mathrm{I}\cdot}{\vec{a}} = \overset{\mathrm{II}\cdot}{\vec{a}} + \vec{u}\times\vec{a} \tag{VI,117}$$

holds for arbitrary vectors, where $\overset{\mathrm{I}\cdot}{\vec{a}}$ is the time derivative in System I, we also have:

$$
\begin{aligned}
\vec{u} &= (\bar{a}\dot{\beta}+ab\dot{\gamma})\vec{i}'_1 + (a\dot{\beta}-\bar{a}b\dot{\gamma})\vec{i}'_2 + (\dot{\alpha}+\bar{b}\dot{\gamma})\vec{i}'_3 = \\
&= (\bar{c}\dot{\beta}+bc\dot{\alpha})\vec{i}_1 + (-c\dot{\beta}+b\bar{c}\dot{\alpha})\vec{i}_2 + (\dot{\gamma}+\bar{b}\dot{\alpha})\vec{i}_3
\end{aligned}
\tag{VI,118}
$$

Besides, it follows from (VI,117)

$$\overset{\mathrm{I}\cdot}{\vec{u}} = \overset{\mathrm{II}\cdot}{\vec{u}} + \vec{u}\times\vec{u} = \overset{\mathrm{II}\cdot}{\vec{u}} \tag{VI,119}$$

i.e., it is irrelevant in which of the two systems of reference $\vec{u}$ is differentiated.

From now on the system II, i.e., $\left\{\bar{i}_1, \bar{i}_2, \bar{i}_3\right\}$ is identified with the system of main axes. In the following, the "system of main axes" is understood to be that Cartesian system whose instantaneous coordinate lines are parallel to the main axes of inertia of the satellite.

(6.3112) Earth' system: Fixation of the Eulerian angles:

The Eulerian matrix treated in (6.3111) connects two Cartesian systems. It is quite obvious that the tensor components in Section (6.13) were given in spherical coordinates; it is, therefore, necessary to transform to Cartesian coordinates. Viz., the introduction of the following system is expedient:

$$\begin{aligned}
&\vec{i}_1': \text{ from 0 to the origin of the } \lambda \text{ counting in the equatorial} \\
&\qquad \text{plane; i. e., to } \varphi = 90^\circ,\ \lambda = 0^\circ \\
&\vec{i}_2': \text{ from 0 to } \varphi = 90^\circ,\ \lambda = 90^\circ \\
&\vec{i}_3': \text{ from 0 to the north pole; i. e., } \varphi = 0^\circ,\ \lambda \text{ not determined.}
\end{aligned} \tag{VI,12o}$$

Then we have:

$$\begin{aligned}
\vec{e}_r &= \vec{e}^r = (\vec{i}_1'\cdot\cos\lambda + \vec{i}_2'\cdot\sin\lambda)\cdot\sin\varphi + \vec{i}_3'\cos\varphi \\
\vec{e}_\varphi &= r^2\vec{e}^\varphi = (\vec{i}_1'.\cos\lambda + \vec{i}_2'\cdot\sin\lambda)\cdot r\cdot\cos\varphi - \vec{i}_3'\cdot r\cdot\sin\varphi \\
\vec{e}_\lambda &= r^2\sin^2\varphi\cdot\vec{e}^\lambda = (-\vec{i}_1'\cdot\sin\lambda + \vec{i}_2'\cdot\cos\lambda)\cdot r\cdot\sin\varphi
\end{aligned} \tag{VI,121 a}$$

and

$$\begin{aligned}
\vec{i}_1' &= \vec{e}^r\cdot\cos\lambda\cdot\sin\varphi + r(\vec{e}^\varphi\cos\varphi\cdot\cos\lambda - \vec{e}^\lambda\sin\varphi\cdot\sin\lambda) \\
\vec{i}_2' &= \vec{e}^r\cdot\sin\lambda\cdot\sin\varphi + r(\vec{e}^\varphi\cos\varphi\cdot\sin\lambda + \vec{e}^\lambda\sin\varphi\cdot\cos\lambda) \\
\vec{i}_3' &= \vec{e}^r\cos\varphi - r\vec{e}^\varphi\sin\varphi
\end{aligned} \tag{VI,121 b}$$

respectively, or, in a compact way

$$\vec{e}_\rho = b_\rho{}^\sigma\vec{i}_\sigma' \qquad \vec{e}^\rho = b^{\rho\sigma}\vec{i}_\sigma' \qquad \vec{i}_\rho' = c_{\rho\sigma}\vec{e}^\sigma \tag{VI,121}$$

On integration of (VI,53 a,b) the Eulerian angles and the coordinates of the center of mass are obtained as functions of time. With

$$\vec{i}_\rho = a_{\mu\rho} c_{\mu\sigma} \vec{e}^\sigma \tag{VI,122}$$

the satellite's position relative to the spherical coordinate trihedral presented in (VI,39) is given in a very illustrative manner.

(6.3113) Orbit system: Fixation of Eulerian Angles:

Cartesian coordinates are somehow introduced in the orbit system, e. g. in a way mentioned at the beginning of section (6.15). We shall use here another system, i.e., *)

$$\vec{i}'_1 := +r\vec{e}^\lambda \sin\varphi \qquad \vec{i}'_2 := -r\vec{e}^\varphi \qquad \vec{i}'_3 := +\vec{e}^r \tag{VI,123}$$

The instantaneous position of the satellite is then obtained either immediately from Fig. 2 or with the held of

$$\vec{i}_\rho = +a'_{3\rho}\vec{e}^r - a'_{2\rho} r\vec{e}^\varphi + a'_{1\rho} r\vec{e}^\lambda \sin\varphi \tag{VI,126}$$

---

*) If the unit vectors defined in this way are regarded as the System II (change of designation! $\vec{i}' \rightarrow \vec{i}$) and those defined according to (VI, 120) as the System I of Fig. 2, the transformation matrix is obtained from (VI,115) with:

$$\alpha := 90^\circ + \lambda \qquad \beta := +\varphi \qquad \gamma := 0 \tag{VI,124}$$

which, of course, leads to (VI,121). With the help of (VI,118) we immediately obtain $\vec{u}_B$:

$$\begin{aligned}\vec{u}_B &= +\dot{\lambda}(\vec{e}^r\cos\varphi - r\vec{e}^\varphi \sin\varphi) + r\dot{\varphi}\vec{e}^\lambda \sin\varphi = \\ &= +\dot{\varphi}(-\vec{i}'_1 \cdot \sin\lambda + \vec{i}'_2 \cdot \cos\lambda) + \dot{\lambda}\vec{i}'_3\end{aligned} \tag{VI,125}$$

where the $\vec{i}'_\rho$ again represent (VI,120), i. e., System II!

where, in order to distinguish from the transformation coefficients (VI,122) of the earth's system we write here $a'_{\rho\sigma}$ instead of $a_{\rho\sigma}$ as was the case in (VI,114) and (VI,115).

(6.3114) Representation of $\vec{J}_s \cdot \dot{\vec{u}} + \vec{u} \times \vec{J}_s \cdot \vec{u} = \vec{N}$ in the System of Main Axes:

This term occurs in (VI,48 b) with $\vec{u} = \vec{u}_t$ and in (VI,62 b) with $\vec{u} = \vec{u}_s$. Because of

$$\vec{J}_s = J^{\rho\sigma}\vec{i}_{\rho\sigma} = J^{11}\vec{i}_{11} + J^{22}\vec{i}_{22} + J^{33}\vec{i}_{33} \qquad \text{(VI,127)}$$

$$\begin{aligned} \dot{\vec{u}} = \dot{u}^{\rho}\vec{i}_{\rho} &= (\bar{c}\beta + bc\alpha + \bar{b}c\dot{\alpha}\beta + b\bar{c}\dot{\alpha}\dot{\gamma} - c\beta\dot{\gamma})\vec{i}_1 + \\ &+ (-c\beta + b\bar{c}\alpha + \bar{b}\bar{c}\dot{\alpha}\beta - bc\dot{\alpha}\dot{\gamma} - \bar{c}\beta\dot{\gamma})\vec{i}_2 + \\ &+ (\gamma + \bar{b}\alpha - b\dot{\alpha}\beta)\vec{i}_3 \end{aligned} \qquad \text{(VI,128)}$$

and

$$\vec{N} = N^{\rho}\vec{i}_{\rho}$$

immediately the well-known system

$$\begin{aligned} J^{11}\dot{u}^1 + (J^{33} - J^{22})u^2u^3 &= N^1 \\ J^{22}\dot{u}^2 + (J^{11} - J^{33})u^1u^3 &= N^2 \\ J^{33}\dot{u}^3 + (J^{22} - J^{11})u^1u^2 &= N^3 \end{aligned} \qquad \text{(VI,129)}$$

or

$$\dot{u}^{\rho} + B_{\rho}u^{\sigma}u^{\varkappa} = \frac{N^{\rho}}{J^{\rho\rho}} \qquad \text{(VI,129 a)}$$

with

$$B_1 := \frac{J^{33} - J^{22}}{J^{11}} \qquad B_2 := \frac{J^{11} - J^{33}}{J^{22}} \qquad B_3 := \frac{J^{22} - J^{11}}{J^{33}} \qquad \text{(VI,130)}$$

results. Solving with respect to the second-time derivatives yields:

$$\begin{aligned} \ddot{\alpha} = &-\bar{b}c\bar{c}(B_1+B_2)\dot{\alpha}^2 + \frac{\bar{b}}{b}(B_1c^2 - B_2\bar{c}^2 - 1)\dot{\alpha}\dot{\beta} - c\bar{c}(B_1+B_2)\dot{\alpha}\dot{\gamma} + \\ &+ \frac{1}{b}(1 + B_1c^2 - B_2\bar{c}^2)\dot{\beta}\dot{\gamma} + \frac{N_1\bar{c}c}{J^{11}b} + \frac{N_2\bar{c}c}{J^{22}b} \end{aligned} \qquad \text{(VI,131 a)}$$

$$\ddot{\beta} = +b\bar{b}(B_2c^2-B_1\bar{c}^2)\dot{\alpha}^2 + \bar{b}c\bar{c}(B_1+B_2)\dot{\alpha}\dot{\beta} + b(B_2c^2-B_1\bar{c}^2-1)\dot{\alpha}\dot{\gamma} + + c\bar{c}(B_1+B_2)\dot{\beta}\dot{\gamma} + \frac{N_1}{J^{11}}\cdot\bar{c} - \frac{N_2}{J^{22}}\cdot c \tag{VI, 131 b}$$

$$\ddot{\gamma} = +c\bar{c}((B_1+B_2)\bar{b}^2-B_3b^2)\dot{\alpha}^2 + \frac{1}{b}(1+(B_2\bar{c}^2-B_1c^2)\bar{b}^2+B_3(c^2-\bar{c}^2)b^2)\cdot\dot{\alpha}\dot{\beta} + + \bar{b}c\bar{c}(B_1+B_2)\dot{\alpha}\dot{\gamma} + B_3c\bar{c}\dot{\beta}^2 + \frac{\bar{b}}{b}(B_2\bar{c}^2-B_1c^2-1)\dot{\beta}\dot{\gamma} + \frac{N_3}{J^{33}} - \frac{\bar{b}}{b}(\frac{N_1}{J^{11}}\cdot c+\frac{N_2}{J^{22}}\cdot\bar{c}) \tag{VI,131 c}$$

(6.3115) Earth's system: Treatment of the Remaining Terms:

It remains to give the components of $\vec{M}_s$, i. e., the right-hand side of (VI,53 b) in the system of main axes. For this purpose we give the following summarizing review:

$$\hat{R}\binom{\mu}{1} = R^{(\mu)}_{\rho}\vec{e}^{\rho} \qquad \hat{R}\binom{\mu}{2} = R^{(\mu)}_{\rho\sigma}\vec{e}^{\rho\sigma} \qquad \hat{R}\binom{\mu}{3} = R^{(\mu)}_{\rho\sigma\varkappa}\vec{e}^{\rho\sigma\varkappa} \tag{VI,132}$$

$$\vec{S}_4 = \int(\vec{s}\circ\vec{s}\circ\vec{s}\circ\vec{s})dm = \vec{i}_{\alpha\beta\gamma\delta}\int s^{\alpha}s^{\beta}s^{\gamma}s^{\delta}dm = :S^{\alpha\beta\gamma\delta}\vec{i}_{\alpha\beta\gamma\delta} \tag{VI,133 a}$$

$$\vec{S}_3 = \int(\vec{s}\circ\vec{s}\circ\vec{s})dm = \vec{i}_{\alpha\beta\gamma}\int s^{\alpha}s^{\beta}s^{\gamma}dm = :S^{\alpha\beta\gamma}\vec{i}_{\alpha\beta\gamma} \tag{VI,133 b}$$

$$\vec{S}_2 = \int(\vec{s}\circ\vec{s})dm = \vec{i}_{\alpha\beta}\int s^{\alpha}s^{\beta}dm = :S^{\alpha\beta}\vec{i}_{\alpha\beta} = = \vec{I}\int\vec{s}^2dm - \vec{J}_s = (S\delta^{\alpha\beta}-J^{\alpha\beta})\vec{i}_{\alpha\beta} \text{ with } S: = \int\vec{s}^2dm \tag{VI,133 c}$$

therefore:

$$S^{11} = S - J^{11} \quad \text{etc.} \tag{VI,133 d}$$

Furthermore it follows from (VI,114) and (VI,121):

$$\vec{e}^{\rho} = b^{\rho\sigma}a_{\sigma\varkappa}\vec{i}^{\varkappa} \qquad \vec{e}_{\rho} = b_{\rho}{}^{\sigma}a_{\sigma\varkappa}\vec{i}^{\varkappa} \tag{VI,134}$$

Hence

$$(\vec{S}_2\cdot\hat{R}\binom{\mu}{1})x\vec{e}_{\mu} = (S^{\alpha\beta}\vec{i}_{\alpha\beta}\cdot R^{(\mu)}_{\rho}b^{\rho\sigma}a_{\sigma\varkappa}\vec{i}^{\varkappa})xb_{\mu}{}^{s}a_{st}\vec{i}^{t} = = S^{\alpha\beta}R^{(\mu)}_{\rho}b^{\rho\sigma}a_{\sigma\beta}b_{\mu}{}^{s}a_{st}\delta^{123}_{\alpha t\varkappa}\vec{i}^{\varkappa} \tag{VI,135 a}$$

Since we have:

$$\vec{i}_\alpha x \vec{i}_\beta = \delta^{123}_{\alpha\beta\gamma}\vec{i}^\gamma \qquad \text{(VI,136)}$$

with

$$\begin{aligned}\delta^{123}_{\alpha\beta\gamma}: &= +1, \text{ if } (\alpha\beta\gamma) \text{ is an even permutation of } (123)\\ &= -1, \text{ if } (\alpha\beta\gamma) \text{ is an odd permutation of } (123)\\ &= \ 0, \text{ if } (\alpha\beta\gamma) \text{ is no permutation of } (123)\end{aligned} \qquad \text{(VI,137)}$$

Furthermore:

$$\begin{aligned}(\vec{S}_3\cdot\cdot\vec{R}^{(\mu)}_{(2)})x\vec{e}_\mu &= (S^{\alpha\beta\gamma}\vec{i}_{\alpha\beta\gamma}\cdot\cdot R^{(\mu)}_{\rho\sigma}b^{\rho b}a_{br}b^{\sigma a}a_{as}\vec{i}^{rs})xb_\mu{}^m a_{mn}\vec{i}^n =\\ &= S^{\alpha\beta\gamma}R^{(\mu)}_{\rho\sigma}b^{\rho b}a_{b\gamma}b^{\sigma a}a_{a\beta}b_\mu{}^m a_{mn}\delta_{\alpha n\varkappa}\vec{i}^\varkappa\end{aligned} \qquad \text{(VI,135 b)}$$

$$\begin{aligned}(\vec{S}_4\cdot\cdot\cdot\vec{R}^{(\mu)}_{(3)})x\vec{e}_\mu &= (S^{\alpha\beta\gamma\delta}\vec{i}_{\alpha\beta\gamma\delta}\cdot\cdot\cdot R^{(\mu)}_{\rho\sigma\varkappa}b^{\rho a}a_{ar}b^{\sigma b}a_{bs}b^{\varkappa c}a_{ct}\vec{i}^{rst})xb_\mu{}^m a_{mn}\vec{i}^n =\\ &= S^{\alpha\beta\gamma\delta}R^{(\mu)}_{\rho\sigma\varkappa}b^{\rho a}b^{\sigma b}b^{\varkappa c}a_{a\delta}a_{b\gamma}a_{c\beta}b_\mu{}^m a_{mn}\delta_{\alpha nt}\vec{i}^t\end{aligned} \qquad \text{(VI,135 c)}$$

Thus, the summation of (VI,135 a) to(VI,135 c) yields $\vec{M}_s$ in (VI,53 b) and $\vec{N}$ in (VI,129),respectively, with x1, $x\frac{1}{2}$ and $x\ \frac{1}{6}$.

(6.3116) Orbit system: Treatment of the Remaining Terms:

In the moment equation (VI,62 b) the expression

$$\vec{N} = \vec{M}_s - \vec{J}_s\cdot\dot{\vec{u}}_B - \vec{u}_B x\vec{J}_s\cdot\vec{u}_B + 2\vec{u}_s x\vec{S}_2\cdot\vec{u}_B \qquad \text{(VI,138)}$$

remains to be represented in terms of the components of the system of main axes. Summary:

$$\vec{i}'_\rho = a'_{\rho\sigma}\vec{i}^\sigma \qquad \vec{i}_\rho = a'_{\sigma\rho}\vec{i}'^\sigma \qquad \text{(VI,114)}$$

$$\vec{u}_B = u'^{\rho}_B\vec{i}'_\rho \qquad \text{(VI,125)}$$

$$u_B'^{1} = +\dot{\varphi} \qquad u_B'^{2} = +\dot{\lambda}\cdot\sin\varphi \qquad u_B'^{3} = +\dot{\lambda}\cdot\cos\varphi \tag{VI,125}$$

According to (VI,119) we have:

$$\dot{\vec{u}}_B = \vec{u}_B = u_B'^{\rho}\vec{i}'_{\rho} \tag{VI,139}$$

Hence:

$$-\vec{J}_s\cdot\dot{\vec{u}}_B = -J^{\alpha\beta}\vec{i}_{\alpha\beta}\cdot u_B'^{\rho}a'_{\rho\sigma}\vec{i}^{\sigma} = -J^{\alpha\sigma}u_B'^{\rho}a'_{\rho\sigma}\vec{i}_{\alpha} \tag{VI,140}$$

$$\begin{aligned} -\vec{u}_B x\vec{J}_s\cdot\vec{u}_B &= +u_B'^{\mu}J^{\beta\gamma}\ \vec{i}_{\beta}x\vec{i}_{\varkappa}\ o\vec{i}_{\gamma}\cdot u_B'^{\rho}a'_{\rho\sigma}\vec{i}^{\sigma}a'_{\mu\varkappa} = \\ &= +u_B'^{\mu}u_B'^{\rho}a'_{\rho\sigma}\delta^{123}_{\beta\varkappa\alpha}J^{\beta\sigma}a'_{\mu\varkappa}\vec{i}^{\alpha} \end{aligned} \tag{VI,141}$$

$$\begin{aligned} +2\vec{u}_s x\vec{S}_2\cdot\vec{u}_B &= 2u_s^{\mu}S^{\beta\gamma}\ \vec{i}_{\mu}x\vec{i}_{\beta}\ o\vec{i}_{\gamma}\cdot u_B'^{\rho}a'_{\rho\sigma}\vec{i}^{\sigma} = \\ &= 2u_B'^{\rho}a'_{\rho\sigma}\delta^{123}_{\mu\beta\alpha}u_s^{\mu}S^{\beta\sigma}\vec{i}^{\alpha} \end{aligned} \tag{VI,142}$$

with:

$$u_s^1 = +\dot{\beta}\bar{c} + \dot{\alpha}bc \qquad u_s^2 = -\dot{\beta}c + \dot{\alpha}b\bar{c} \qquad u_s^3 = +\dot{\gamma} + \dot{\alpha}\bar{b} \tag{VI,118}$$

The equations for $\vec{M}_s$ can directly be taken over from (VI,135) if the $b^{\rho\sigma}$ and $c_{\rho\sigma}$ of (VI,121) are given by

$$\begin{aligned} &c'_{13} = +r\cdot\sin\varphi \qquad &&c'_{22} = -r \qquad &&c'_{31} = +1 \\ &b'^{13} = +1 &&b'^{22} = -\frac{1}{r} &&b'^{31} = +\frac{1}{r\cdot\sin\varphi} \end{aligned} \tag{VI,143}$$

The rest of the $c_{\rho\sigma}$ and $b^{\rho\sigma}$ vanish. (VI,138) is obtained by summing up (VI,14o), (VI,141), (VI,142) and (VI,135).

$$\vec{e}^{\rho} = b'^{\rho\sigma}\vec{i}'_{\sigma} \qquad \vec{i}'_{\rho} = c'_{\rho\sigma}\vec{e}^{\sigma} \tag{VI,121}$$

$$\vec{e}^{\rho} = b'^{\rho\sigma}a'_{\sigma\varkappa}\vec{i}^{\varkappa} \qquad \vec{i}_{\rho} = a'_{\sigma\rho}c'_{\sigma\varkappa}\vec{e}^{\varkappa} \tag{VI,123 a}$$

(6.3117) The Equation of Motion of the Center of Mass:

To obtain the coordinates of the center of mass as functions of time the integration of

$$\vec{b}_s = \vec{g}(\vec{r}_s) + \frac{1}{2m} \vec{S}_2 \cdot\cdot \vec{B}_{(2)} + \frac{1}{6m} \vec{S}_3 \cdot\cdot\cdot \vec{B}_{(3)} + \cdots \qquad \text{(VI,53 a)}$$

is necessary. Of course, here is no reason of giving the components in the system of main axes. In section (6.13), $\vec{g}$ and the $\vec{B}_{(k)}$ are given in terms of spherical coordinates whose use is quite obvious under these circumstances. In the following the index "s" is omitted in r_s, φ_s, λ_s in order to simplify denotation; hence, r, φ, λ are the instantaneous spherical coordinates of the satellite's center of mass.

$$\begin{aligned}\vec{b}_s = \vec{r}_s &= (r - r\dot{\varphi}^2 - r\dot{\lambda}^2 \sin^2\varphi)\vec{e}^r + (r^2\varphi + 2r\dot{r}\dot{\varphi} - r^2\dot{\lambda}^2 \sin\varphi \cdot \cos\varphi)\vec{e}^\varphi + \\ &+ (r^2\lambda \cdot \sin^2\varphi + 2r\dot{r}\dot{\lambda} \cdot \sin^2\varphi + 2r^2\dot{\varphi}\dot{\lambda} \cdot \sin\varphi \cdot \cos\varphi)\vec{e}^\lambda\end{aligned} \qquad \text{(VI,144)}$$

$$\vec{g}(\vec{r}_s) = \left(\frac{\alpha}{r^2} + \frac{3}{r^4}(\beta - \gamma \cdot \cos^2\varphi)\right)\vec{e}^r - \frac{2\gamma}{r^5}\vec{e}^\varphi \sin\varphi \cdot \cos\varphi \qquad \text{(VI,16)}$$

According to whether the moment equation is solved in the earth's or in the orbit system different Eulerian angles, i. e., different transformations are obtained. Since the resulting expressions are, however, equal, $a_{\rho\sigma}$, $a'_{\rho\sigma}$, etc. will not be distinguished in the following.

$$i_\rho = a_{r\rho} c_{rs} e^s \qquad \text{(VI,122)}$$

$$\begin{aligned}\frac{1}{2m} \vec{S}_2 \cdot\cdot \vec{B}_{(2)} &= \frac{1}{2m} S^{\alpha\beta} a_{r\alpha} c_{rn} a_{s\beta} c_{sm} \vec{e}^{nm} \cdot\cdot B^{\rho\sigma\varkappa} \vec{e}_{\rho\sigma\varkappa} = \\ &= \frac{1}{2m} S^{\alpha\beta} a_{r\alpha} c_{r\sigma} a_{s\beta} c_{s\rho} B^{\rho\sigma\varkappa} \vec{e}_\varkappa\end{aligned} \qquad \text{(VI,145 a)}$$

$$\frac{1}{6m} \vec{S}_3 \cdot\cdot\cdot \vec{B}_{(3)} = \frac{1}{6m} S^{\alpha\beta\gamma} a_{r\alpha} a_{s\beta} a_{t\gamma} c_{ra} c_{sb} c_{tc} \vec{e}^{abc} \cdot\cdot\cdot B^{\mu\rho\sigma\varkappa} \vec{e}_{\mu\rho\sigma\varkappa} = \qquad \text{(VI,145 b)}$$

$$= \frac{1}{6m} S^{\alpha\beta\gamma} a_{r\alpha} a_{s\beta} a_{t\gamma} c_{r\sigma} c_{s\rho} c_{t\mu} B^{\mu\rho\sigma\varkappa} \vec{e}_{\varkappa} \tag{VI,145 b}$$

(6.312) Specialisation on a Satellite Moving on a Circular Orbit:

For the numerical evaluation of the equations of motion, which will be discussed in (6.4), the case of a satellite moving along a circular orbit in the equatorial plane was taken as basis. Such an orbit is a solution of the equation of the center of mass' motion (VI,53 a), if there one puts

$$\beta = 0 = \gamma \qquad \vec{B}_{(2)} = \vec{0} = \vec{B}_{(3)} \tag{VI,146}$$

Then holds:

$$r(t) = R \qquad \varphi(t) = 90^{0} \tag{VI,147}$$

and therefore in (VI,144):

$$\vec{b}_s = -R\dot{\lambda}^2 \vec{e}^r + R^2 \ddot{\lambda} \vec{e}^{\lambda} = +\frac{\alpha}{R^2} \vec{e}^r \tag{VI,148}$$

hence

$$\ddot{\lambda} = 0 \qquad \dot{\lambda} = \sqrt{-\frac{\alpha}{R^3}} = \sqrt{+\frac{\Gamma \cdot m_E}{R^3}} = \text{const.} \tag{VI,149}$$

From (VI,125) follows:

$$\vec{u}_B = +\dot{\lambda} \dot{\vec{i}}_2' \qquad \dot{\vec{u}}_B = +\ddot{\lambda} \dot{\vec{i}}_2' = \vec{o} \tag{VI,150}$$

yielding immediately

$$-\vec{J}_s \cdot \dot{\vec{u}}_B = \vec{o} \tag{VI,140}$$

$$-\vec{u}_B x \vec{J}_s \cdot \vec{u}_B = +\lambda^2 a_{2\sigma}' a_{2\varkappa}' J^{\beta\sigma} \delta_{\beta\varkappa\alpha}^{123} \vec{i}^{\alpha} = \tag{VI,141}$$

$$= +\dot{\lambda}^2\Big\{a'_{22}a'_{33}(J^{22}-J^{33})\vec{i}^1 + a'_{23}a'_{21}(J^{33}-J^{11})\vec{i}^2 + + a'_{21}a'_{22}(J^{11}-J^{22})\vec{i}^3\Big\} \qquad \text{(VI,141)}$$

$$+2\vec{u}_s x\vec{S}_2\cdot\vec{u}_B = 2\dot{\lambda}a'_{2\sigma}u^\mu_s S^{\beta\sigma}\delta^{123}_{\mu\beta\alpha}\vec{i}^\alpha = +2\dot{\lambda}\Big\{(u^2_s S^{33}a'_{23}-u^3_s S^{22}a'_{22})\vec{i}^1 + + (u^3_s S^{11}a'_{21}-u^1_s S^{33}a'_{23})\vec{i}^2 + (u^1_s S^{22}a'_{22}-u^2_s S^{11}a'_{21})\vec{i}^3\Big\} \qquad \text{(VI,142)}$$

with

$$u^1_s = +\dot{\beta}\bar{c}+\dot{\alpha}bc \qquad u^2_s = -\dot{\beta}c+\dot{\alpha}b\bar{c} \qquad u^3_s = +\dot{\gamma}+\dot{\alpha}\bar{b} \qquad \text{(VI,118)}$$

$\vec{M}_s$ is obtained with

$$\vec{R}\begin{pmatrix}r\\1\end{pmatrix} = -\frac{3\alpha}{R^3}\vec{e}^r \qquad \vec{R}\begin{pmatrix}\varphi\\1\end{pmatrix} = \vec{o} \qquad \vec{R}\begin{pmatrix}\lambda\\1\end{pmatrix} = \vec{o} \qquad \text{(VI,151)}$$

from (VI, 135):

$$(\vec{S}_2\cdot\vec{R}\begin{pmatrix}\mu\\1\end{pmatrix})x\vec{e}_\mu = S^{\alpha\beta}R^{(r)}_{\rho=1}b'^{1\sigma}a'_{\sigma\beta}b'^{s}_{1}a'_{st}\delta^{123}_{\alpha t\varkappa}\vec{i}^\varkappa$$

From (VI,143) comes

$$b'^{13} = +1 \qquad b'^{22} = -\frac{1}{R} \qquad b'^{31} = +\frac{1}{R} \qquad \text{(VI,152 a)}$$

Because of

$$\vec{e}_\rho = g_{\rho\sigma}\vec{e}^\sigma$$

and

$$g_{11} = +1 \qquad g_{22} = +R^2 = g_{33}$$

follows from (VI, 121)

$$b'^{\sigma}_{\rho} = g_{\rho\varkappa}b'^{\varkappa\sigma}$$

$$b'^{3}_{1} = +1 \qquad b'^{2}_{2} = -R \qquad b'^{1}_{3} = +R \qquad \text{(VI,152 b)}$$

Hence:

$$\vec{M}_s = R_1^{(r)} b_1'^{1} b_1'^{3} S^{\alpha\beta} a'_{3\beta} a'_{3t} \delta^{123}_{\alpha t\varkappa} \vec{i}^{\varkappa} = -\frac{3\alpha}{R^3}\Big\{ a'_{32}a'_{33}(S^{22}-S^{33})\vec{i}^1 + \qquad \text{(VI,153)}$$

$$+ a'_{33}a'_{31}(S^{33}-S^{11})\vec{i}^2 + a'_{31}a'_{32}(S^{11}-S^{22})\vec{i}^3\Big\}$$

Furthermore, by reason of (VI,133 d):

$$(S^{\rho\rho}-S^{\sigma\sigma}) = (J^{\sigma\sigma}-J^{\rho\rho})$$

Because of (VI,138) $\vec{N}$ in (VI,129) and (VI,131) then reads:

$$N_1 = +(J^{22}-J^{33})(\dot{\lambda}^2 a'_{22}a'_{23}+\frac{3\alpha}{R^3}a'_{32}a'_{33}) + 2\dot{\lambda}(u_s^2 S^{33}a'_{23}-u_s^3 S^{22}a'_{22})$$

$$N_2 = +(J^{33}-J^{11})(\dot{\lambda}^2 a'_{23}a'_{21}+\frac{3\alpha}{R^3}a'_{33}a'_{31}) + 2\dot{\lambda}(u_s^3 S^{11}a'_{21}-u_s^1 S^{33}a'_{23}) \qquad \text{(VI,154)}$$

$$N_3 = +(J^{11}-J^{22})(\dot{\lambda}^2 a'_{21}a'_{22}+\frac{3\alpha}{R^3}a'_{31}a'_{32}) + 2\dot{\lambda}(u_s^1 S^{22}a'_{22}-u_s^2 S^{11}a'_{21})$$

With this all members in the moment equation (VI,129) are known (u^ρ there means, of course, u_s^ρ). In the next paragraphs some ways to solve this equation will be discussed.

(6.32) Explicit Calculation of the first three coefficients of the Lie Series Solution of Eq.(VI,131)

As far as the numerical evaluation of Eq.(VI,131) is concerned three possible ways offer themselves.

1. Repeated application of the D-operator ($Z = \sum_0^{\infty} \frac{t^i}{i!} D^i z$) yields very complex results for $i > 3$, such that we have to restrict ourselves to few terms; thus, we have to choose a

small step length $\Delta t = t_2 - t_1$ from which, on the other hand, an increase of the calculation time arises. Furthermore, the truncation error might be considerable.

2. Derivation of recurrence formulas for $D^i z$; in this case, one may perhaps choose a relatively large step length.
3. The third approach starts from a representation of the solution in terms of main part and a "perturbation integral" to be evaluated by iteration:

$$Z = e^{tD_1} z + \sum_{o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!} \left[D_2 D_z^\alpha \right]_{\bar{a}} d\tau \; .$$

In the present report, the method of recurrence formulas is treated in extenso.

In this section the repeated application of the D-operator will be treated.

The equations to be studied have the following form:

$$\ddot{\alpha} = -\dot{\alpha}\dot{\gamma}\bar{c}c(B_1 + B_2) - \dot{\alpha}\dot{\beta}\,\frac{(\bar{b}+B_2\bar{c}^2\bar{b}-c^2B_1\bar{b})}{b} - \frac{\dot{\beta}\dot{\gamma}}{b}(-1 + B_2\bar{c}^2 - c^2B_1) -$$
$$- \dot{\alpha}^2 c\bar{c}\bar{b}(B_1 + B_2) + \frac{N_1}{I_1}\frac{c}{b} + \frac{N_2}{I_2}\frac{\bar{c}}{b} \qquad \text{(VI,155a)}$$

$$\ddot{\beta} = \dot{\alpha}^2 b\bar{b}(-B_1\bar{c}^2 + B_2c^2) + \dot{\alpha}\dot{\beta}c\bar{c}\bar{b}(B_1 + B_2) - b\dot{\alpha}\dot{\gamma}\left[1 + B_1\bar{c}^2 - B_2c^2\right] +$$
$$+ \dot{\beta}\dot{\gamma}(B_1 + B_2)\, c\bar{c} + \frac{N_1}{I_1}\bar{c} - \frac{N_2}{I_2}c \qquad \text{(VI,155b)}$$

and

$$\ddot{\gamma} = \dot{\beta}^2 B_3 c\bar{c} + \dot{\alpha}^2 c\bar{c}(-B_3b^2 + \bar{b}^2(B_1 + B_2)) + \frac{\dot{\alpha}\dot{\beta}}{b}(1 + b^2B_3(c^2 - \bar{c}^2) +$$
$$+ \bar{b}^2(B_2\bar{c}^2 - c^2B_1)) + \dot{\alpha}\dot{\gamma}\bar{b}c\bar{c}(B_1 + B_2) + \dot{\beta}\dot{\gamma}(B_2\bar{c}^2 - B_1c^2 - 1)\frac{\bar{b}}{b} +$$
$$+ \frac{N_3}{I_3} - \frac{N_1}{I_1}\frac{c\bar{b}}{b} - \frac{N_2}{I_2}\frac{\bar{c}}{b}\bar{b} \qquad \text{(VI,155c)}$$

where α, γ, β are the Eulerian angles, N_i the external torques, and the B_i abbreviations for:

$$B_1 = \frac{T_{23}}{I_1}, \quad B_2 = \frac{T_{31}}{I_2}, \quad B_3 = \frac{T_{12}}{I_3}, \quad T_{ij} = T_i - T_j$$

The I_i are the main moments of inertia and $T_i = \int X_i^2 \, dm$

With the Eulerian matrix A

$$A = \begin{pmatrix} \bar{c}\bar{a} - ca\bar{b}, & -\bar{c}a\bar{b} - c\bar{a}, & ab \\ \bar{c}a + c\bar{a}\bar{b}, & \bar{c}\bar{a}\bar{b} - ca, & -\bar{a}b \\ cb \quad , & \bar{c}b, & \bar{b} \end{pmatrix} \qquad \text{(VI,156)}$$

where

$$\begin{array}{lll} a = \sin\alpha & b = \sin\beta & c = \sin\gamma \\ \bar{a} = \cos\alpha & \bar{b} = \cos\beta & \bar{c} = \cos\gamma \end{array} \qquad \text{(VI,157)}$$

The torques N_i read:

$$\begin{aligned} N_1 &= u^2(3a_{32}a_{33} - a_{22}a_{23})T_{23} + 2u(T_3\dot{\theta}_2 a_{23} - T_2 a_{22}\dot{\theta}_3) \\ N_2 &= u^2(3a_{31}a_{33} - a_{21}a_{23})T_{31} + 2u(T_1 a_{21}\dot{\theta}_3 - T_3 a_{23}\dot{\theta}_1) \qquad \text{(VI,158)} \\ N_3 &= u^2(3a_{31}a_{32} - a_{21}a_{22})T_{12} + 2u(-T_1 a_{21}\dot{\theta}_2 + T_2 a_{22}\dot{\theta}_1) \end{aligned}$$

For our purpose we rewrite Eq.(VI,155)

$$\ddot{\alpha} = f_1 = \sum_1^6 f_{1,\varrho}, \quad \ddot{\beta} = f_2 = \sum_1^6 f_{2,\varrho}, \quad \ddot{\gamma} = f_3 = \sum_1^7 f_{3,\varrho} \qquad \text{(VI,159)}$$

where

$$\begin{array}{ll} f_{11} = -g_{11}\dot{\alpha}^2 & f_{14} = g_{14}\dot{\beta}\dot{\gamma} \\ f_{12} = g_{12}\dot{\alpha}\dot{\beta} & f_{15} = \frac{N_1}{I_1}\frac{c}{b} \\ f_{13} = -g_{13}\dot{\alpha}\dot{\gamma} & f_{16} = \frac{N_2}{I_2}\frac{\bar{c}}{b} \end{array} \qquad \text{(VI,160a)}$$

$$g_{11} = \bar{b}c\bar{c}(B_1 + B_2) \qquad g_{13} = c\bar{c}(B_1 + B_2)$$
$$g_{12} = \frac{\bar{b}}{b}(B_1c^2 - 1 - B_2\bar{c}^2) \qquad g_{14} = \frac{1}{b}(1 + B_1c^2 - B_2\bar{c}^2) \tag{VI,160b}$$

$$f_{21} = g_{21}\dot{\alpha}^2 \qquad f_{24} = g_{24}\dot{\beta}\dot{\gamma}$$
$$f_{22} = g_{22}\dot{\alpha}\dot{\beta} \qquad f_{25} = \frac{N_1}{I_1}\bar{c} \tag{VI,161a}$$
$$f_{23} = g_{23}\dot{\alpha}\dot{\gamma} \qquad f_{26} = -\frac{N_2}{I_2}c$$

$$g_{21} = b\bar{b}(B_2c^2 - B_1\bar{c}^2) \qquad g_{23} = b(B_2c^2 - B_1\bar{c}^2 - 1)$$
$$g_{22} = g_{11} \qquad g_{24} = g_{13} \tag{VI,161b}$$

$$f_{31} = g_{31}\dot{\alpha}^2 \qquad f_{34} = g_{34}\dot{\beta}\dot{\gamma}$$
$$f_{32} = g_{32}\dot{\alpha}\dot{\beta} \qquad f_{35} = g_{35}\dot{\beta}^2 \tag{VI,162a}$$
$$f_{33} = g_{33}\dot{\alpha}\dot{\gamma} \qquad f_{36} = \frac{N_3}{I_3}$$
$$f_{37} = -\bar{b}(f_{15} + f_{16})$$

$$g_{31} = c\bar{c}\left[(B_1 + B_2)\bar{b}^2 - B_3b^2\right]$$
$$g_{32} = \frac{1}{b}\left[1 + \bar{b}^2(B_2\bar{c}^2 - B_1c^2) + b^2B_3(c^2 - \bar{c}^2)\right] \tag{VI,162b}$$
$$g_{33} = g_{11}, \qquad g_{34} = \frac{\bar{b}}{b}(B_2\bar{c}^2 - B_1c^2 - 1), \qquad g_{35} = B_3c\bar{c}$$

With the Eqs.(VI,160b), (VI,161b) and (VI,162b) we obtain the relations

$$g_{11} = \bar{b}g_{13} \qquad g_{21} = \bar{b}(g_{23} + b)$$
$$g_{12} = \bar{b}(g_{14} - \frac{2}{b}) \qquad g_{34} = -\bar{b}g_{14} \tag{VI,163}$$

According to Chapt.I the formal solution of Eq.(VI,159) reads

$$\alpha(t) = \sum_{0}^{\infty} \frac{t^{\varrho}}{\varrho!} D^{\varrho}\alpha_o, \qquad \beta(t) = \sum_{0}^{\infty} \frac{t^{\varrho}}{\varrho!} D^{\varrho}\beta_o$$

$$\gamma(t) = \sum_{0}^{\infty} \frac{t^{\varrho}}{\varrho!} D^{\varrho}\gamma_o \tag{VI,164}$$

The operator D reads in our case

$$D = \dot{\alpha}\frac{\partial}{\partial\alpha} + \dot{\beta}\frac{\partial}{\partial\beta} + \dot{\gamma}\frac{\partial}{\partial\gamma} + f_1\frac{\partial}{\partial\dot{\alpha}} + f_2\frac{\partial}{\partial\dot{\beta}} + f_3\frac{\partial}{\partial\dot{\gamma}} \tag{VI,165}$$

Obviously one has

$$\begin{array}{lll} D^{o}\alpha_o = \alpha(t=t_o), & D^{o}\beta_o = \beta(t=t_o), & D^{o}\gamma_o = \gamma(t=t_o) \\ D^{1}\alpha_o = f_{1,o} & D^{1}\beta_o = f_{2,o} & D^{1}\gamma_o = f_{3,o} \\ D^{2}\alpha_o = D^{1}f_{1,o} & D^{2}\beta_o = D^{1}f_{2,o} & D^{2}\gamma_o = D^{1}f_{3,o} \end{array} \tag{VI,166a}$$

With Eq.(VI,159) we obtain

$$D^1 f_1 = \sum_{1}^{6} D^1 f_{1,\varrho,o}; \quad D^1 f_2 = \sum_{1}^{6} D^1 f_{2,\varrho,o}; \quad D^1 f_3 = \sum_{1}^{7} D^1 f_{3,\varrho,o} \tag{VI,166b}$$

$$\begin{aligned} Df_{11,o} &= \left[h_3(\dot{\beta}bc\bar{c} - h_1\dot{\gamma}\bar{b})\dot{\alpha} - 2g_{11}f_1 \right]\dot{\alpha} \\ Df_{12,o} &= \frac{1}{b}(2h_3\dot{\gamma}\bar{b}c\bar{c} + \dot{\beta}\frac{1+h_2}{b})\dot{\alpha}\dot{\beta} + g_{12}h_{12} \\ Df_{13,o} &= -h_1h_3\dot{\alpha}\dot{\gamma}^2 - g_{13}h_{13} \\ Df_{14,o} &= \frac{1}{b}(2g_{13}\dot{\gamma} - \bar{b}\dot{\beta}g_{14})\dot{\beta}\dot{\gamma} + g_{14}h_{23} \\ Df_{15,o} &= \frac{1}{I_1b^2}\left[a_{31}DN_1 + N_1(\dot{\gamma}b\bar{c} - \dot{\beta}\bar{b}c) \right] \\ Df_{16,o} &= \frac{1}{I_2b^2}\left[a_{32}DN_2 - N_2(\dot{\gamma}bc + \dot{\beta}\bar{b}\bar{c}) \right] \end{aligned} \tag{VI,167a}$$

$$Df_{21,o} = \left\{ \left[h_4 \dot{\beta} (\bar{b}^2 - b^2) + 2h_3 \dot{\gamma} b\bar{b}c\bar{c} \right] \dot{\alpha} + g_{21} f_1 \right\} \dot{\alpha}$$

$$Df_{22,o} = h_3 (h_1 \dot{\gamma} \bar{b} - \dot{\beta} bc\bar{c}) \dot{\alpha}\dot{\beta} + g_{22} h_{12}$$

$$Df_{23,o} = (2b\dot{\gamma} g_{13} + \frac{\bar{b}}{b} \dot{\beta} g_{23}) \dot{\alpha}\dot{\gamma} + g_{23} h_{13}$$

$$Df_{24,o} = h_1 h_3 \dot{\beta} \dot{\gamma}^2 + g_{13} h_{23} \qquad \text{(VI,167b)}$$

$$Df_{25,o} = \frac{1}{I_1} (\bar{c} DN_1 - c\dot{\gamma} N_1)$$

$$Df_{26,o} = - \frac{1}{I_2} (c DN_2 + \bar{c} \dot{\gamma} N_2)$$

$$Df_{31,o} = \left\{ \left[h_1 \dot{\gamma} (h_3 \bar{b}^2 - B_3 b^2) + 2b\bar{b}c\bar{c}\dot{\beta}(h_3 + B_3) \right] \dot{\alpha} + 2g_{31} f_1 \right\} \dot{\alpha}$$

$$Df_{32,o} = - \frac{1}{b} \left[\dot{\beta}\bar{b}(g_{32} + 2b(h_2 + B_3 h_1)) + \right.$$
$$\left. + \dot{\gamma}\ 2(g_{31} - B_3 b^2 c\bar{c}) \right] \dot{\alpha}\dot{\beta} + g_{32} h_{12}$$

$$Df_{33,o} = h_3 (h_1 \dot{\gamma} \bar{b} - \dot{\beta} bc\bar{c}) \dot{\alpha}\dot{\gamma} + g_{33}$$

$$Df_{34,o} = \frac{1}{b} (g_{14} - g_{11}) \dot{\beta}\dot{\gamma} + g_{34} h_{23} \qquad \text{(VI,167c)}$$

$$Df_{35,o} = \left\{ h_1 \dot{\gamma}\dot{\beta} + 2c\bar{c} f_2 \right\} \dot{\beta} B_3$$

$$Df_{36,o} = \frac{1}{I_3} DN_3$$

$$Df_{37,o} = \frac{1}{b} \left[\frac{1}{b} \left\{ \frac{N_1}{I_1} (\dot{\beta} c - \dot{\gamma} b\bar{b}\bar{c}) + \frac{N_2}{I_2} (\dot{\beta}\bar{c} + \dot{\gamma} b\bar{b}c) \right\} - \right.$$
$$\left. - \bar{b} (\frac{c}{I_1} DN_1 + \frac{\bar{c}}{I_2} DN_2) \right]$$

where

$h_1 = \bar{c}^2 - c^2$ $\qquad$ $h_3 = B_1 + B_2$

$h_2 = B_2 \bar{c}^2 - B_1 c^2$ $\qquad$ $h_4 = B_2 c^2 - B_1 \bar{c}^2$

$$h_{11} = f_1\dot{\beta} + f_2\dot{\alpha} \qquad h_{13} = f_1\dot{\gamma} + f_3\dot{\alpha}$$

$$h_{23} = f_2\dot{\gamma} + f_3\dot{\beta}$$

For DN_i $(i = 1,2,3)$ we obtain

$$\begin{aligned}
DN_{1,o} = +u\Big\{&+3u(T_2 - T_3)(+a_{32}Da_{33} + a_{33}Da_{32}) + 2(+T_3a_{23}D\dot{\theta}_2 - \\
&- T_2a_{22}D\dot{\theta}_3) + \Big[+T_3(+2\dot{\theta}_2 + ua_{22}) - uT_2a_{22}\Big]Da_{23} - \\
&- \Big[+T_2(+2\dot{\theta}_3 + ua_{23}) - uT_3a_{23}\Big]Da_{22}\Big\} \\
DN_{2,o} = +u\Big\{&+3u(T_3 - T_1)(+a_{33}Da_{31} + a_{31}Da_{33}) + 2(+T_1a_{21}D\dot{\theta}_3 - \\
&- T_3a_{23}D\dot{\theta}_1) + \Big[+T_1(+2\dot{\theta}_3 + ua_{23}) - uT_3a_{23}\Big]Da_{21} - \\
&- \Big[+T_3(+2\dot{\theta}_1 + ua_{21}) - uT_1a_{21}\Big]Da_{23}\Big\} \\
DN_{3,o} = +u\Big\{&+3u(T_1 - T_2)(+a_{31}Da_{32} + a_{32}Da_{31}) + 2(+T_2a_{22}D\dot{\theta}_1 - \\
&- T_1a_{21}D\dot{\theta}_2) + \Big[+T_2(+2\dot{\theta}_1 + ua_{21}) - uT_1a_{21}\Big]Da_{22} - \\
&- \Big[+T_1(+2\dot{\theta}_2 + ua_{22}) - uT_2a_{22}\Big]Da_{21}\Big\}
\end{aligned} \tag{VI,167d}$$

$$\begin{aligned}
D\dot{\theta}_{1\rho} &= (+f_1bc + f_2\bar{c}) + \dot{\alpha}(b\bar{c}\dot{\gamma} + \bar{b}c\dot{\beta}) - \dot{\beta}c\dot{\gamma} = +\dot{\gamma}\dot{\theta}_2 + \\
&\quad + (+f_1bc + f_2\bar{c}) + \dot{\alpha}\bar{b}c\dot{\beta} \\
D\dot{\theta}_{2\rho} &= (+f_1b\bar{c} - f_2c) + \dot{\alpha}(\bar{b}\bar{c}\dot{\beta} - bc\dot{\gamma}) - \dot{\beta}\bar{c}\dot{\gamma} = -\dot{\gamma}\dot{\theta}_1 + \\
&\quad + (+f_1b\bar{c} - f_2c) + \dot{\alpha}\bar{b}\bar{c}\dot{\beta} \\
D\dot{\theta}_{3\rho} &= (+f_1\bar{b} + f_3) - \dot{\alpha}b\dot{\beta}
\end{aligned} \tag{VI,167e}$$

$$\begin{aligned}
Da_{21,o} &= D(a\bar{c}) + D(\bar{a}\bar{b}c) = +(+\dot{\alpha}\bar{a}\bar{c} - \dot{\gamma}ac) + (-\dot{\alpha}a\bar{b}c - \bar{a}bc\dot{\beta} + \bar{a}\bar{b}\bar{c}\dot{\gamma}) = \\
&= +\dot{\alpha}a_{11} + \dot{\gamma}a_{22} - \bar{a}bc\dot{\beta} \\
Da_{22,o} &= -D(ac) + D(\bar{a}\bar{b}\bar{c}) = -(+\dot{\alpha}\bar{a}c + a\bar{c}\dot{\gamma}) + (-\dot{\alpha}a\bar{b}\bar{c} - \bar{a}b\bar{c}\dot{\beta} - \bar{a}\bar{b}c\dot{\gamma}) = \\
&= +\dot{\alpha}a_{12} - \dot{\gamma}a_{21} - \bar{a}b\bar{c}\dot{\beta}
\end{aligned}$$

$$Da_{23,o} = -D(\bar{a}b) = -(-ab\dot{\alpha} + \bar{a}\bar{b}\dot{\beta}) = +\dot{\alpha}a_{13} - \bar{a}\bar{b}\dot{\beta}$$

$$Da_{31,o} = +D(bc) = +(+\dot{\beta}\bar{b}c + \dot{\gamma}b\bar{c}) = +\dot{\gamma}a_{32} + \bar{b}c\dot{\beta} \qquad \text{(VI,167f)}$$

$$Da_{32,o} = +D(b\bar{c}) = +(+\dot{\beta}\bar{b}\bar{c} - \dot{\gamma}bc) = -\dot{\gamma}a_{31} + \bar{b}\bar{c}\dot{\beta}$$

$$Da_{33,o} = +D(\bar{b}) = -b\dot{\beta}$$

Further applications of the operator D yield very complex expressions.

(6.33) <u>The solution of the Eq.(VI,155) by means of Lie series making use of recurrence formulas</u>

We now replace the system (VI,155) of three second-order differential equations by the following system of six first-order differential equations:

$$\dot{Z}_j \equiv Z_{j+3}$$

$$\ddot{Z}_j \equiv \dot{Z}_{j+3} = f_j = \sum_{i=1}^{5} n_i d_{ji} + \sum_{i=6}^{8} \bar{n}_i d_{ji} \qquad \text{(VI,168)}$$

$$(j = 1,2,3)$$

where use has been made of the following designations and abbreviations:

$$\alpha = Z_1, \qquad \beta = Z_2, \qquad \gamma = Z_3$$

$$\dot{\alpha} = Z_4, \qquad \dot{\beta} = Z_5, \qquad \dot{\gamma} = Z_6 \qquad \text{(VI,169)}$$

as well as

$$n_1 = \dot{\alpha}\dot{\gamma}, \qquad n_2 = \dot{\alpha}\dot{\beta}, \qquad n_3 = \dot{\beta}\dot{\gamma}, \qquad n_4 = \dot{\alpha}^2,$$

$$n_5 = \dot{\beta}^2, \qquad \bar{n}_6 = N_1, \qquad \bar{n}_7 = N_2, \qquad \bar{n}_8 = N_3 \qquad \text{(VI,170)}$$

The d_{ij} are given by:

$$d_{11} = -c\bar{c}\ (B_1 + B_2)$$

$$d_{12} = -\frac{\bar{b}}{b}(1 + B_2\bar{c}^2 - c^2B_1)$$

$$d_{13} = -\frac{1}{b}(B_2\bar{c}^2 - c^2B_1 - 1) \qquad \text{(VI,171)}$$

$$d_{14} = -c\bar{c}\bar{b}(B_1 + B_2)$$

$$d_{16} = \frac{1}{I_1}\frac{c}{b}$$

$$d_{17} = \frac{1}{I_2}\frac{c}{b}$$

$$d_{21} = -b\left[1 + B_1\bar{c}^2 - B_2c^2\right]$$

$$d_{22} = c\bar{c}\bar{b}(B_1 + B_2) = -d_{11}$$

$$d_{23} = -d_{11}$$

$$d_{24} = b\bar{b}(-B_1\bar{c}^2 + B_2c^2) \qquad \text{(VI,172)}$$

$$d_{26} = \frac{\bar{c}}{I_1}$$

$$d_{27} = -\frac{c}{I_2}$$

$$d_{31} = -d_{14}$$

$$d_{32} = \frac{1}{b}(1 + b^2B_3(c^2 - \bar{c}^2))$$

$$d_{33} = \frac{\bar{b}}{b}(B_2\bar{c}^2 - B_1c^2 - 1)$$

$$d_{34} = c\bar{c}(-B_3b^2 + \bar{b}^2(B_1 + B_2)) \qquad \text{(VI,173)}$$

$$d_{35} = B_3c\bar{c}$$

$$d_{36} = \frac{1}{I_3}, \qquad d_{37} = -\frac{c\bar{b}}{I_1b}, \qquad d_{38} = -\frac{\bar{c}\bar{b}}{I_2b}$$

The formal solution of the system is given by:

$$Z_\sigma = e^{tD}z_\sigma = \sum_{\varrho=0}^{\infty}\frac{t^\varrho}{\varrho!}D^\varrho z_\sigma \quad (\sigma = 1,2,3) \qquad \text{(VI,174a)}$$

and

$$\dot{Z}_\sigma = \sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} D^{\varrho+1} z_\sigma \qquad \text{(VI,174b)}$$

Now we attempt to derive recurrence formulas connecting higher order powers of the D-operator with lower ones; for this purpose we write $D^{\varrho+2} z_\sigma$ in the following form:

$$D^{\varrho+2} z_i = D^\varrho (D^2 z_i) = D^\varrho f_i = D^\varrho \Big(\sum_{i=1}^{5} n_i d_{ji} + \sum_{i=6}^{8} \bar{n}_i d_{ji}\Big) =$$

$$= \sum_{j_1=0}^{\varrho} \binom{\varrho}{j_1} \left\{ D^{j_1} n_i D^{\varrho-j_1} d_{ji} + D^{j_1} \bar{n}_i\, D^{\varrho-j_1} d_{ji} \right\} =$$

$$= \sum_{j_1=0}^{\varrho} \binom{\varrho}{j_1} \left\{ \left[\sum_{j_2=0}^{j_1} \binom{j_1}{j_2} (D^{j_2} z_{\lambda_1}\, D^{j_1-j_2} z_{\lambda_2}) \right] D^{\varrho-j_1} d_{ji} + \right.$$

$$\left. + D^{j_1} N_\sigma D^{\varrho-j_1} d_{ji} \right\} \qquad \text{(VI,175c)}$$

where $j_1 \leq \varrho$, $j_2 \leq \varrho$

$\underline{D^{\varrho-j_1} d_{ji}}$:

$$\underline{D^{\varrho-j_1} d_{11}} = -(B_1 + B_2)\, D^{\varrho-j_1}\, c\bar{c} \qquad \text{(VI,176)}$$

$$D^{\varrho-j_1} c\bar{c} = D^{j_4} c\bar{c} = \sum_{j_5}^{j_4} \binom{j_4}{j_5} D^{j_5} c D^{j_4-j_5}\, \bar{c} \qquad \text{(VI,177)}$$

$$D^{j_5}\, c = +D^{j_5-1} (\bar{c} z_6) = \sum_{j_6}^{j_5-1} \binom{j_5-1}{j_6} D^{j_6}\, \bar{c} D^{j_5-1-j_6} z_6 \qquad \text{(VI,178a)}$$

$$D^{j_4-j_5}\, \bar{c} = D^{j_4-j_5-1} (c z_6) =$$

$$= \sum_{j_6}^{j_4-j_5-1} \binom{j_4-j_5-1}{j_6} D^{j_6} c D^{j_4-j_5-1-j_6}\, z_6 \qquad \text{(VI,178b)}$$

where $j_6 \leq j_5 - 1$, $j_4 - j_5 - 1 - j_6 \leq \varrho$, $j_5 - j_1 - j_6 \leq \varrho$ (VI,179)

$D^{\varrho-j_1}d_{12}$:

$$D^{\varrho-j_1}d_{12} = -D^{\varrho-j_1}\frac{\bar b}{b} - B_2 D^{\varrho-j_1}\left(\frac{\bar b\bar c}{b}\right)^2 + B_1 D^{\varrho-j_1}\left(\frac{\bar b c^2}{b}\right) \quad (VI,180)$$

$$D^{\varrho-j_1}\frac{\bar b}{b} = \sum_{j_3}^{\varrho-j_1}\binom{\varrho-j_1}{j_3} D^{j_3}\frac{1}{b} D^{\varrho-j_1-j_3}\bar b \quad (VI,181)$$

$$D^{j_3}\frac{1}{b} = -D^{j_3-1}\left(\frac{\bar b}{b^2}z_5\right) = -D^{j_3-1}(\bar b b^{-1}b^{-1}z_5) =$$

$$= -\sum_{j_4}^{j_3-1}\binom{j_3-1}{j_4} D^{j_4}(\bar b b^{-1})D^{j_3-1-j_4}(b^{-1}z_5) \quad (VI,182)$$

$$D^{j_4}(\bar b b^{-1}) = \sum_{j_5}^{j_4}\binom{j_4}{j_5} D^{j_5}\bar b D^{j_4-j_5} b^{-1} \quad (VI,182a)$$

$$D^{j_3-1-j_4}(b^{-1}z_5) = \sum_{j_6}^{j_3-1-j_4}\binom{j_3-1-j_4}{j_6} D^{j_6}b^{-1}D^{j_3-1-j_4-j_6}z_5 \quad (VI,182b)$$

$$D^{\varrho-j_1-j_3}\bar b = D^{l_1}\bar b = -D^{l_1-1}(bz_5) =$$

$$= -\sum_{l_2}^{l_1-1}\binom{l_1-1}{l_2} D^{l_2}bD^{l_1-1-l_2}z_5 \quad (VI,183a)$$

$$D^{l_1}b = D^{l_1-1}(\bar b z_5) = \sum_{l_2}^{l_1-1}\binom{l_1-1}{l_2} D^{l_2}\bar b D^{l_1-1-l_2} z_5 \quad (VI,183b)$$

$$l_1 - 1 - l_2 \leq \varrho,\ j_4 \leq \varrho,\ j_3 - 1 - j_4 \leq \varrho \quad (VI,184)$$

$$D^{\varrho-j_1}\left(\frac{\bar b}{b}\bar c^2\right) = \sum_{j_3}^{\varrho-j_1}\binom{\varrho-j_1}{j_3} D^{j_3}\frac{\bar b}{b} D^{\varrho-j_1-j_3}\bar c^2 \quad (VI,185)$$

$D^{j_3}\frac{\bar b}{b}$: see (VI,181)

$$D^{\varrho-j_1-j_3}\bar c^2 = D^{l_1}(\bar c\bar c) = \sum_{l_2}^{l_1}\binom{l_1}{l_2} D^{l_2}\bar c D^{l_1-l_2}\bar c \quad (VI,186)$$

$$D^{l_2}\ \bar{c}\text{: see (VI,178)}$$

$$D^{\varrho-j_1}\left(\frac{\bar{b}c^2}{b}\right) = \sum_{j_3}^{\varrho-j_1}\binom{\varrho-j_1}{j_3} D^{j_3}\frac{\bar{b}}{b} D^{\varrho-j_1-j_3} c^2 \qquad \text{(VI,187)}$$

$$D^{j_3}\frac{\bar{b}}{b}\text{: see (VI,181)}$$

$$D^{\varrho-j_1-j_3} c^2 = D^{l_1}(cc) = \sum_{l_2}^{l_1}\binom{l_1}{l_2} D^{l_1} c\, D^{l_1-l_2} c \qquad \text{(VI,188)}$$

$$D^{l_1}\ c\text{: see (VI,178)}$$

$\underline{D^{\varrho-j_1}\ d_{13}}$: $$-B_2 D^{\varrho-j_1}\frac{\bar{c}^2}{b} + \cdot B_1 D^{\varrho-j_1}\frac{c^2}{b} + D^{\varrho-j_1}\frac{1}{b} \qquad \text{(VI,189)}$$

$$D^{\varrho-j_1}\frac{\bar{c}^2}{b} = \sum_{j_3}^{\varrho-j_1}\binom{\varrho-j_1}{j_3} D^{j_3}\frac{1}{b} D^{\varrho-j_1-j_3}\bar{c}^2 \qquad \text{(VI,190)}$$

$$D^{j_3}\frac{1}{b}\text{: see (VI,182)}$$

$$D^{\varrho-j_1-j_3}\bar{c}^2\text{: see (VI,186)}$$

$$D^{\varrho-j_1}\frac{c^2}{b} = \sum_{j_3}^{\varrho-j_1}\binom{\varrho-j_1}{j_3} D^{j_3}\frac{1}{b} D^{\varrho-j_1-j_3} c^2 \qquad \text{(VI,191)}$$

$$D^{j_3}\frac{1}{b}\text{: see (VI,182)}$$

$$D^{\varrho-j_1-j_3} c^2\text{: see (VI,188)}$$

$$D^{\varrho-j_1}\frac{1}{b}\text{: see (VI,182)}$$

$\underline{D^{\varrho-j_1}\ d_{14}}$: $$-(B_1 + B_2) D^{\varrho-j_1}(\bar{b}c\bar{c}) = D^{\varrho-j_1}\ d_{14} \qquad \text{(VI,192)}$$

$$D^{\varrho-j_1}(\bar{b}c\bar{c}) = \sum_{j_3}^{\varrho-j_1}\binom{\varrho-j_1}{j_3} D^{j_3}\bar{b} D^{\varrho-j_1}(c\bar{c}) \qquad \text{(VI,193)}$$

$D^{j_3}\,\bar{b}$: see (VI,183)

$D^{\varrho-j_1}\,c\bar{c}$: see (VI,177)

$\underline{D^{\varrho-j_1}\,d_{15}}$:

$$D^{\varrho-j_1}\,d_{16} = \frac{1}{I_1}\,D^{\varrho-j_1}\,\frac{c}{b} \tag{VI,194}$$

$$D^{\varrho-j_1}\,\frac{c}{b} = \sum_{j_3}^{\varrho-j_1}\binom{\varrho-j_1}{j_3}\,D^{j_3}\,c\,D^{\varrho-j_1}\,\frac{1}{b} \tag{VI,195}$$

$D^{j_3}\,c$: see (VI,178)

$D^{\varrho-j_1}\,\frac{1}{b}$: see (VI,182)

$\underline{D^{\varrho-j_1}\,d_{17}}$:

$$D^{\varrho-j_1}\,d_{17} = \frac{1}{I_2}\,D^{\varrho-j_1}\,\frac{\bar{c}}{b} \tag{VI,196}$$

$$D^{\varrho-j_1}\,\frac{\bar{c}}{b} = \sum_{j_3}^{\varrho-j_1}\binom{\varrho-j_1}{j_3}\,D^{j_3}\,\bar{c}\,D^{\varrho-j_1-j_3}\,\frac{1}{b} \tag{VI,197}$$

$D^{j_3}\,\bar{c}$: see (VI,178)

$D^{\varrho-j_1-j_3}\,\frac{1}{b}$: see (VI,182)

$\underline{D^{\varrho-j_1}\,d_{21}}$:

$$D^{\varrho-j_1}\,d_{21} = -D^{\varrho-j_1}b - B_1D^{\varrho-j_1}b\bar{c}^2 + B_2D^{\varrho-j_1}bc^2 \tag{VI,198}$$

$D^{\varrho-j_1}\,b$: see (VI,183)

$$D^{\varrho-j_1}\,b\bar{c}^2 = \sum_{j_3}^{\varrho-j_1}\binom{\varrho-j_1}{j_3}D^{j_3}\,b\,D^{\varrho-j_1-j_3}\,\bar{c}^2 \tag{VI,199}$$

D^{j_3} b: see (VI,183)

$D^{\varrho-j_1-j_3}$ $\bar{c}^2$: see (VI,186)

$$D^{\varrho-j_1}(bc^2) = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3} b D^{\varrho-j_1-j_3} c^2 \qquad \text{(VI,200)}$$

D^{j_3} b: see (VI,183)

$D^{\varrho-j_1-j_3}$ c^2: see (VI,188)

$\underline{D^{\varrho-j_1} d_{22}}$:

$$D^{\varrho-j_1} d_{22} = -D^{\varrho-j_1}(\bar{b}d_{11}) \qquad \text{(VI,201)}$$

$\underline{D^{\varrho-j_1} d_{23}}$: $$D^{\varrho-j_1} d_{23} = -D^{\varrho-j_1} d_{11} \qquad \text{(VI,202)}$$

$\underline{D^{\varrho-j_1} d_{24}}$: $$D^{\varrho-j_1} d_{24} = -B_1 D^{\varrho-j_1}(b\bar{b}\bar{c}^2) + B_2 D^{\varrho-j_1}(b\bar{b}c^2) \qquad \text{(VI,203)}$$

$$D^{\varrho-j_1}(b\bar{b}\bar{c}^2) = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3}(b\bar{b}) D^{\varrho-j_1-j_3} \bar{c}^2 \qquad \text{(VI,204)}$$

D^{l_1} $\bar{c}^2$: see (VI,186)

$$D^{j_3}(b\bar{b}) = \sum_{j_4}^{j_3} \binom{j_3}{j_4} D^{j_4} b\, D^{j_3-j_4} \bar{b} \qquad \text{(VI,205)}$$

D^{j_4} b: see (VI,183)

$D^{j_3-j_4}$ $\bar{b}$: see (VI,183)

$\underline{D^{\varrho-j_1} d_{26}}$: $D^{\varrho-j_1} d_{26} = \frac{1}{I_1} D^{\varrho-j_1} \bar{c}$ see (VI,178) (VI,206)

$\underline{D^{\varrho-j_1} d_{27}}$:

$$- \frac{1}{I_2} D^{\varrho-j_1} c\text{: see (VI,178)} \tag{VI,207}$$

$\underline{D^{\varrho-j_1} d_{31}}$:

$$D^{\varrho-j_1} d_{31} = -D^{\varrho-j_1} d_{14} \tag{VI,208}$$

$\underline{D^{\varrho-j_1} d_{32}}$:

$$D^{\varrho-j_1} \frac{1}{b} + B_3 D^{\varrho-j_1} (bc^2) - B_3 D^{\varrho-j_1} (b\bar{c}^2) \tag{VI,209}$$

$D^{\varrho-j_1} \frac{1}{b}$: see (VI,180)

$$D^{\varrho-j_1} (bc^2) = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3} b D^{\varrho-j_1-j_3} c^2 \tag{VI,210}$$

$D^{j_3} b$: see (VI,183)

$D^{\varrho-j_1-j_3} c^2$: see (VI,188)

$\underline{D^{\varrho-j_1} d_{33}}$:

$$B_2 D^{\varrho-j_1} \frac{\bar{b}\bar{c}^2}{b} - B_1 D^{\varrho-j_1} \frac{\bar{b}c^2}{b} - D^{\varrho-j_1} \frac{\bar{b}}{b} = D^{\varrho-j_1} d_{33} \tag{VI,211}$$

$D^{\varrho-j_1} \frac{\bar{b}\bar{c}^2}{b}$: see (VI,185)

$D^{\varrho-j_1} \frac{\bar{b}c^2}{b}$: see (VI,187)

$D^{\varrho-j_1} \frac{\bar{b}}{b}$: see (VI,181)

$\underline{D^{\varrho-j_1} d_{34}}$: $D^{\varrho-j_1} d_{34} = -B_3 D^{\varrho-j_1} (c\bar{c}b^2) + B_1 D^{\varrho-j_1} (c\bar{c}\bar{b}^2) +$

$$+ B_2 D^{\varrho-j_1} (c\bar{c}\bar{b}^2) \tag{VI,212}$$

$$D^{\varrho-j_1}(c\bar{c}b^2) = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3}(c\bar{c})\, D^{\varrho-j_1-j_3}\, b^2 \qquad \text{(VI,213)}$$

$D^{j_3}(c\bar{c})$: see (VI,178)

$$D^{l_1} b^2 = \sum_{l_2}^{l_1} \binom{l_1}{l_2} D^{l_2}\, b\, D^{l_1-l_2}\, b \qquad \text{(VI,214)}$$

D^{l_2} b: see (VI,183)

$$D^{\varrho-j_1}(c\bar{c}b^2): = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3}\, c\bar{c}\, D^{\varrho-j_1-j_3}\, b^2 \qquad \text{(VI,215)}$$

$$D^{\varrho-j_1-j_3}\, b^2 = \sum_{j_4}^{\varrho-j_1-j_3} \binom{\varrho-j_1-j_3}{j_4} D^{j_4}\, b\, D^{\varrho-j_1-j_3}\, b \qquad \text{(VI,216)}$$

D^{j_4} b: see (VI,183)

D^{j_3} $c\bar{c}$: see (VI,177)

$D^{\varrho-j_1}$ d_{35}:

$$D^{\varrho-j_1}\, d_{35} = B_3 D^{\varrho-j_1}(c\bar{c}) \text{ see (VI,177)} \qquad \text{(VI,217)}$$

$D^{\varrho-j_1}$ d_{36}:

$$D^{\varrho-j_1}\, d_{36} = 0 \qquad \text{(VI,218)}$$

$D^{\varrho-j_1}$ d_{37}:

$$D^{\varrho-j_1}\, d_{37} = -\frac{1}{I_1} D^{\varrho-j_1} \frac{c\bar{b}}{b} \qquad \text{(VI,219)}$$

$$D^{\varrho-j_1} \frac{c\bar{b}}{b} = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3}\, c\, D^{\varrho-j_1-j_3} \frac{\bar{b}}{b} \qquad \text{(VI,220)}$$

D^{j_3} c: see (VI,178)

$D^{\varrho-j_1-j_3} \frac{\bar{b}}{b}$: see (VI,181)

$D^{\varrho-j_1} d_{38}$: $-\frac{1}{I_2} D^{\varrho-j_1} \frac{\bar{c}\bar{b}}{b}$ (VI,221)

$$D^{\varrho-j_1} \frac{\bar{c}\bar{b}}{b} = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{\varrho-j_1-j_3} \bar{c} \, D^{j_3} \frac{\bar{b}}{b} \qquad \text{(VI,222)}$$

$D^{j_3} \frac{\bar{b}}{b}$: see (VI,181)

$D^{l_1} \bar{c}$: see (VI,178)

The recurrence formulas for $D^{j_1} N_i$ $(i=1,2,3)$ can be derived in analogy to the expressions presented above.

$D^{j_1} N_i$:

$D^{j_1} N_1$:

$$D^{j_1} N_1 = u^2 T_{23} D^{j_1}(3a_{32}a_{33} - a_{22}a_{23}) - 2uT_2 D^{j_1}(a_{22}\dot{\theta}_3) +$$

$$+ 2uT_3 D^{j_1}(a_{23}\dot{\theta}) \qquad \text{(VI,223)}$$

$$D^{j_1}(a_{32}a_{33}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{32} D^{j_1-j_3} a_{33} \qquad \text{(VI,224)}$$

$$D^{j_3} a_{32} = D^{j_3}(\bar{c}b) = \sum_{j_4}^{j_3} \binom{j_3}{j_4} D^{j_4} \bar{c} \, D^{j_3-j_4} b \qquad \text{(VI,225)}$$

$D^{j_4} \bar{c}$: see (VI,178); $D^{j_3-j_4} b$: see (VI,183)

$D^{j_1-j_3} a_{33} = D^{j_1-j_3} \bar{b}$: see (VI,183) (VI,226)

$$D^{j_1}(a_{22}a_{23}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{22} \, D^{j_1-j_3} a_{23} \qquad \text{(VI,227)}$$

$$D^{j_3} a_{22} = D^{j_3}(\bar{c}a + c\bar{a}\bar{b}) = D^{j_3}(\bar{c}a) + D^{j_3}(c\bar{a}\bar{b}) \qquad \text{(VI,228)}$$

$$D^{j_3}(\bar{c}a) = \sum_{j_4}^{j_3} \binom{j_3}{j_4} D^{j_4} \bar{c} D^{j_3-j_4} a \tag{VI,229}$$

$D^{j_4} \bar{c}$: see (VI,178),

$$D^{j_3-j_4} a = D^{l_1} a = D^{l_1-1}(\bar{a}z_4) = \\ = \sum_{l_2}^{l_1-1} \binom{l_1-1}{l_2} D^{l_2} \bar{a}D^{l_1-1-l_2} z_4 \tag{VI,230}$$

$$D^{l_1} \bar{a} = -D^{l_1-1}(az_4) = -\sum_{l_2}^{l_1-1} \binom{l_1-1}{l_2} D^{l_2} aD^{l_1-1-l_2} z_4 \tag{VI,231}$$

$$D^{j_3}(\bar{a}\bar{b}c) = \sum_{j_4}^{j_3} \binom{j_3}{j_4} D^{j_4} (\bar{a}\bar{b}) D^{j_3-j_4} c \tag{VI,232}$$

$D^{j_3-j_4} c$: see (VI,178)

$$D^{j_4}(\bar{a}\bar{b}) = \sum_{j_5}^{j_4} \binom{j_4}{j_5} D^{j_5} \bar{a}D^{j_4-j_5} \bar{b} \tag{VI,233}$$

$D^{j_5} \bar{a}$: see (VI,231) and (VI,232)

$D^{j_4-j_5} \bar{b}$: see (VI,183) (VI,234)

$$D^{j_2-j_3} a_{23} = -D^{j_2-j_3} \bar{a}b = -\sum_{j_4}^{j_2-j_3} \binom{j_2-j_3}{j_4} D^{j_4} \bar{a}D^{j_2-j_4} \tag{VI,235}$$

$D^{j_4} \bar{a}$: see (VI,234), $D^{j_2-j_3-j_4} b$: see (VI,183) (VI,236)

$$D^{j_1}(a_{22}\dot{\theta}_3) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{22} D^{j_1-j_3} \dot{\theta}_3 \tag{VI,237}$$

$D^{j_3} a_{22}$: see (VI,227) (VI,238)

$$D^{j_1-j_3}\,\dot{\theta}_3 = D^{j_1-j_3}(z_4\bar{b} + z_6) =$$

$$= \sum_{j_4}^{j_1-j_3} \binom{j_1-j_3}{j_4} D^{j_4} z_4 D^{j_1-j_3-j_4}\,\bar{b} + D^{j_1-j_3} z_6 \qquad (VI,239)$$

$$D^{j_1-j_3-j_4}\,\bar{b}\text{: see (VI,183)} \quad j_4 \leq \varrho,\ j_1-j_3 \leq \varrho \qquad (VI,240)$$

$$D^{j_1}(a_{23}\dot{\theta}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3}\, a_{23}\, D^{j_1-j_3}\,\dot{\theta}_2 \qquad (VI,241)$$

$$D^{j_3}\, a_{23}\text{: see (VI,175)} \qquad (VI,242)$$

$$D^{j_1-j_3}\,\dot{\theta}_2 = D^{j_1-j_3}(z_4\bar{c}b - z_5 c) =$$

$$= \sum_{j_4}^{j_1-j_3} \binom{j_1-j_3}{j_4} \left\{ D^{j_1-j_3}(z_4\bar{c}b) + D^{j_1-j_3}(z_5 c) \right\} \qquad (VI,243)$$

$$D^{j_1-j_3}(z_4\bar{c}b) = \sum_{j_5}^{j_1-j_3} \binom{j_1-j_3}{j_5} D^{j_5}(\bar{c}b) D^{j_1-j_3-j_5}\, z_4 \qquad (VI,244)$$

$$j_1-j_3-j_5 \leq \varrho$$

$$D^{j_5}(\bar{c}b) = \sum_{j_6}^{j_5} \binom{j_5}{j_6} D^{j_6}\,\bar{c}\, D^{j_5-j_6}\, b \qquad (VI,245)$$

$$D^{j_6}\,\bar{c}\text{: see (VI,178);} \quad D^{j_5-j_6}\, b\text{: see (VI,183)} \qquad (VI,246)$$

$$D^{j_1-j_3}(z_5 c) = \sum_{j_4}^{j_1-j_3} \binom{j_1-j_3}{j_4} D^{j_4}\, z_5\, D^{j_1-j_3-j_4}\, c \qquad (VI,247)$$

$$D^{j_4}\, c\text{: see (VI,178)} \qquad (VI,248)$$

$$j_4 \leq \varrho$$

$D^{j_1}N_2$:

$$D^{j_1}N_2 = D^{j_1}\Big\{(3a_{31}a_{33} - a_{21}a_{23})\,u^2T_{31} +$$

$$+ 2u(T_1a_{21}\dot{\theta}_3 - T_3a_{23}\dot{q})\Big\} \qquad (VI,249)$$

$$D^{j_1}(a_{31}a_{33}) = \sum_{j_3}^{j_1}\binom{j_1}{j_3} D^{j_3}a_{31}\, D^{j_1-j_3}a_{33} \qquad (VI,250)$$

$$D^{j_3}a_{31} = D^{j_3}(cb) = \sum_{j_4}^{j_3}\binom{j_3}{j_4} D^{j_4}c\, D^{j_3-j_4}b \qquad (VI,251)$$

$D^{j_4}c$: see (VI,178); $D^{j_3-j_4}b$: see (VI,183) (VI,252)

$D^{j_1-j_3}a_{33}$: see (VI,226) (VI,253)

$$D^{j_1}(a_{22}a_{23}) = \sum_{j_3}^{j_1}\binom{j_1}{j_3} D^{j_3}a_{22}\, D^{j_1-j_3}a_{23} \qquad (VI,254)$$

$D^{j_1-j_3}a_{23}$: see (VI,235) (VI,255)

$D^{j_3}a_{22}$: see (VI,228) (VI,256)

$$D^{j_3}(a_{21}\dot{\theta}_3) = \sum_{j_4}^{j_3}\binom{j_3}{j_4} D^{j_4}a_{21}\, D^{j_3-j_4}\dot{\theta}_3 \qquad (VI,257)$$

$D^{j_3-j_4}\dot{\theta}_3$: see (VI,239) (VI,258)

$$D^{j_4}a_{21} = D^{j_4}(\bar{c}a + c\bar{a}\bar{b}) =$$

$$= \sum_{j_5}^{j_4}\binom{j_4}{j_5}\; D^{j_4}(\bar{c}a) + D^{j_4}(c\bar{a}\bar{b}) \qquad (VI,259)$$

$D^{j_4}(\bar{c}a)$: see (VI,229); $D^{j_4}(c\bar{a}\bar{b})$: see (VI,230) (VI,260)

$$D^{j_1}(a_{23}\dot{\theta}_1) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{23} D^{j_1-j_3} \dot{\theta}_1 \qquad \text{(VI,261)}$$

$$D^{j_3} a_{23}\text{: see (VI,235)} \qquad \text{(VI,262)}$$

$$D^{j_1-j_3} \dot{\theta}_1 = D^{j_1-j_3}(z_4 cb + z_5\bar{c}) =$$

$$= D^{j_1-j_3}(z_4 cb) + D^{j_1-j_3}(z_5\bar{c}) \qquad \text{(VI,263)}$$

$$D^{j_1-j_3}(z_4 cb) = \sum_{j_4}^{j_1-j_3} \binom{j_1-j_3}{j_4} D^{j_1-j_3} z_4 D^{j_1-j_3-j_4}(cb) \qquad \text{(VI,264)}$$

$$j_1-j_3 \leq \varrho$$

$$D^{j_1-j_3-j_4}(cb) = D^{l_1}(cb) = \sum_{l_2}^{l_1} \binom{l_1}{l_2} D^{l_2} c D^{l_1-l_2} b \qquad \text{(VI,265)}$$

$$D^{l_2} c\text{: see (VI,178);} \quad D^{l_1-l_2} b\text{: see (VI,183)} \qquad \text{(VI,266)}$$

$$D^{j_1-j_3}(z_5\bar{c}) = \sum_{j_4}^{j_1-j_3} \binom{j_1-j_3}{j_4} D^{j_4} z_5 D^{j_1-j_3-j_4} \bar{c} \qquad \text{(VI,267)}$$

$$j_4 \leq \varrho$$

$$D^{j_1-j_3-j_4} \bar{c}\text{: see (VI,178)} \qquad \text{(VI,268)}$$

$D^{j_1}N_3$:

$$D^{j_1}N_3 = D^{j_1}\Big\{(3a_{31}a_{32} - a_{21}a_{22})u^2T_{12} +$$

$$+ 2u(-T_1a_{21}\dot{\theta}_2 + T_2a_{22}\dot{\theta}_1)\Big\} \qquad \text{(VI,269)}$$

$$D^{j_1}(a_{31}a_{32}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{31} D^{j_1-j_3} a_{32} \qquad \text{(VI,270)}$$

$$D^{j_3} a_{31}\text{: see (VI,251);} \quad D^{j_1-j_3} a_{32}\text{: see (VI,225)} \qquad \text{(VI,271)}$$

$$D^{j_1}(a_{21}a_{22}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{21} D^{j_1-j_2} a_{22} \qquad \text{(VI,272)}$$

$D^{j_3} a_{21}$: see (VI,259); $D^{j_1-j_2} a_{22}$: see (VI,227) (VI,273)

$$D^{j_1}(a_{21}\dot{\theta}_2) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{21} D^{j_1-j_3} \dot{\theta}_2 \qquad \text{(VI,274)}$$

$D^{j_3} a_{21}$: see (VI,259); $D^{j_1-j_3} \dot{\theta}_2$: see (VI,243) (VI,275)

$$D^{j_1}(a_{22}\dot{\theta}_1) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{22} D^{j_1-j_3} \dot{\theta}_1 \qquad \text{(VI,276)}$$

$D^{j_3} a_{22}$: see (VI,228); $D^{j_1-j_3} \dot{\theta}_1$: see (VI,263) (VI,277)

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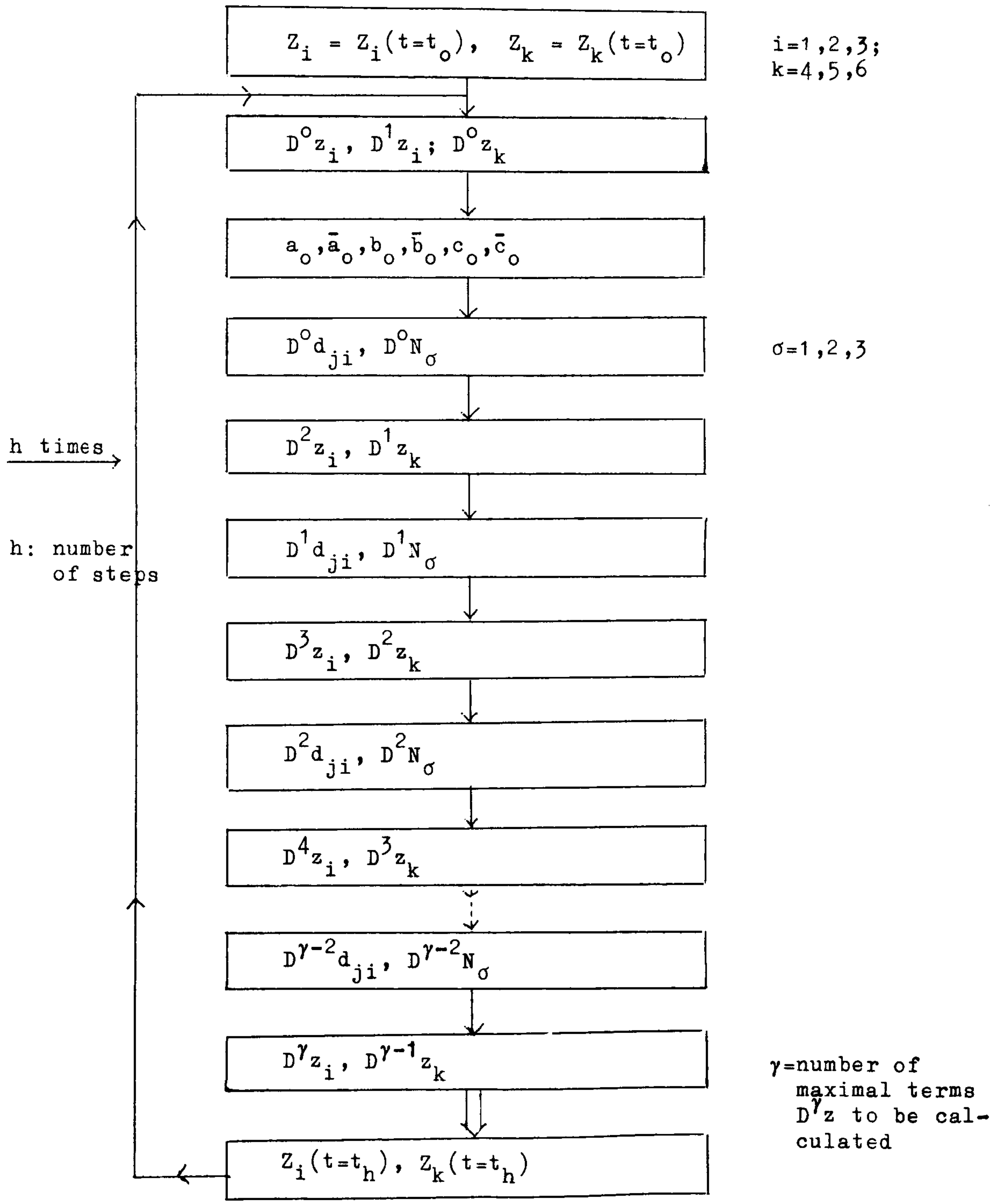

(6.34) Application of the iteration method to the equations (VI,155)

Essentially we have to solve the following system

$$\begin{aligned} \dot z_1 &= z_2 & \dot z_4 &= f_2 \\ \dot z_2 &= f_1 & \dot z_5 &= z_6 \\ \dot z_3 &= z_4 & \dot z_6 &= f_3 \end{aligned} \tag{VI,278}$$

For a circular orbit the explicit form of f_i $(i=1,2,3)$ is given in Eq.(VI,155). The operator D appearing in Eq.(III,74) reads in that case

$$D = \sum_{i=1}^{6} \dot z_i \frac{\partial}{\partial z_i} \tag{VI,279}$$

We proceed now to the construction of $\hat z(t)$ (see Eq.(III,85)). This problem is from the numerical point of view a very important one, because the computer time and the maximum accuracy depend very strongly on the choice of $\hat z(t)$. We are not experts on finding such effective functions, let alone on finding the most effective one. Nevertheless we would like to propose such a function $\hat z(t)$. We choose

$$\hat z(t) = \sum_{\varrho=0}^{N} \frac{t^\varrho}{\varrho!} D^\varrho z(t=t_o).$$

Here $D^\varrho z$ can be calculated either by recurrence formulas (see Sect. (6.33)) or by applying the operator D explicitly (see Sect.(6.32)). The latter procedure yields very complex expressions for $\varrho > 2$.

According to Eq.(III,75) the operator ${}_\lambda D_2$ reads in our case

$${}_\lambda D_2 = \sum_{i=1}^{6} (\dot z_i(t) - {}_\lambda\dot{\hat z}_i(t)) \frac{\partial}{\partial z_i} \tag{VI,280}$$

In the interest of clarity we describe the first iteration in more detail. For this iteration $\lambda = o$. Using the denotation $\frac{\partial}{\partial z_i} = \partial_i$, ${}_{\lambda=o}D_2$ reads explicitly

$$
{}_oD_2 = (\dot z_1 - {}_o\dot{\hat z}_1)\partial_1 + (\dot z_2 - {}_o\dot{\hat z}_2)\partial_2 + (\dot z_3 - {}_o\dot{\hat z}_3)\partial_3 + (\dot z_4 - {}_o\dot{\hat z}_4)\partial_4 +
$$
$$
+ (\dot z_5 - {}_o\dot{\hat z}_5)\partial_5 + (\dot z_6 - {}_o\dot{\hat z}_6)\partial_6 = (z_2 - {}_o\dot{\hat z}_1)\partial_1 + (f_1 - {}_o\dot{\hat z}_2)\partial_2 +
$$
$$
+ (\dot z_4 - {}_o\dot{\hat z}_3)\partial_3 + (f_2 - {}_o\dot{\hat z}_4)\partial_4 + (z_6 - {}_o\dot{\hat z}_5)\partial_5 + (f_3 - {}_o\dot{\hat z}_6)\partial_6
$$

(VI,281)

According to Chapter I we have

$$
z(t) = D^o z(t=t_o) + tD^1 z(t=t_o) + \frac{t^2}{2} D^2 z(t=t_o) + \sum_{\varrho=3}^{\infty} \frac{t^\varrho}{\varrho!} D^\varrho z(t=t_o) =
$$
$$
= \hat z(t) + \sum_{\varrho=3}^{\infty} \frac{t^\varrho}{\varrho!} D^\varrho z(t=t_o), \text{ i.e.,}
$$

we choose for $\hat z(t)$

$$
\hat z(t) = D^o z(t=t_o) + tD^1 z(t=t_o) + \frac{t^2}{2} D^2 z(t=t_o) \qquad \text{(VI,282)}
$$

With the Eqs.(VI,278) and (VI,279) we obtain for $\hat z = {}_o\hat z$

$$
\begin{aligned}
{}_o\hat z_1(t) &= z_1(t=t_o) + tz_2(t=t_o) + \frac{t^2}{2!} f_1 \\
{}_o\hat z_2(t) &= z_2(t=t_o) + tf_1(t=t_o) + \frac{t^2}{2!} D^1 f_1 \\
{}_o\hat z_3(t) &= z_3(t=t_o) + tz_4(t=t_o) + \frac{t^2}{2!} f_2 \\
{}_o\hat z_4(t) &= z_4(t=t_o) + tf_2(t=t_o) + \frac{t^2}{2!} D^1 f_2 \\
{}_o\hat z_5(t) &= z_5(t=t_o) + tz_6(t=t_o) + \frac{t^2}{2!} f_3 \\
{}_o\hat z_6(t) &= z_6(t=t_o) + tf_3(t=t_o) + \frac{t^2}{2!} D^1 f_3
\end{aligned}
\qquad \text{(VI,283)}
$$

With the Eqs.(VI,281), (VI,282) and (VI,283) ${}_oD_2$ reads

$$\begin{aligned}
{}_oD_2 = {} & (z_2(t) - z_2(t=t_o) - tf_1(t=t_o))\partial_1 + \\
& + (f_1(t) - f_1(t=t_o) - tD^1 f_1(t=t_o))\partial_2 + \\
& + (z_4(t) - z_4(t=t_o) - tf_2(t=t_o))\partial_3 + \\
& + (f_2(t) - f_2(t=t_o) - tD^1 f_2(t=t_o))\partial_4 + \\
& + (z_6(t) - z_6(t=t_o) - tf_3(t=t_o))\partial_5 + \\
& + (f_3(t) - f_3(t=t_o) - tD^1 f_3(t=t_o))\partial_6
\end{aligned} \qquad \text{(VI,284)}$$

Putting $\alpha \leqq 2$ Eq.(III,85) reads

$$\begin{aligned}
{}_1g_i = {} & h_i({}_oz_i(t),t) + \int_{t_o}^{t} \left[{}_oD_2D^1 z_i\right]_{{}_o\hat{z}_i(\tau),\tau} d\tau + \\
& + \int_{t_o}^{t} (t-\tau) \left[{}_oD_2D^2 z_i\right]_{{}_o\hat{z}_i(\tau),\tau} d\tau, \text{ where}
\end{aligned} \qquad \text{(VI,285)}$$

in our case $h({}_o\hat{z}_i(t),t)$ is given by

$$\begin{aligned}
h({}_oz_1(t),t) &= {}_o\dot{\hat{z}}_1(t) \\
h({}_oz_1(t),t) &= f_1({}_o\hat{z}, {}_o\hat{z}_2, \ldots\ldots, {}_o\hat{z}_6) \\
h({}_oz_3(t),t) &= {}_o\dot{\hat{z}}_3(t) \\
h({}_oz_4(t),t) &= f_2({}_o\hat{z}_1, {}_o\hat{z}_2, \ldots, {}_o\hat{z}_6) \\
h({}_oz_5(t),t) &= {}_o\dot{\hat{z}}_5(t) \\
h({}_oz_6(t),t) &= f_3({}_o\hat{z}_1, {}_o\hat{z}_2, \ldots, {}_o\hat{z}_6)
\end{aligned} \qquad \text{(VI,286)}$$

and for $\left[{}_oD_2D^1 z_i\right]_{{}_o\hat{z}_i(\tau),\tau}$ we obtain the relations

$$\left[{}_oD_2D^1z_1\right]_{{}_o\hat{z}_i(\tau),\tau} = \left[{}_oD_2z_2\right]_{{}_o\hat{z}_i(\tau),\tau} =$$

$$= \left[f_1(z(t),t) - f_1(z(t=t_o),t_o) - tDf_1\right]_{{}_o\hat{z}_i(\tau),\tau} =$$

$$= f_1({}_o\hat{z}_1(\tau),\ {}_o\hat{z}_2(\tau),\ \dots,\ {}_o\hat{z}_6(\tau)) -$$

$$- f_1(z_1(t=t_o),\ \dots,\ z_6(t=t_o),t_o) - \tau\left[Df_1\right]_{{}_o\hat{z}_i(\tau),\tau}$$

$$\left[{}_oD_2D^1z_2\right]_{{}_o\hat{z}_i(\tau),\tau} = \left[{}_oD_2f_1\right]_{{}_o\hat{z}_i(\tau),\tau}$$

$$\text{- -} \qquad \text{(VI,287)}$$

$$\left[{}_oD_2D^1z_6\right]_{{}_o\hat{z}_i(\tau),\tau} = \left[{}_oD_2f_3\right]_{{}_o\hat{z}_i(\tau),\tau}$$

and for $\left[{}_oD_2D^2z_i\right]_{{}_o\hat{z}_i(\tau),\tau}$ we obtain

$$\left[{}_oD_2D^2z_1\right]_{{}_o\hat{z}_i(\tau),\tau} = \left[{}_oD_2f_1\right]_{{}_o\hat{z}_i(\tau),\tau} \qquad \text{(VI,288)}$$

$$\text{- -}$$

$$\left[{}_oD_2D^2z_6\right]_{{}_o\hat{z}_i(\tau),\tau} = \left[{}_oD_2D^1f_3\right]_{{}_o\hat{z}_i(\tau),\tau}$$

With Eqs.(VI,285), (VI,286), (VI,287) and (VI,288) ${}_1g_i$ reads

$${}_1g_1 = {}_o\dot{\hat{z}}_1(t) + \int_{t_o}^{t} (f_1({}_o\hat{z}_1(\tau),\ {}_o\hat{z}_2(\tau),\ \dots,\ {}_o\hat{z}_6(\tau),\tau) -$$

$$- f_1(z_1(t=t_o)\ \dots,\ z_6(t=t_o),t_o) -$$

$$- \tau\left[Df_1\right]_{{}_o\hat{z}_i(\tau),\tau})d\tau +$$

$$+ \int_{t_o}^{t} (t-\tau)\left[{}_oD_2f_1\right]_{{}_oz_i(\tau),\tau} d\tau$$

or

$$ {}_1g_1 = {}_o\dot{\hat{z}}_1(t) + \int_{t_o}^{t} (f_1({}_o\hat{z}(\tau),\tau) - f_1(z(t=t_o) - $$

$$ - \tau\left[Df_1\right]_{{}_o\hat{z}_i(\tau),\tau})d\tau + \int_{t_o}^{t} (t-\tau)\left[{}_oD_2f_1\right]_{{}_o\hat{z}_i(\tau),\tau} d\tau $$

$$ {}_1g_2 = f_1({}_o\hat{z}(t),t) + \int_{t_o}^{t} \left[{}_oD_2f_1\right]_{{}_o\hat{z}\ (\tau),\tau} d\tau + $$

$$ + \int_{t_o}^{t} (t-\tau) \left[{}_oD_2Df_1\right]_{{}_o\hat{z}(\tau),\tau} d\tau $$

$$ {}_1g_3 = {}_o\dot{\hat{z}}_3(t) + \int_{t_o}^{t} (f_2({}_o\hat{z}(\tau),\tau) - f_2(z(t=t_o)) - $$

$$ - \tau\left[Df_2\right]_{{}_o\hat{z}_i(\tau),\tau} d\tau + \int_{t_o}^{t} (t-\tau)\left[Df_2\right]_{{}_o\hat{z}_i(\tau),\tau} d\tau $$

$$ {}_1g_4 = f_2({}_o\hat{z}(t),t) + \int_{t_o}^{t} \left[{}_oD_2f_2\right]_{{}_o\hat{z}(\tau),\tau} d\tau + $$

$$ + \int_{t_o}^{t} (t-\tau) \left[{}_oD_2Df_2\right]_{{}_o\hat{z}(\tau),\tau} d\tau $$

$$ {}_1g_5 = {}_o\dot{\hat{z}}_5(t) + \int_{t_o}^{t} (f_3({}_o\hat{z}(\tau),\tau - f_3(z(t=t_o)) - $$

$$ - \tau\left[Df_3\right]_{{}_o\hat{z}_i(\tau),\tau})d\tau + \int_{t_o}^{t} (t-\tau)\left[Df_3\right]_{{}_o\hat{z}_i(\tau),\tau} d\tau $$

$$ {}_1g_6 = f_3({}_o\hat{z}(t),t) + \int_{t_o}^{t} \left[Df_3\right]_{{}_o\hat{z}_i(\tau),\tau} d\tau + $$

$$ + \int_{t_o}^{t} (t-\tau) \left[{}_oD_2Df_3\right]_{{}_o\hat{z}_i(\tau),\tau} d\tau $$

With Eq.(III,73) we have

$$ {}_1\hat{z}_i(t) = z_i(t=t_o) + \int_{t_o}^{t} g(\tau)\, d\tau \qquad (i=1,2,\ldots,6) $$

The second iteration ($\lambda = 1$) yields for ${}_{\lambda=2}g_{i=1}$, e.g.,

$$ {}_2g_1 = {}_1\dot{\hat{z}}_1 + \int_{t_o}^{t} \Big(f_1({}_1\hat{z}(\tau),\tau) - f_1(z(t=t_o) - $$

$$ - \tau \Big[Df_1 \Big]_{{}_1\hat{z}_i(\tau),\tau} \Big) d\tau + \int_{t_o}^{t} (t-\tau) \Big[{}_1D_2 f_1 \Big]_{{}_1\hat{z}_i(\tau),\tau} \, d\tau $$

and for $\lambda = \varrho+1$ (i.e., ϱ iterations)

${}_\lambda g_i$ reads for $i = 1$

$$ {}_{\varrho+1}g_1 = {}_\varrho\dot{\hat{z}}_1 + \int_{t_o}^{t} \Big(f_1({}_\varrho\hat{z}(\tau),\tau) - f_1(z(t=t_o) - $$

$$ - \tau \Big[Df_1 \Big]_{{}_\varrho\hat{z}_i(\tau),\tau} \, d\tau + \int_{t_o}^{t} (t-\tau) \Big[{}_\varrho D_2 f_1 \Big]_{{}_\varrho\hat{z}_i(\tau),\tau} \, d\tau $$

In our case (see Sect.(6.32)) the integrations appearing in the above formulas can be carried out analytically.

(6.4) The Numerical Evaluation

(6.41) Now, the problem of solving (VI,131) with $\vec{N}$ from (VI,154) was given. For the purpose of information, Lie series

$$\begin{aligned} \alpha(t_o+t) &= \alpha_o + t\cdot D\alpha_o + \frac{t^2}{2}\cdot D^2\alpha_o + \frac{t^3}{6}\cdot D^3\alpha_o + \dots \\ \dot{\alpha}(t_o+t) &= \qquad D\alpha_o + t\cdot D^2\alpha_o + \frac{t^2}{6}\cdot D^3\alpha_o + \dots \end{aligned} \qquad \text{(VI,289)}$$

etc., broken off after a few terms, were considered, see (6.32). Unfortunately, by lack of time, we could not start an iteration process corresponding to (6.34) with (VI,289). The computation of $D^3\alpha_o$ was already rather difficult (see (VI,166 b)); hence the direct computation of the next terms $D^k\alpha_o$ would not be recommendable, but it might be suitable to use recurrence formulas (see (6.33)) together, with $D^k\alpha_o$, $k = 0,1,2,3$. Using such an approximation even one iteration step may be sufficient to provide a very satisfactory result.

(6.42) The fact that, when using (VI,131) with (VI,154) to describe the satellite's attitude motion, the singularities $b = \sin\beta = 0$ coincided with the two points where the satellite should be at rest give rise to a number of serious objections: Simply speaking, the better the stabilization, the worse the corresponding result. Nevertheless, we solved these equations, since (a) we would not renounce this most intuitive choice of Euler angles (β is the deviation of a satellite symmetrical to $\vec{i}_3$ from the local vertical), (b) we had no time for another procedure, as, e. g., the introduction of the Cayley variables, and (c) we considered it to be an improbable event that the satellite should approach the critical domains around $\beta = 0, \pi$ closely enough, just in our examples. These facts and the very small step length caused by them ($\frac{1}{20}$ degree of the angle in the orbit, i. e., more than 7000 analytical continuations per orbital revolution) made the results appear somewhat doubtful; but Lie series long before this proved to be very suitable in such critical cases.

(6.43) After some other computations described below, we started calculating the motion of a satellite with $J^{11} = J^{22} = 96030\ kg \cdot m^2$, $J^{33} = 60\ kg \cdot m^2$, height 1667 km about the earth's surface. The initial values are $\alpha_o = 0$, $\beta_o = 2{,}5^o$, $\gamma_o = 45^o$, $\dot{\alpha}_o = \dot{\gamma}_o = 5\ ^o/min$, $\dot{\beta}_o = 0$. $\beta(\mu)$ and $\dot{\beta}(\mu) = \frac{d\beta}{dt}$ with $\mu := (t-t_o) \cdot /\vec{u}_B/$ are plotted in Fig. 3 from chapter VI (instead of $\dot{\beta}$ is given $10\dot{\beta}$). The figure shows that β is bounded by $0 < \beta < 3{,}5^o$. We interpret this as a stabilization effect. The behavior in time of β suggests a resonance effect with the motion in the orbit (a period of two orbital revolutions), but as we presumed an exact circular orbit no coupling of the two motions is possible. But it is just this periodic behavior which makes us believe that the results are astonishingly good, under the circumstrances given. It is by no means understandable that a wrong result should produce such a clear period and reproduce the initial value of β: $\beta(\mu=4\pi) = \beta(0) = 2{,}5^o$.

(6.44) At first, the "horizontal" case was treated in the test calculations: $\alpha_o = \gamma_o = 0 = \dot{\alpha}_o = \dot{\beta}_o = \dot{\gamma}_o$, $\beta = 90^o$: The first coefficients of the Lie series solution vanish, and the assumption that this result should also hold for all following terms is straightforward. Consequently, all quantities would have to remain unchanged. Accordingly, this case was used to study the error propagation as a function of the step size (2 to 6 revolutions). The result was that the step size should not exceed $0{,}5^o$ and possibly be considerably smaller, if the solution approaches a singularity. For the rest of calculations we usually chose a step size of $0{,}05^{\circ}$.

(6.45) As a second example - strictly speaking, for the purpose of information - we considered a satellite with $J_{11} = J_{22} = 10363$ kg·m^2, $J_{33} = 60$ kg·m^2 and $\alpha_o = \gamma_o = 0 = \dot{\beta}_o = \dot{\gamma}_o$, $\beta_o = 45^{\circ}$, $\dot{\alpha}_o = 15$ $^{\circ}$/min. The axis of symmetry pendulated between $\beta = 5^{\circ}$ and $\beta = 175^{\circ}$, and $\dot{\alpha}$ and $\dot{\gamma}$ showed the expected increasing behavior close to the singularities: $\dot{\alpha} = (cu_s^1 + \bar{c}u_s^2)/b$, $\dot{\gamma} = -\bar{b}\dot{\alpha} + u_s^3$; if we assume that also close to the singularities the satellite's behavior is "physically meaningful", i. e., that $\vec{u}_s^2$ remains restricted to reasonable values, $\dot{\alpha}$ increases strongly because of $b = \sin\beta \longrightarrow 0$ and $\dot{\gamma}$ tends to $-\dot{\alpha}$. On the basis of these results the case discussed in (6.13) was then calculated.*

D. Floriani is indebted to Dr. Knapp, Docent in the Institute for Computation Techniques, University of Innsbruck, for a number of valuable discussions.

* Tables of numerical results are available upon request from Chief Applied Mathematics (RRA), Research Division, NASA Headquarters.

(6.5) Appendix; Conclusion

First we will deal with some more papers concerning the attitude motion of satellites.

Bounds on the Librations of a Symmetrical Satellite (Ref. 128): In this paper the case of a rigid, symmetrical satellite moving in a circular orbit was taken as basis. This corresponds to our specialization in (6.312) with $\vec{u}_B^2 = n^2$, $J_{33} = C$, $J_{11} = A = J_{22}$. Two systems are used: the orbit system $\left\{\vec{1},\vec{2},\vec{3}\right\}$ (equal to $\left\{\vec{i}_3',\pm\vec{i}_1',\mp\vec{i}_2'\right\}$ from (VI,123)) and the system of main axis $\left\{\vec{1b},\vec{2b},\vec{3b}\right\}$ (our system II $\left\{\vec{i}_1,\vec{i}_2,\vec{i}_3\right\}$ from VI, Fig. 2). To avoid singularities in points of interest two sets of variables are used for transformation: (a) φ,Θ,Ψ: $\cos\varphi = \mp a_{22}'$, $\cos\Theta = \pm a_{11}'$ in (VI,126), $\Psi = \gamma$; (b) Θ_1,Θ_2,Ψ: $\cos\Theta_1 = \pm a_{12}'$, $\cos\Theta_2 = a_{31}'$, in (VI,126), $\Psi = \gamma$. The Hamiltonian describing the rotation is splitted up in one term R_2, containing the angular velocities, and another term U (the "dynamical potential") free of them. In the following by help of U these values (φ,Θ) or (Θ_1,Θ_2), resp., are sought, for which the axis of symmetry is at rest in $\left\{\vec{1},\vec{2},\vec{3}\right\}$. Such an $\left(\frac{\text{spin speed}}{\text{orbit ang. vel.}}\right.$ against $\left.\frac{C}{A} = \frac{J_{33}}{J_{11}}\right)$ - - diagramm is subdivided into seven domains with different nature of stability. These seven cases are graphically discussed.

Tumbling Motions of an Artificial Satellite (Ref. 129): In this work the motion of a satellite with moving internal parts effected by gravity gradient torques is treated. These torques cause relative motions of the individual parts and produce internal friction destroying rotational energy. Thus the satellite's tumbling motion by and by decreases until he is captured into librational motion. This type of motion may occur after separation from the booster rocket or after collision with a meteorite. As a simple model a satellite is considered with two internal, symmetrical inertial wheels, one arranged to rotate about the body's main axis of inertia, the other about a transverse principal axis. To investigate the satellite's motion a kind of perturbational method is used, since the direct digital solution of the equations of motion seems to be very costly in time and in round - off errors. Thus one obtains a set of non-linear differential equations for the averaged motion the integration of which is essential easier. Then results are discussed; it is, e. g., found that, for tumbling angular rates greater than three times the mean orbit angular rate, the time to capture increases as the cube of the initial rate.

Stability of Damped Mechanical Systems (Ref. 13o): The question for the stability behaviour of damped, mechanical systems is of great importance, for the investigation of the motion of space vehicles. In the work considered here for some systems their Hamiltonians prove to be useful "test-functions" for application of Ljapunov's method. In the important case of gyroscopic systems there is a principal difference between the Hamiltonian belonging to and the total energy. After a theorem demonstrable by Ljapunov's method three corollaries are stated. The second of them, e. g., gives the important result that the behavior in stability does not depend upon the magnitude or analytical form of the power on the premises demanded.

On the Stability of a Body with Connected Moving Parts (Ref. 131): The stability behaviour of bodies with holonomically constrained moving parts, i.e., mechanical systems with internal damping, is investigated. After some preliminary definitions the equations of motion of such a system are considered. Assumption is, that the attitude motions do not effect the trajectory of the center of mass. By help of the direct method of Ljapunov a general method to determine stability problems is discussed and first applied to the stability of damped mechanical systems, then to free nongyroscopic systems. Necessary and sufficient conditions for (asymptotic) stability are given. Some special cases are discussed as illustration to the theorems derived. More in detail the case of a spinning asymmetrical body damped by a control moment gyro is treated. The general theory is also applicable to non-linear systems, and it gives bounds on the configuration space convergence regions.

Analytical Methods for Practical Investigations on Attitude-Controlled Satellites (Ref. 132): Assuming the orbit of the centre of mass to be an ellipse in an invariant plane torques acting on the satellite are discussed as follows: gravitational torques (but only in the lowest approximation as used by us in (6.312)), aerodynamic torques, magnetic torques (due to a permanent magnet), hysteresis damping (linear approach combined with a sort of perturbation method). For the followig the differences between the satellite's actual attitude and the attitude desired are supposed to be small. These deviations are taken as variables in the equations of motion given in the following. Expansion to these small quantities simplifies the equations. Finally, one obtains in the Eulerian angles three linear differential equations of second order with periodic coefficients (t is replaced by the satellite's true anomally). In the next paragraphs, a method for analytical treatment of such equations is described and then used to investigate (a) the stability of angular motion, (b) the complete solution for the stationary angular motion along the orbit, (c) the response of the satellite to deterministic and (d) to random disturbances. The first German satellite, 625-A-1, is taken as example.

Nonlinear Resonances Affecting Gravity Gradient Stability (Ref. 133): This work deals with the influence of non-linear resonances on the attitude librations of an undamped rigid satellite. The only torques considered are due to gravitational effects. Further assumptions are: (a) for the gravity potential $U(\vec{r}) = \frac{\alpha}{r}$ holds exactly (yields $\beta = \gamma = 0$ in (VI,14)), (b) the orbit of the satellite's centre of mass is a planar ellipse with small eccentricity, $e<0{,}1$ (leads in (VI,25 a) to $\vec{B}_{(k)} = \vec{0}$ for $k>1$; this corresponds to our specialization in (6.312) or Ref. 115 (Scient. Rep. 15), respectively). First a simplified equation of motion for the satellite's attitude is given and the stability behavior as a function of different mass distributions is discussed. Two main groups of resonance effects are distinguished, viz. "internal" (for exact circular orbits) and "external" ($0<e<0{,}1$) resonance. In the second section the Hamilton function used to describe the attitude motions is discussed, where for the total potential energy V the approache $V: = \int U(\vec{r})dm(\vec{r}) = U_s \cdot m - \frac{1}{2}\vec{S}_2 \cdot\cdot \vec{A}_{(1)}$ (see (6.131)) is taken. With some more assumptions concerning angles and momenta the authors obtain linearized equations of motion solved in section III by means of an averaging method and canonical transformations. Then the behavior at internal near-resonance is treated in section IV and near external resonance in section V.

Stability of the Planar Librational Motion of a Satellite in an Elliptic Orbit (Ref. 134): The paper numerically investigates the bounds that must be placed on a disturbance applied to a gravity gradient stabilized satellite of arbitrary shape in an (exact) elliptic orbit such that it will librate and not tumble. In section II the equations of motion are derived, by the help of Lagrange's formalism. The approach for the total potential energy is equal to that in Ref. 133. The formula for the total kinetic energy follows immediately from the assumption of a <u>planar</u> librational motion and seems us to be as doubtful as this assumption, because it leads to curious consequences, also immediately *). The investigations on stability are carried out in the phase space (section III). The limits e_{max} for the eccentricity e imply periodic solutions, discussed in section IV. In section V the results are summarized: e. g., the analysis points out that there is a limit to the value of orbit eccentricity, dependent on the satellite geometry, for which stable librational motion is possible; it appears that a large value of inertia parameter (i. e. a slender satellite) and a small value of eccentricity would help to ensure stability; for $0{,}38 < e$ gravity gradient stabilization is not possible.

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*) Formula (1) yields $u^1 = u^3 = 0$ in (VI,118). This means, $bc\dot{\alpha} = -\bar{c}\dot{\beta}$, $bc\dot{\gamma} = \overline{bc}\dot{\beta}$ (denotation from (VI,116); β is equal to Ψ in Ref. 134). Let us consider a satellite, symmetrical to $\vec{i}_3$ from VI, Fig. 2. There, the choice of starting point for the counting of γ is irrelevant. It follows then that the <u>planar</u> librational motion $\dot{\beta} \neq 0$ yields $\dot{\alpha} \neq 0$, i. e., the satellite does not remain in the orbital plane. Thus, the term "planar libr. motion" has to be defined rather widely. Another strange consequence is $\dot{\alpha} = 0$, $\dot{\gamma} = 0$ for $\dot{\beta} = 0$, i.e., the satellite changes the sense of its rotation about its axis of symmetry reaching points of greatest β (= the satellite turns back to the local vertical; these points β_{max} exist according to the assumption, that the motion is bounded).

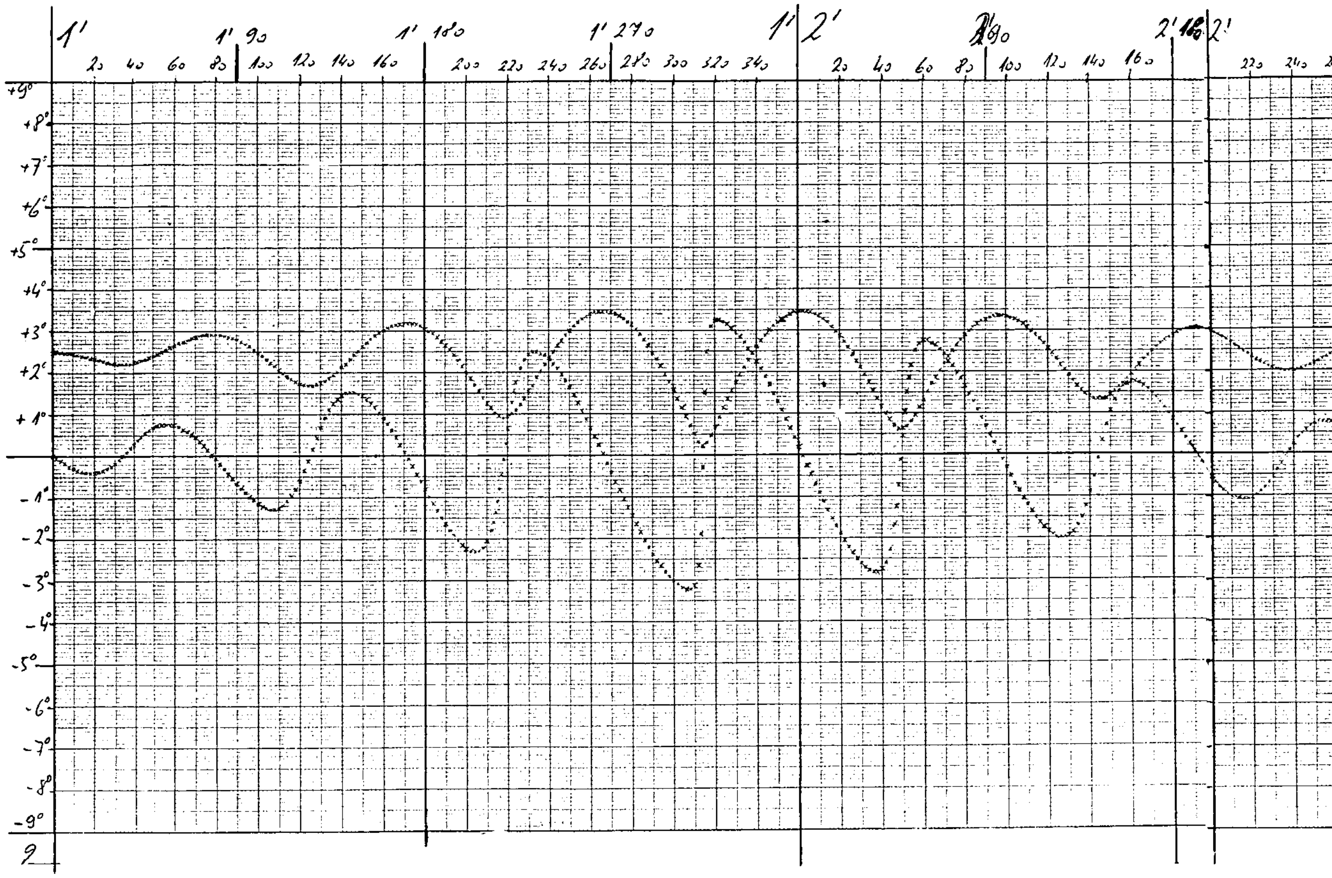
The Magnetic Torque Acting on Artificial Satellites (Ref. 135):
This paper gives a general survey of the influence of magnetic fields on the orientation of satellites. After a remark on the importance of such magnetic effects first the motion of the angular momentum vector is discussed. The torques are subdivided into two classes: first-order perturbations, gravitational and magnetic (for the satellite as magnetic dipole) torques, and second-order perturbations as eddy currents, magnetic hysteresis, atmospheric drag and internal vibration. The latter are by one or two orders of magnitude smaller, generally. For the case of a rotationally symmetrical satellite the equation of motion is given: the gravitational torques again correspond to our specialization in (6.312); the magnetic torques are split up into two parts, one of which is due to the permanent magnetic dipole moment of the satellite, and the other one to the induced dipole moment. Then, the gravitational and magnetic torques are averaged over (a) one precessional period,(b) one orbital revolution. By means of results of Explorer XI, Tiros I and SR I it is shown, amongother conclusions, that this method of averaging thoroughly yields useful results. In the second part, the decreasing of the angular momentum due to eddy currents and hysteresis torques is discussed. Aerodynamic braking torques are neglected.

Untersuchung von magnetisch geregelten, erdnahen Satelliten (Ref. 136) / Investigations on Magnetically Stabilized Satellites in low Orbits:
In the main, this work is a detailed representation of Ref. 132 by the same authors. A larger part explains in detail the analytical method used to solve these linearized equations of motion. By restriction to deviations from the desired orientation of the satellite (taken as reference system) small enough, this method can be more effective than

the numerical integration of the (linearized) equations of motion, by experience of the authors. Then considerations on the approximate derivation of the gravitational, aerodynamic and magnetic torques are given in full detail. For the rest, see under Ref. 132.

Ueber eine Methode zur Loesung des klassischen Vielkoerperproblems mit Hilfe von Lie-Reihen / On a Method of Solving the Classical Many-Body Problem Using Lie Series (Ref. 137): The paper starts with a short explanation of the notion of Lie series and treats then in (II) the splitting-up of the Lie operator. In (III) general remarks on the method of recursion formulas follow while (IV) deals in a detailed manner with the rather simple problem of deriving recursion formulas for Lie series. (V) gives recursion formulas for the restricted three body problem. The paper contains some remarks which we do not understand. For example, on page 222, the unfavorable influence of a too small step length on the accuracy of the result is undervalued, obviously. On page 223, we find also some strange ideas: the author seems to believe (on page 229 he formulated this statement more precisely) that it is "not possible to give the solution exactly" by help of the iteration method elaborated. The study of Ref. 9-12 could clarify this misunderstanding. The process of numerical computation gives also rise to inaccuracies, inevitably, such that the break-off error is certainly not the only error. Also to page 229: In principle, one can always take such an approximation function that only some few iteration steps are necessary. The same difficulties appear on page 223. In contrast to these critical remarks, we completely agree with the remark on the combination of the two methods.

Our experience concerning the two computation methods in question may be summarized as follows: Already for simple differential equations as, e. g., the Mathieu equation (Ref. 6) it is rather troublesome to code the recurrence formulas and it is possible to attain the limits of capacity of little computers (e. g., store capacity). The time needed to compute higher terms by the help of recurrence formulas may increase very rapidly so that an iterative method will be throughout competent (it depends upon the quantity also for "pure algebraic operations"). Moreover the iteration method provides an error estimation and therefore the possibility of changing the step length, automatically. If one has no suitable approximation a combination of recurrence and iteration method will be advantageous: One avoids the unpleasant repeated application of the Lie operator, calculates a rather suitable approximation by means of recurrence formulas and starts the iteration process thereby; once or two steps may then be enough, generally. The expense in coding may be expected to correspond to the accuracy of the result. Sometimes, this procedure can be more effective than the splitting up of an approach suggested by physical reasons, in spite of its formal elegance.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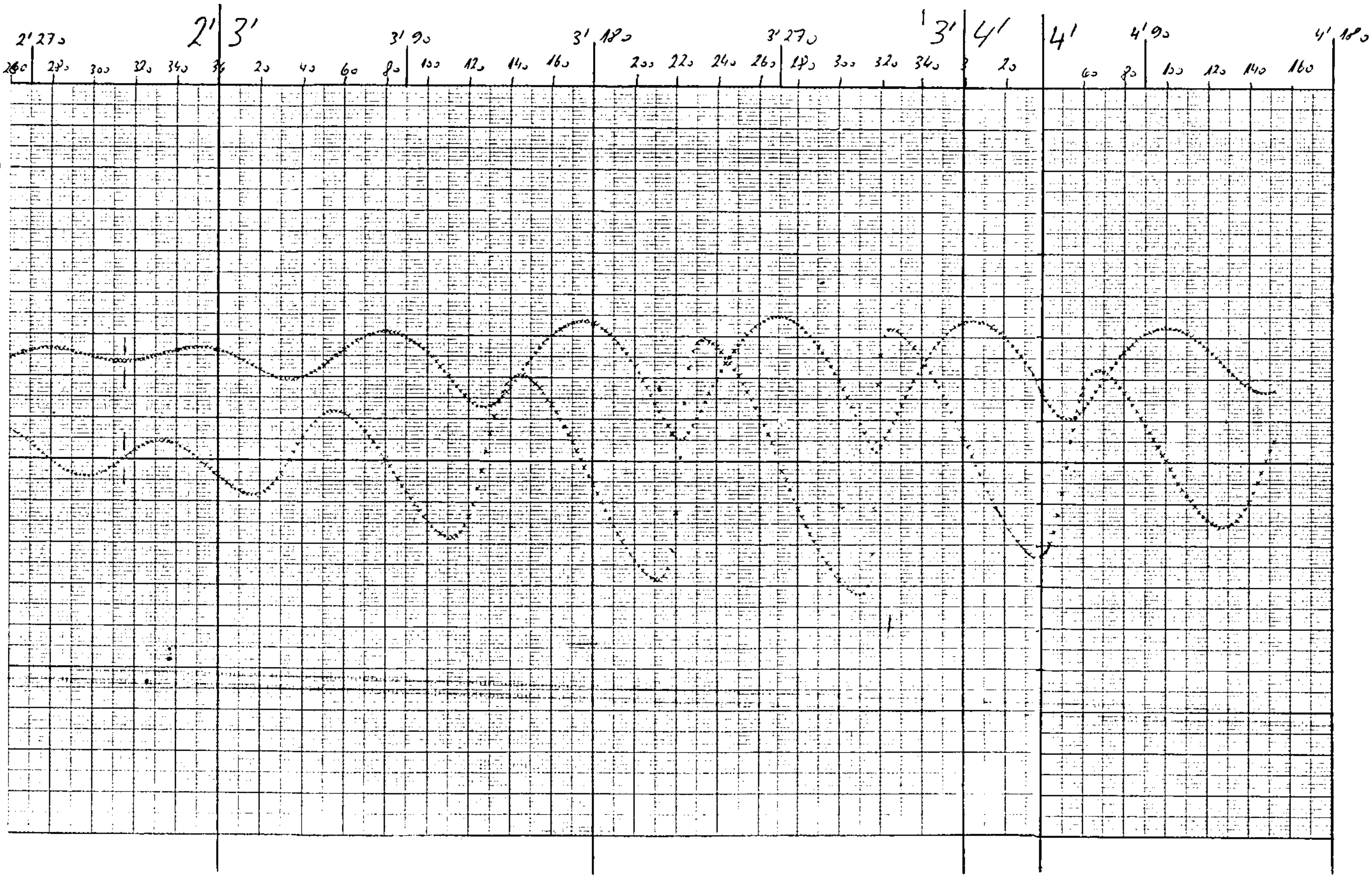
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Conclusion

As already mentioned in (6.41) the approximation used by us probably lies at the lower limit of utilizability. The process of combining recursion and iteration methods which we proposed there is supposed to increase the coding input considerably, but it is very likely that no method of solving the Eulerian gyroscope equations having such complex moments is free of remarkable troubles. If the improvements proposed are realized it should, however, be possible to compute the satellite's motion in a satisfactory way. Compared to the solution used by us the computer time probably will be increased in a considerable manner, but this fact should play a minor part if big computers are used.

If one wants to keep the Eulerian angles unchanged the singularities at $\beta = 0$ and $\beta = \pi$ can be avoided by introducing a second set of Eulerian angles such that the two axes corresponding to the singularities are normal to one another; this can, e. g., be achieved by a simple relabeling of the unit vectors of the systems of main axes.

The formulas for the influence of the gravitational field on the satellite's motion, which we have developed in (6.1), are supposed to yield these effects exactly enough so as to take account of other moments, too (see (6.12)). Possibly, also other terms neglected in the potential formula (VI,14) will play a role, and its series expansion will only be necessary up to the second term; according to the above indicated method the series expansion is certainly easy to perform. Moreover, we want to point to the fact that in the literature which we reviewed no expansion of $g(\vec{r})$ reaching so far could be found.

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