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Adaptive Telemetry Systems

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ABSTRACT

In most space communication systems the energy available to the spacecraft is limited. This paper is an effort to reduce the energy required by the spacecraft for a given overall probability of error by employing feedback from the ground station to the spacecraft.

Three memoryless systems employing feedback are compared with a memoryless system without feedback and it is shown that for the same energy level, feedback improves the performance of a system as far as the overall probability of error is concerned.

The signals considered are orthogonal, phase-coherent, and perturbed by white Gaussian noise. The conclusions, however, could be extended to non-phase-coherent signals and also Rayleigh noise.

ERROR REDUCTION IN A TELEMETRY SYSTEM USING ADAPTIVE FEEDBACK

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CHAPTER I

INTRODUCTION

In most telemetry systems the prime considerations are always the reduction of the overall probability of error, lowering of the expected value of the energy required for communication, and an increase in the information rate. These three factors are not quite independent and depending on the purpose for which a particular telemetry system is designed any one of these requirements will dominate over the others. As an example consider a space communication system where the energy available to the spacecraft is small due to weight and space limitations whereas the ground station has access to the most powerful transmitters. In this system the important requirement would be to minimize the spacecraft energy for a given overall probability of error or stating it differently to minimize the overall probability of error for a given energy level. It seems natural to assume that some form of feedback from the ground station to the spacecraft will be necessary to reduce the overall probability of error.

CHAPTER II

A COMMUNICATION SYSTEM WITHOUT FEEDBACK

For the purpose of this paper we shall consider only the performance of memoryless systems with the number of retransmissions arbitrarily truncated. We shall consider the transmission of phase-coherent orthogonal signals through a channel perturbed by additive white Gaussian noise. In other words, we shall deal with a group of signals where the cross-correlation between any two signals is zero, and the power spectral density of noise is a constant. If $s_i(t)$ is the signal transmitted with period = T seconds and $n(t)$ is the additive noise then the received signal $x(t)$ is given by,

$$x(t) = s_i(t) + n(t)$$

Let us assume without loss of generality the spacecraft wants to transmit one of $M = 2^n$ orthogonal, equiprobable and equal-energy signals.

$$P(s_i) = \frac{1}{M} \qquad i = 1,2,3,\ldots,M$$

$$\int_0^T s_i(t)s_j(t)dt = \begin{cases} 0 & i \neq j \\ E_s & i = j \end{cases}$$

The noise power spectral density for white Gaussian noise is given by,

$$S_{nn}(\omega) = \frac{N_o}{2} \qquad -\infty < \omega < \infty$$

Assume $s_1(t)$ was transmitted by the spacecraft. If we use the maximum likelihood decision scheme as shown in Figure 1, the probability that y_1 is the largest output is given by,[2]

$$P(C|s_1) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(y_1-E_s)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} \left(\int_{-\infty}^{y_1} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} dy\right)^{M-1} dy_1$$

where $\sigma^2 = \frac{N_o E_s}{2}$

Substituting $\alpha = \frac{y_1}{E_s}$ and $\beta = \frac{y}{E_s}$, for equiprobable signals the probability of correct decision is given by,[4]

$$P(C) = \int_{-\infty}^{\infty} f(\alpha - \sqrt{E_s}) \left(\int_{-\infty}^{\alpha} f(\beta)\, d\beta\right)^{M-1} d\alpha$$

where $f(\alpha) \triangleq \frac{1}{\sqrt{\pi N_o}} e^{-\frac{\alpha^2}{N_o}}$

A convenient plot of $P(C)$ vs. $\frac{E_b}{N_o}$ for various values of $M = 2^n$ is included here for reference[3] (Figure 2) where $E_b = \frac{E_s}{n}$ = energy per bit. Note, however, that if the number of signals M is not an exponent of 2, to get the energy per bit E_b, $n = \log_2 M$.

Now suppose the spacecraft retransmits the original signal, (i.e., simple redundancy) in our case s_1. If the second transmission is considered by itself then the probability of correct decision is the same as before ($P(C)$); however, when the two received signals x_1, x_2, are added to

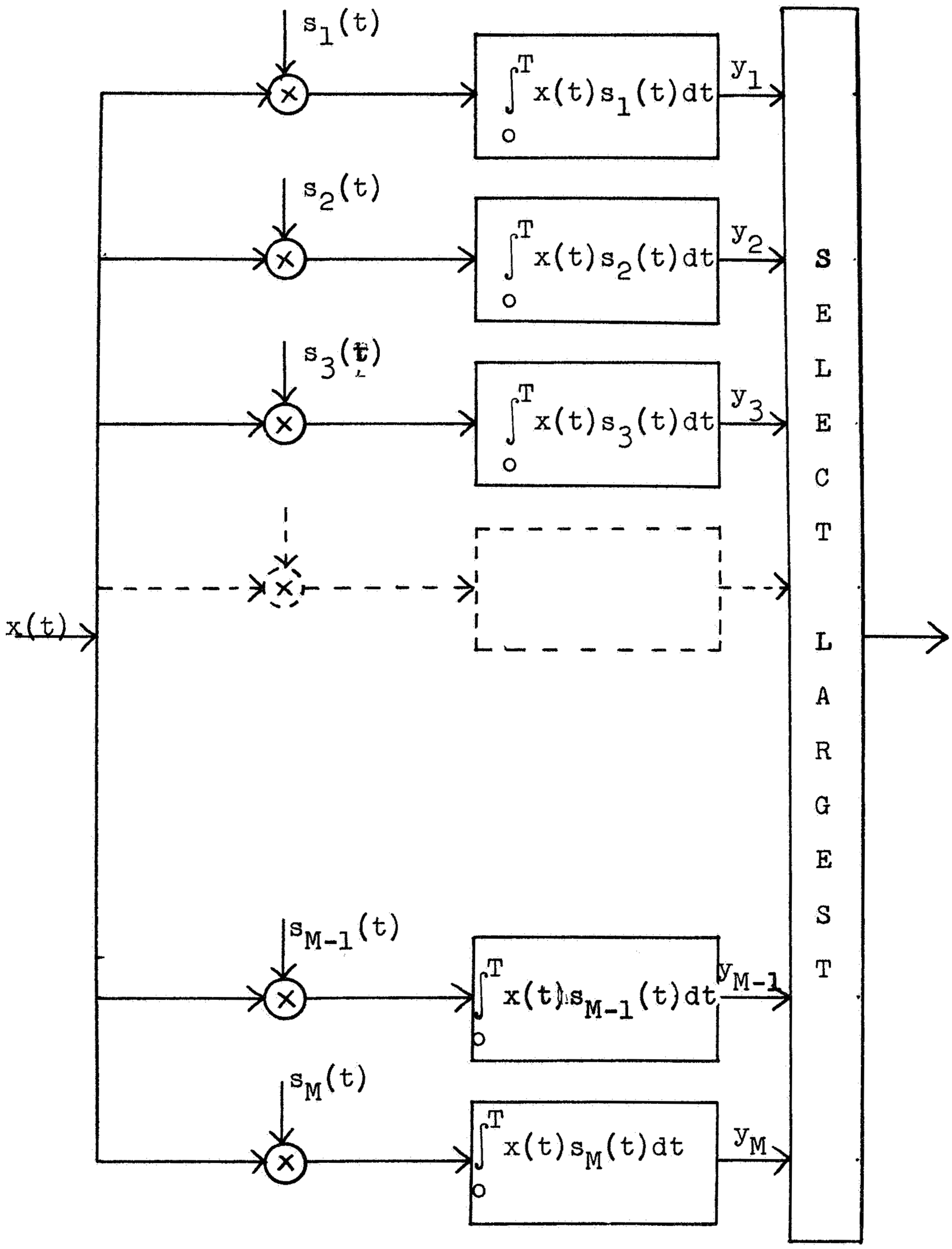

FIGURE 1. Maximum Likelihood Decision Scheme

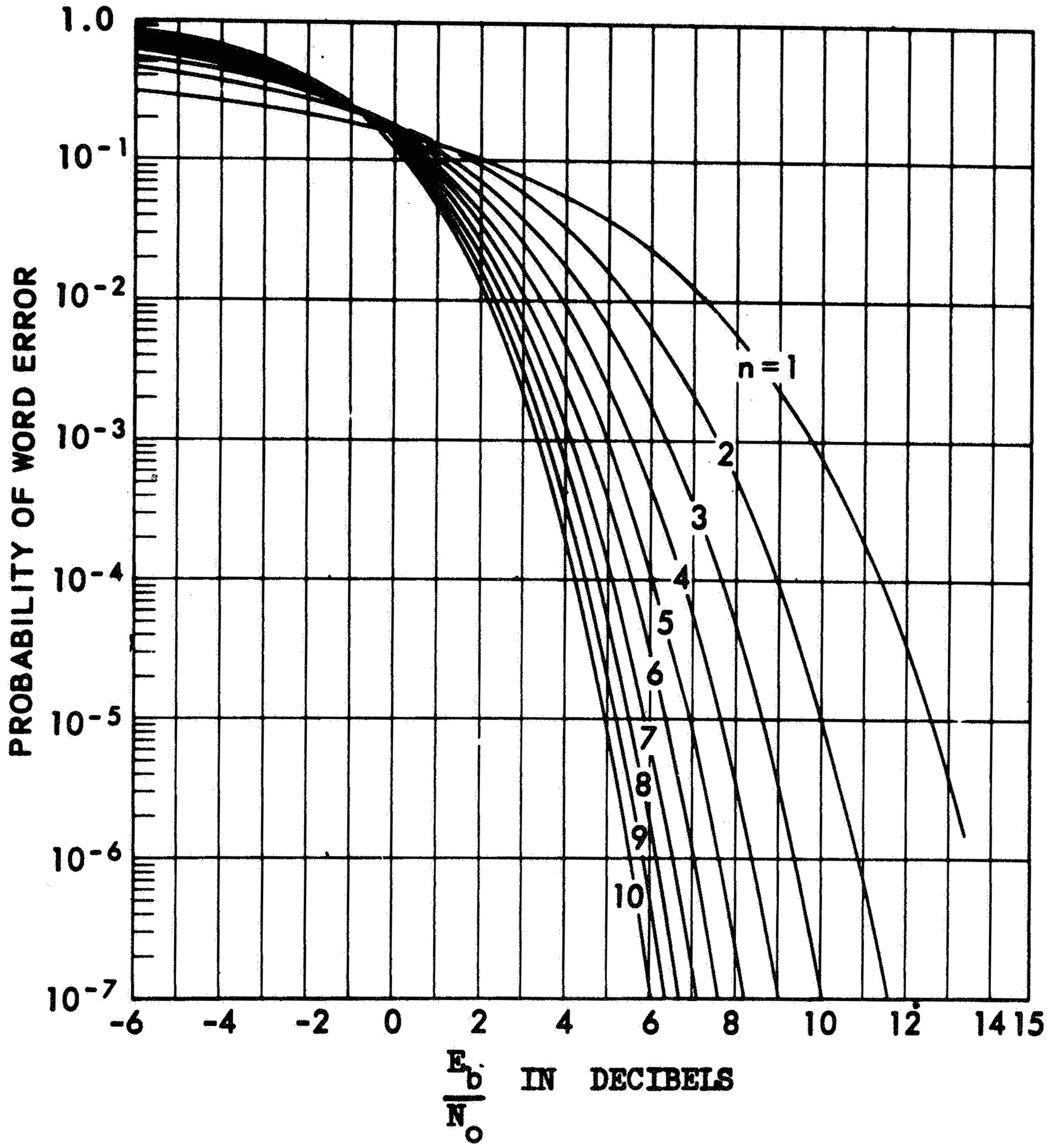

FIGURE 2. Word-Error Probability Curves for Gaussian Noise

form the input to the maximum liklihood receiver the probability of correct decision is greatly improved. When $x_1(t) + x_2(t) = 2s_1(t) + n_1(t) + n_2(t)$ is the new input to the maximum likelihood receiver the probability of correct decision is given by,

$$P_2(C) = \int_{-\infty}^{\infty} f(\alpha - \sqrt{2E_s}) \left(\int_{-\infty}^{\alpha} f(\beta)d\beta\right)^{M-1} d\alpha$$

where as before $f(\alpha) \triangleq \frac{1}{\sqrt{\pi N_o}} e^{-\frac{\alpha^2}{N_o}}$

As expected this probability is equal to the probability of correct decision if the spacecraft had transmitted the signal with twice the energy in one interval alone. For both these procedures the overall probability of error, $P(\epsilon) = 1 - P_2(C) = P_2(\epsilon)$ and the expected value of the energy required is $2E_s$.

CHAPTER III

COMMUNICATION SYSTEM WITH SIMPLE FEEDBACK

Two forms of feedback are available to us, 1) Decision Feedback, and 2) Information Feedback.

1) Decision Feedback as the name implies requires the receiver on the ground to decide when the transmission is to be repeated or terminated and what is going to be the final decision.

2) Information Feedback means the spacecraft transmitter decides when and what to retransmit upon receiving feedback from the ground station.

The key word for our purpose shall be adaptive feedback meaning a combination of both of the above to give the lowest probability of error. We shall first consider a simple adaptive feedback scheme as follows. This shall be referred to as Scheme 1 for comparison purposes:

1. Scheme 1

First Transmission: Spacecraft transmits 1 of M equiprobable, equal-energy orthogonal signals through a channel perturbed by white Gaussian noise. The receiver on the ground stores the information and employs maximum likelihood detection to decode the correct signal. The decoded signal is fed back to the spacecraft in a negligibly short period of time through a noisefree path. Since the ground station has unlimited energy compared to the spacecraft we are justified in assuming the feedback loop is virtually error-free and takes very little time compared to the spacecraft transmission period, T.

Second Transmission: The spacecraft decides whether the receiver on the ground made the correct decision or not. Depending on the outcome of this, the spacecraft transmits 1 of 2 equiprobable, equal-energy orthogonal signals s_a, s_b. The signals s_a, s_b may or may not be in the original set of M signals. For our purpose we shall not include these in the original set of M signals. The signal s_a corresponds to the message "accept the decision" and s_b corresponds to "reject the decision." The receiver on the ground decodes the message and feeds it back to the spacecraft through the error-free path. If the outcome was s_a the ground receiver accepts the signal decoded in the first transmission as the final decision and waits for the next message. Similarly, the spacecraft starts transmitting the next message on the list.

Third Transmission: If the outcome was s_b the spacecraft retransmits the original 1 of M orthogonal signals. The receiver on the ground adds the information stored during the first interval to this third interval and again employs maximum likelihood detection to decode the transmitted signal. For this particular truncated system, the ground receiver accepts this as the final decision and awaits the next message. Similarly the spacecraft stops retransmitting and goes on to the next message. Scheme 1 is represented schematically in Figure 3.

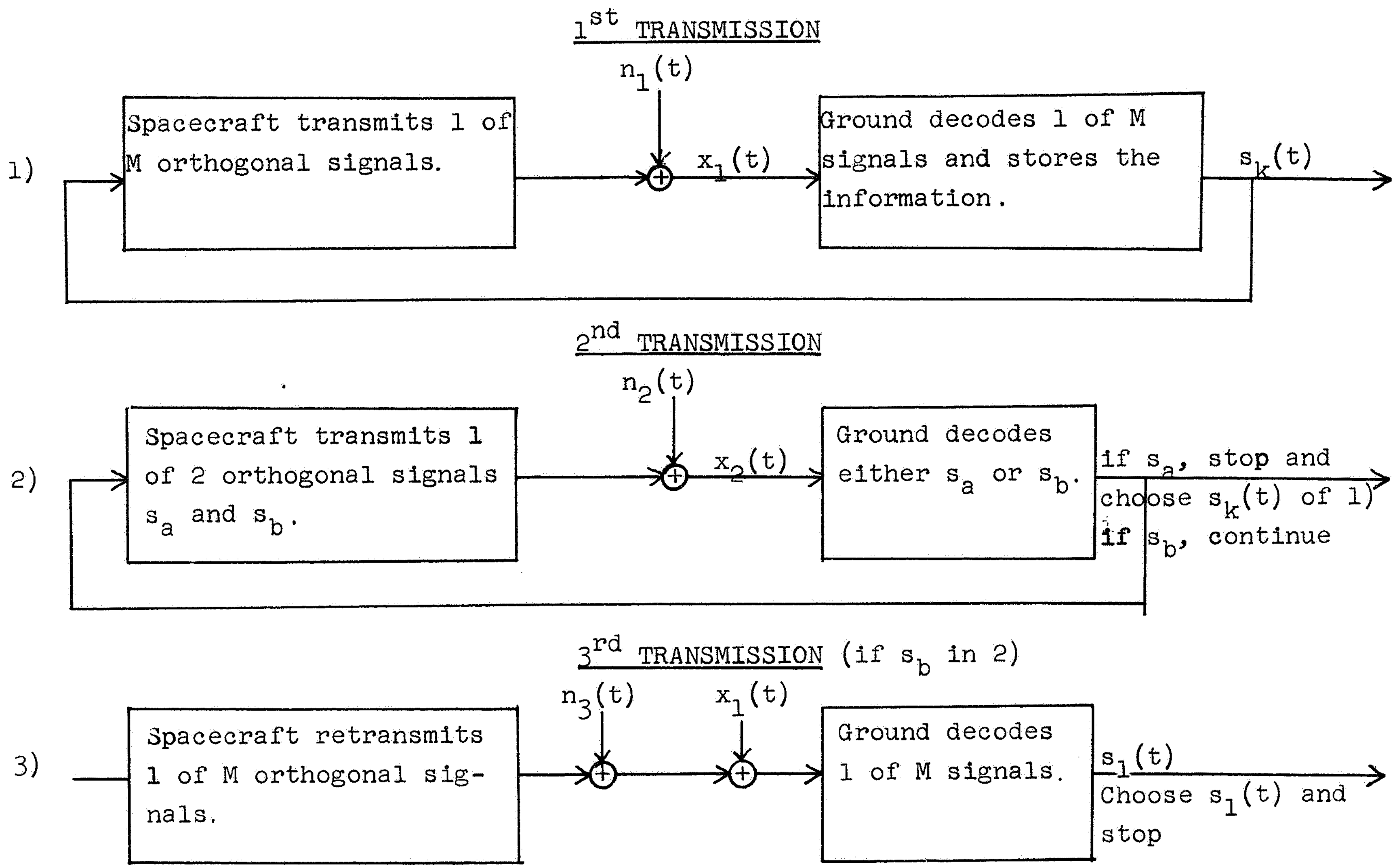

FIGURE 3. Scheme 1.

Let p_1 = probability of error in the first interval consisting of 1 of M orthogonal signals.

$\bar{p}_1 = 1 - p_1$

q_1 = probability of error in the second interval consisting of 1 of 2 orthogonal signals.

$q_1 = 1 - q_1$

p_2 = probability of error for 1 of M orthongonal signals for the first and the third intervals added together with the original signal present twice.

$\bar{p}_2 = 1 - p_2$

The overall probability or error can be obtained from the probability tree shown in Figure 4.

$$\langle P(\epsilon)\rangle = p_1 q_1 + p_1 \bar{q}_1 p_2 + \bar{p}_1 q_1 p_2$$

and the expected value of the required energy $\langle E_s \rangle =$ $(3 - p_1 q_1 - \bar{p}_1 \bar{q}_1) E_s$ which is approximately given by, $\langle E_s \rangle \stackrel{\circ}{=} 2E_s$ when $q_1 < p_1 < 0.1$.

Comparing the system without feedback with Scheme 1 we get the following results:

$\frac{E_s}{N_o}$ in db	Overall Error-Probability $\langle P(\epsilon)\rangle$	
	Scheme 1 with feedback	**System** without feedback
10.39 db, for M = 32	4.9×10^{-6}	4×10^{-5}
10.83 db, for M = 64	2.51×10^{-6}	3×10^{-5}
11.2 db, for M = 128	1.41×10^{-6}	1.1×10^{-5}

For both the above systems the expected value of the required energy is $2E_s$. For each value of M, the value of $\frac{E_s}{N_o}$

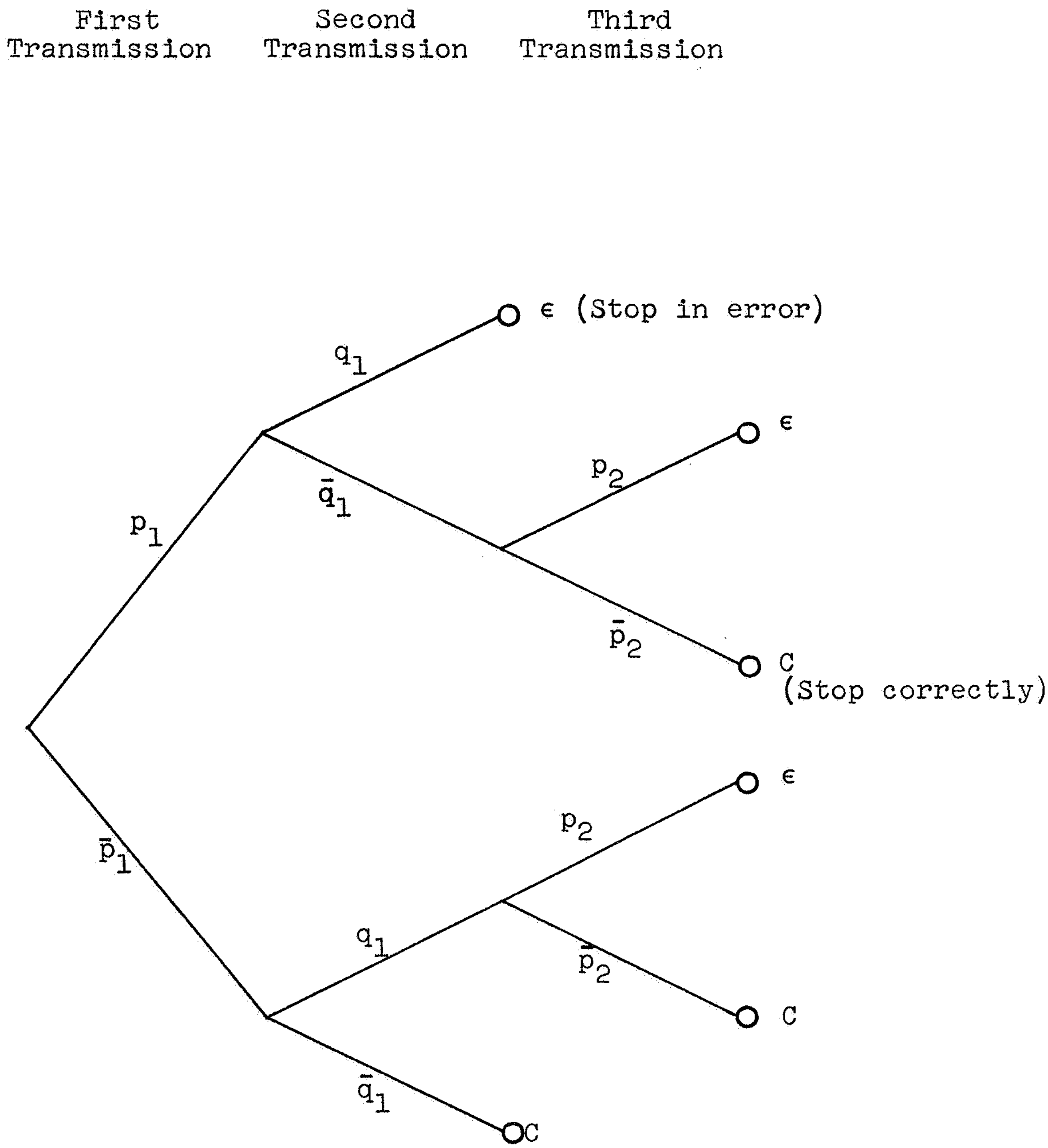

FIGURE 4. Probability of Error Tree for Scheme 1

corresponds to the word error probability of 0.01, (i.e., $p_1 = 0.01$)

The above figures show a vast improvement in the overall probability of error for the same expected value of signal energy by introducing simple feedback. The reason for improvement lies in the fact that Scheme 1 through the use of adaptive feedback allows us to transmit <u>1 out of 2 orthogonal signals</u> during the second interval instead of <u>1 out of M orthogonal signals</u> which has a higher probability of error, i.e., $p_1 > q_1$.

CHAPTER IV

FEEDBACK COMMUNICATION SYSTEM WITH A MORE COMPLEX SCHEME

In Chapter II we found an expression for the probability of correct decision based on the largest output of the maximum likelihood receiver.

$$P^{(1)}\ (C) = \int_{-\infty}^{\infty} f(\alpha - \sqrt{E_s})\ \left(\int_{-\infty}^{\alpha} f(\beta)d\beta\right)^{M-1} d\alpha$$

where as before $f(\alpha) \triangleq \frac{1}{\sqrt{\pi N_o}}\ e^{-\frac{\alpha^2}{N_o}}$

If we allow ourselves to include the largest two outputs of the maximum likelihood receiver instead of just one, the probability of error is reduced. We obtain a still lower probability of error if we pick the top three outputs instead of one.

Parallel to the discussion in Chapter II, the probability that the second largest output of the maximum likelihood receiver corresponds to the transmitted signal is given by,

$$P^{(2)}(C) = (M-1) \int_{-\infty}^{\infty} f(\alpha) \int_{-\infty}^{\alpha} f(\beta - \sqrt{E_s}) \left(\int_{-\infty}^{\beta} f(\gamma)d\gamma\right)^{M-2} d\beta d\alpha.$$

Similarly the probability that the third largest output corresponds to the correct signal is given by,

$$P^{(3)}(C) = (M-1)(M-2) \int_{-\infty}^{\infty} f(\alpha) \int_{-\infty}^{\alpha} f(\beta) \int_{-\infty}^{\beta} f(\gamma - \sqrt{E_s}) \left(\int_{-\infty}^{\gamma} f(\eta)d\eta\right)^{M-3} d\gamma d\beta d\alpha$$

The probability of correct decision, (i.e., one of the top three outputs of the maximum likelihood receiver corresponds to the transmitted signal) is given by,

$$P(C) = P^{(1)}(C) + P^{(2)}(C) + P^{(3)}(C).$$

For $M = 64$, $P^{(1)}(C)$, $P^{(2)}(C)$, $P^{(3)}(C)$ and $P(C)$ are listed[1] in Table 1. The probability of error, $P(\epsilon) = 1 - P(C)$, is considerably reduced.

We shall make use of this fact in the following adaptive feedback scheme which shall be referred to as Scheme 2.

1. Scheme 2

First Transmission: Spacecraft transmits 1 of M equiprobable, equal-energy orthogonal signals through a channel perturbed by white Gaussian noise. One of these M signals, s_b, shall correspond to the message "retransmit." The ground receiver stores the information and employs maximum likelihood detection to select the three largest outputs (excluding y_b corresponding to s_b). Rename the three most likely signals s_1, s_2, s_3 in descending order of probability to correspond to the top three outputs. All three signals are fed back to the spacecraft through the error-free path.

Second Transmission: Spacecraft, upon deciding whether the ground made the correct decision or not, transmits 1 of 4 orthogonal signals s_1, s_2, s_3, and s_b. If the correct signal is included in s_1, s_2, and s_3 the spacecraft retransmits

Error Rate $1 - P^{(1)}(C)$	$P^{(1)}(C)$	$P^{(2)}(C)$	$P^{(3)}(C)$	$P(C)$
2×10^{-1}	0.799878	0.091618	0.0382	0.929696
1×10^{-1}	0.900082	0.055153	0.0191	0.974335
1×10^{-2}	0.990011	0.007499	0.00147	0.998980
1×10^{-3}	0.998999	0.000860	0.000096	0.999955
1×10^{-4}	0.999900	0.000092	0.000006	0.999998

TABLE 1. Probability of Correct Detection for the First, the Second, and the Third Largest Output of the Receiver.

the original signal. If the correct signal is not included in s_1, s_2, and s_3 the spacecraft transmits s_b. The receiver on the ground stores the information and decodes the message, call it s_i ($i = 1,2,3$, and b). If $s_i = s_b$ then the ground station feeds it back to the spacecraft through the error-free path. If $s_i \neq s_b$ then the receiver on the ground adds the previously stored first, and the second intervals together and uses maximum likelihood detection to obtain s_j ($j = 1,2,3$ and b), which is fed back to the spacecraft. If $s_j \neq s_b$ the ground receiver accepts s_j as the final decision and waits for the next message. Similarly the spacecraft moves on to the next message on the list.

Third Transmission: If $s_i = s_b$ or $s_j = s_b$ the spacecraft retransmits the original 1 of M orthogonal signals. The ground receiver adds the stored first and third intervals and uses maximum likelihood detection to obtain the final decision. Both the spacecraft and the ground receiver move on to the next message. For a schematic representation see Figure 5.

Let $\bar{p}_1$ = probability that the correct signal is in the top three of M orthogonal signals

$p_1 = 1 - \bar{p}_1$

q_1 = probability of error for 1 of 4 orthogonal signals

$\bar{q}_1 = 1 - q_1$

r_1 = probability of error for 1 of 4 orthogonal signals for two intervals added together with the original signal present only <u>once</u>.

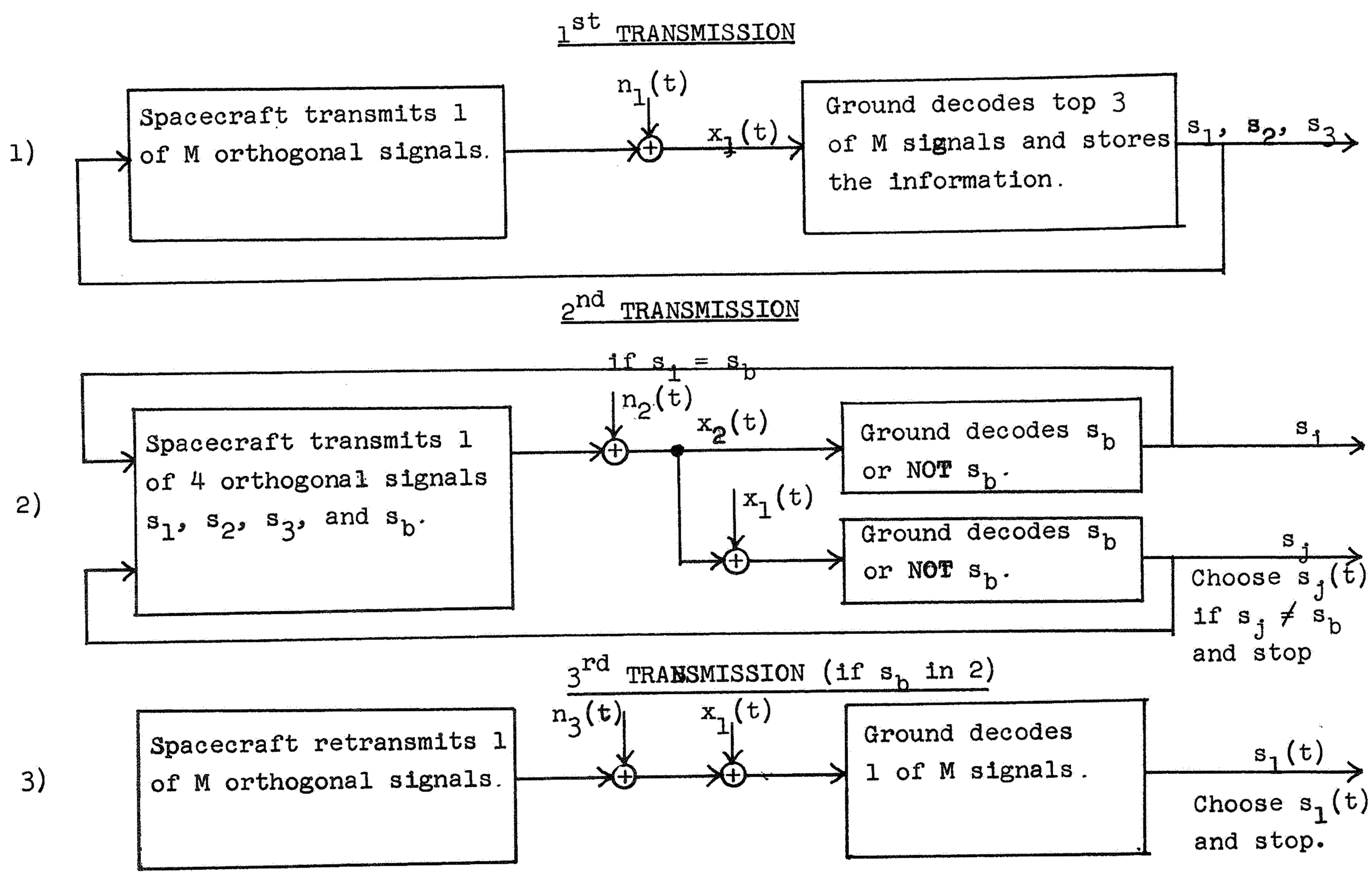

FIGURE 5. Scheme 2

$\bar{r}_1 = 1 - r_1$

r_2 = probability of error for 1 of 4 orthogonal signals for two intervals added together with the original signal present twice.

$\bar{r}_2 = 1 - r_2$

p_2 = probability of error for 1 of M orthogonal signals for two intervals added together with the original signal present twice.

$\bar{p}_2 = 1 - p_2$

The overall probability of error is obtained from the error tree in Figure 6.

$$<P(\epsilon)> = p_1q_1r_1 + p_1q_1\bar{r}_1p_2 + p_1\bar{q}_1p_2 + \frac{1}{3}\bar{p}_1q_1p_2 + \frac{2}{3}\bar{p}_1(\frac{2}{3}q_1 + \bar{q}_1)r_2 + \frac{1}{3}\bar{p}_1(\frac{2}{3}q_1 + \bar{q}_1)r_2p_2$$

and the overall expected energy required is

$$<E_s> = (3 - \bar{p}_1\bar{q}_1\bar{r}_2 - \frac{2}{3}\bar{p}_1\bar{q}_1\bar{r}_2 - p_1q_1r_1)\, E_s$$
$$\doteq 2E_s \text{ if } q_1 < p_1 < r_1 < 0.1.$$

Comparing the system without feedback with Scheme 1 and Scheme 2 we get the following results

	Overall Probability of Error $<P(\epsilon)>$		
	Scheme 2	Scheme 1	No Feedback
$M = 64$, $\frac{E_s}{N_o}$ in db = 10.83 db	4.97×10^{-7}	2.51×10^{-6}	3×10^{-5}

The expected value of the required energy is $2E_s$ for the above three systems. The value of $\frac{E_s}{N_o}$ corresponds to the word error probability of 0.01, (i.e., $P^{(1)}(C) = 0.99$).

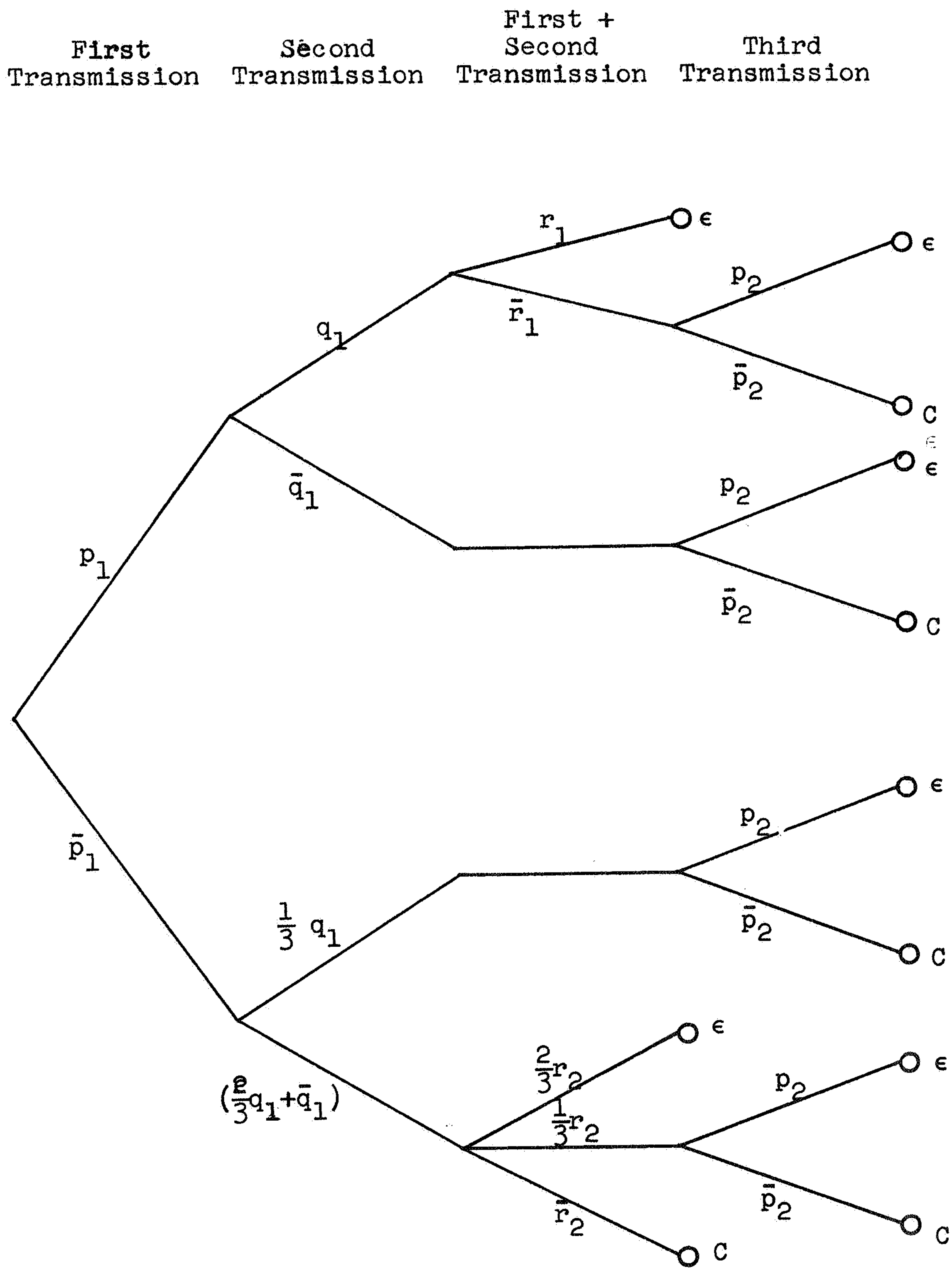

FIGURE 6. Probability of Error Tree for Scheme 2.

The above results clearly show a considerable improvement over Scheme 1 with simple feedback. Reduction in error can be attributed to many factors; the most important one being the fact that in Scheme 1 the spacecraft transmits an "accept" or "reject" signal during the second interval whereas in Scheme 2 the spacecraft retransmits the original signal itself instead of the "accept" signal if the receiver made a correct decision, otherwise a "retransmit" singal which is equivalent to the "reject" signal. The other error reductions come from choosing the top three signals instead of the top signal which has a higher probability of error.

2. Scheme 3

The probability of error is significantly reduced by slightly modifying Scheme 2. During the Second Transmission if the output of the maximum likelihood receiver after adding the first and the second interval does not agree with the output for the second interval by itself, request the spacecraft to retransmit the signal, i.e., if $s_j \neq s_i$ feed back s_D. Therefore during the Third Transmission if $s_i = s_b$ or $s_j = s_b$ or $s_j \neq s_i$ the spacecraft retransmits the original 1 of M orthogonal signals. The overall probability of error in this case is given by the probability tree in Figure 7.

$$\langle P(\epsilon)\rangle = \frac{1}{3}\,p_1q_1r_1 + \frac{2}{3}\,p_1q_1r_1p_2 + p_1q_1\bar{r}_1p_2 + p_1\bar{q}_1p_2 + \frac{1}{3}\,\bar{p}_1q_1p_2 + \frac{2}{9}\,\bar{p}_1q_1r_2 + \frac{2}{3}\,\bar{p}_1q_1p_2\left(\frac{2}{3}\,r_2 + \bar{r}_2\right) + \bar{p}_1\bar{q}_1r_2p_2$$

and the overall expected energy required is,

$$\langle E_s\rangle = \left(3 - \bar{p}_1\bar{q}_1\bar{r}_2 - \frac{2}{9}\,\bar{p}_1q_1r_2 - \frac{1}{3}\,p_1q_1r_1\right)E_s$$
$$\cong 2E_s \text{ if } q_1 < p_1 < r_1 < 0.1$$

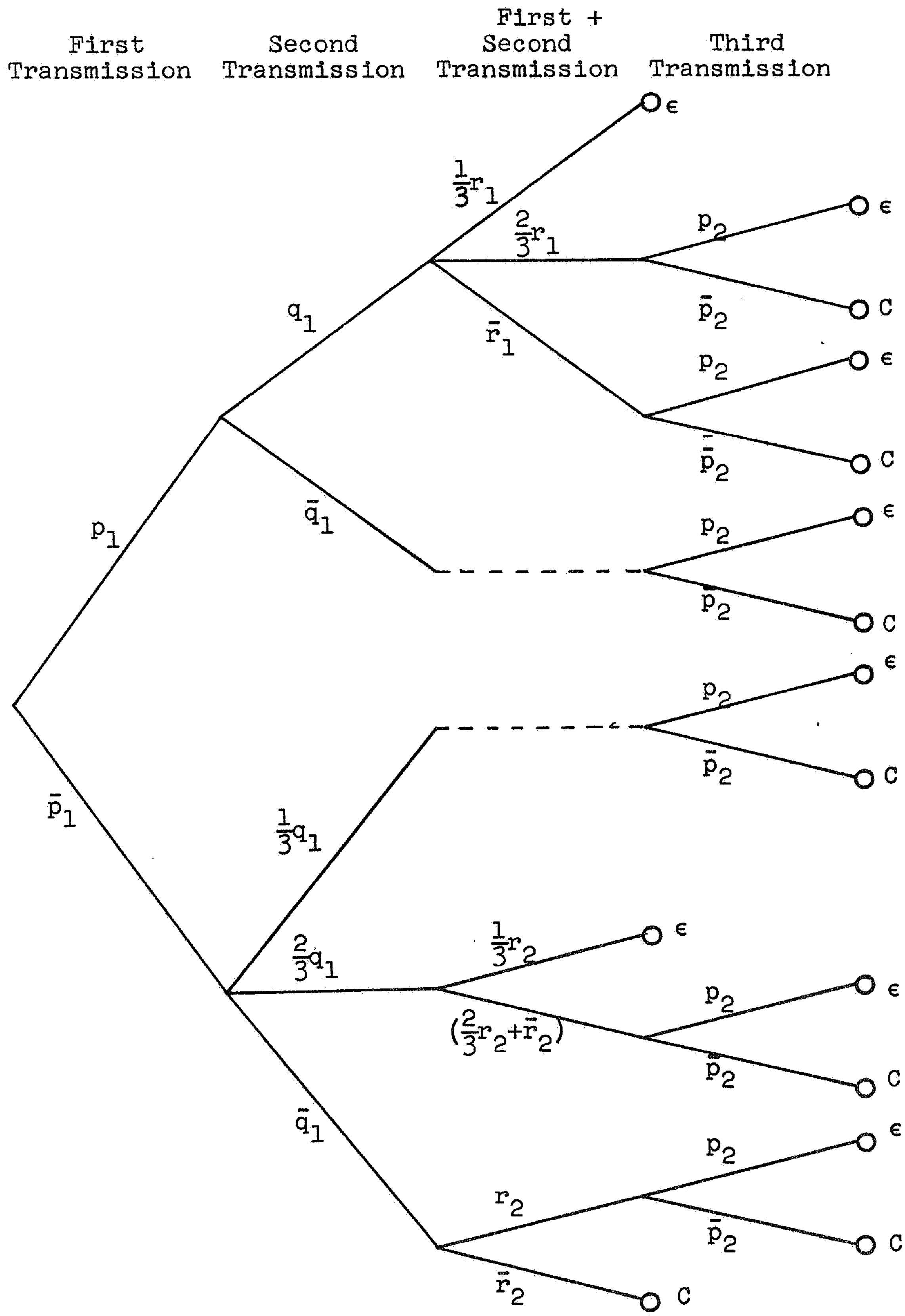

FIGURE 7. Probability of Error Tree for Scheme 3.

where p_1, q_1, r_1, r_2, p_2 are the same as in Scheme 2.

Comparing all three Schemes with the system without feedback we get the following results.

	Overall Error-Probability $\langle P(\epsilon)\rangle$			
M = 64	Scheme 3	Scheme 2	Scheme 1	No feedback
$\frac{E_s}{N_o}$ in db=11.03db	5.63×10^{-8}	4.97×10^{-7}	2.51×10^{-6}	3×10^{-5}
$\frac{E_s}{N_o}$ in db=8.53db	1.59×10^{-4}	2.54×10^{-4}	8.37×10^{-4}	4.2×10^{-3}

The values 11.03db and 8.53db of $\frac{E_s}{N_o}$ in db correspond to the word-error probabilities of 1×10^{-2} and 1×10^{-1} respectively. The expected value of the required energy is $2E_s$ as in previous comparisons.

Of the 3 feedback schemes considered, Scheme 3, (i.e., the modified form of Scheme 2) is definitely the best as far as the overall probability of error is concerned.

CHAPTER V

CONCLUSION

Altogether we have considered 3 feedback schemes and compared their performance with a system without feedback. Some of the assumptions made at the beginning may be discarded. For all the feedback schemes we have arbitrarily truncated the number of transmissions to three, but this need not be the case. All the conclusions reached would be unaffected if the truncation was eliminated. It appears that we would obtain similar results if we consider non-phase-coherent orthogonal codes or for that matter biorthogonal codes where a reduction in hardware is important. This analysis could also easily be extended to Rayleigh noise.

We have ignored the feedback time required by the ground station to transmit a signal back to the spacecraft. This was justified on the basis that the ground station has infinite energy compared to the spacecraft energy. Even if the feedback time was not negligible when compared to the spacecraft transmission time T we can easily get around this problem by employing a different format. We may arrange for the spacecraft to transmit the next signal while the previous signal is still being fed back to the spacecraft and decoded.

All the systems were compared on an equal energy basis, however, it should not be very difficult to compare them on an equal error-rate basis. It would require the programming of various error functions and the error trees. The results should be consistent with the equal energy analysis.

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