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SUMMARY

The classical least squares curve fitting method is used to determine the frequency, amplitude, damping ratio, phase angle, and zero offset of both a one- and two-degree-of-freedom system from free oscillation data.

The method is applied to a number of experimental transients with good results. Where possible, comparisons are made with the results of other methods. The least-squares method is found to be particularly useful in the analysis of two-degree-of-freedom systems where other techniques are difficult or impossible to apply.

INTRODUCTION

In wind-tunnel testing when the dynamic stability of a configuration is one of the parameters to be determined, either free-oscillation or forced-oscillation techniques can be used. The free-oscillation technique is easier to implement and was used recently at Ames Research Center in testing axi-symmetric hammerhead models (ref. 1). A typical model was mounted in the wind tunnel on a free-oscillation balance and a disturbance in the model attitude was introduced by rotating the model and then quickly releasing it. A continuous signal proportional to the model attitude was obtained and later passed through an analog to digital converter to provide discrete measurements at equal time intervals. When this record corresponds to a one-degree-of-freedom system, there are several methods commonly used to obtain the damping ratio and natural frequency. One of these methods (called here amplitude response method) is to use the amplitude response curve (ref. 2) as indicated in figure 1. This curve is obtained from the Fourier transform of the free-oscillation transient. Another method (called here peak amplitude method) is to plot the values of the peak amplitude on a semilog plot and then calculate the damping ratio as indicated in figure 2.

During the reduction of the data from these tests it was found the methods mentioned above failed to yield consistent data for a number of transients. These transients all showed evidence of multiple-mode oscillations. In an effort to find a data analysis technique that could be applied to transients of this type, a least-squares method was developed. It is the purpose of this paper to describe this method which will analyze either a one- or two-degree-of-freedom system rapidly and accurately, and to describe how the results obtained by this method compare with the results obtained by the amplitude response and peak amplitude methods.

NOTATION

A	amplitude of the envelope of curve 1 at $t = 0$, same units as $y(t)$
B	amplitude of the envelope of curve 2 at $t = 0$, same units as $y(t)$
C	zero offset, same units as $y(t)$
f	phase angle of curve 1, radians
g	phase angle of curve 2, radians
n	number of data points
p	natural frequency of curve 1, radians/sec
q	natural frequency of curve 2, radians/sec
t	time, sec
$y(t)$	amplitude of transient
$y_m(t)$	measured values of transient
$y_c(t)$	calculated values of transient
α	$-\zeta_1 p$, 1/sec
β	$-\zeta_2 q$, 1/sec
ζ_1	damping ratio of curve 1
ζ_2	damping ratio of curve 2

DESCRIPTION OF THE LEAST-SQUARES METHOD

The response of the one- and two-degree-of-freedom transient is assumed to be of the form

$$y(t) = Ae^{\alpha t} \sin(pt + f) + C \tag{1}$$

and

$$y(t) = Ae^{\alpha t} \sin(pt + f) + Be^{\beta t} \sin(qt + g) + C \tag{2}$$

The problem is to determine the constants in these equations so that the equation best describes, in a least-squares sense, a set of experimental data.

Appendix A is a detailed description of the derivation of the equations needed to solve for the unknowns in equation (1) or (2) by the LSM (Least Squares Method). The analysis results in a matrix equation that can be solved by normal matrix techniques. To apply this method to a two-degree-of-freedom system, first assign a vector $(A_o, \alpha_o, p_o, f_o, C_o, B_o, \beta_o, q_o, g_o)$ of initial estimates for all the unknowns. Using these values, calculate both y (denoted by y_c) and the partial derivatives of y with respect to each of the unknowns $(\partial y/\partial A, \partial y/\partial \alpha, \ . \ . \ . \ , \partial y/\partial g)$ at times corresponding to each of the n data points. After doing the indicated summations the following matrix equation for corrections to the initial estimates is obtained.

$$\begin{bmatrix} \sum_{i=1}^{n}\left(\frac{\partial y}{\partial A}\right)_i^2 & \sum\left(\frac{\partial y}{\partial A}\right)_i\left(\frac{\partial y}{\partial \alpha}\right)_i & \cdots & \sum\left(\frac{\partial y}{\partial A}\right)_i\left(\frac{\partial y}{\partial g}\right)_i \\ \sum\left(\frac{\partial y}{\partial A}\right)_i\left(\frac{\partial y}{\partial \alpha}\right)_i & \sum\left(\frac{\partial y}{\partial \alpha}\right)_i^2 & \cdots & \sum\left(\frac{\partial y}{\partial \alpha}\right)_i\left(\frac{\partial y}{\partial g}\right)_i \\ \vdots & \vdots & & \vdots \\ \sum\left(\frac{\partial y}{\partial A}\right)_i\left(\frac{\partial y}{\partial g}\right)_i & \sum\left(\frac{\partial y}{\partial \alpha}\right)_i\left(\frac{\partial y}{\partial g}\right)_i & \cdots & \sum\left(\frac{\partial y}{\partial g}\right)_i^2 \end{bmatrix} \cdot \begin{bmatrix} \Delta A \\ \Delta \alpha \\ \vdots \\ \Delta g \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\left(\frac{\partial y}{\partial A}\right)_i\left(y_m - y_c\right)_i \\ \sum\left(\frac{\partial y}{\partial \alpha}\right)_i\left(y_m - y_c\right)_i \\ \vdots \\ \sum\left(\frac{\partial y}{\partial g}\right)_i\left(y_m - y_c\right)_i \end{bmatrix}$$

After solving these nine equations for the corrections $(\Delta A, \Delta \alpha, \ . \ . \ . \ , \Delta g)$, they are added to the original estimates of the constants and the process is repeated. This iteration process is continued until the corrections are arbitrarily small (assuming convergence) at which time the estimate vector contains values of the constants which best describe the experimental data.

This procedure does not guarantee convergence. It has been found, however, that if reasonable initial estimates can be made of the constants (in particular the p and q), then convergence is assured and is very rapid.

PRESENTATION OF RESULTS

One Degree of Freedom

The transients of the three models selected are plotted in figure 3. They were chosen because they are representative of the range of damping ratios encountered during the actual testing program described in reference 1.

The peak amplitude and amplitude response plots of each transient ar shown in figures 4 and 5, respectively. The Share Library Program, AAHAN3, mode 3 (ref. 3) was used to calculate the amplitude response of the transien (i.e., Fourier transforms). This program was converted to MAP for use with Fortran IV programs by the staff of the Computation and Analysis Branch at Ames. The damping ratios of the peak amplitude plots were calculated as indicated in figure 2 while the frequencies were determined from figure 3 by counting the number of cycles in a given time increment. The damping ratios and frequencies of the amplitude response plots were calculated as indicated in figure 1. These results, the results obtained from the LSM, and the normalized standard deviation, or SD, (see section on Error) are recorded in table I. From this table it can be seen that the SD's, calculated with the parameters from the LSM, for the three transients are very small. Since these values are less than the accuracy of the data, the values obtained from the LSM for the frequencies and damping ratios are assumed to be correct. Now, with the parameters of the LSM as the basis, the errors of the other methods are as follows:

Transient A		Transient B		Transient C		Method
p	ζ	p	ζ	p	ζ	
-0.36%	+7.14%	-2.09%	-5.94%	-1.68%	+2.17%	Peak amplitude
-0.06%	+185.60%	+0.17%	+37.95%	+0.20%	+34.35%	Amplitude response

From this, it can be seen that all three methods compare well on frequency but the amplitude response method gives damping ratios that do not agree with those obtained from the other two methods.

The solutions obtained from the LSM have been plotted against the experimental data in figure 6. It can be seen from this plot that there is very good agreement between the generated curves and the transient data.

Two Degree of Freedom

The three transients chosen for the two-degree-of-freedom system have been plotted in figure 7. They were chosen to give a variety of two-degree-of-freedom cases in which the applicability of peak amplitude and amplitude response methods becomes doubtful. The peak amplitude plots are as shown in figure 8, and the fact that no straight line can adequately represent the peak values is apparent. The amplitude response plots are shown in figure 9, and although the frequencies are well defined, the bandwidth measurement for the second mode of transient E was uncertain. These results, along with those obtained from the LSM, and the normalized standard deviation, or SD, are recorded in table II. Here again the parameters as calculated by the LSM are assumed to be correct for the reason stated previously. With these parameters as a basis, the errors for the amplitude response parameters are as follows:

	Transient D	Transient E	Transient F
p	+0.02%	+0.18%	+0.33%
q	+0.53%	+3.68%	+0.10%
ζ_1	+18.45%	-15.15%	+33.80%
ζ_2	-25.30%	---	+22.10%

From this, it can be seen that although both methods agree well on frequency, they disagree on the damping ratios.

The results of the LSM for transient D are shown in figure 10 where the individual transients, curves 1 and 2, and their sum have been plotted. The sum is plotted against the experimental data. The results obtained for transients E and F are plotted in figures 11 and 12, respectively. It can be seen that there is again good agreement between the experimental data and the theoretical curve for each of the transients.

Error

The present LSM has been programmed to assume that it has successfully arrived at the values of the parameters when the incremental change in all parameters is less than 0.05 percent of the previous value of that parameter.

As an aid in judging how well the theoretical curve fits the experimental data, the normalized standard deviation is computed.

$$SD = \sqrt{\frac{\sum_{i=1}^{n}\left[y_c(t_i) - y_m(t_i)\right]^2}{\sum_{i=1}^{n}\left[y_m(t_i)\right]^2}} \times 100$$

This parameter has been recorded in table I for the one-degree-of-freedom transients and in table II for the two-degree-of-freedom transients. The poorest fit was for transient D, where SD = 1.98. Thus, in all cases, very satisfactory fits were obtained.

Since the second frequency for transient E was not pronounced (see fig. 9), this transient was analyzed also as a one-degree-of-freedom system. The best fit obtainable under this assumption was SD = 2.11, significantly poorer than the two-degree-of-freedom case where SD = 1.29.

CONCLUSIONS

A least squares method has been developed for the analysis of free-oscillation data and a comparison of this technique with the peak amplitude and amplitude response methods has led to the following conclusions:

1. For the one-degree-of-freedom transient data, all three methods give good answers for the frequency but the amplitude response method gives poor values for the damping ratio.

2. For the two-degree-of-freedom transient data, the least squares method gave a good fit for cases where the other methods were not practical to apply.

Ames Research Center
National Aeronautics and Space Administration
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APPENDIX A

MATHEMATICAL DERIVATION OF NUMERICAL SOLUTION

It is assumed that the free-oscillation transients considered in this paper can be modeled as the sum of one or more damped sine waves; that is

$$y(t) = Ae^{\alpha t} \sin(pt + f) + Be^{\beta t}(qt + g) + . . . + C \tag{A1}$$

where t is the time variable; C is a zero offset correction; $A, B, . . .$ are amplitude factors; $\alpha, \beta, . . .$ are functions of the damping ratios; $p, q, . . .$ are frequencies; and $f, g, . . .$ are phase angles.

The problem is to determine these parameters from the transient data such that equation (A1) best describes the transient data in the least squares sense.

The usual method of using the least squares criterion (ref. 4) directly for fitting equation (A1) to the transient, $y(t)$, would result in a set of nonlinear equations for which there is no known analytic solution. However, an iterative method of determining successive corrections to estimates of the parameters can be applied. This method (refs. 4 and 5) which results in linear equations is derived in the following paragraphs.

The derivation will be made for the solution of the equation

$$y(t) = Ae^{\alpha t} \sin(pt + f) + Be^{\beta t} \sin(qt + g) + C \tag{A2}$$

Extending the results to any number of damped sine waves is straightforward.

Equation (A2) can be written in the symbolic form

$$y = y(t) = y(t,A,\alpha,p,f,C,B,\beta,q,g) \tag{A3}$$

To apply the least squares method, we need to express the relationship between the empirical data and equation (A3) as a set of residual equations. Given n data points y_i, $i = 1, 2, . . . , n$, if v_i represents the difference or residual between the data y_i, taken at time t_i, and the value calculated from equation (A2) at time t_i, we obtain the following residual equations:

$$
\left.\begin{aligned}
v_1 &= y(t_1, A, \alpha, \ldots, g) - y_1 \\
&\ldots \\
v_i &= y(t_i, A, \alpha, \ldots, g) - y_i \\
&\ldots \\
v_n &= y(t_n, A, \alpha, \ldots, g) - y_n
\end{aligned}\right\} \qquad (A4)
$$

If we now assume that we can obtain approximate values $A_o, \alpha_o, p_o, f_o, C_o, B_o, \beta_o, q_o, g_o$ for the true parameters $A, \alpha, \ldots, g$, then we can express the true parameters as

$$
\left.\begin{aligned}
A &= A_o + \Delta A & B &= B_o + \Delta B \\
\alpha &= \alpha_o + \Delta\alpha & \beta &= \beta_o + \Delta\beta \\
p &= p_o + \Delta p & q &= q_o + \Delta q \\
f &= f_o + \Delta f & g &= g_o + \Delta g \\
C &= C_o + \Delta C & &
\end{aligned}\right\} \qquad (A5)
$$

where $\Delta A, \Delta\alpha, \ldots, \Delta g$ are corrections to be determined. Substituting equation (A5) in (A4) we obtain the residual equations

$$
\left.\begin{aligned}
v_1 &= y(t_1, A_o + \Delta A, \alpha_o + \Delta\alpha, \ldots, g_o + \Delta g) - y_1 \\
&\ldots \\
v_i &= y(t_i, A_o + \Delta A, \alpha_o + \Delta\alpha, \ldots, g_o + \Delta g) - y_i \\
&\ldots \\
v_n &= y(t_n, A_o + \Delta A, \alpha_o + \Delta\alpha, \ldots, g_o + \Delta g) - y_n
\end{aligned}\right\} \qquad (A6)
$$

For simplicity let us consider only the ith residual equation, which can be written as

$$v_i + y_i = y(t_i, A_o + \Delta A, \alpha_o + \Delta\alpha, \ldots, g_o + \Delta g) \tag{A7}$$

By considering the right side of equation (A7) as a function of $A, \alpha, \ldots, g$ with t constant, we can expand it by Taylor's theorem for a function of several variables (refs. 5 and 6) around the point $(A_o, \alpha_o, \ldots, g_o)$. This expansion is valid since equation (A2) has continuous partial derivatives of all orders with respect to $A, \alpha, \ldots,$ and g.

Taylor's expansion yields

$$v_i + y_i = y(t_i, A_o, \alpha_o, \ldots, g_o) + \sum_{j=1}^{\infty} \frac{1}{j!}\left[\left(\Delta A \frac{\partial}{\partial A} + \Delta\alpha \frac{\partial}{\partial \alpha} + \ldots + \Delta g \frac{\partial}{\partial g}\right)^j y(t, A, \alpha, \ldots, g)\right]_* \tag{A8}$$

where $*$ means that after differentiation the numerical values of the partials are calculated for $t = t_i$, $A = A_o$, $\alpha = \alpha_o$, $p = p_o$, $f = f_o$, $C = C_o$, $B = B_o$, $\beta = \beta_o$, $q = q_o$, $g = g_o$.

Expanding equation (A8) gives

$$v_i + y_i = y(t_i, A_o, \alpha_o, \ldots, g_o) + \Delta A \left(\frac{\partial y}{\partial A}\right)_* + \Delta\alpha \left(\frac{\partial y}{\partial \alpha}\right)_* + \ldots + \Delta g \left(\frac{\partial y}{\partial g}\right)_* + \sum_{j=2}^{\infty} R_j \tag{A9}$$

Since $\sin(pt + f, qt + g) \leq 1$, $\cos(pt + f, qt + g) \leq 1$ for all $(pt + f, qt + g)$; and αt, βt, A, and B are bounded; it is clear that the R_j consist of sums of terms R_{jk} of the order

$$R_{jk} = \text{order}\left[\max(1, A, B)\max(\Delta A^2, \Delta A\ \Delta\alpha, \ldots \Delta A\ \Delta g, \Delta\alpha^2, \Delta\alpha\ \Delta p, \ldots, \Delta p^2, \ldots, \Delta f^2, \ldots, \Delta g^2)\right] \tag{A10}$$

Since all the $\Delta a, \Delta\alpha, \ldots, \Delta g$ are considerably less than 1, and all the partials are bounded, all the R_j approach zero as j gets large. This result, together with the fact that all the partials exist and are continuous in the range $0 \leq t \leq 1$ is a sufficient condition (ref. 7) for equations (A8) and (A9) to be valid.

If we now approximate $y(t_i, A_o + \Delta A, \alpha_o + \Delta\alpha, \ldots, g_o + \Delta g)$ by dropping the second and higher order terms of equation (A9),

$$v_i + y_i = y(t_i, A_o, \alpha_o, \ldots, g_o) + \Delta A \left(\frac{\partial y}{\partial A}\right)_* + \Delta\alpha \left(\frac{\partial y}{\partial \alpha}\right)_* + \ldots + \Delta g \left(\frac{\partial y}{\partial g}\right)_* \qquad \text{(A11)}$$

or

$$v_i = \Delta A \left(\frac{\partial y}{\partial A}\right)_* + \Delta\alpha \left(\frac{\partial y}{\partial \alpha}\right)_* + \ldots + \Delta g \left(\frac{\partial y}{\partial g}\right)_* + y(t_i, A_o, \alpha_o, \ldots, g_o) - y_i \qquad \text{(A12)}$$

If we let $\gamma_i = y(t_i, A_o, \alpha_o, \ldots, g_o) - y_i$, equation (A12) becomes

$$v_i = \Delta A \left(\frac{\partial y}{\partial A}\right)_* + \Delta\alpha \left(\frac{\partial y}{\partial \alpha}\right)_* + \ldots + \Delta g \left(\frac{\partial y}{\partial g}\right)_* + \gamma_i , \quad i = 1, 2, \ldots, n \qquad \text{(A13)}$$

Equation (A13) now represents a set of n residual equations which are linear in the corrections $\Delta A, \Delta\alpha, \ldots, \Delta g$ and can thus be solved for the corrections by a least squares technique.

The least squares criterion is that $\sum_{i=1}^{n} v_i^2$ will be a minimum (ref. 7). This criterion will be satisfied when the first partials with respect to all the unknowns, evaluated at the point $(A_o, \alpha_o, \ldots, g_o)$, are equal to zero. Thus for the partial with respect to ΔA:

$$0 = \frac{\partial}{\partial \Delta A} \sum_{i=1}^{n} v_i^2 = \sum_{i=1}^{n} \frac{\partial v_i^2}{\partial \Delta A}$$

Substituting equation (A13) for v_i becomes

$$0 = \sum_{i=1}^{n} \frac{\partial}{\partial \Delta A} \left[\Delta A \left(\frac{\partial y}{\partial A}\right)_* + \Delta\alpha \left(\frac{\partial y}{\partial \alpha}\right)_* + \ldots + \Delta g \left(\frac{\partial y}{\partial g}\right)_* + \gamma_i\right]^2$$

or

$$0 = \sum_{i=1}^{n} \left[\left(\frac{\partial y}{\partial A}\right)_* \left(\frac{\partial y}{\partial A}\right)_* \Delta A + \left(\frac{\partial y}{\partial A}\right)_* \left(\frac{\partial y}{\partial \alpha}\right)_* \Delta\alpha + \cdots \right.$$

$$\left. + \left(\frac{\partial y}{\partial A}\right)_* \left(\frac{\partial y}{\partial g}\right)_* \Delta g + \left(\frac{\partial y}{\partial A}\right)_* \gamma_i \right]$$

Distributing the summation and taking the constants $\Delta A, \Delta\alpha, \ldots, \Delta g$ outside the summations gives

$$-\sum_{i=1}^{n} \left(\frac{\partial y}{\partial A}\right)_* \gamma_i = \angle \sum_{i=1}^{n} \left(\frac{\partial y}{\partial A}\right)_*^2 + \Delta\alpha \sum_{i=1}^{n} \left(\frac{\partial y}{\partial A}\right)_* \left(\frac{\partial y}{\partial \alpha}\right)_* + \cdots$$

$$+ \Delta g \sum_{i=1}^{n} \left(\frac{\partial y}{\partial A}\right)_* \left(\frac{\partial y}{\partial g}\right)_* \tag{A14}$$

Proceeding similarly for each of the other corrections $\Delta\alpha, \Delta p, \ldots, \Delta g$ results in nine linear equations in the nine unknown corrections. Numerical values for the corrections can now be obtained by matrix techniques (ref. 8). If we arrange into an array the coefficients of the unknowns in the set of nine equations derived in the same manner as equation (A14) we obtain the square matrix U of order 9:

$$U = \begin{pmatrix} \sum_{i=1}^{n} \left(\frac{\partial y}{\partial A}\right)_*^2 & \sum_{i=1}^{n} \left(\frac{\partial y}{\partial A}\right)_* \left(\frac{\partial y}{\partial \alpha}\right)_* & \cdots & \sum_{i=1}^{n} \left(\frac{\partial y}{\partial A}\right)_* \left(\frac{\partial y}{\partial g}\right)_* \\ \sum_{i=1}^{n} \left(\frac{\partial y}{\partial \alpha}\right)_* \left(\frac{\partial y}{\partial A}\right)_* & \sum_{i=1}^{n} \left(\frac{\partial y}{\partial \alpha}\right)_*^2 & \cdots & \sum_{i=1}^{n} \left(\frac{\partial y}{\partial \alpha}\right)_* \left(\frac{\partial y}{\partial g}\right)_* \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{n} \left(\frac{\partial y}{\partial g}\right)_* \left(\frac{\partial y}{\partial A}\right)_* & \sum_{i=1}^{n} \left(\frac{\partial y}{\partial g}\right)_* \left(\frac{\partial y}{\partial \alpha}\right)_* & \cdots & \sum_{i=1}^{n} \left(\frac{\partial y}{\partial g}\right)_*^2 \end{pmatrix} \tag{A15}$$

We now define a column vector V as the left hand members of the nine equations

$$V = \begin{pmatrix} -\sum_{i=1}^{n} \left(\frac{\partial y}{\partial A}\right)_* \gamma_i \\ -\sum_{i=1}^{n} \left(\frac{\partial y}{\partial \alpha}\right)_* \gamma_i \\ \cdots \\ -\sum_{i=1}^{n} \left(\frac{\partial y}{\partial g}\right)_* \gamma_i \end{pmatrix}$$

or

$$V = \begin{pmatrix} \sum_{i=1}^{n} \left(\frac{\partial y}{\partial A}\right)_* \left[y_i - y(t_i, A_o, \alpha_o, \ldots, g_o)\right] \\ \sum_{i=1}^{n} \left(\frac{\partial y}{\partial \alpha}\right)_* \left[y_i - y(t_i, A_o, \alpha_o, \ldots, g_o)\right] \\ \cdots \\ \sum_{i=1}^{n} \left(\frac{\partial y}{\partial g}\right)_* \left[y_i - y(t_i, A_o, \alpha_o, \ldots, g_o)\right] \end{pmatrix} \tag{A16}$$

We now wish to determine the vector X such that UX = V. Jordon's method (ref. 9) is used to reduce U to I, the identity matrix, through a series of elementary transformations, which when applied to V result in X. The vector X now represents the corrections:

$$X = \begin{pmatrix} \Delta A \\ \Delta\alpha \\ \cdot \\ \cdot \\ \cdot \\ \Delta g \end{pmatrix} \tag{A17}$$

Since this result was obtained by a truncation of the Taylor series expansion in equation (A11), X contains only approximate values for the corrections. We, therefore, form a new set of approximate values A_1, α_1, . . . , g_1 by adding the computed corrections to the initial estimates

$$\left.\begin{aligned} A_1 &= A_o + \Delta A \\ \alpha_1 &= \alpha_o + \Delta\alpha \\ &\cdot\;\cdot\;\cdot \\ g_1 &= g_o + \Delta g \end{aligned}\right\} \tag{A18}$$

For notational convenience we now let A_1 be called A_o, α_1 be called α_o, . . . , and g_1 be called g_o, and return to equation (A12). We now proceed as before to obtain an improved approximation to the true parameters A, α, . . . , g. This process of iteration is repeated until there is no change (to the degree of accuracy obtainable from the input data) in the parameters A, α, . . . , g in equation (A18) in two subsequent iterations.

By using the computer program based on this analysis, we have shown that this iterative process does converge rapidly under a specific range of conditions. The primary condition is that the initial estimate of the frequencies be close to the true values. The choice of first and last data points presented to the program is not critical because of the parameters included for phase angles. However, including data containing extraneous forces or vibrations, or data with nonstationary parameters within the time period used will definitely limit or negate convergence to meaningful values.

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TABLE I.- COMPARISON OF RESULTS FOR ONE-DEGREE-OF-FREEDOM TRANSIENTS

Transient	Method	Frequency $(p/2\pi)$ Hz	Damping ratio $(\zeta=-\alpha/p)$	Amplitude (A)	Phase angle (f) radians	Zero offset (C)	Normalized standard deviation (SD) (%)
A	Peak amplitude	16.6	0.0018				
	Amplitude response	16.65	.0048				
	Least squares	16.66	.00168	1.9801	0.0761	-0.0044	0.20
B	Peak amplitude	17.3	.0088				
	Amplitude response	17.70	.0129				
	Least squares	17.67	.00935	.8634	-.0069	.0138	.78
C	Peak amplitude	19.8	.0165				
	Amplitude response	20.30	.0217				
	Least squares	20.26	.01615	-1.715	.0428	.0257	1.42

TABLE II.- COMPARISON OF RESULTS FOR TWO-DEGREE-OF-FREEDOM TRANSIENTS

Transient	Method	Frequency		Damping ratio		Amplitude		Phase angle		Zero offset (C)	Normalized standard deviation (SD)
		Curve 1 $(p/2\pi)$ Hz	Curve 2 $(q/2\pi)$ Hz	Curve 1 $(\zeta_1=-\alpha/p)$	Curve 2 $(\zeta_2=-\beta/q)$	Curve 1 (A)	Curve 2 (B)	Curve 1 (f) radians	Curve 2 (g) radians		
D	Peak amplitude	---	---	---	---						
	Amplitude response	25.50	32.45	0.0088	0.0157						
	Least squares	25.46	32.28	.00743	.02097	1.0161	1.3805	-0.8603	0.3161	-0.0147	1.98
E	Peak amplitude	---	---	---	---						
	Amplitude response	21.70	23.55	.0235	---						
	Least squares	21.66	24.45	.02769	.01546	-1.9502	-.4288	.0017	2.499	.0009	1.29
F	Peak amplitude	---	---	---	---						
	Amplitude response	45.20	102.20	.0149	.0069						
	Least squares	45.05	102.10	.01114	.00565	.6614	-1.7834	.4784	.0429	-.0141	1.57

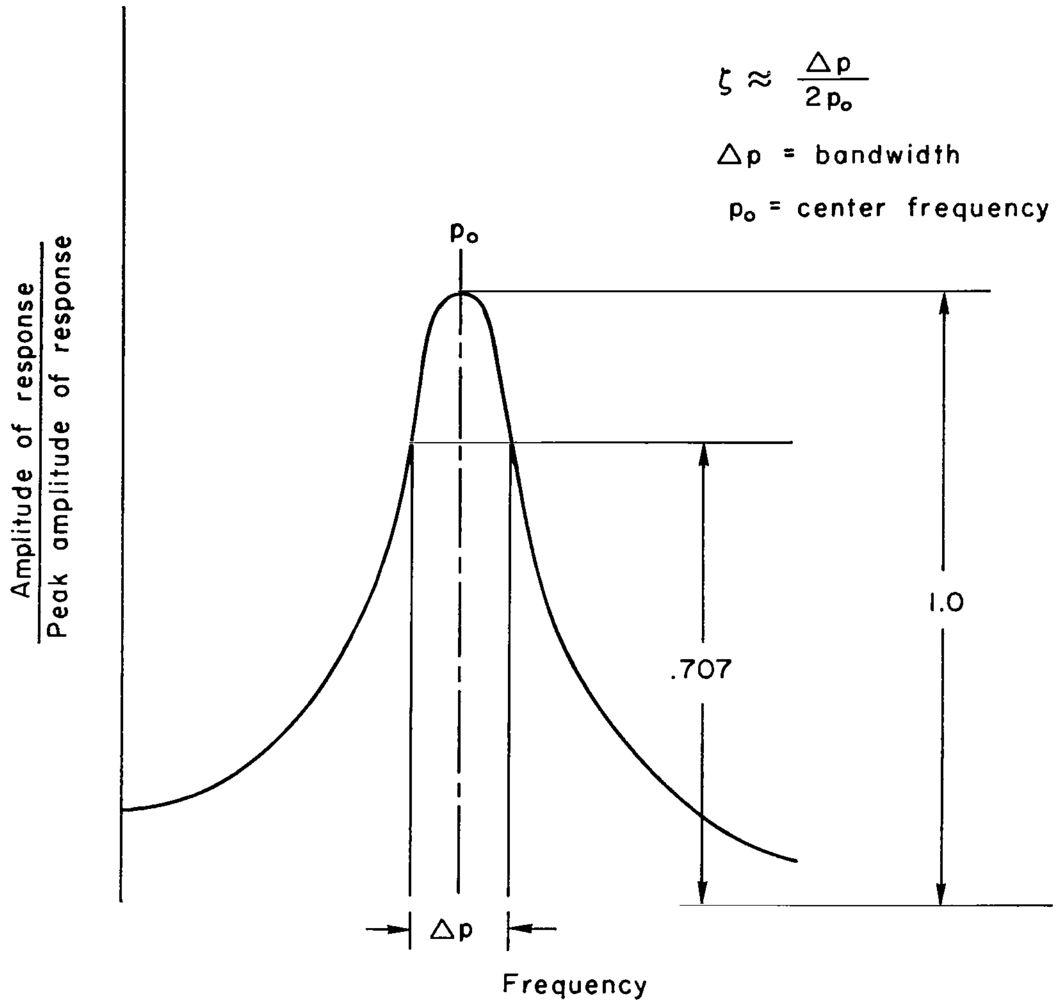

Figure 1.- An amplitude response curve.

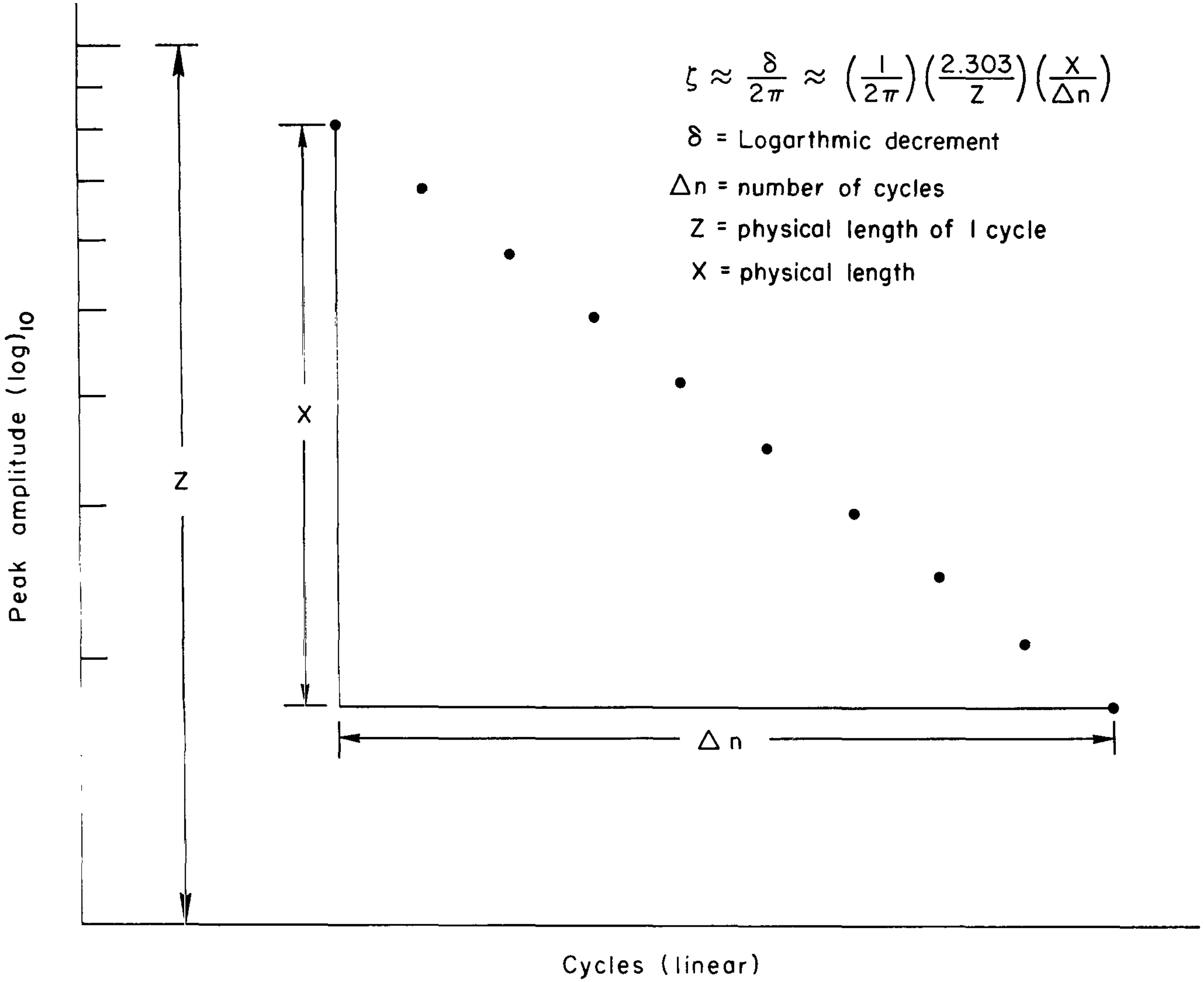

Figure 2.- Plot of peak amplitudes.

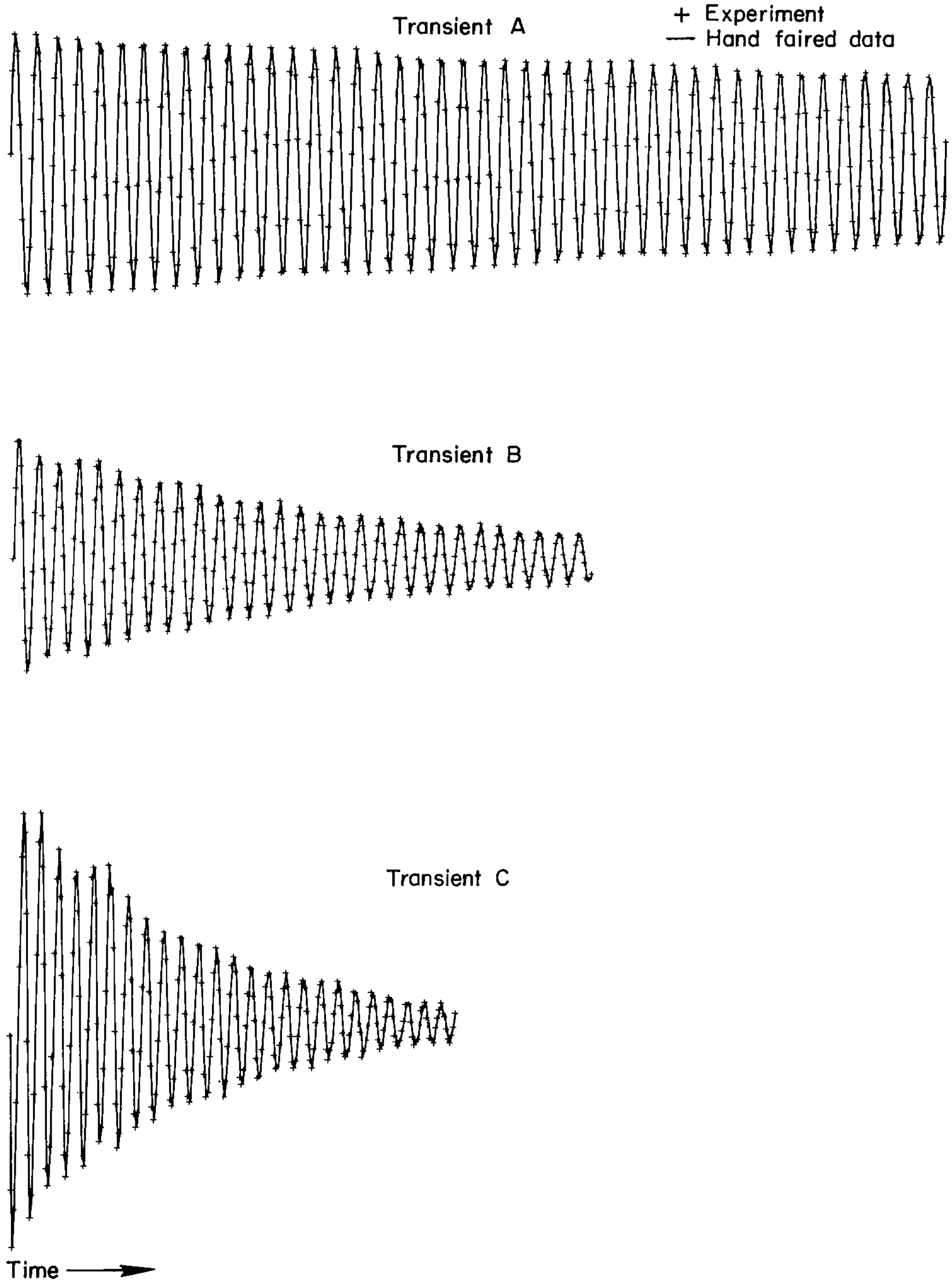

Figure 3.- Transients used in the one-degree-of-freedom analysis.

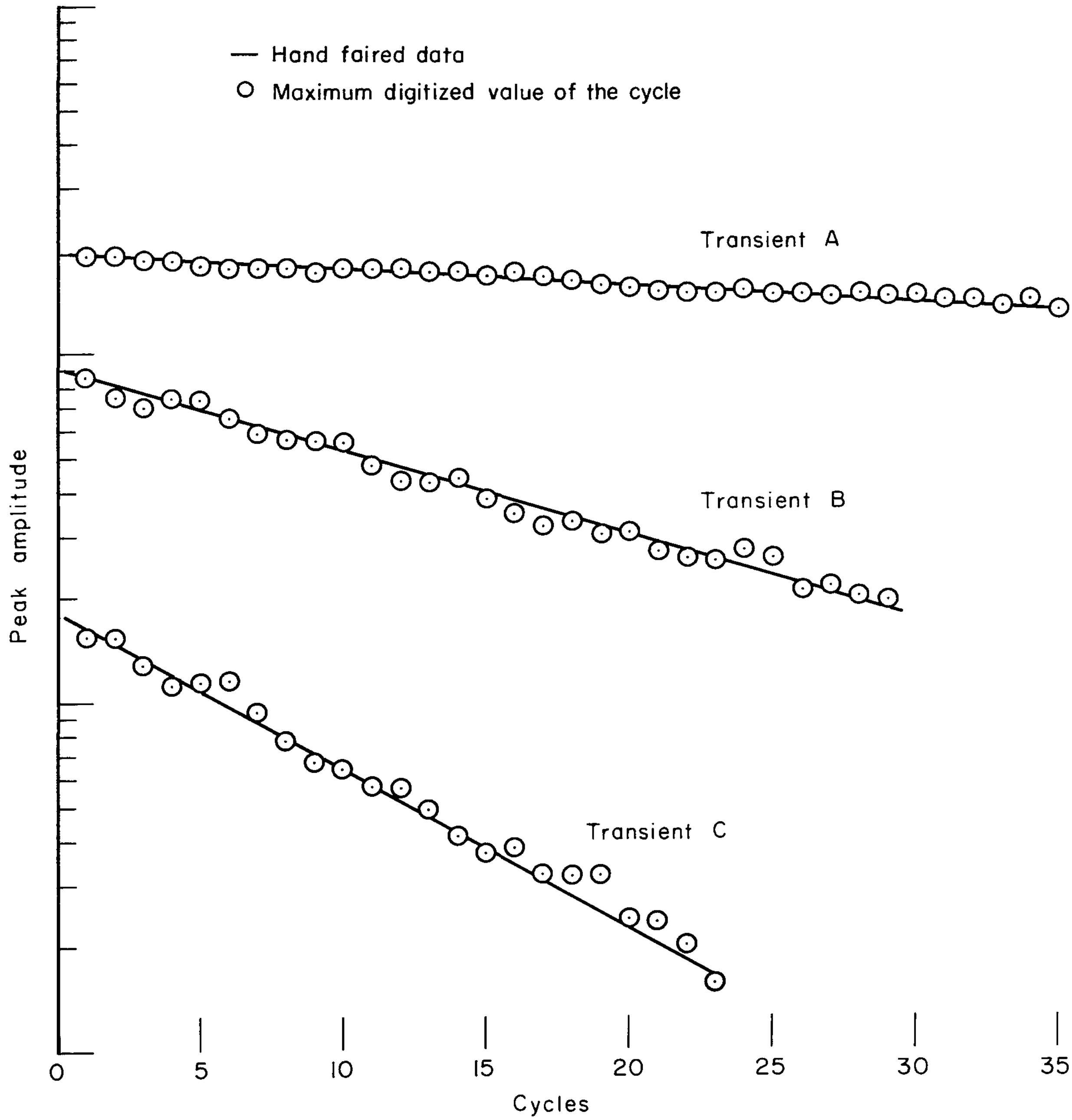

Figure 4.- Peak amplitude of transients used in single-degree-of-freedom analysis.

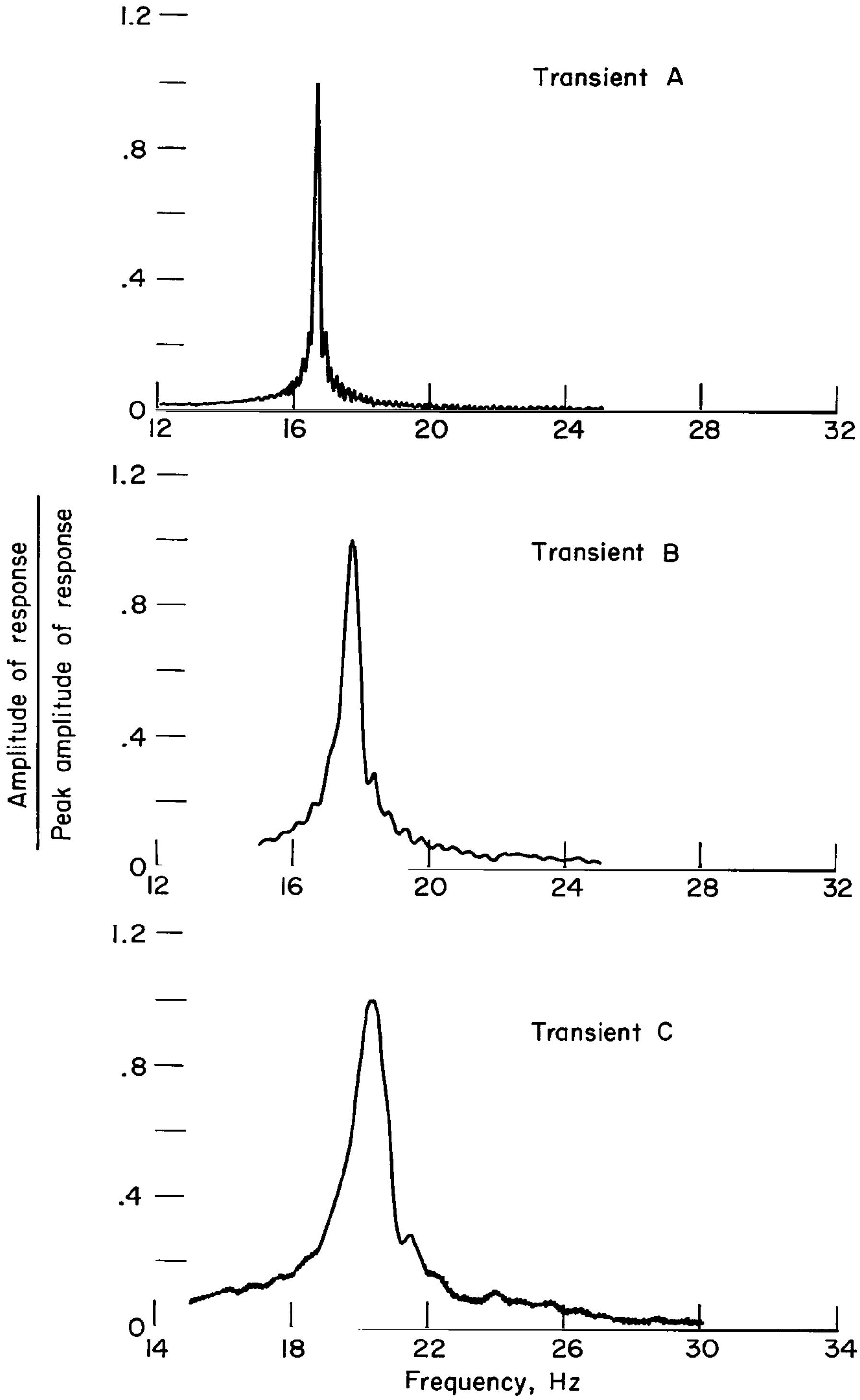

Figure 5.- Amplitude response of transients used in single-degree-of-freedom analysis.

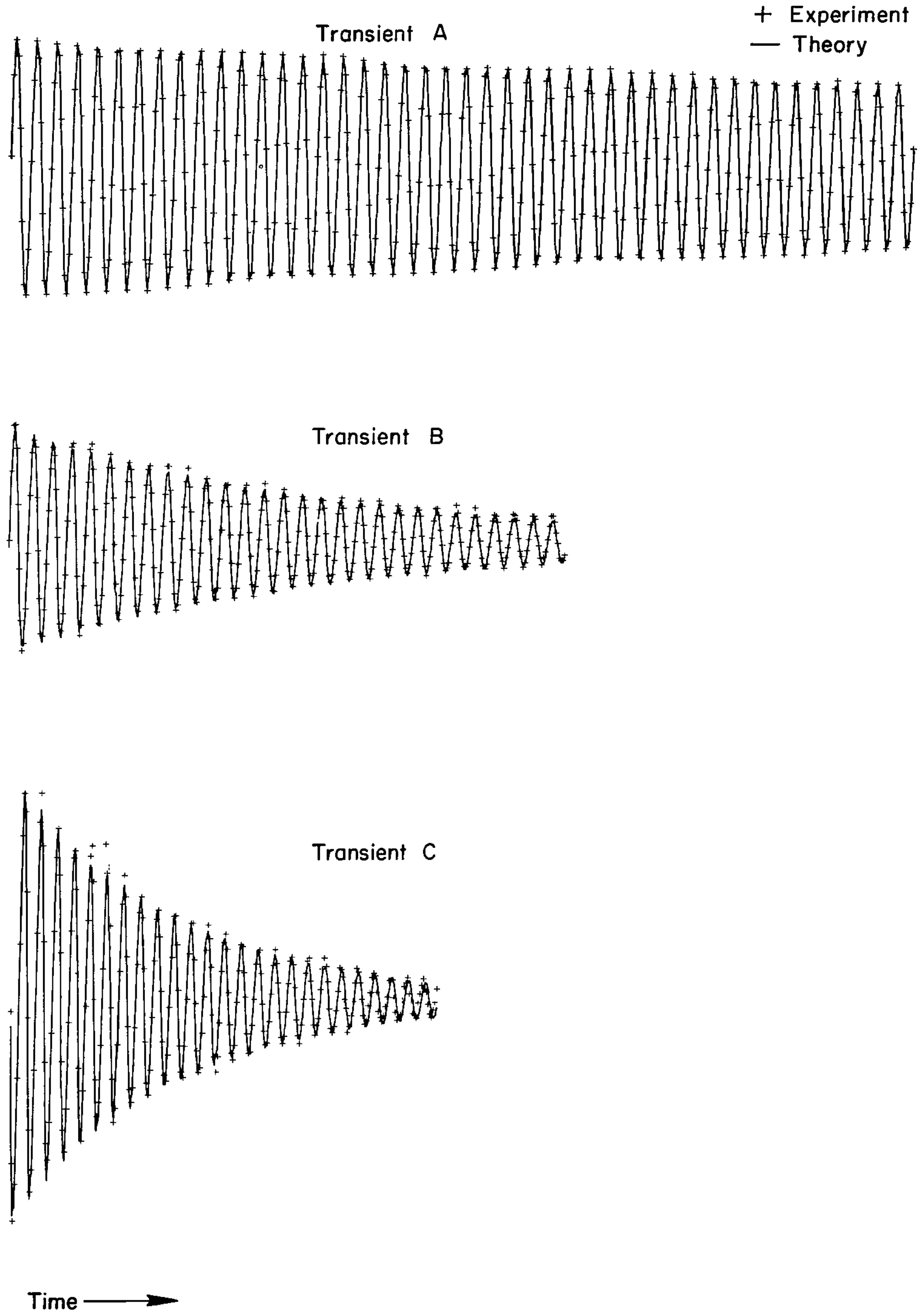

Figure 6.- Results of the transients as calculated from the LSM.

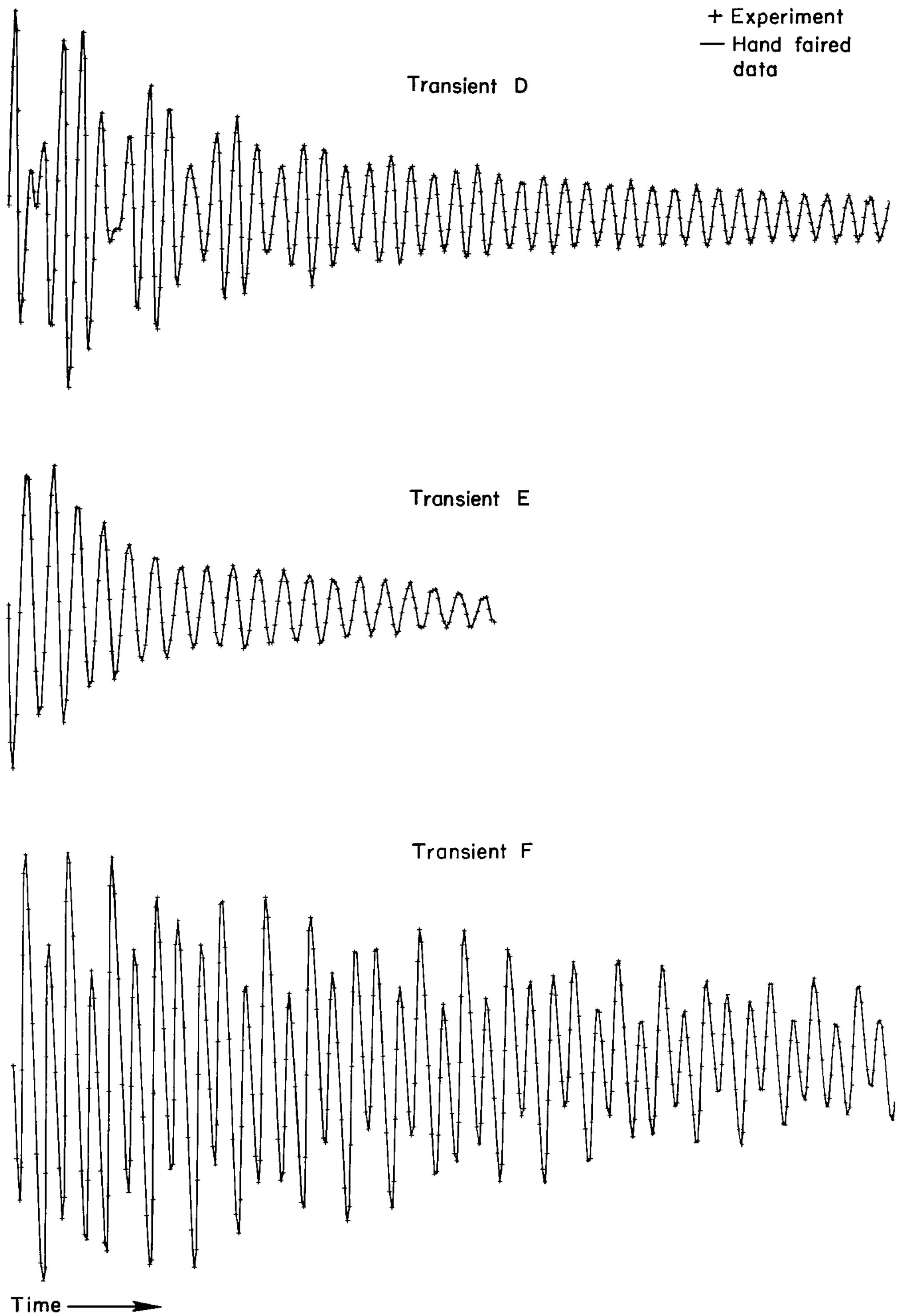

Figure 7.- Transients used in the two-degree-of-freedom analysis.

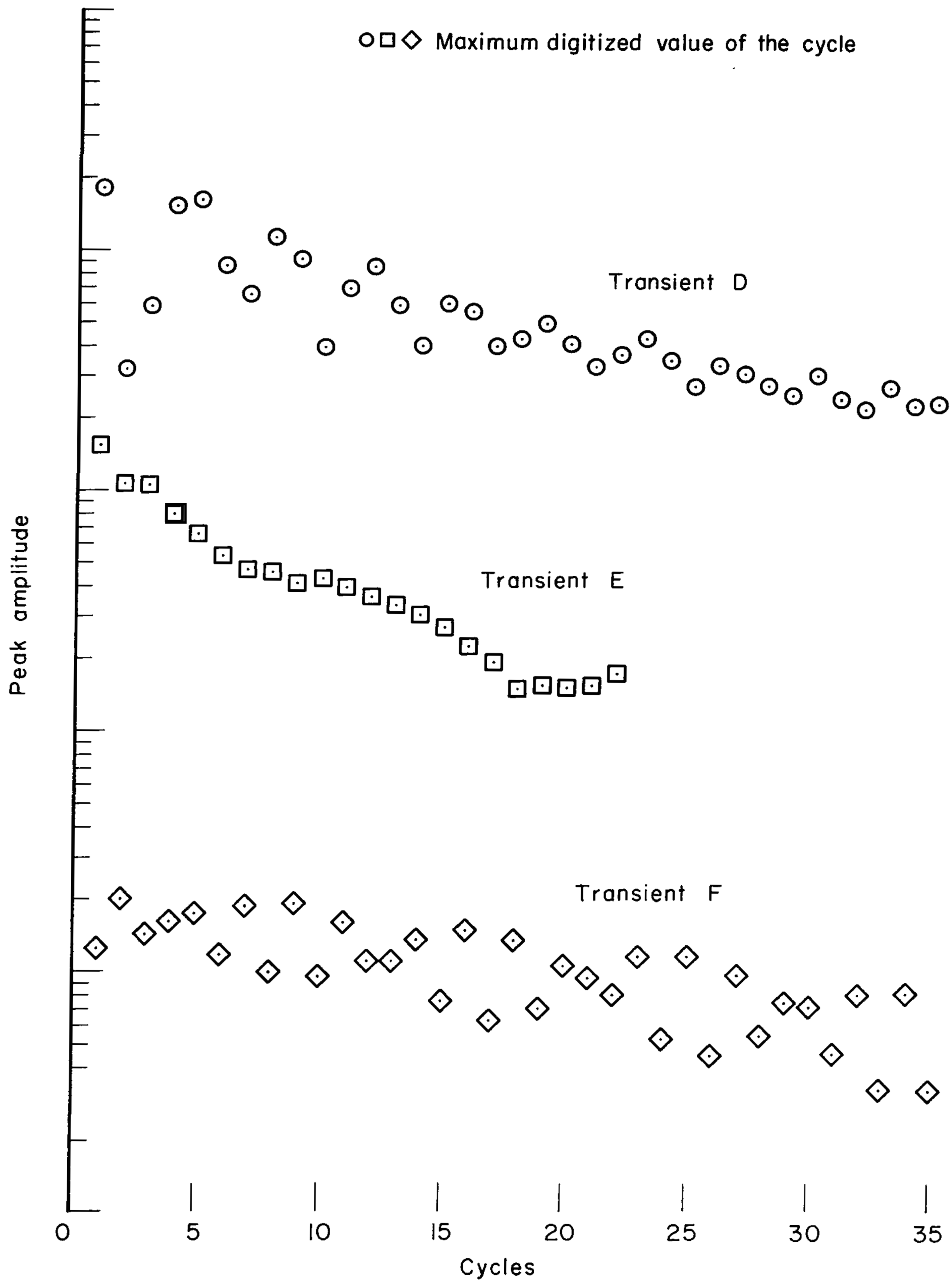

Figure 8.- Peak amplitude of transients used in the two-degree-of-freedom analysis.

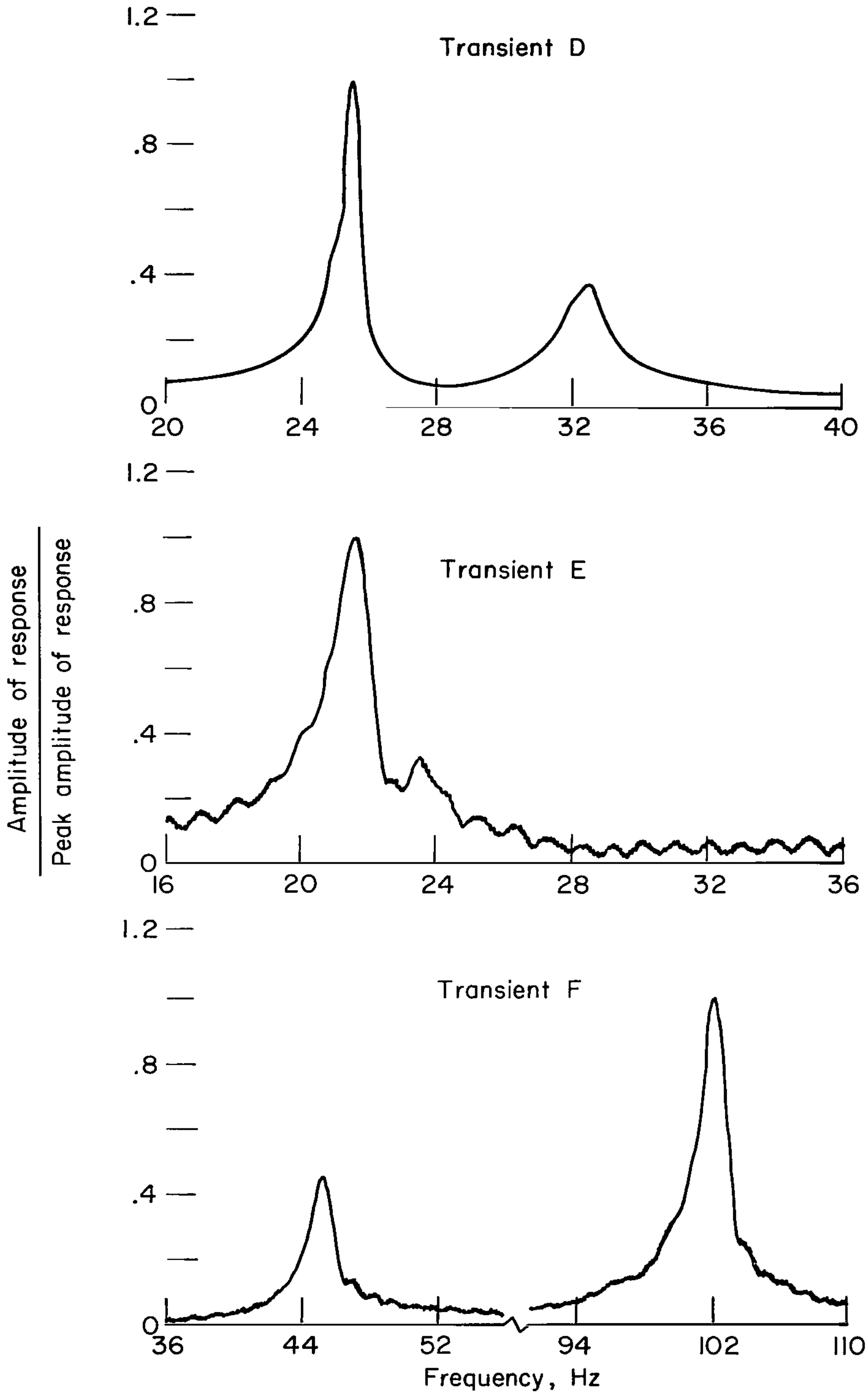

Figure 9.- Amplitude response of transients used in two-degree-of-freedom analysis.

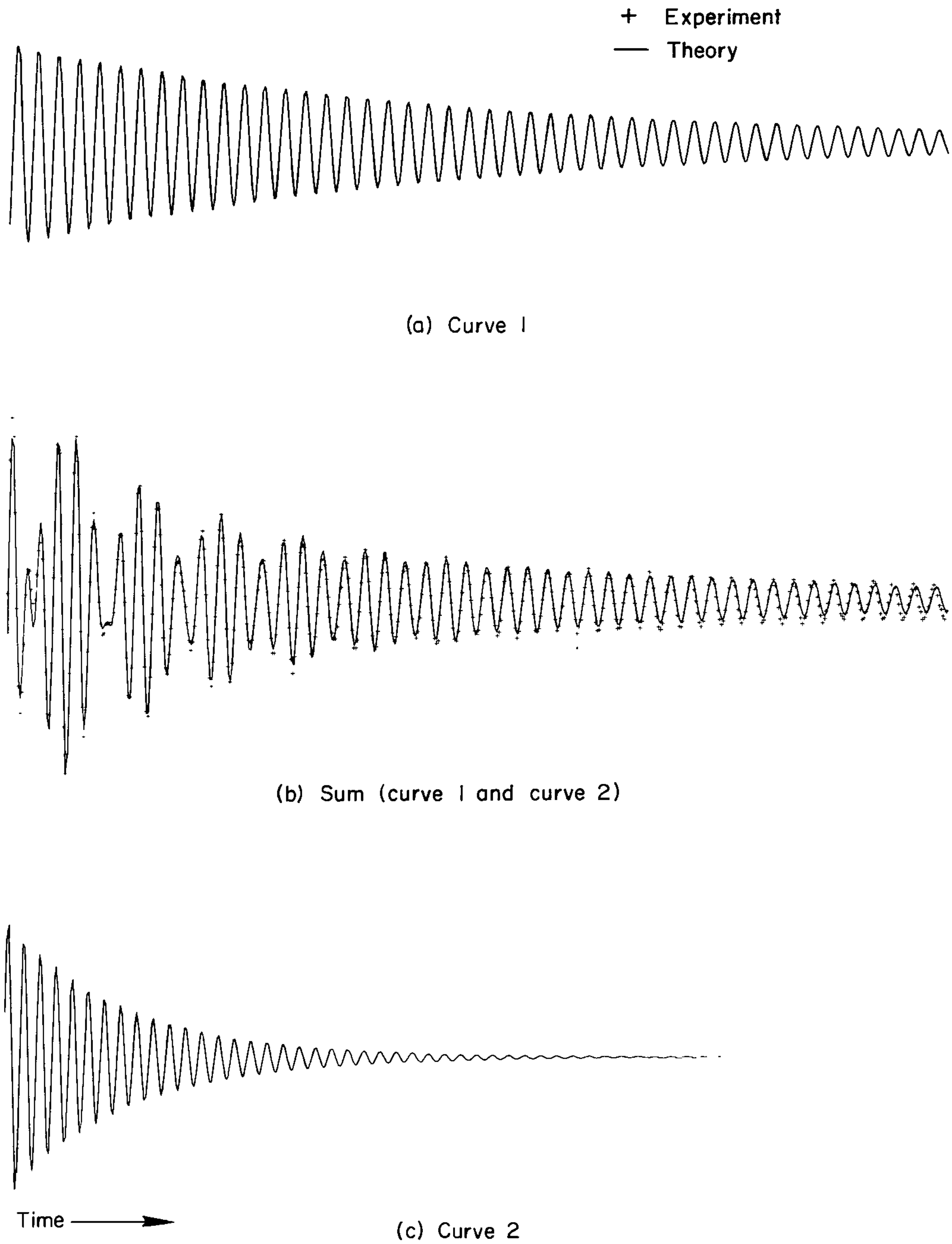

Figure 10.- Results of transient D as calculated from the LSM.

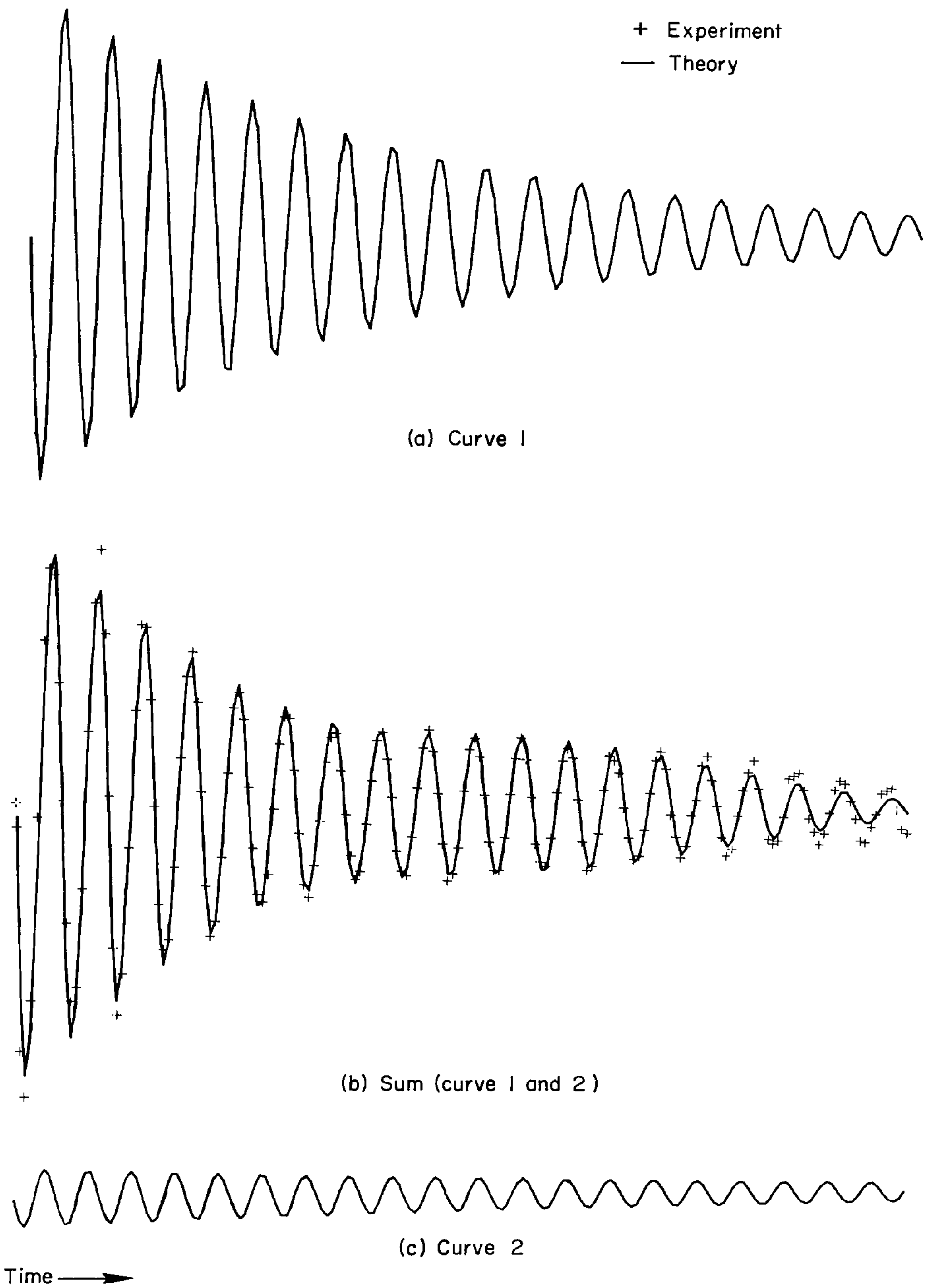

Figure 11.- Results of transient E as calculated from the LSM.

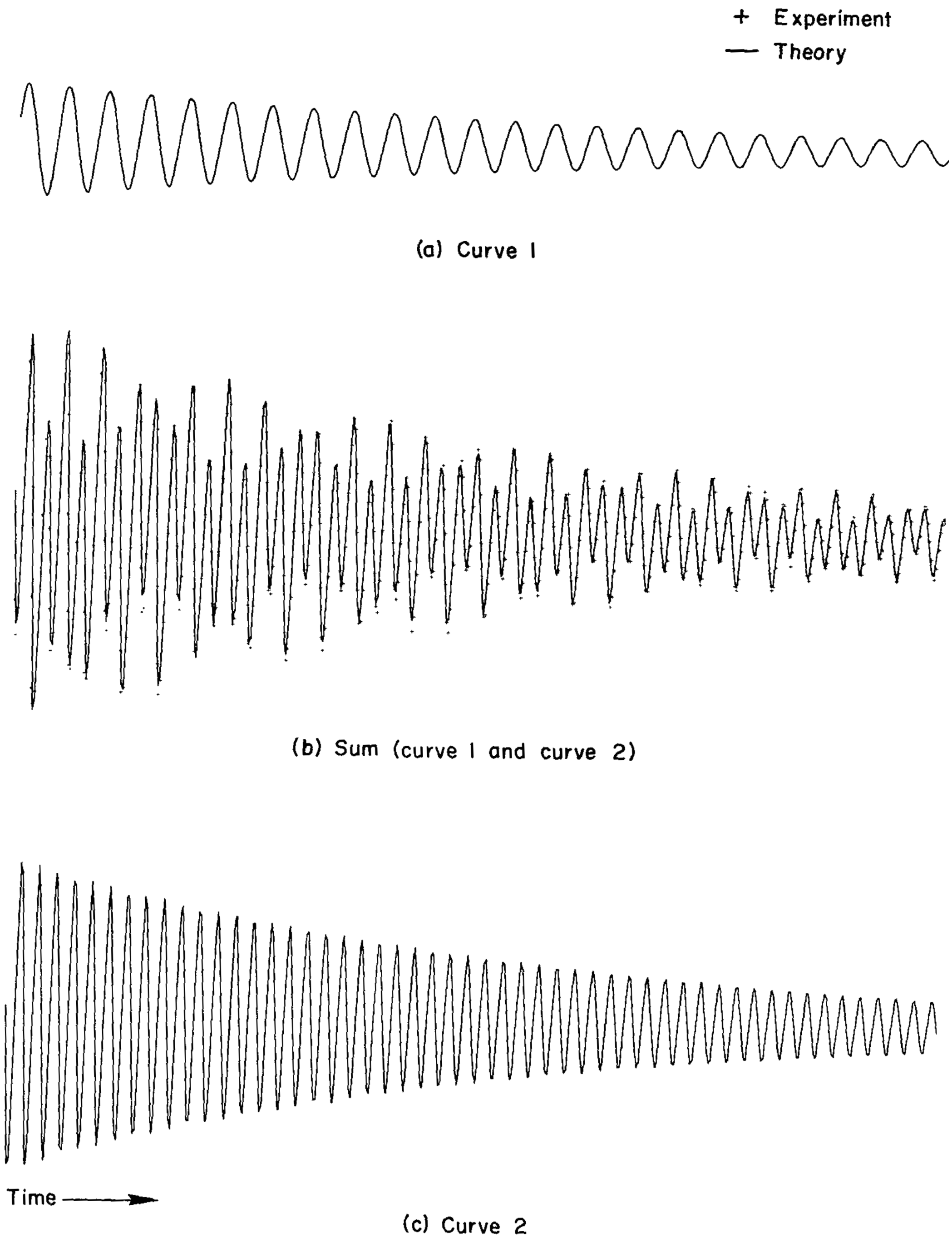

Figure 12.- Results of transient F as calculated from the LSM.

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