NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

*Technical Memorandum No. 33-306*

# *Analysis of Propellant Slosh Dynamics and Generation of an Equivalent Mechanical Model For Use in Preliminary Voyager Autopilot Design Studies*

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December 1, 1966

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

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## ABSTRACT

The linearized dynamic equations of propellant motion in the regime dominated by gravity force have been developed for a cylindrical tank approximation of the Lunar Excursion Module Descent Propulsion System (LEMDS) propellant tanks. An equivalent mechanical model consisting of a fixed mass and an infinite series of springs and masses has been generated. A computer program has been developed to calculate the parameters required to describe the equivalent mechanical model for an assumed spacecraft flying an assumed mission profile. These parameters are presented as a function of time for three mission profiles and can be used for preliminary analysis of *Voyager* closed-loop autopilot designs. Transfer functions for the propellant dynamics are also presented.

## I. INTRODUCTION

Propellant sloshing is a potential source of spacecraft disturbances that may adversely affect the stability of the attitude control system during both powered and unpowered phases of flight.

Sloshing refers to the free surface oscillations of a fluid in a partially filled tank. These oscillations result from lateral and longitudinal displacements or angular motions of the spacecraft. The magnitude of the free surface oscillations, and hence the forces and moments acting on the vehicle as a result of propellant motion, depends upon several factors. The most significant parameters that influence propellant slosh are the following:

1. Tank geometry
2. Propellant properties
3. Slosh damping
4. Depth of propellant in the tank
5. Acceleration field
6. Perturbed motion of the spacecraft

Since current *Voyager* spacecraft designs envisage liquid propulsion systems with propellant loads of up to 60% of the total planetary vehicle mass, sloshing can be expected to be a potential source of disturbances.

Various theoretical and analytical investigations have shown that sloshing waves can be large enough to produce forces whose magnitude and phase can cause instabilities in closed-loop control systems unless more liquid damping than that provided by the wiping action of the propellant on the tank walls is made available. The addition of baffles to the tanks will provide the additional damping, if required, and limit the resultant slosh to an acceptable value.

Analysis of the coupled response of the spacecraft attitude control system and propellant free surface oscillations requires either an equivalent mechanical model for the fluid system or a transfer function for the forces and moments produced by the propellant motion in response to an input excitation. The effort reported herein has addressed itself to the matter of obtaining the aforementioned analytical tools for use in the analysis of control system dynamics.

# II. ANALYSIS

## A. Statement of the Problem

The determination of the forced oscillations of a fluid that is assumed to be inviscid and irrotational in a tank of arbitrary shape consists of a solution for a velocity potential function that satisfies Laplace's equation in the region of space occupied by the fluid and, in addition, satisfies certain boundary conditions on the surfaces enclosing the region occupied by the fluid. Since the situation consists of a boundary value problem in differential equations, the solution can become extremely difficult to obtain in closed form if the geometry of the problem cannot be simplified so as to show symmetry in a standard coordinate dimension, i.e., Cartesian, spherical, or cylindrical.

The LEMDS propellant tanks are the subject of the current investigation. The LEMDS tanks consist of two hemispherical end caps with a cylindrical section between. Since this geometry does not lead to a simple mathematical description of the bounding surface, an equivalent tank with symmetry in cylindrical coordinates and a flat bottom was constructed as a means for approximating the slosh properties of the actual tank. The equivalent cylindrical tank has the properties that its diameter is equivalent to the diameter of the cylindrical section of the LEMDS tank and its internal volume is equivalent to the internal volume of the LEMDS tank. A graphical representation of this approximation is shown in Fig. 1.

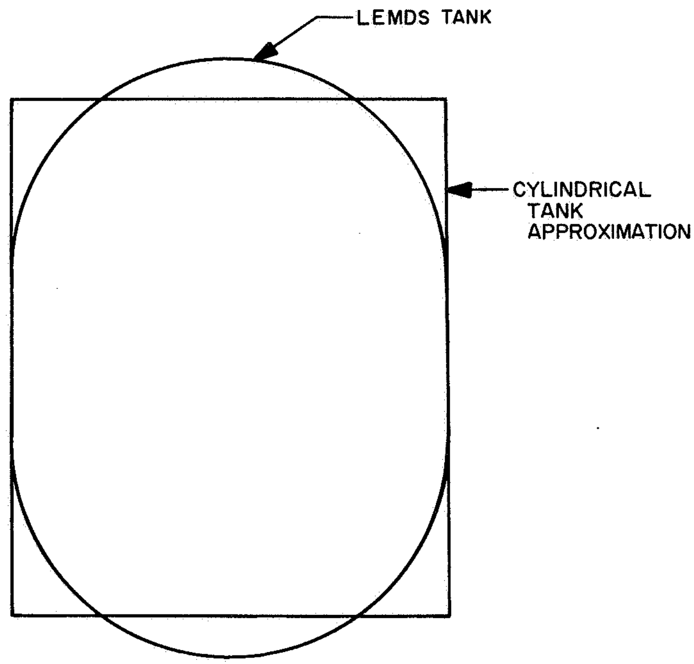


**Fig. 1. Cylindrical tank approximation of LEMDS propellant tank**

The analysis to date has been concerned with propellant motion during the powered phases of the mission. An analysis relative to the propellant–autopilot control loop interactions during the cruise portions of the flight presents a more difficult problem but will subsequently be investigated.

## B. Regime of Validity of Analysis

It is of interest to determine *a priori* what the relevant forces are that will influence the fluid motion so that irrelevant forces can be neglected in the dynamic equations, therefore simplifying the problem. Viscous forces will be neglected since the Reynolds number for the assumed tanks is of the order of $10^5$. Surface tension forces can, however, become important in low-gravity environments. An estimation of the Bond number (ratio of gravity to capillary forces) furnishes information on the relative importance of surface tension and gravity forces in the problem at hand. The Bond number based on the tank radius is given by

$$B_0 = \frac{\rho g r^2}{\sigma g_c} \tag{1}$$

For the LEMDS tanks, $r = 2.25$ ft, $\rho_{N_2O_4} = 90.48$ lbm/ft$^3$, and $\sigma = 2.18 \times 10^{-3}$ lbf/ft.

Therefore,

$$B_0 = 2.1 \times 10^5 \left(\frac{g}{g_c}\right) = 2.1 \times 10^5 \left(\frac{F}{M}\right) \tag{2}$$

Therefore, the thrust/mass ratio of the spacecraft would have to be of the order of $10^{-5}$ g's before the Bond number of an oxidizer tank would approach unity. Although surface tension may well influence the hydrodynamics for Bond numbers on the order of 100, it can still be seen that for any reasonable thrust/mass ratio (say, greater than $10^{-3}$ g's) one would not expect surface tension to exert any detectable influence upon the propellant motion during the powered phases of the mission. Therefore, surface tension forces have also been neglected in the current analysis.

In addition to the disclaimers above, it must be asserted that the analysis completed to date treats only the case where the liquid responds to a purely lateral input excitation. This problem has been studied extensively and reported in the literature. The case of liquid response to a longitudinal excitation has also been explored, but to a lesser degree. The case of combined lateral, longitudinal, and pitching motion is a more difficult analytical problem and has not been treated to a significant degree. Current effort is, however, being directed toward developing the analysis necessary to handle the problem of combined motion.

### C. *Construction of the Fluid Dynamic Solution*

The problem at hand can be stated then, as: determine a velocity potential function $\phi$ that satisfies the following conditions:

$$\nabla^2\phi = 0 \quad \text{in} \quad \begin{cases} 0 \leqq r \leqq a \\ 0 \leqq \theta \leqq 2\pi \\ -h \leqq z \leqq 0 \end{cases} \tag{3}$$

subject to the boundary conditions that

$$\frac{\partial \phi}{\partial r} = 0 \qquad \text{on} \qquad r = a \tag{4}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on} \qquad z = -h \tag{5}$$

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = -\ddot{u}_0 x \qquad \text{on the free surface} \tag{6}$$

The complete solution for this problem is presented in Appendix A. Pertinent results will be presented here.

In Appendix A, the solution for the velocity potential function of the *mn* mode of slosh that satisfies the fixed boundary conditions was found from Eq. (A-48) to be

$$\phi_{mn} = [C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta)]\, J_m(\lambda_{mn} r)\frac{\cosh[\lambda_{mn}(z+h)]}{\cosh(\lambda_{mn} h)}\, T_{mn}(t) \tag{7}$$

where $T_{mn}(t)$ is an unknown function of time, which depends upon the input excitation, through the free surface boundary conditions. The most general solution is, then,

$$\phi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}$$

In certain cases, all but one or two of the *mn* modes may be neglected. In the case of a lateral input excitation, experiment has shown that the first asymmetric mode (associated with the values of $m = 1$, $n = 1$) will be the predominant mode. Less than 3% of the total sloshing mass is usually associated with the higher-order ($n = 2, 3, \cdots$) modes.

If the lateral excitation is assumed to be sinusoidal, it can be represented as

$$u_0 = \epsilon_{0\omega}\cos(\omega t) \tag{8}$$

The unknown function of time then becomes

$$T_{1n} = \frac{\epsilon_0 \omega^3 F_{1n}}{p_{1n}^2 - \omega^2}\cos(\omega t) \tag{9}$$

and the velocity potential becomes

$$\phi = \sum_{n=1}^{\infty} \frac{2\epsilon_0 \omega^3 a}{(\lambda_{1n}^2 a^2 - 1)(p_{1n}^2 - \omega^2)} \frac{J_1(\lambda_{1n} r)}{J_1(\lambda_{1n} a)} \frac{\cosh[\lambda_{1n}(z+h)]}{\cosh(\lambda_{1n} h)} \cos\theta \cos(\omega t) \tag{10}$$

where only the first asymmetric mode has been considered.

In Eq. (10), above, $p_{1n}$ is the natural circular slosh frequency and is given by

$$p_{mn} = [g\lambda_{mn} \tanh(\lambda_{mn} h)]^{1/2} \tag{11}$$

and $\omega$ is the frequency of the forcing function. It is apparent from inspection of Eq. (10) that, as $\omega$ approaches $p$, the solution becomes unbounded. Actually, in the region $|p| - |\omega| < \epsilon$ where $\epsilon$ is a small number the linearized theory breaks down and the current analysis will not apply.

### D. Construction of an Equivalent Mechanical Model

An equivalent mechanical model can be developed by performing the following operations. If we take Euler's equation in the form

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}(\nabla\phi \cdot \nabla\phi) + \dot{u}_0 x + \dot{v}_0 y + \dot{w}_0 z + gz + \frac{p}{\rho} = 0 \tag{12}$$

and, linearizing by assuming that the squares of the velocity terms can be neglected and, also, that $\dot{v}_0 = \dot{w}_0 = 0$, solve for the pressure term, we get

$$p(t) = \rho\left(\frac{-\partial\phi}{\partial t} - \ddot{x}_0 r \cos\theta - gz\right) \tag{13}$$

If the static pressure term $gz$ is neglected, we can take the Laplace transform of Eq. (13) to obtain

$$p(s) = -\rho\,[s\phi(s) + s^2 r \cos\theta\, x_0(s)] \tag{14}$$

Further, taking the transform of the velocity potential function (Eq. 7), we get

$$\phi(s) = \left[\frac{-2as^3 x_0(s)}{(s^2 + p^2) J_1(\lambda_n a)(\lambda_n^2 a^2 - 1)}\right] J_1(\lambda_n r) \frac{\cosh[\lambda_n(z+h)]}{\cosh(\lambda_n h)} \cos\theta \tag{15}$$

Substituting Eq. (15) into Eq. (14) and integrating the $x$-component of the pressure force over the wetted surface of the tank, we obtain the transform of the sum of the forces in the $x$-direction:

$$F_x(s) = \int_{z=-h}^{0} \int_{\theta=0}^{2\pi} p(t)\, a \cos\theta \, d\theta \, dz \tag{16}$$

or, performing the indicated operations,

$$F_x(s) = -ms^2 x_0(s) + m\left[\frac{2s^4 x_0(s) \tanh(\lambda_n h)}{\lambda_n h (s^2 + p^2)(\lambda_n^2 a^2 - 1)}\right] \tag{17}$$

Similarly, by integrating the infinitesimal pressure forces multiplied by their perpendicular distances from the $y$-axis, we obtain the total moment around the $y$-axis:[1]

$$M_0(s) = \int_{r=0}^{a} \int_{\theta=0}^{2\pi} p(s) r^2 \cos\theta \, d\theta \, dr + \int_{\theta=0}^{2\pi} \int_{z=-h}^{0} p(s)\, za \cos\theta \, d\theta \, dr \tag{18}$$

[1]As shown in Fig. A-3.

where the first integrand is evaluated at $z = -h$ and the second at $r = a$. Therefore, we obtain

$$M_0(s) = m\left(\frac{h}{2} - \frac{a^2}{4h}\right) s^2 x_0(s) + m s^4 x_0(s) \left\{\frac{2\,[2 - \cosh(\lambda_n h)]}{\lambda_n^2 h\,(\lambda_n^2 a^2 - 1)(s^2 + p^2)\cosh(\lambda_n h)}\right\} \tag{19}$$

Since experimental studies have shown that the fluid velocity decreases exponentially with depth (a fact that is verified by the exponential dependence of the velocity potential function upon depth through the hyperbolic functions, see Eq. 15) one might suspect that the fluid dynamics of propellant slosh in response to a lateral input excitation might be represented by a fixed mass and an infinite series of oscillating spring masses. Such a system is shown schematically in Fig. 2. The infinite series of spring masses can be chosen so that each represents a single slosh mode and the series can, therefore, be truncated at any desired accuracy. In practice it has been found that usually only one spring mass is required since only a small percentage of the total slosh mass is associated with higher-order modes.

If one develops the equations of motion for this spring–mass system, the following expressions are obtained (in transform notation):

$$F_x(s) = s^2 x_0(s)\,(m_0 + \Sigma\, m_n) + s^4 \sum \frac{m_n x_0(s)}{s^2 + \omega_n^2} \qquad n = 1, 2, \cdots \infty \tag{20}$$

$$M_0(s) = -s^2 x_0(s)\left[m_0 d_0 + \sum m_n\left(d_n + \frac{g}{\omega_n^2}\right)\right] + s^4 x_0(s) \sum \frac{m_n\left(d_n + \frac{g}{\omega_n^2}\right)}{s^2 + \omega_n^2} \tag{21}$$

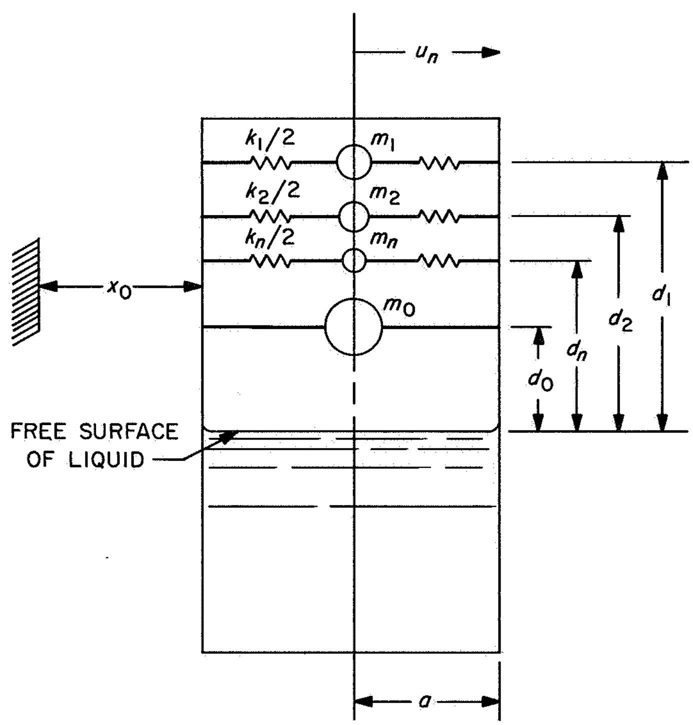


**Fig. 2. Equivalent spring-mass system for propellant slosh in a cylindrical tank**

Comparison of Eq. (17) with Eq. (20) and Eq. (19) with Eq. (21) shows that, for dynamic similarity between the fluid system and the equivalent mechanical system to exist, the following relationships relative to the parameters describing the mechanical model must be satisfied:

$$\omega_n = p_n \tag{22}$$

$$m_n = mA_n \tag{23}$$

$$m_0 = m(1 - \Sigma A_n) \tag{24}$$

$$d_n = \frac{2[1 - \cosh(\lambda_n h)]}{\lambda_n \sinh(\lambda_n h)} \tag{25}$$

$$d_0 = \frac{B - \Sigma B_n}{1 - \Sigma A_n} \tag{26}$$

where

$$A_n = \frac{2 \tanh(\lambda_n h)}{\lambda_n h(\lambda_n^2 a^2 - 1)}$$

and $B$ and $B_n$ are defined by Eqs. (A-138) and (A-137) respectively. If the parameters of the mechanical model are so constructed that Eqs. (22) through (26) above are satisfied, then the model will, in the limit of infinite series, exactly represent the assumed fluid system.

## E. *Generation of Slosh Model for Typical Voyager Missions*

Since the parameters describing the equivalent mechanical model are functions of tank geometry, propellant properties, depth of propellant in the tank, and the applied acceleration vector, one must hypothesize some gross spacecraft properties and a mission model before the propellant dynamics can be simulated. One must then numerically "fly" the assumed mission to determine how the propellant slosh model parameters vary with burn time during the various maneuvers.

An IBM 1620 computer program has been developed to perform the calculations necessary to "fly" an assumed mission and determine the parameters necessary to describe spacecraft slosh model dynamics as a function of time from ignition during a series of propulsive maneuvers. One output of the program is in the form of punched cards that contain the parameters relating to the spacecraft characteristics and equivalent mechanical slosh model data as a function of time. These data have been used as input to computer programs to analyze the interactions of total spacecraft dynamics with closed-loop autopilot designs.

## F. *Discussion of Results*

Figures 3 through 11 show the results of these calculations for a typical spacecraft midcourse maneuver of 200 m/sec. The spacecraft characteristics assumed for these computations are typical for a 1973 mission with a 3,000-lb capsule. The following specific characteristics were assumed:

| | Mass, lbm |
|---|---|
| Spacecraft bus and payload | 2,500 |
| Spacecraft propulsion | 15,000 |
| Flight capsule | 3,000 |
| Total planetary vehicle | 20,500 |

It was further assumed that 12,914 lbm of propellant was tanked and that this propellant was contained in four

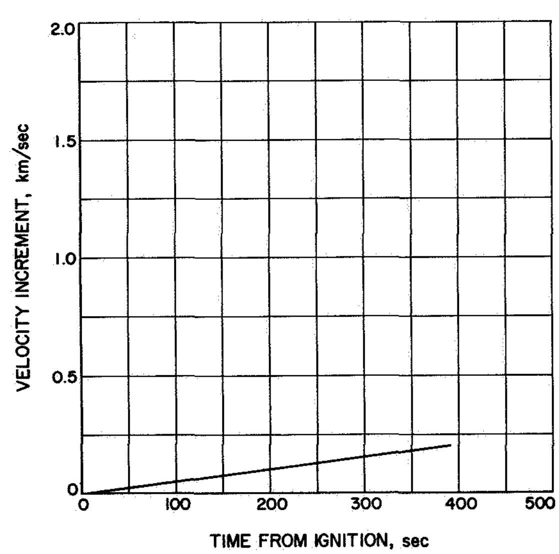


**Fig. 3. Velocity increment, midcourse maneuver**

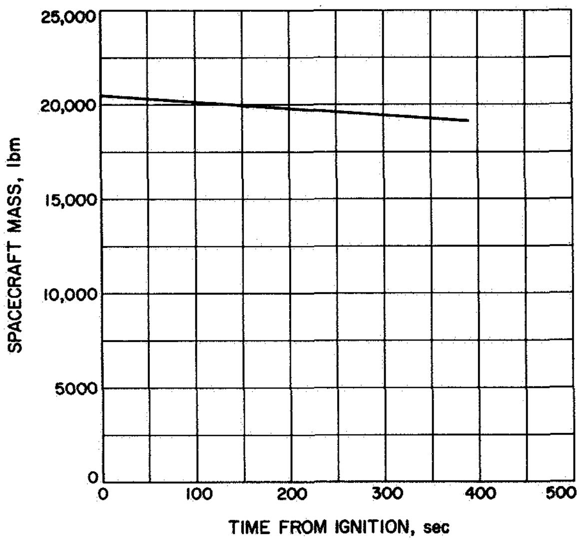


**Fig. 4. Spacecraft mass (midcourse maneuver)**

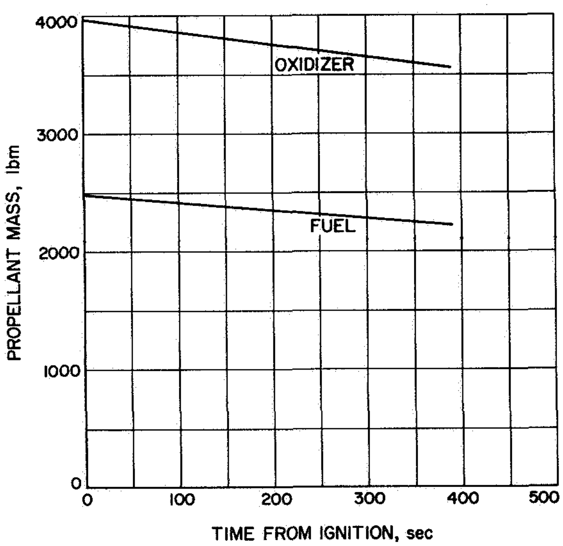


**Fig. 5. Fuel and oxidizer mass (midcourse maneuver)**

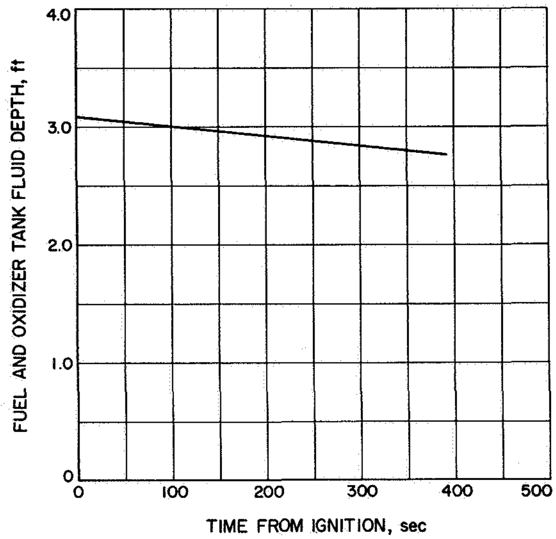


**Fig. 6. Fuel and oxidizer tank fluid depth (midcourse maneuver)**

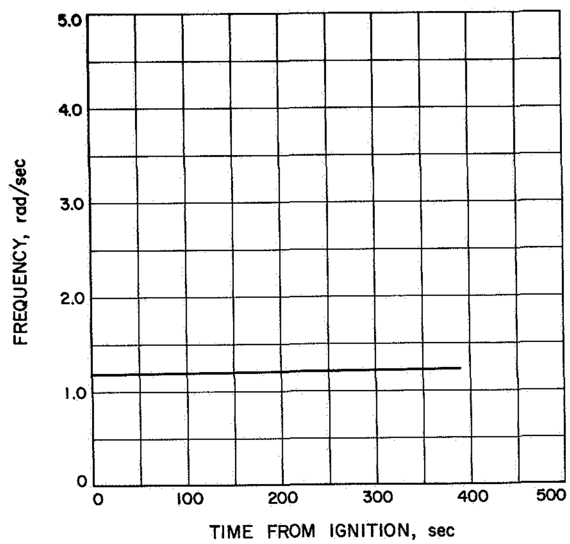


**Fig. 7. Fuel and oxidizer tank natural slosh frequencies (midcourse maneuver)**

equal-volume cylindrical tanks of 51-in. diameter (two oxidizer and two fuel). These tanks constitute a cylindrical tank approximation of the LEMD tankage. It was assumed that the midcourse maneuver was performed by the LEMD engine operating at 1,050 lbf, 1.6 mixture ratio, and an $I_{sp}$ of 302 lbf-sec/lbm. The 12,914 lbm of propellant represents a mass fraction of 0.86. The propellants assumed were a 50–50 blend of hydrazine and unsymmetrical-dimethylhydrazine with a density of 56.78 lbm/ft$^3$ and nitrogen tetroxide with a density of 90.48 lbm/ft$^3$.

Figure 3 shows midcourse velocity increment as a function of time; Fig. 4, spacecraft mass versus time from ignition. Figure 5 depicts residual fuel and oxidizer mass during the maneuver, and Fig. 6 plots fuel and oxidizer tank fluid depth, showing that the tanks are being depleted at approximately the same rate with the assumed mixture ratio and propellant densities. Figure 7 shows the fuel and oxidizer tank natural frequencies. It is interesting to note that, for the assumed conditions, the frequencies of the tanks are identical. This is to be expected since the natural frequency for a given mode is a function of the tank diameter, acceleration field, and fluid depth only, i.e., independent of density (see Eq. 11). Since the assumed conditions were equal diameter tanks and

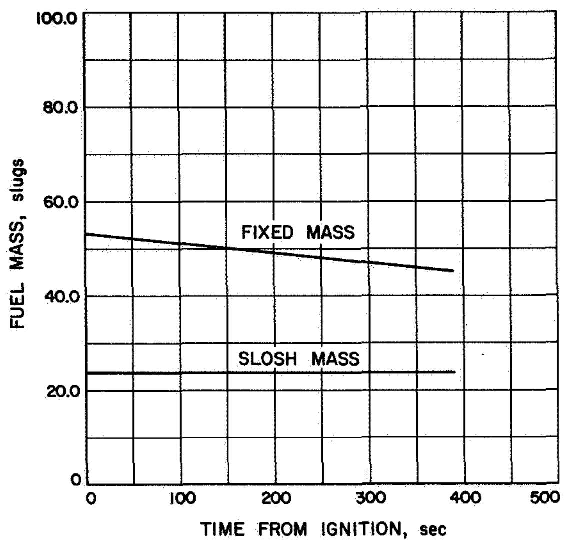


**Fig. 8. Fixed and slosh fuel mass (midcourse maneuver)**

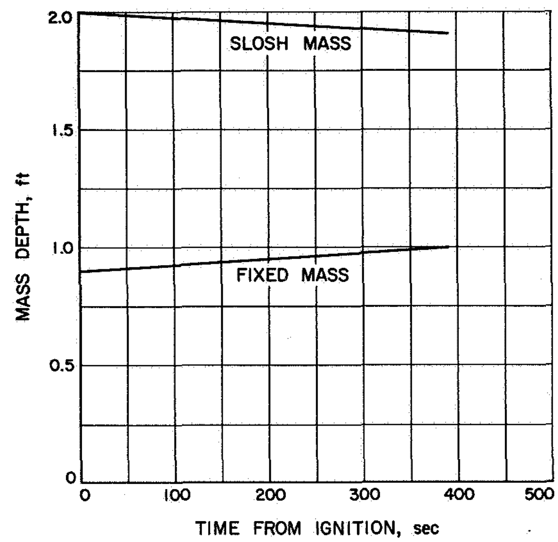


**Fig. 10. Fixed and slosh fuel mass depth (midcourse maneuver)**

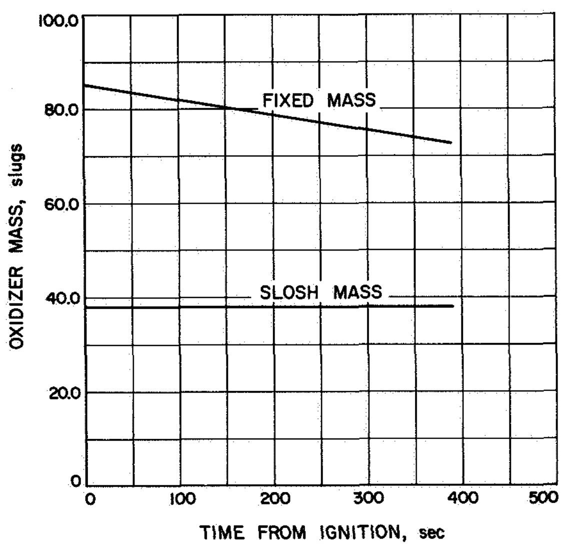


**Fig. 9. Fixed and slosh oxidizer mass (midcourse maneuver)**

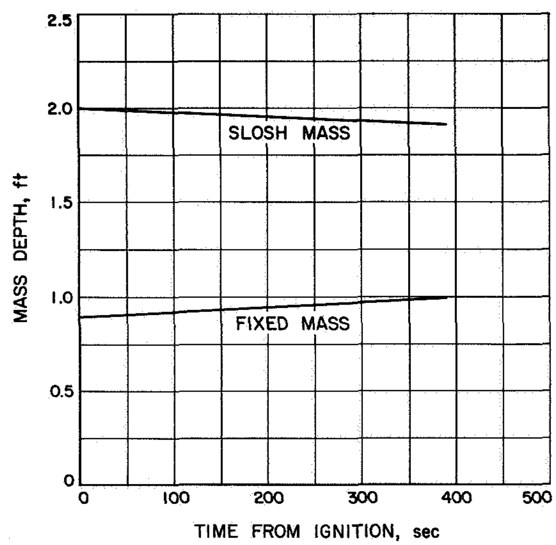


**Fig. 11. Fixed and slosh oxidizer mass depth (midcourse maneuver)**

equal volumetric mixture ratio, the natural frequencies of the fuel and oxidizer tanks are identical. It can also be seen that the frequencies change very little during the midcourse maneuver. This results from the fact that during the midcourse maneuver only about 10% of the total propellant load is expended and, therefore, the tank fluid depths change very little while the acceleration increases by less than 8%. Further, these effects tend to cancel each other. Figures 8 and 9 show the values of the fixed and slosh masses as a function of time. It can be seen from these plots that the slosh mass remains approximately constant while the fixed mass is depleted. This corresponds to the physical observation that, for relatively deep liquid levels, most of the fluid motion takes place near the free surface while the fluid near the bottom of the tank is nearly quiescent. Thus the mechanical model represents, at least qualitatively, this particular physical phenomenon. Figures 10 and 11 show the location of the attachment points for the fixed and slosh masses in the fuel and oxidizer tanks respectively.

Figures 12 through 20 plot these same parameters for the retromaneuver. A retromaneuver of approximately

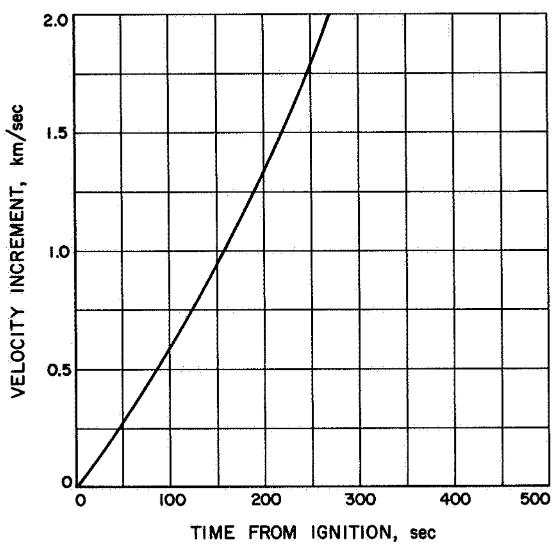


**Fig. 12. Velocity increment, retromaneuver**

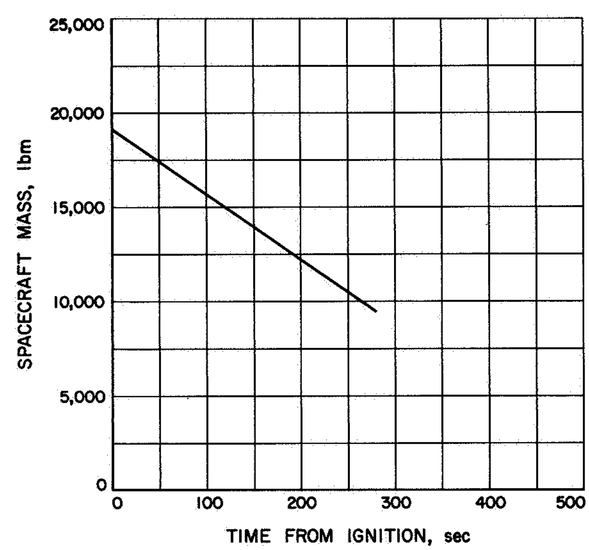


**Fig. 13. Spacecraft mass (retromaneuver)**

2.0 km/sec is shown. The spacecraft mass at retroignition was assumed to be the residual mass at midcourse thrust termination. In this case, however, the thrust was assumed to be 10,500 lbf. It should be noticed that the natural frequencies of the propellant tanks increase from about 1.22 rad/sec at the end of the midcourse maneuver to 3.87 rad/sec at the start of the retromaneuver. Inspection of Eq. (11) shows that the natural frequency is proportional to the square root of the acceleration. Since thrust has been increased by an order of magnitude for the retromaneuver, one would expect the frequency to increase by the root of ten. It can also be noticed that the natural frequency of the tanks exhibits a maximum of 4.25 rad/sec at about 175 sec into the retromaneuver. Figures 8, 9, 17, and 18 show that the propellant slosh and fixed masses are equal at the end of the midcourse maneuver and the start of the retromaneuver, even though the acceleration has increased by a factor of ten. Inspection of Eq. (23) shows that these masses are independent of acceleration. Figures 19 and 20 show the locations of the fixed and slosh masses for the assumed retromaneuver.

Figures 21 through 29 plot the same parameters as a function of time for a 200-m/sec orbit trim at 1,050 lbf with the capsule off.

Appendix B contains additional data for two other assumed missions. The first case given in Appendix B is an assumed "maximum" case, where a full propellant load in the LEMDS tanks and a 10,000-lbm capsule were assumed. The second case represents a mode where the 1973 spacecraft would be forced to perform the orbit insertion maneuver with the capsule separated.

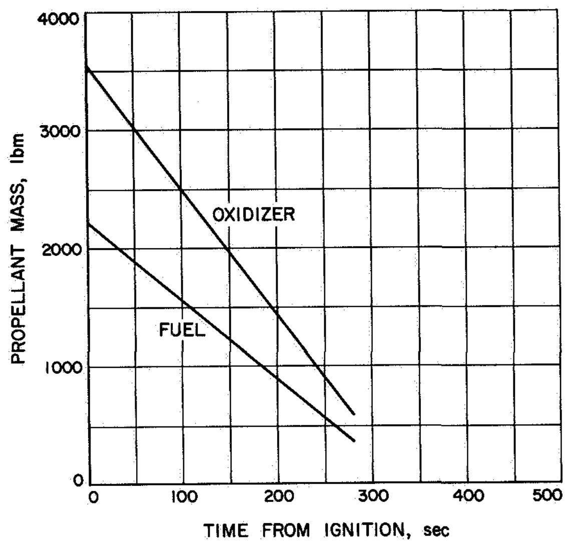


**Fig. 14. Fuel and oxidizer mass (retromaneuver)**

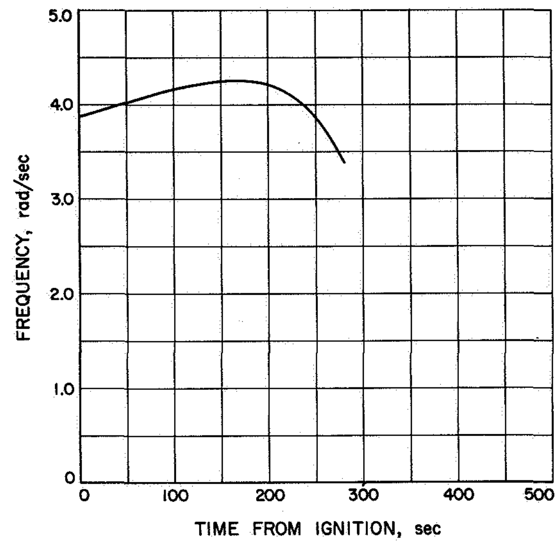


**Fig. 16. Fuel and oxidizer tank natural slosh frequencies (retromaneuver)**

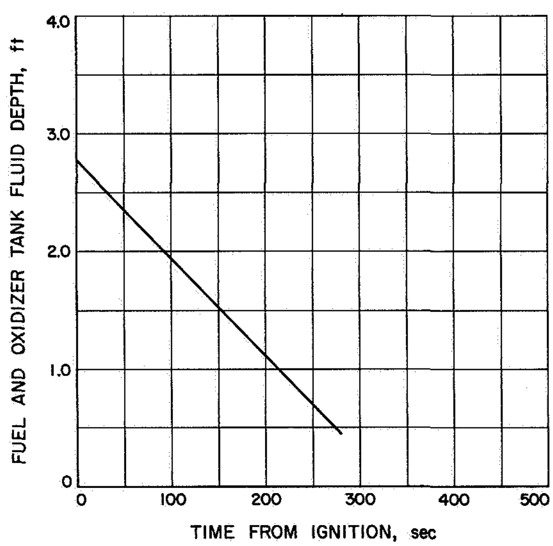


**Fig. 15. Fuel and oxidizer tank fluid depth (retromaneuver)**

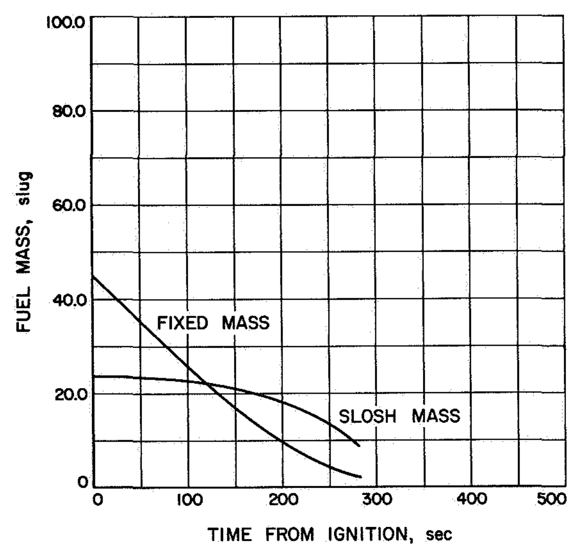


**Fig. 17. Fixed and slosh fuel mass (retromaneuver)**

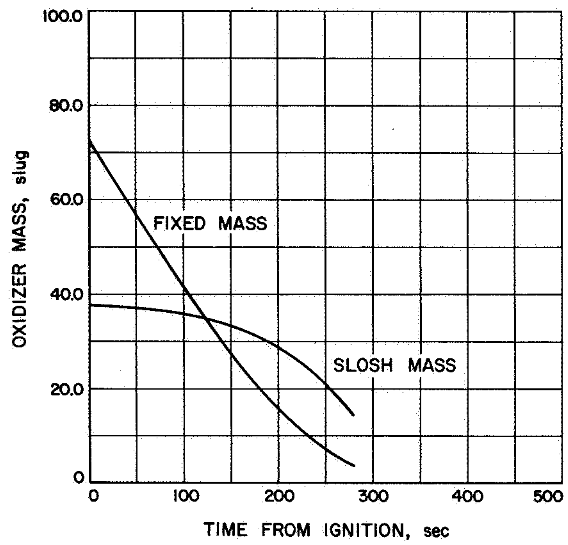


**Fig. 18. Fixed and slosh oxidizer mass (retromaneuver)**

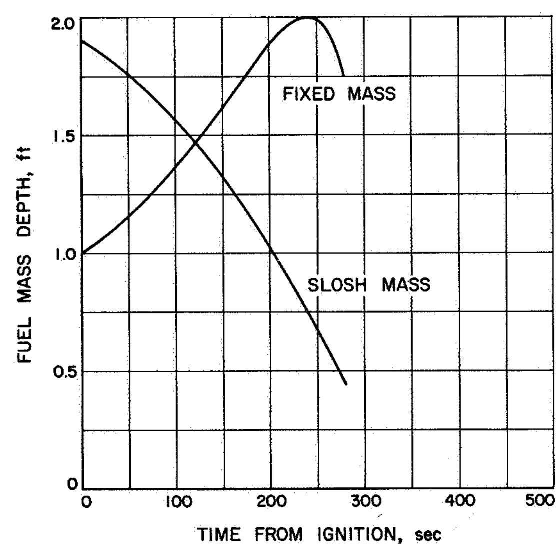


**Fig. 19. Fixed and slosh fuel mass depth (retromaneuver)**

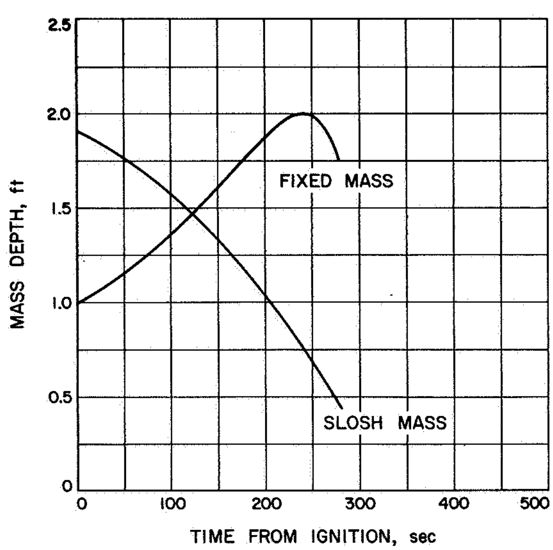


**Fig. 20. Fixed and slosh oxidizer mass depth (retromaneuver)**

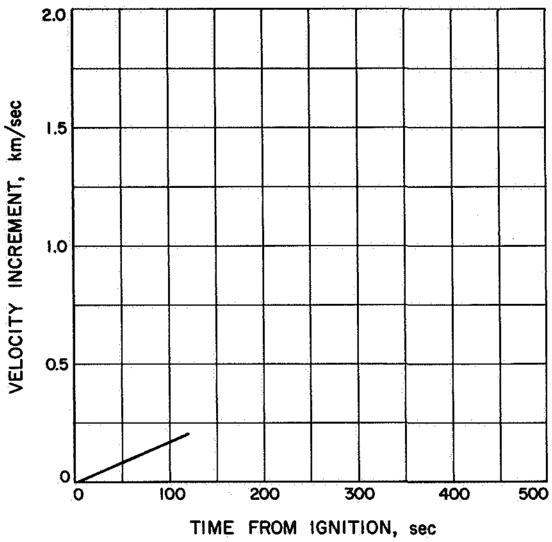


Fig. 21. Velocity increment, orbit trim maneuver

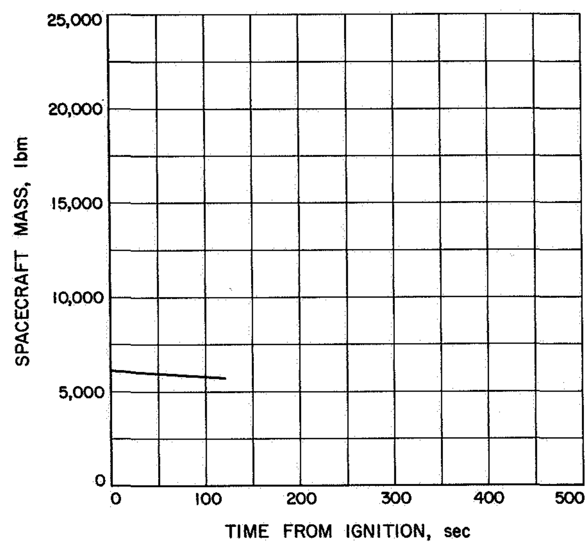


Fig. 22. Spacecraft mass (orbit trim maneuver)

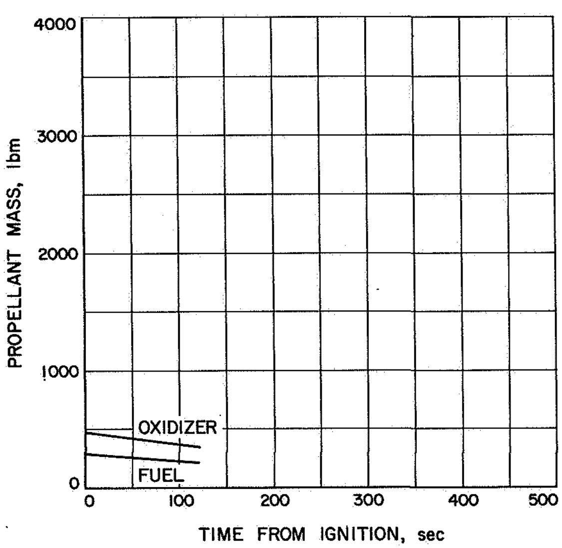


Fig. 23. Fuel and oxidizer mass (orbit trim maneuver)

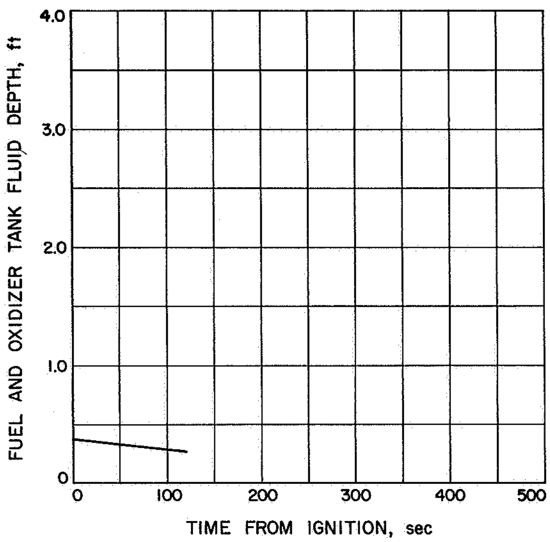


**Fig. 24. Fuel and oxidizer tank fluid depth (orbit trim maneuver)**

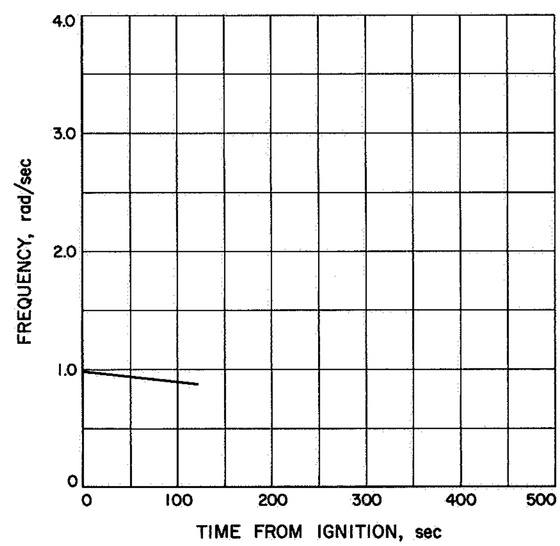


**Fig. 25. Fuel and oxidizer tank natural slosh frequencies (orbit trim maneuver)**

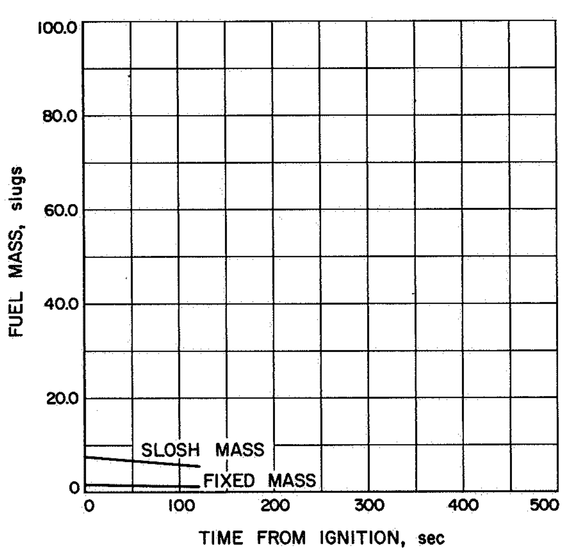


**Fig. 26. Fixed and slosh fuel mass (orbit trim maneuver)**

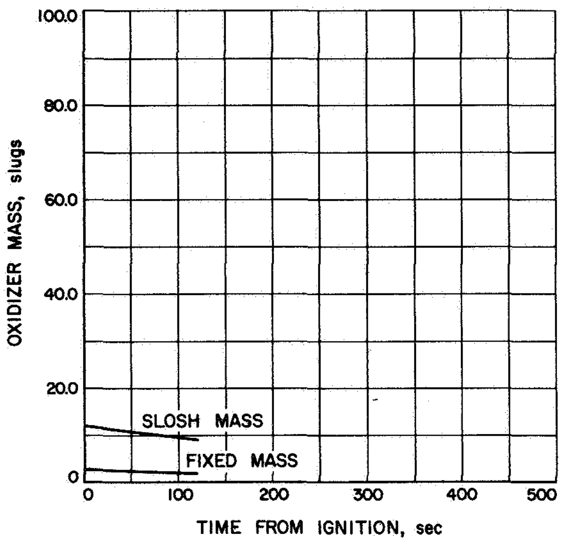


**Fig. 27. Fixed and slosh oxidizer mass (orbit trim maneuver)**

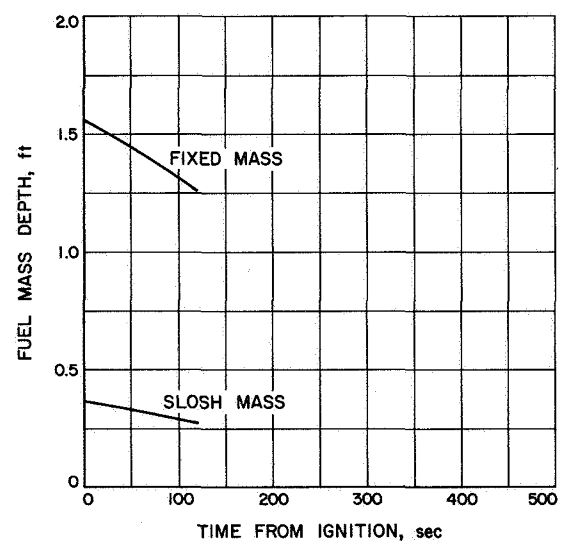


**Fig. 28. Fixed and slosh fuel mass depth (orbit trim maneuver)**

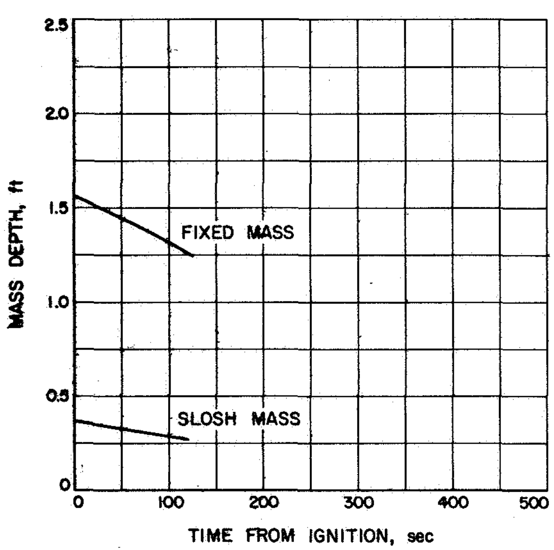


**Fig. 29. Fixed and slosh oxidizer mass depth (orbit trim maneuver)**

# III. CONCLUSIONS

The linearized dynamic equations of propellant motion in the regime dominated by gravity force have been developed for a cylindrical tank approximation of the Lunar Excursion Model Descent Propulsion System tankage. An equivalent mechanical model consisting of a fixed mass and an infinite series of springs and masses has been generated. This model will, in the limit of infinite series, duplicate the forces and moments on a tank resulting from propellant slosh in response to a lateral input motion and will closely approximate most fluid dynamic situations when truncated to only two terms.

The natural frequency of the propellant motion is shown to be dependent upon the tank diameter, acceleration magnitude, and fluid depth; it is found to be independent of fluid density. The maximum natural frequency of propellant motion for a nominal 1973 *Voyager* spacecraft, flying a nominal mission profile, is found to be less than 4.25 rad/sec ($\approx$ 0.68 cps) and to prevail during the orbit-insertion maneuver. The minimum natural frequency for these nominal conditions is approximately 1 rad/sec ($\approx$ 0.16 cps) and occurs at the end of orbit trim.

The propellant slosh mass is found to be essentially independent of acceleration and dependent upon tank radius, residual propellant mass per tank, and, consequently, the depth of the propellant in a given tank.

A computer program has been developed to calculate the parameters required to describe the equivalent mechanical model for an assumed spacecraft flying an assumed mission profile. Output from this program, in the form of punched cards, can be used in preliminary analysis of *Voyager* closed-loop autopilot designs.

Transfer functions for the propellant dynamics have been developed and are presented in Appendix A.

# NOMENCLATURE

### *Symbols*

| | |
|---|---|
| $a$ | cylindrical tank radius |
| $\mathbf{a}_0$ | acceleration of tank relative to a fixed coordinate system |
| $B_0$ | Bond number based on tank radius |
| $C$ | undetermined coefficient |
| $d_0, d_1$ | location of rigid and first mode sloshing masses |
| $\mathbf{F}$ | body force per unit mass |
| $f$ | unspecified function |
| $F_x$ | force in $x$ direction |
| $g$ | longitudinal acceleration magnitude |
| $h$ | fluid depth |
| $\mathbf{i}$ | unit vector in $x$ direction |
| $\mathbf{j}$ | unit vector in $y$ direction |
| $J_m$ | Bessel function of the first kind of order $m$ |
| $\mathbf{k}$ | unit vector in $z$ direction |
| $k_n$ | spring constant of $n$th mode sloshing mass spring |
| $M$ | moment |
| $m$ | total fluid mass or separation constant |
| $m_0$ | rigid mass |
| $m_1$ | sloshing mass (first mode) |
| $p$ | fluid pressure or natural fluid oscillation frequency |
| $\mathbf{q}$ | velocity of fluid relative to tank |
| $r$ | radial tank coordinate |
| $T$ | unknown function of time; kinetic energy |
| $t$ | time |
| $u_n$ | velocity of the $n$th sloshing mass relative to the tank |
| $u_r, u_\theta, u_z$ | radial, circumferential, and longitudinal fluid velocity components |

## NOMENCLATURE (Cont'd)

| | |
|---|---|
| $u_0$ | velocity of tank in the $x$ direction |
| $V$ | potential energy |
| $\mathbf{V}_0$ | velocity of tank relative to a fixed coordinate system |
| $v_0$ | velocity of tank in $y$ direction |
| $w_0$ | velocity of tank in $z$ direction |
| $x$ | $x$ coordinate |
| $y$ | $y$ coordinate |
| $Y_m$ | Bessel function of the second kind of order $m$ |
| $z$ | distance from free surface (positive direction, downward into fluid) |
| $\epsilon$ | small displacement or a small number |
| $\eta$ | surface wave height above undisturbed surface |
| $\theta$ | circumferential coordinate |
| $\lambda$ | separation constant |
| $\rho$ | density |
| $\phi$ | velocity potential |
| $\omega$ | tank excitation frequency |

### Subscripts

| | |
|---|---|
| $m$ | $m$th order slosh mode in the $\theta$ direction |
| $n$ | $n$th order slosh mode in the $r$ direction |

# APPENDIX A

## Derivation of Equations for Fluid Dynamics in a Cylindrical Tank and Generation of an Equivalent Mechanical Model

### I. FREE SLOSHING

The field equation can be written as

$$\nabla^2 \phi = 0 \quad \text{in} \quad \begin{cases} 0 \leqq r \leqq a \\ 0 \leqq \theta \leqq 2\pi \\ -h \leqq z \leqq 0 \end{cases} \tag{A-1}$$

where $\phi$ is the velocity potential function. The directional derivative of $\phi$ at any point in the fluid is the velocity of the fluid in that direction at the given point.

On the free surface of the liquid, both dynamic and kinematic conditions must be satisfied. The origin of the cylindrical coordinate system is taken to be situated at the undisturbed free surface, as shown in Fig. A-1.

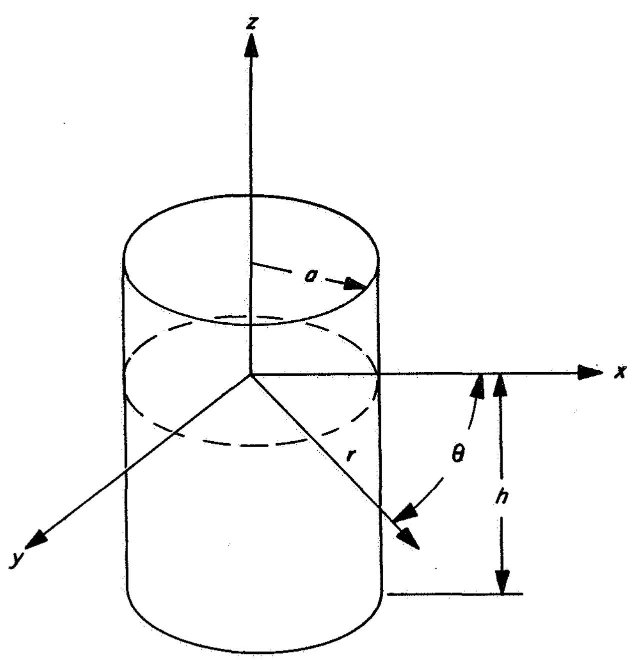


**Fig. A-1. Circular cylindrical tank of radius *a*, partially filled to an arbitrary depth *h***

Writing Euler's equation, we have

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \tfrac{1}{2}(u_r^2 + u_\theta^2 + u_z^2) + gz = F(t) \tag{A-2}$$

where $u_r$, $u_\theta$, $u_z$ are the components of velocity in the $r$, $\theta$, and $z$ directions, respectively.

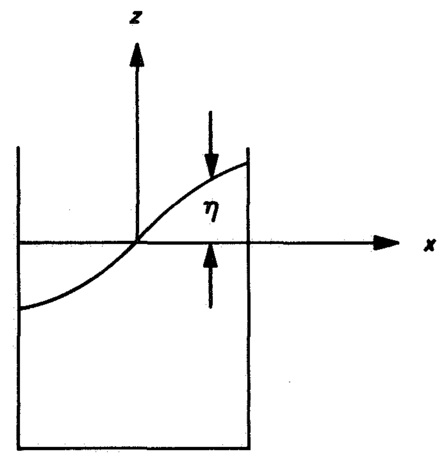


**Fig. A-2. Surface wave in cylindrical tank**

If we denote by $\eta\,(r, \theta, t)$ the vertical displacement of the free surface above the undisturbed free surface (Fig. A-2) and set the pressure on the free surface equal to zero, we have the following:

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}\left(u_r^2 + u_\theta^2 + u_z^2\right) + g\eta\,(r, \theta, t) = F\,(t) \tag{A-3}$$

Since the velocities of the fluid are assumed to be small, the problem can be linearized by neglecting the squares of the velocity terms. If, in addition, the $F(t)$ term is absorbed into the definition of $\phi$, Eq. (A-3) can be rewritten as follows:

$$\frac{\partial \phi}{\partial t} + g\eta\,(r, \theta, t) = 0 \tag{A-4}$$

or

$$\eta\,(r, \theta, t) = -\frac{1}{g}\left(\frac{\partial \phi}{\partial t}\right) \tag{A-5}$$

If Eq. (A-5) is evaluated on $z = 0$ instead of $z = \eta\,(r, \theta, t)$, the evaluation of the equation becomes more tractable and the error introduced by this approximation is consistent with the error introduced by neglecting the velocity-squared terms. Therefore,

$$\eta\,(r, \theta, t) = -\frac{1}{g}\left(\frac{\partial \phi}{\partial t}\right)_{z=0} \tag{A-6}$$

where

$$\phi = \phi\,(r, \theta, z, t)$$

If it is assumed that the normal $z$ component of the fluid velocity at the free surface is equal to the normal component of velocity of the surface itself, then we can write the following:

$$\frac{\partial \eta}{dt} \approx \left(\frac{\partial \phi}{\partial z}\right)_{z=0} \tag{A-7}$$

which, from Eq. (A-6), requires that

$$-\frac{1}{g}\left(\frac{\partial^2 \phi}{\partial t^2}\right)_{z=0} = \left(\frac{\partial \phi}{\partial z}\right)_{z=0} \tag{A-8}$$

or

$$\left(\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}\right)_{z=0} = 0 \tag{A-9}$$

Eq. (A-9) is the linearized condition that the free surface of the fluid must obey.

In addition, since there can be no fluid motion normal to the tank wall at the tank boundaries, the following boundary conditions must be satisfied:

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{on} \quad r = a \tag{A-10}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h \tag{A-11}$$

The problem can be stated, therefore, as: determine $\phi$ such that

$$\nabla^2 \phi = 0 \quad \text{in} \quad \begin{cases} 0 \leqq r \leqq a \\ 0 \leqq \theta \leqq 2\pi \\ -h \leqq z \leqq \eta \end{cases} \qquad \text{(A-1) (repeated)}$$

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{on} \quad r = a \qquad \text{(A-10) (repeated)}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h \qquad \text{(A-11) (repeated)}$$

$$\left(\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}\right)_{z=0} = 0 \qquad \text{(A-9) (repeated)}$$

The solution of Eq. (A-1) is obtained by the method of separation of variables.[2]

Assume that solutions for the velocity potential function exist as products of functions of the independent variables alone. Therefore,

$$\phi\,(r, \theta, z, t) = f_1\,(r)\,f_2\,(\theta)\,f_3\,(z)\,T\,(t) \tag{A-12}$$

Now, since

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

and

$$x = r \cos \theta$$

$$y = r \sin \theta$$

the field equation becomes, in cylindrical coordinates,

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{A-13}$$

[2]See, for instance: Wylie, C. R., Jr., *Advanced Engineering Mathematics,* 2nd ed., McGraw-Hill Book Co., Inc., 1960, Ch. 9, pp. 363–372.

where

$$\frac{\partial^2\phi}{\partial r^2} = f_1''(r)\, f_2(\theta)\, f_3(z)\, T(t) \tag{A-14}$$

$$\frac{\partial\phi}{\partial r} = f_1'(r)\, f_2(\theta)\, f_3(z)\, T(t) \tag{A-15}$$

$$\frac{\partial^2\phi}{\partial \theta^2} = f_1(r)\, f_2''(\theta)\, f_3(z)\, T(t) \tag{A-16}$$

$$\frac{\partial^2\phi}{\partial z^2} = f_1(r)\, f_2(\theta)\, f_3''(z)\, T(t) \tag{A-17}$$

Putting Eqs. (A-14) through (A-17) into Eq. (A-13) yields

$$\nabla^2\phi = f_1''(r)\, f_2(\theta)\, f_3(z)\, T(t) + \frac{1}{r}\left[f_1'(r)\, f_2(\theta)\, f_3(z)\, T(t)\right] + \frac{1}{r^2}\left[f_1(r)\, f_2''(\theta)\, f_3(z)\, T(t)\right]$$
$$+ f_1(r)\, f_2(\theta)\, f_3''(z)\, T(t) = 0 \tag{A-18}$$

Dividing both sides of Eq. (A-18) by $\phi = f_1 f_2 f_3 T$, we get

$$\frac{f_1''(r)}{f_1(r)} + \frac{1}{r}\frac{f_1'(r)}{f_1(r)} + \frac{1}{r^2}\frac{f_2''(\theta)}{f_2(\theta)} + \frac{f_3''(z)}{f_3(z)} = 0 \tag{A-19}$$

Rearranging,

$$\frac{1}{f_1(r)}\left[f_1''(r) + \frac{f_1'(r)}{r}\right] + \frac{1}{r^2}\frac{f_2''(\theta)}{f_2(\theta)} = \frac{-f_3''(z)}{f_3(z)} \tag{A-20}$$

The left-hand side of Eq. (A-20) is clearly independent of $z$, since it consists entirely of functions of $r$ and $\theta$; therefore, the right-hand side must also be independent of $z$ (in spite of its appearance). This can be true only if both sides of Eq. (A-20) are equal to a constant, say $(-\lambda^2)$. Therefore,

$$\frac{f_3''(z)}{f_3(z)} = \lambda^2 \tag{A-21}$$

Consequently,

$$f_3''(z) - \lambda^2 f_3(z) = 0 \tag{A-22}$$

If $\lambda^2 \neq 0$, Eq. (A-22) has a solution of the form

$$f_3 = C_1 \sinh(\lambda z) + C_2 \cosh(\lambda z) \tag{A-23}$$

Rewriting Eq. (A-20), multiplying by $r^2$, and rearranging, we have

$$\frac{r^2}{f_1(r)}\left[f_1''(r) + \frac{f_1'(r)}{r}\right] + \lambda^2 r^2 = \frac{-f_2''(\theta)}{f_2(\theta)} \tag{A-24}$$

The argument applied to Eq. (A-20) can now be applied to Eq. (A-24), and $f_2''(\theta)/f_2(\theta)$ can be set equal to a constant, say $m^2$.

For the case where $m^2 \neq 0$, we have

$$f_2''(\theta) + m^2 f_2(\theta) = 0 \qquad (m \neq 0) \tag{A-25}$$

Equation (A-25) has a solution of the form

$$f_2 = C_3 \cos m\theta + C_4 \sin m\theta \tag{A-26}$$

After solving for $f_3(z)$ and $f_2(\theta)$, we are left with the following:

$$\frac{r^2}{f_1(r)}\left[f_1''(r) + \frac{f_1'(r)}{r}\right] + \lambda^2 r^2 = m^2 \tag{A-27}$$

Rearranging Eq. (A-27), we get

$$r^2 f_1''(r) + r f_1'(r) + (\lambda^2 r^2 - m^2) f_1(r) = 0 \tag{A-28}$$

The equation above is recognized as Bessel's equation[3] of order $m$ with parameter $\lambda$. Equation (A-28) has a solution of the form

$$f_1(r) = C_5 J_m(\lambda r) + C_6 Y_m(\lambda r) \tag{A-29}$$

where $J_m(\lambda r)$ is the Bessel function of the first kind of order $m$ and $Y_m$ is the Bessel function of the second kind of order $m$.

In addition to those solutions already found, there are additional trivial solutions which can be obtained by setting the separation constants equal to zero. Consequently, from Eq. (A-22) with $\lambda = 0$, we get

$$f_3(z) = C_7 z + C_8 \qquad (\lambda = 0) \tag{A-30}$$

Similarly, from Eq. (A-25), with $m = 0$, we obtain

$$f_2(\theta) = C_9\theta + C_{10} \qquad (m = 0) \tag{A-31}$$

also, from Eq. (A-28), if $\lambda = 0$ and $m \neq 0$,

$$r^2 f_1''(r) + r f_1'(r) - m^2 f_1(r) = 0 \tag{A-32}$$

Equation (A-32) can be recognized as the so-called equation of Euler, where the order of the independent variable coefficient of each term is equal to the order of the derivative of each term.

If we change the independent variable from $r$ to, say, $x$ by means of the substitution

$$r = e^x \qquad \text{or} \qquad x = \ln r$$

then

$$f_1' = \frac{df_1}{dr} = \frac{df_1}{dx}\frac{dx}{dr} = \frac{1}{r}\frac{df_1}{dx} \tag{A-33}$$

$$f_1'' = \frac{d}{dr}\left(\frac{1}{r}\frac{df_1}{dx}\right) = \frac{1}{r^2}\frac{d^2 f_1}{dx^2} - \frac{1}{r^2}\frac{df_1}{dx} \tag{A-34}$$

Substituting Eqs. (A-33) and (A-34) into Eq. (A-32), we get

$$r^2\left(\frac{1}{r^2}\frac{d^2 f_1}{dx^2} - \frac{1}{r^2}\frac{df_1}{dx}\right) + r\left(\frac{1}{r}\frac{df_1}{dx}\right) - m^2 f_1 = 0 \tag{A-35}$$

which reduces to

$$\frac{d^2 f_1}{dx^2} - m^2 f_1 = 0 \tag{A-36}$$

Equation (A-36) has the solution

$$f_1 = C_{11} e^{mx} + C_{12} e^{-mx} \tag{A-37}$$

[3]See, for instance: Wylie, op. cit., Ch. 10, p. 410.

Substituting $x = \ln r$ into Eq. (A-37) yields

$$f_1 = C_{11}e^{m \ln r} + C_{12}e^{-m \ln r} \tag{A-38}$$

$$f_1 = C_{11}e^{(\ln r)m} + C_{12}e^{(\ln r)-m} \tag{A-39}$$

$$f_1 = C_{11}r^m + C_{12}r^{-m} \qquad (\lambda = 0), (m \neq 0) \tag{A-40}$$

If $\lambda = 0$ and $m = 0$,

$$\frac{d^2 f_1}{dx^2} = 0 \tag{A-41}$$

Therefore,

$$f_1 = C_{13} \ln r + C_{14} \tag{A-42}$$

Summarizing, the most general solution for $\phi$ is a linear combination of Eqs. (A-23), (A-26), (A-29), (A-30), (A-31), (A-40), and (A-42), which are repeated here for convenience.

$$f_3 = C_1 \sinh(\lambda z) + C_2 \cosh(\lambda z) \qquad \lambda \neq 0 \tag{A-23 repeated}$$

$$f_2 = C_3 \cos(m\theta) + C_4 \sin(m\theta) \qquad m \neq 0 \tag{A-26 repeated}$$

$$f_1 = C_5 J_m(\lambda r) + C_6 Y_m(\lambda r) \qquad \lambda \neq 0, m \neq 0 \tag{A-29 repeated}$$

$$f_3 = C_7 z + C_8 \qquad \lambda = 0 \tag{A-30 repeated}$$

$$f_2 = C_9 \theta + C_{10} \qquad m = 0 \tag{A-31 repeated}$$

$$f_1 = C_{11} r^m + C_{12} r^{-m} \qquad \lambda = 0, m \neq 0 \tag{A-40 repeated}$$

$$f_1 = C_{13} \ln r + C_{14} \qquad \lambda = 0, m = 0 \tag{A-42 repeated}$$

Since $Y_m(\lambda r)$ approaches $-\infty$ as $r$ approaches zero, $C_6$ must be zero in order for the solutions to be finite at the origin. Also, since $\ln r$ approaches $-\infty$ as $r$ approaches zero and, similarly, $r^{-m}$ approaches $\infty$ as $r$ approaches zero, it must be required that $C_{12}$ and $C_{13}$ be zero for the solution to be finite at the origin. For the potential function to be single-valued, it must be required that $C_9 = 0$.

To satisfy the boundary condition that $\partial\phi/\partial z = 0$ on $z = -h$, it must be required that $C_7 = 0$ and, further, that

$$\frac{df_3}{dz} = 0 \qquad \text{on} \qquad z = -h \tag{A-43}$$

Equation (A-43) requires that

$$C_1 \cosh(-\lambda h) + C_2 \sinh(-\lambda h) = 0 \tag{A-44}$$

but $\cosh(-\lambda h) = \cosh(\lambda h)$ and $\sinh(-\lambda h) = -\sinh(\lambda h)$. Therefore,

$$C_1 \cosh(\lambda h) - C_2 \sinh(\lambda h) = 0 \tag{A-45}$$

or

$$C_1 = C_2 \frac{\sinh(\lambda h)}{\cosh(\lambda h)}$$

It follows that

$$f_3 = C_2 \frac{\sinh(\lambda h)}{\cosh(\lambda h)} \sinh(\lambda z) + C_2 \cosh(\lambda z)$$

$$f_3 = C_2 \frac{\cosh \lambda (h + z)}{\cosh(\lambda h)} \tag{A-46}$$

To satisfy the condition that $\partial\phi/\partial r = 0$ on $r = a$, it must be required that $C_{11} = 0$, and further that $df_1/dr = 0$ when $r = a$. This must require that

$$\frac{d}{dr}[J_m(\lambda r)]_{r=a} = 0 \tag{A-47}$$

(from which, the values of $\lambda_{mn}$ can be solved for).

Therefore, the solution to the field equation that satisfies the fixed boundary conditions is

$$\phi_{mn} = [C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta)]\, J_m(\lambda_{mn} r) \frac{\cosh[\lambda_{mn}(z+h)]}{\cosh(\lambda_{mn} h)} T_{mn}(t) \tag{A-48}$$

where $T_{mn}(t)$ is still an unknown function of time.

Recalling that the free surface condition was found to be

$$\left(\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right)_{z=0} = 0 \quad \text{(neglecting surface tension)} \tag{A-49}$$

then

$$\frac{\partial\phi}{\partial t} = [C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta)]\, J_m(\lambda_{mn} r) \frac{\cosh[\lambda_{mn}(z+h)]}{\cosh(\lambda_{mn} h)} \dot{T}_{mn}(t)$$

and

$$\frac{\partial^2\phi}{\partial t^2} = [C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta)]\, J_m(\lambda_{mn} r) \frac{\cosh[\lambda_{mn}(z+h)]}{\cosh(\lambda_{mn} h)} \ddot{T}_{mn}(t) \tag{A-50}$$

Also,

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=0} = [C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta)]\, J_{mn}(\lambda_{mn} r) \frac{\sinh(\lambda_{mn} h)\,\lambda_{mn}}{\cosh(\lambda_{mn} h)} T_{mn}(t)$$

or

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=0} = [C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta)]\, [J_{mn}(\lambda_{mn} r)]\; \lambda_{mn}\tanh(\lambda_{mn} h)\, T_{mn}(t) \tag{A-51}$$

Therefore, Eq. (A-49) can be written in the form

$$[C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta)]\, J_m(\lambda_{mn} r)\; [\ddot{T}_{mn}(t) + g\,\lambda_{mn}\tanh(\lambda_{mn} h)\, T_{mn}] = 0 \tag{A-52}$$

Equation (A-52) reduces to

$$\ddot{T}_{mn}(t) + g\lambda_{mn}\tanh(\lambda_{mn} h)\, T_{mn} = 0 \tag{A-53}$$

which is of the form

$$\ddot{T}_{mn}(t) + p^2 T_{mn} = 0 \tag{A-54}$$

where $p$ is the natural frequency of oscillation for the liquid and is given by

$$p = [g\lambda_{mn}\tanh(\lambda_{mn} h)]^{1/2} \tag{A-55}$$

Equation (A-54) is valid in the region where surface tension forces may be neglected.

## II. SMALL-AMPLITUDE FORCED OSCILLATIONS

Knowing the velocity potential function for a cylindrical tank partially filled with a liquid, we can now determine the linear response of the liquid when the tank is translated harmonically.

Let $\mathbf{q}$ represent the velocity of the fluid relative to the tank and $\mathbf{V}_0$ the velocity of the tank relative to a fixed coordinate system. Euler's equation for the motion of the liquid in a tank subjected to an acceleration can be written in vector form as

$$\rho\left(\frac{D\mathbf{q}}{Dt} + \mathbf{a}_0\right) = \rho\mathbf{F} - \nabla p \tag{A-56}$$

where

$$\mathbf{a}_0 = \frac{d}{dt}(\mathbf{V}_0) \tag{A-57}$$

and $\mathbf{F}$ is the body force on the liquid, per unit mass. Here,

$$\mathbf{a}_0 = \dot{\mathbf{V}}_0 = \dot{u}_0\mathbf{i} + \dot{v}_0\mathbf{j} + \dot{w}_0\mathbf{k} \tag{A-58}$$

Also,

$$\mathbf{a}_0 = \nabla(\dot{u}_0 x + \dot{v}_0 y + \dot{w}_0 z) \tag{A-59}$$

and

$$\mathbf{F} = \nabla(-gz) \tag{A-60}$$

Expanding Eq. (A-56), we get

$$\rho\left[\frac{\partial\mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\,\mathbf{q} + \nabla(\dot{u}_0 x + \dot{v}_0 y + \dot{w}_0 z)\right] = \rho\nabla(-gz) - \nabla p \tag{A-61}$$

Making use of the vector relationship that

$$\nabla(\mathbf{q}\cdot\mathbf{q}) = 2(\mathbf{q}\cdot\nabla)\,\mathbf{q} + 2\mathbf{q}\,\mathbf{X}\,(\nabla\,\mathbf{X}\,\mathbf{q}) \tag{A-62}$$

and $\nabla\,\mathbf{X}\,\mathbf{q} = 0$ if the fluid is inviscid and irrotational, we have

$$(\mathbf{q}\cdot\nabla)\,\mathbf{q} = \tfrac{1}{2}\nabla(\mathbf{q}\cdot\mathbf{q}) \tag{A-63}$$

Using Eq. (A-63), we can rewrite Eq. (A-61) in the form

$$\frac{\partial\mathbf{q}}{\partial t} + \tfrac{1}{2}\nabla(\mathbf{q}\cdot\mathbf{q}) + \nabla(\dot{u}_0 x + \dot{v}_0 y + \dot{w}_0 z) + \nabla(gz) + \frac{\nabla p}{\rho} = 0 \tag{A-64}$$

which is equivalent to

$$\frac{\partial}{\partial t}(\nabla\phi) + \tfrac{1}{2}\nabla(\mathbf{q}\cdot\mathbf{q}) + \nabla(\dot{u}_0 x + \dot{v}_0 y + \dot{w}_0 z) + \nabla(gz) + \nabla\int\frac{dp}{\rho} = 0 \tag{A-65}$$

Rewriting Eq. (A-65), we get

$$\nabla\left[\frac{\partial\phi}{\partial t} + \tfrac{1}{2}(\mathbf{q}\cdot\mathbf{q}) + \dot{u}_0 x + \dot{v}_0 y + \dot{w}_0 z + gz + \int\frac{dp}{\rho}\right] = 0 \tag{A-66}$$

Now, if the gradient of a function is zero throughout a region, the function must certainly be a constant (spatially) throughout the region or, at most, a function of time. It follows then, that

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}(\mathbf{q}\cdot\mathbf{q}) + \dot{u}_0 x + \dot{v}_0 y + \dot{w}_0 z + gz + \int\frac{dp}{\rho} = F(t) \tag{A-67}$$

where $F(t)$ is an arbitrary function of time. For incompressible fluids, Eq. (A-67) takes the form

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}(\nabla\phi \cdot \nabla\phi) + \dot{u}_0 x + \dot{v}_0 y + \dot{w}_0 z + gz + \frac{p}{\rho} = 0 \tag{A-68}$$

where $F(t)$ has been absorbed into the definition of $\phi(x, y, z, t)$.

The kinematic condition arising from the requirement that a fluid particle on the free surface $z = \eta(x, y, t)$ moves with the velocity of the free surface gives

$$\frac{D\eta}{Dt} = \frac{\partial \phi}{\partial z} \tag{A-69}$$

or

$$\frac{\partial \eta}{\partial t} + \mathbf{q} \cdot \nabla\eta = \frac{\partial \phi}{\partial z} \tag{A-70}$$

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = \frac{\partial \phi}{\partial z} \tag{A-71}$$

Equations (A-68) and (A-71) can be linearized and combined to form the free surface condition for the case where the tank is translated in the $x$ direction only. Therefore, $\dot{v}_0 = \dot{w}_0 = 0$; also, assuming $p = 0$ and linearizing by neglecting the squares and products of the velocity components, we get

$$\frac{\partial \phi}{\partial t} + \dot{u}_0 x + g\eta = 0 \qquad \text{on} \qquad z = 0 \tag{A-72}$$

and

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \qquad \text{on} \qquad z = 0 \tag{A-73}$$

Solving Eq. (A-72) for $\eta$ and substituting the value for $\partial\eta/\partial t$ into Eq. (A-73), we get

$$\left(\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}\right)_{z=0} = -\ddot{u}_0 x \tag{A-74}$$

Equation (A-74) is, therefore, the linearized free surface condition.

The velocity potential function was found from Eq. (A-48) to be

$$\phi_{mn} = [C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta)]\, J_m(\lambda_{mn} r)\frac{\cosh[\lambda_{mn}(z + h)]}{\cosh(\lambda_{mn} h)}\, T_{mn}(t) \tag{A-75}$$

It can be shown that the solution for Eq. (A-75) that is obtained by letting $C_{mn} = 0$ and $D_{mn} \neq 0$ is the same as the solution when $D_{mn} = 0$ and $C_{mn} \neq 0$ if the coordinates are rotated 90 deg. Therefore, Eq. (A-75) can be rewritten as

$$\phi_{mn} = \cos(m\theta)\, J_m(\lambda_{mn} r)\frac{\cosh[\lambda_{mn}(z + h)]}{\cosh(\lambda_{mn} h)}\, T_{mn}(t) \tag{A-76}$$

where the coefficient $C_{mn}$ has been absorbed into the still arbitrary definition of $T_{mn}(t)$.

Now, when $m = 1$, we have the solution for the first asymmetric mode of lateral sloshing. Therefore,

$$\phi = \sum_{n=1}^{\infty} T_{1n}(t)\, J_1(\lambda_{1n} r)\frac{\cosh[\lambda_{1n}(z + h)]}{\cosh(\lambda_{1n} h)}\cos\theta \tag{A-77}$$

Consequently,

$$\left(\frac{\partial \phi}{\partial t}\right)_{z=0} = \sum_{n=1}^{\infty} \dot{T}_{1n}(t)\, J_1(\lambda_{1n} r)\cos\theta \tag{A-78}$$

and

$$\left(\frac{\partial^2\phi}{\partial t^2}\right)_{z=0} = \sum_{n=1}^{\infty} \ddot{T}_{1n}(t)\, J_1(\lambda_{1n}r)\cos\theta \tag{A-79}$$

Also,

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=0} = \sum_{n=1}^{\infty} T_{1n}(t)\, J_1(\lambda_{1n}r)\,\lambda_{1n}\tanh(\lambda_{1n}h)\cos\theta \tag{A-80}$$

Putting Eqs. (A-79) and (A-80) into Eq. (A-74), and noting that in cylindrical coordinates $x = r\cos\theta$, we get

$$\sum_{n=1}^{\infty} J_1(\lambda_{1n}r)\cos\theta\,[\ddot{T}_{1n} + g\lambda_{1n}\tanh(\lambda_{1n}h)\,T_{1n}] = \ddot{u}_0 r\cos\theta \tag{A-81}$$

To solve for the unknown function of time $T_{1n}(t)$, we first expand the radial coordinate in terms of a Bessel series, as follows:

Assume that

$$r = \sum_{n=1}^{\infty} F_n J_1(\lambda_{1n}r) \tag{A-82}$$

For Eq. (A-82) to be true, it can be shown that

$$F_n = \frac{2a}{(\lambda^2_{1n}a^2 - 1)\, J_1(\lambda_{1n}a)} \tag{A-83}$$

and, therefore, Eq. (A-82) becomes

$$r = \sum_{n=1}^{\infty}\left[\frac{2a}{(\lambda^2_{1n}a^2 - 1)}\right]\frac{J_1(\lambda_{1n}r)}{J_1(\lambda_{1n}a)} \tag{A-84}$$

Introducing Eq. (A-82) into Eq. (A-81),we get

$$\sum_{n=1}^{\infty} J_1(\lambda_{1n}r)\cos\theta\,[\ddot{T}_{1n} + g\lambda_{1n}\,tanh\,(\lambda_{1n}h)\,T_{1n}] = -\ddot{u}_0\cos\theta\sum_{n=1}^{\infty}\left[\frac{2a}{(\lambda^2_{1n}a^2 - 1)}\right]\frac{J_1(\lambda_{1n}r)}{J_1(\lambda_{1n}a)}$$

$$= -\ddot{u}_0\cos\theta\sum_{n=1}^{\infty} F_n J_1(\lambda_{1n}r) \tag{A-85}$$

Equating coefficients of $J_1(\lambda_{1n}r)$ in Eq. (A-85), we get

$$[\ddot{T}_{1n} + g\lambda_{1n}\tanh(\lambda_{1n}h)\,T_{1n}] = -\ddot{u}_0 F_n \qquad n = 1, 2, \cdots, \infty \tag{A-86}$$

or

$$\ddot{T}_{1n} + p^2_{1n} T_{1n} = -\ddot{u}_0 F_n \tag{A-87}$$

where

$$p^2_{1n} = g\lambda_{1n}\tanh(\lambda_{1n}h) \tag{A-88}$$

If the tank motion is harmonic, i.e.,

$$x_0 = \epsilon_0 \sin\omega t \tag{A-89}$$

then

$$\dot{x}_0 = u_0 = \epsilon_0\omega\cos\omega t \tag{A-90}$$

and

$$\ddot{u}_0 = \epsilon_0 \omega^3 \cos \omega t \tag{A-91}$$

Then Eq. (A-87) becomes

$$\ddot{T}_{1n} + p_{1n}^2 \omega^2 T_{1n} = -\epsilon_0 F_n \omega^3 \cos \omega t \tag{A-92}$$

Equation (A-92) has the *steady state* solution

$$T_{1n} = \frac{\epsilon_0 \omega^3 F_n}{p_{1n}^2 - \omega^2} \cos \omega t \tag{A-93}$$

as long as $p_{1n} \neq \omega$, i.e., the driving frequency is not the same as the resonant frequency. Therefore, Eq. (A-77) becomes

$$\phi = \sum_{n=1}^{\infty} \frac{2\epsilon_0 \omega^3 a}{(\lambda_{1n}^2 a^2 - 1)(p_{1n}^2 - \omega^2)} \frac{J_1(\lambda_{1n} r)}{J_1(\lambda_{1n} a)} \frac{\cosh[\lambda_{1n}(z+h)]}{\cosh(\lambda_{1n} h)} \cos\theta \cos(\omega t) \tag{A-94}$$

## III. EQUIVALENT MECHANICAL MODEL

From Eq. (A-85), we have the following series of equations:

$$J_1(\lambda_{1n} r) \cos\theta \,[\ddot{T}_{1n} + g\lambda_{1n} \tanh(\lambda_{1n} h)\, T_{1n}] = -\,\ddot{u}_0 \cos\theta \left[\frac{2a}{(\lambda_{1n}^2 a^2 - 1)}\right] \frac{J_1(\lambda_{1n} r)}{J_1(\lambda_{1n} a)} \qquad n = 1, 2, \cdots, \infty \tag{A-95}$$

which can be simplified to

$$\ddot{T}_{1n} + g\lambda_{1n} \tanh(\lambda_{1n} h)\, T_{1n} = -\,\ddot{u}_0 \frac{2a}{J_1(\lambda_{1n} a)(\lambda_{1n}^2 a^2 - 1)} \qquad n = 1, 2, \cdots, \infty \tag{A-96}$$

Dropping the subscript 1 and noting that $\ddot{u}_0 = \dddot{x}_0$ and $p_n^2 = g\lambda_n \tanh(\lambda_n h)$, we have

$$\ddot{T}_{1n} + p_{1n}^2 T_{1n} = -\,\dddot{x}_0 \frac{2a}{(\lambda_n^2 a^2 - 1) J_1(\lambda_n a)} \tag{A-97}$$

Taking the Laplace transform of Eq. (A-97), we get

$$s^2 \mathcal{L}[T_n(t)] - sT_n(0) - \dot{T}_n(0) + p_n^2 \mathcal{L}[T_n(t)] = \frac{-\{s^3 \mathcal{L}[x_0(t)] - s^2 \mathcal{L}[x_0(0)] - s\dot{x}_0(0) - \ddot{x}_0(0)\}\, 2a}{J_1(\lambda_n a)(\lambda_n^2 a^2 - 1)} \tag{A-98}$$

Let

$$T_n(0) = 0; \dot{T}_n(0) = 0$$

and

$$x_0(0) = 0; \dot{x}_0(0) = 0; \ddot{x}_0(0) = 0$$

Then

$$s^2 \mathcal{L}[T_n(t)] + p_n^2 \mathcal{L}[T_n(t)] = \frac{-2as^3 \mathcal{L}[x_0(t)]}{J_1(\lambda_n a)(\lambda_n^2 a^2 - 1)} \tag{A-99}$$

Solving for the transform of the unknown function of time, we have

$$\mathcal{L}[T_n(t)] = \frac{-2as^3\mathcal{L}[x_0(t)]}{(s^2+p_n^2)J_1(\lambda_n a)(\lambda_n^2 a^2 - 1)} \tag{A-100}$$

Equation (A-68) can be rewritten in linearized form as

$$\frac{\partial \phi}{\partial t} + \ddot{x}_0 r \cos\theta + gz + \frac{p}{\rho} = 0 \tag{A-101}$$

The Laplace transform of Eq. (A-101) is

$$s\mathcal{L}[\phi(t)] - \phi(0) + s^2 r\cos\theta\,\mathcal{L}[x_0(t)] - s\cos\theta\,x_0(0) - s\cos\theta\,\dot{x}_0(0) + \frac{1}{\rho}\mathcal{L}[p(t)] = 0$$

Let

$$\phi(0) = 0;\; x_0(0) = 0;\; \dot{x}_0(0) = 0 \tag{A-102}$$

Then

$$s\mathcal{L}[\phi(t)] + s^2 r\cos\theta\,\mathcal{L}[x_0(t)] + \frac{1}{\rho}\mathcal{L}[p(t)] = 0 \tag{A-103}$$

Solving for the transform of the pressure term, we get

$$\mathcal{L}[p(t)] = -\rho\,\{s\mathcal{L}[\phi(t)] + s^2 r\cos\theta\,\mathcal{L}[x_0(t)]\} \tag{A-104}$$

The Laplace transform of Eq. (A-77) is

$$\mathcal{L}[\phi(t)] = \mathcal{L}[T_n(t)]\,J_1(\lambda_n r)\,\frac{\cosh[\lambda_n(z+h)]}{\cosh(\lambda_n h)}\cos\theta \tag{A-105}$$

Substituting Eq. (A-100) into Eq. (A-105), we get

$$\mathcal{L}[\phi(t)] = \left\{\frac{-2as^3\mathcal{L}[x_0(t)]}{(s^2+p_n^2)J_1(\lambda_n a)(\lambda_n^2 a^2-1)}\right\} J_1(\lambda_n r)\frac{\cosh[\lambda_n(z+h)]}{\cosh(\lambda_n h)}\cos\theta \tag{A-106}$$

Substituting Eq. (A-106) into Eq. (A-104), we get

$$\mathcal{L}[p(t)] = -\rho\left\{\frac{-2as^4\mathcal{L}[x_0(t)]}{(s^2+p_n^2)J_1(\lambda_n a)(\lambda_n^2 a^2-1)}\right\} J_1(\lambda_n r)\frac{\cosh[\lambda_n(z+h)]}{\cosh(\lambda_n h)}\cos\theta - \rho s^2 r\cos\theta\,\mathcal{L}[x_0(t)] \tag{A-107}$$

The Laplace transform of the resultant force in the $x$-direction is obtained by integrating Eq. (A-107) evaluated at $r = a$ over the wetted sides of the tank. Therefore, the transform of the resultant force in the $x$-direction is

$$\mathcal{L}[F_x(t)] = \int_{z=-h}^{0}\int_{\theta=0}^{2\pi} \mathcal{L}[p(t)]_{r=a}\cos\theta\,a\,d\theta\,dz \tag{A-108}$$

or

$$\mathcal{L}[F_x(t)] = \int_{z=-h}^{0}\int_{\theta=0}^{2\pi}\left(-1 + \left[\frac{2s^2}{(s^2+p_n^2)(\lambda_n^2 a^2-1)}\right]\left\{\frac{\cosh[\lambda_n(z+h)]}{\cosh(\lambda_n h)}\right\}\right)\rho a^2 s^2\,\mathcal{L}[x_0(t)]\cos^2\theta\,d\theta\,dz \tag{A-109}$$

Making use of the definite integrals

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \pi$$

$$\int_{-h}^{0} dz = h$$

$$\int_{-h}^{0} \frac{\cosh[\lambda_n (z+h)]}{\cosh(\lambda_n h)} dz = \frac{\sinh[\lambda_n (z+h)]}{\lambda_n \cosh(\lambda_n h)} \bigg|_{-h}^{0}$$

$$= \frac{1}{\lambda_n} \tanh(\lambda_n h)$$

Eq. (A-109) becomes

$$\mathcal{L}[F_x(t)] = -\pi\rho a^2 h s^2 \mathcal{L}[x_0(t)] + \pi\rho a^2 h \left\{ \frac{2s^4 \mathcal{L}[x_0(t)] \tanh(\lambda_n h)}{\lambda_n h (s^2 + p_n^2)(\lambda_n^2 a^2 - 1)} \right\} \quad \text{(A-110)}$$

but $\rho\pi a^2 h = m$, the mass of fluid in the tank. Therefore,

$$\mathcal{L}[F_x(t)] = -ms^2 \mathcal{L}[x_0(t)] + m \left\{ \frac{2s^4 \mathcal{L}[x_0(t)] \tanh(\lambda_n h)}{\lambda_n h (s^2 + p_n^2)(\lambda_n^2 a^2 - 1} \right\} \quad \text{(A-111)}$$

Having found the Laplace transform of the forces acting on the tank, we can proceed to find the moment, say around the $y$ axis (see Fig. A-3).

The moments about the $y$ axis, composed of moments due to pressure forces on the bottom and sides of the tank, are given by

$$M_0(t) = \int_{r=0}^{a} \int_{\theta=0}^{2\pi} r\cos\theta \, (pr\, d\theta\, dr) \Big|_{z=-h} + \int_{\theta=0}^{2\pi} \int_{z=-h}^{0} z\cos\theta \, (pa\, d\theta\, dz) \Big|_{r=a} \quad \text{(A-112)}$$

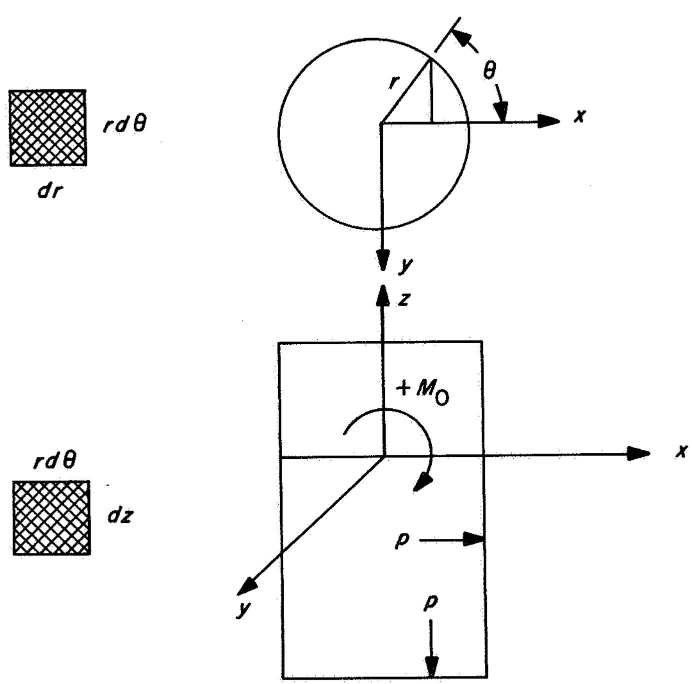


**Fig. A-3. Geometry of moment calculation for cylindrical tank**

Therefore,

$$\mathcal{L}[M_0(t)] = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2 \cos\theta \,\mathcal{L}[p(t)] d\theta\, dr \,\Big|_{z=-h} + \int_{\theta=0}^{2\pi}\int_{z=-h}^{0} za\cos\theta\,\mathcal{L}[p(t)] dz\, d\theta \,\Big|_{r=a} \tag{A-113}$$

Substituting Eq. (A-107) into Eq. (A-113), we get

$$\mathcal{L}[M_0(t)] = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos\theta\left\{\frac{2pas^4\mathcal{L}[x_0(t)]J_1(\lambda_n r)}{(s^2+p_n^2)J_1(\lambda_n a)(\lambda_n^2 a^2-1)}\cdot\frac{\cosh[\lambda_n(0)]}{\cosh(\lambda_n h)}\cos\theta - ps^2 r\cos\theta\,\mathcal{L}[x_0(t)]\right\} d\theta\, dr$$
$$+\int_{\theta=0}^{2\pi}\int_{z=-h}^{0} za\cos\theta\left\{\frac{2pas^4\,\mathcal{L}[x_0(t)]J_1(\lambda_n a)\cosh[\lambda_n(z+h)]\cos\theta}{(s^2+p_n^2)J_1(\lambda_n a)(\lambda_n^2 a^2-1)\cosh(\lambda_n h)} - ps^2 a\cos\theta\,\mathcal{L}[x_0(t)]\right\} dz\, d\theta \tag{A-114}$$

By making use of the definite integrals

$$\int_{-h}^{0} z\, dz = -\frac{h^2}{2}$$

$$\int_{0}^{a} r^3\, dr = \frac{a^4}{4}$$

$$\int_{0}^{2\pi} \cos^2\theta\, d\theta = \pi$$

$$\int_{0}^{a} \frac{r^2 J_1(\lambda_n r)\, dr}{J_1(\lambda_n a)} = \frac{a}{\lambda_n^2 2}$$

$$\int_{-h}^{0} \frac{z\cosh[\lambda_n(z+h)]}{\cosh(\lambda_n h)} dz = \frac{1-\cosh(\lambda_n h)}{\lambda_n^2\cosh(\lambda_n h)}$$

Eq. (A-114) becomes

$$\mathcal{L}[M_0(t)] = \pi p a^2 s^2\,\mathcal{L}[x_0(t)]\left\{\frac{h^2}{2} + \frac{2s^2[1-\cosh(\lambda_n h)]}{(s^2+p_n^2)(\lambda_n^2 a^2-1)\lambda_n^2\cosh(\lambda_n h)}\right\}$$
$$+\pi p s^2\mathcal{L}[x_0(t)]\left\{\frac{-a^4}{4} + \frac{2a^2 s^2}{\lambda_n^2(s^2+p_n^2)(\lambda_n^2 a^2-1)\cosh(\lambda_n h)}\right\} \tag{A-115}$$

Equation (A-115) may be rewritten as

$$\mathcal{L}[M_0(t)] = m\left(\frac{h}{2}-\frac{a^2}{4h}\right)s^2\mathcal{L}[x_0(t)] + ms^4\mathcal{L}[x_0(t)]\frac{2[2-\cosh(\lambda_n h)]}{\lambda_n^2 h(\lambda_n^2 a^2-1)(s^2+p_n^2)\cosh(\lambda_n h)} \tag{A-116}$$

Therefore, Eqs. (A-111) and (A-116) represent the forces and moments on the tank when the tank is translated laterally with some $x_0(t)$. The transfer functions for the forces and moments can be written as follows:

$$\frac{\mathcal{L}[F_x(t)]}{\mathcal{L}[x_0(t)]} = \frac{F_x(s)}{x_0(s)} = -m\left[s^2 - \frac{2s^4\tanh(\lambda_n h)}{\lambda_n h(s^2+p_n^2)(\lambda_n^2 a^2-1)}\right] \tag{A-117}$$

$$\frac{\mathcal{L}[M_0(t)]}{\mathcal{L}[x_0(t)]} = \frac{M_0(s)}{x_0(s)} = m\left\{\left(\frac{h}{2}-\frac{a^2}{4h}\right)s^2 + \frac{2s^4[2-\cosh(\lambda_n h)]}{\lambda_n^2 h(\lambda_n^2 a^2-1)(s^2+p_n^2)\cosh(\lambda_n h)}\right\} \tag{A-118}$$

Choosing a spring–mass system to represent the fluid dynamic system, let us determine the conditions which must exist for dynamic similarity of the two systems. Consider the spring–mass system shown in Fig. A-4. The kinetic energy can be written as

$$T = \tfrac{1}{2} m_0\dot{x}_0^2 + \tfrac{1}{2}\sum_{n=1}^{\infty} m_n(\dot{x}_0+\dot{u}_n)^2 \tag{A-119}$$

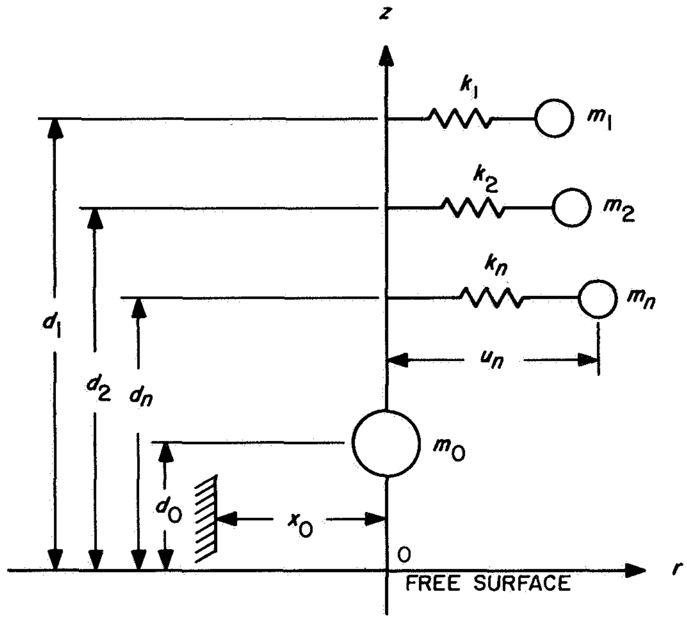


**Fig. A-4. Equivalent spring–mass system**

The potential energy of the system is given as

$$V = \tfrac{1}{2} \sum_{n=1}^{\infty} k_n u_n^2 \tag{A-120}$$

Applying Hamilton's principle to the system, we obtain the relevant Euler equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) - \frac{\partial L}{\partial x_0} = F_x \tag{A-121}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_n}\right) - \frac{\partial L}{\partial u_n} = 0 \tag{A-122}$$

where $L = T - V$.

The equations of motion for the system are thus found to be

$$m_0 \ddot{x}_0 + \sum_{n=1}^{\infty} m_n (\ddot{x}_0 + \ddot{u}_n) = F_x \tag{A-123}$$

$$\sum_{n=1}^{\infty} m_n (\ddot{x}_0 + \ddot{u}_n) + \sum_{n=1}^{\infty} k_n u_n = 0 \tag{A-124}$$

Since we are interested in the force exerted on the spacecraft by the spring–mass system, we have, by Newton's third law,

$$F_x(t) = -m_0 \ddot{x}_0 - \sum_{n=1}^{\infty} m_n (\ddot{x}_0 + \ddot{u}_n) \tag{A-125}$$

Taking moments around point 0, we get

$$M_0(t) = -m_0 d_0 \ddot{x}_0 - \sum m_n d_n (\ddot{x}_0 + \ddot{u}_n) + \sum m_n g u_n \tag{A-126}$$

Taking the Laplace transform of Eq. (A-125),

$$F_x(s) = -m_0 s^2 x_0(s) - \Sigma\left[m_n s^2 x_0(s) + s^2 u_n(s)\right] \tag{A-127}$$

but, from Eq. (A-124),

$$u_n(s) = -\frac{s^2 x_0(s)}{s^2 + \omega_n^2} \tag{A-128}$$

where $\omega_n = (k_n/m_n)^{1/2}$. It follows that

$$F_x(s) = -s^2 x_0(s)(m_0 + \Sigma m_n) + s^4 \sum \frac{m_n x_0(s)}{s^2 + \omega_n^2} \tag{A-129}$$

But, from Eq. (A-111), we had, for the fluid system,

$$F_x(s) = -s^2 x_0(s) m + s^4 \sum \frac{m A_n x_0(s)}{s^2 + p_n^2} \tag{A-130}$$

where $A_n$ is defined by

$$A_n = \frac{2\tanh(\lambda_n h)}{\lambda_n h(\lambda_n^2 a^2 - 1)}$$

Therefore, for dynamic similarity of the forces, we must have

$$p_n = \omega_n$$

$$m = m_0 + \Sigma m_n \tag{A-131}$$

$$m_n = m A_n \tag{A-132}$$

Taking the transform of the moment equation,

$$M_0(s) = -m_0 d_0 s^2 x_0(s) - \Sigma\, m_n d_n\left[s^2 x_0(s) + s^2 u_n(s)\right] + \Sigma\, m_n g u_n(s) \tag{A-133}$$

Using Eq. (A-128), Eq. (A-133) is transformed to

$$M_0(s) = -m_0 d_0 s^2 x_0(s) - \sum m_n d_n\left[s^2 x_0(s) - \frac{s^4 x_0(s)}{s^2 + \omega_n^2}\right] - \sum m_n g \frac{s^2 x_0(s)}{s^2 + \omega_n^2} \tag{A-134}$$

This can be rearranged into the form

$$M_0(s) = -s^2 x_0(s)\left[m_0 d_0 + \sum m_n\left(d_n + \frac{g}{\omega_n^2}\right)\right] + s^4 x_0(s) \sum \frac{m_n(d_n + g/\omega_n^2)}{s^2 + \omega_n^2} \tag{A-135}$$

But, from Eq. (A-116), we had, for the fluid system,

$$M_0(s) = -s^2 x_0(s) m B + s^4 x_0(s)\, m \sum \frac{B_n}{s^2 + p_n^2} \tag{A-136}$$

where

$$B_n = \frac{2(2 - \cosh\lambda_n h)}{\lambda_n^2 h(\lambda_n^2 a^2 - 1)\cosh(\lambda_n h)} \tag{A-137}$$

and

$$B = \left(\frac{a^2}{4h} - \frac{h}{2}\right) \tag{A-138}$$

Therefore, for dynamic similarity of the moments, we must have

$$mB = m_0 d_0 + \sum m_n \left( d_n + \frac{g}{p_n^2} \right) \tag{A-139}$$

$$m \Sigma B_n = \sum m_n \left( d_n + \frac{g}{p_n^2} \right) \tag{A-140}$$

where

$$p_n^2 = \lambda_n g \tanh(\lambda_n h) = \omega_n^2$$

Simultaneous satisfaction of conditions (A-131), (A-132), (A-139), and (A-140) requires that

$$m_n = mA_n \tag{A-141}$$

$$m_0 = m(1 - \Sigma A_n) \tag{A-142}$$

$$d_n = \frac{2[1 - \cosh(\lambda_n h)]}{\lambda_n \sinh(\lambda_n h)} \tag{A-143}$$

$$d_0 = \frac{B - \Sigma B_n}{1 - \Sigma A_n} \tag{A-144}$$

Using Eqs. (A-141 to A-144), a dynamic model that will simulate the propellant slosh phenomena can be constructed.

# APPENDIX B

## Additional Data

### I. "MAXIMUM" PLANETARY VEHICLE

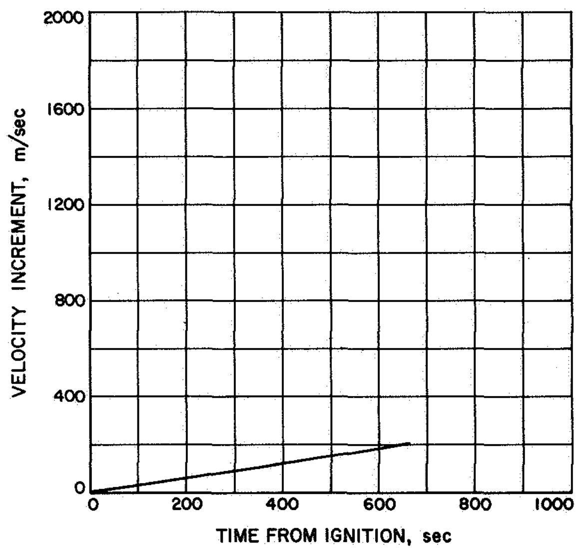


**Fig. B-1. Velocity increment for midcourse maneuver, "maximum" planetary vehicle**

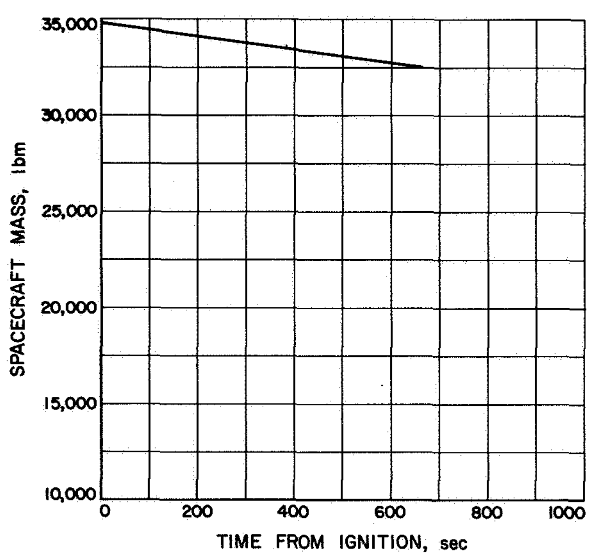


**Fig. B-2. Spacecraft mass, "maximum" planetary vehicle (midcourse maneuver)**

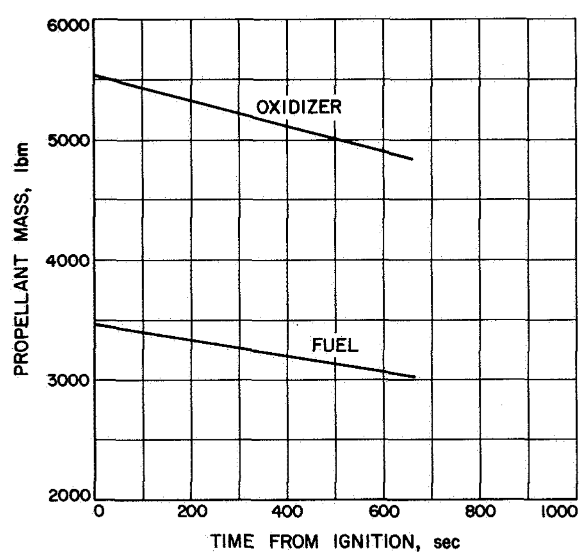


**Fig. B-3. Fuel and oxidizer mass, "maximum" planetary vehicle (midcourse maneuver)**

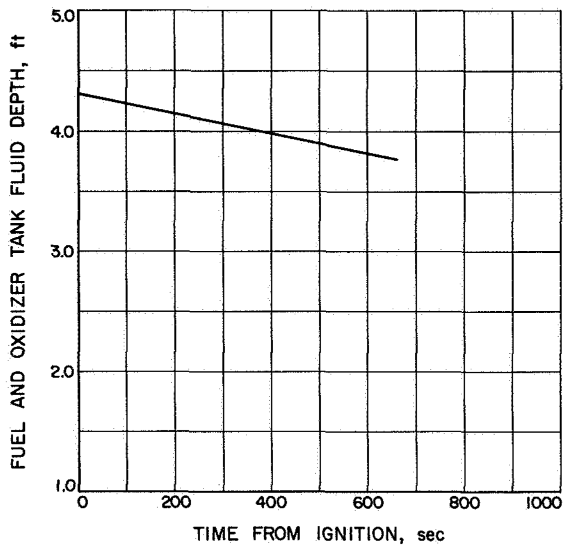


**Fig. B-4. Fuel and oxidizer tank fluid depth, "maximum" planetary vehicle (midcourse maneuver)**

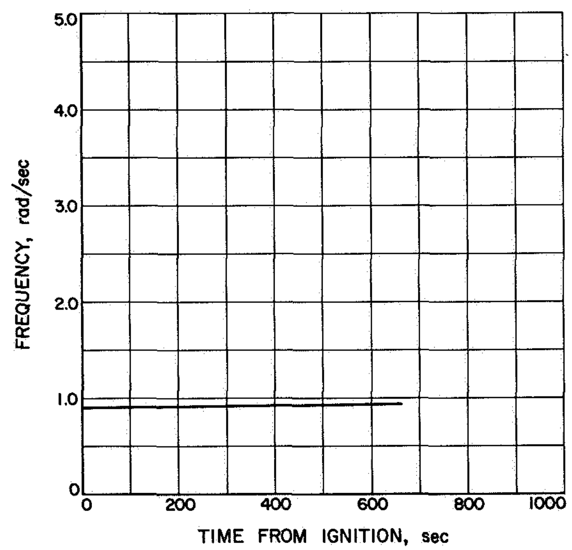


**Fig. B-5. Fuel and oxidizer tank natural slosh frequencies, "maximum" planetary vehicle (midcourse maneuver)**

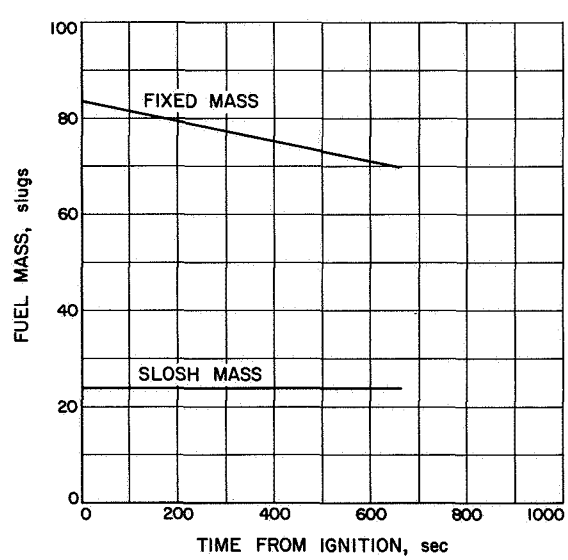


**Fig. B-6. Fixed and slosh fuel mass, "maximum" planetary vehicle (midcourse maneuver)**

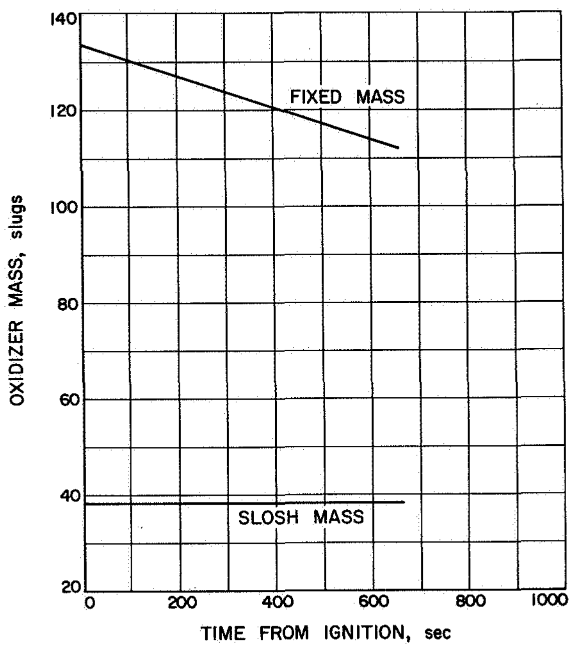


**Fig. B-7. Fixed and slosh oxidizer mass, "maximum" planetary vehicle (midcourse maneuver)**

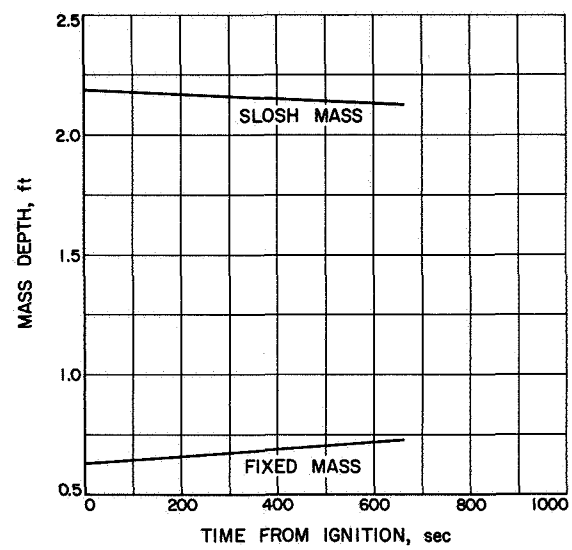


**Fig. B-8. Fixed and slosh fuel mass depth, "maximum" planetary vehicle (midcourse maneuver)**

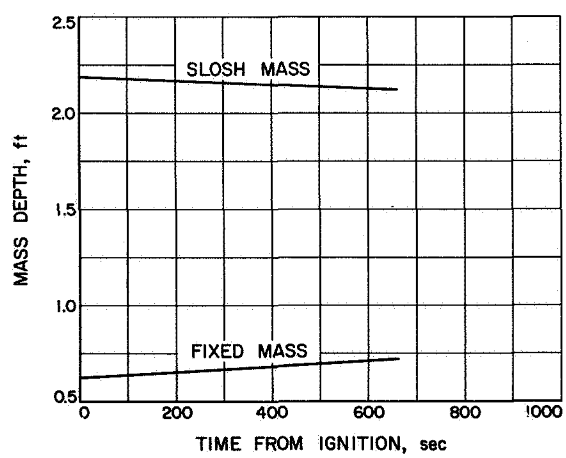


**Fig. B-9. Fixed and slosh oxidizer mass depth, "maximum" planetary vehicle (midcourse maneuver)**

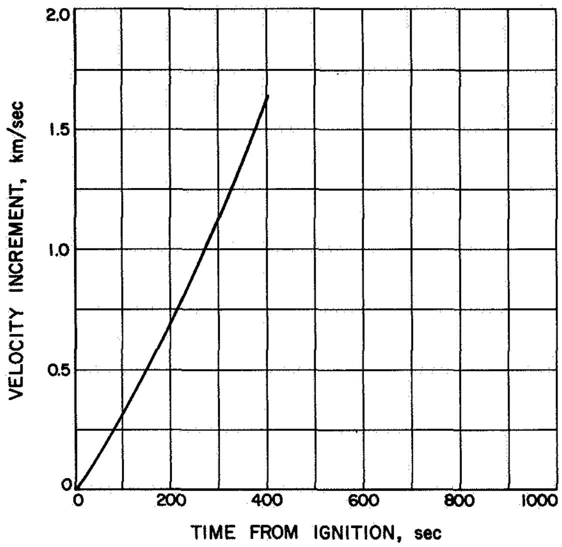


**Fig. B-10. Velocity increment for retromaneuver, "maximum" planetary vehicle**

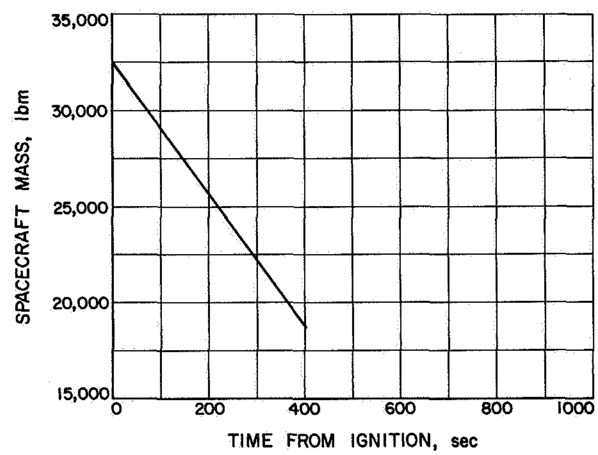


**Fig. B-11. Spacecraft mass, "maximum" planetary vehicle (retromaneuver)**

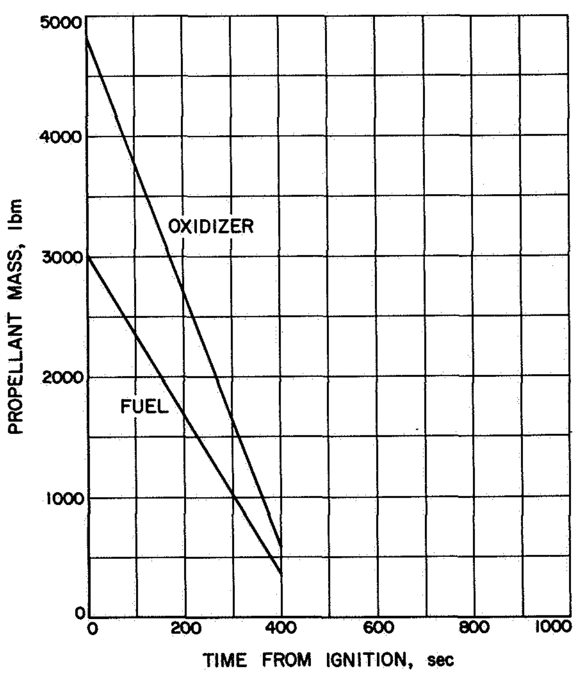


**Fig. B-12. Fuel and oxidizer mass, "maximum" planetary vehicle (retromaneuver)**

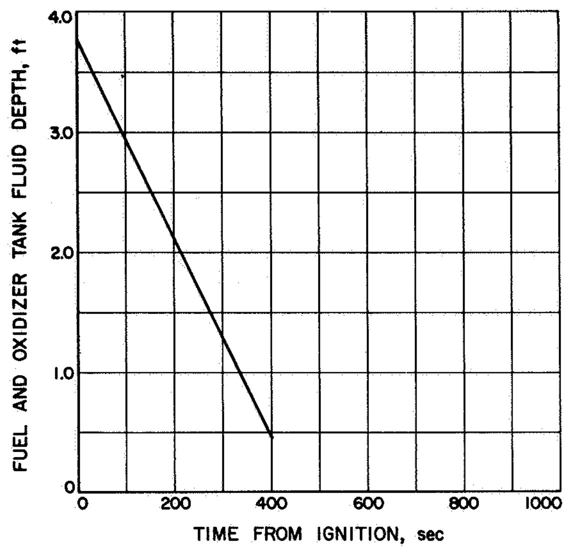


**Fig. B-13. Fuel and oxidizer tank fluid depth, "maximum" planetary vehicle (retromaneuver)**

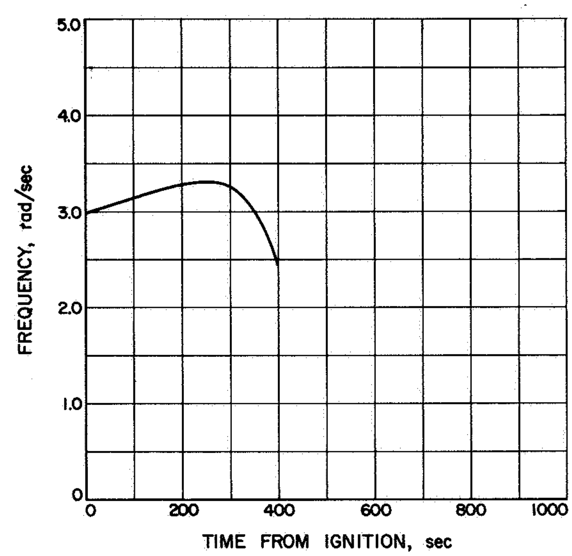


**Fig. B-14. Fuel and oxidizer tank natural slosh frequencies, "maximum" planetary vehicle (retromaneuver)**

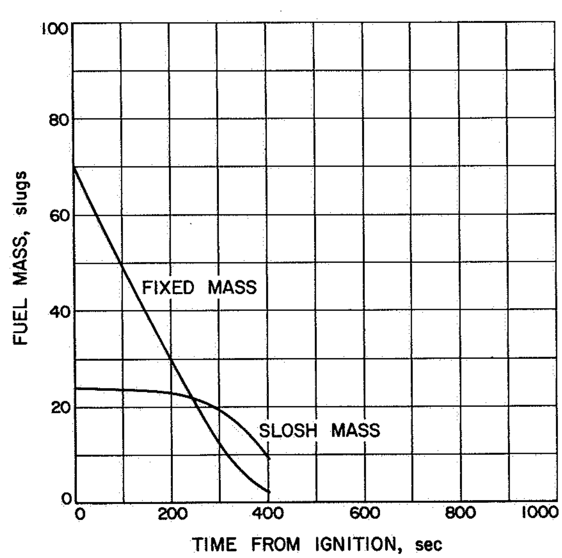


**Fig. B-15. Fixed and slosh fuel mass, "maximum" planetary vehicle (retromaneuver)**

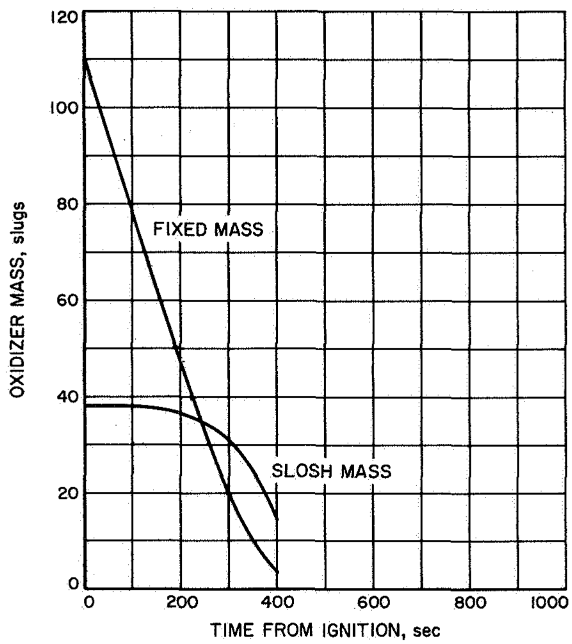


**Fig. B-16. Fixed and slosh oxidizer mass, "maximum" planetary vehicle (retromaneuver)**

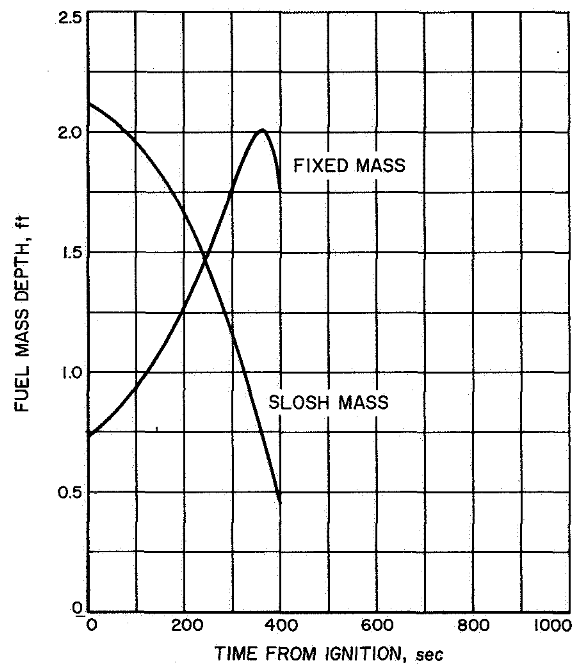


**Fig. B-17. Fixed and slosh fuel mass depth, "maximum" planetary vehicle (retromaneuver)**

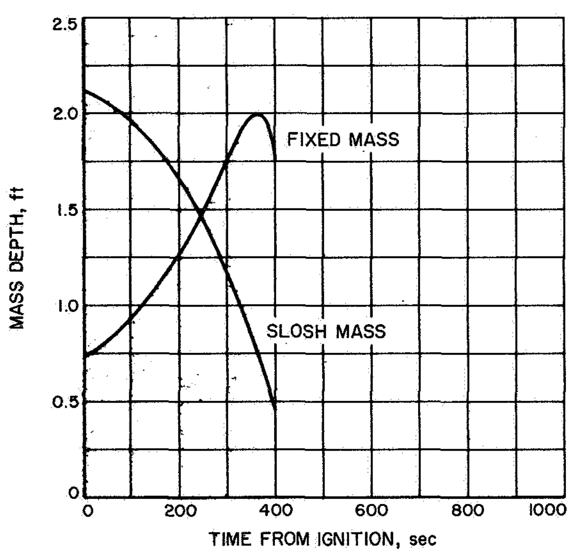


**Fig. B-18. Fixed and slosh oxidizer mass depth, "maximum" planetary vehicle (retromaneuver)**

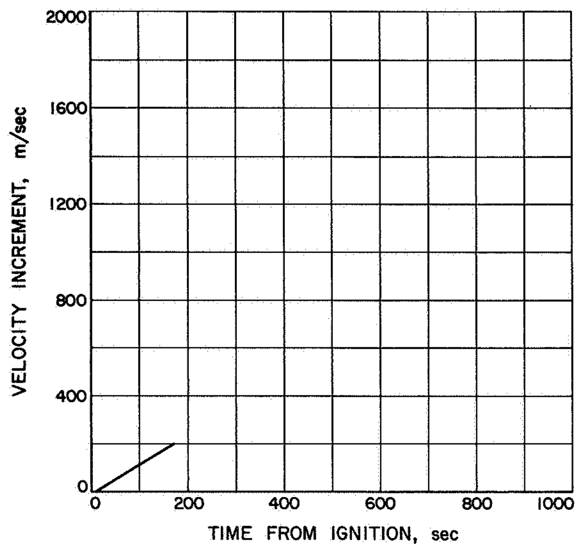


**Fig. B-19. Velocity increment for orbit trim maneuver, "maximum" planetary vehicle**

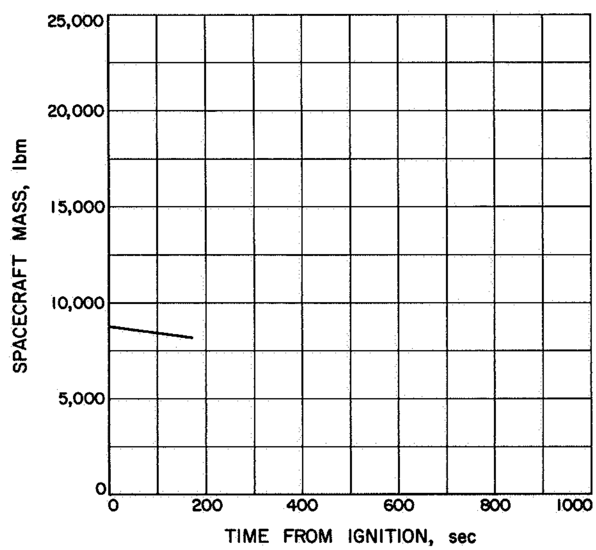


**Fig. B-20. Spacecraft mass, "maximum" planetary vehicle (orbit trim maneuver)**

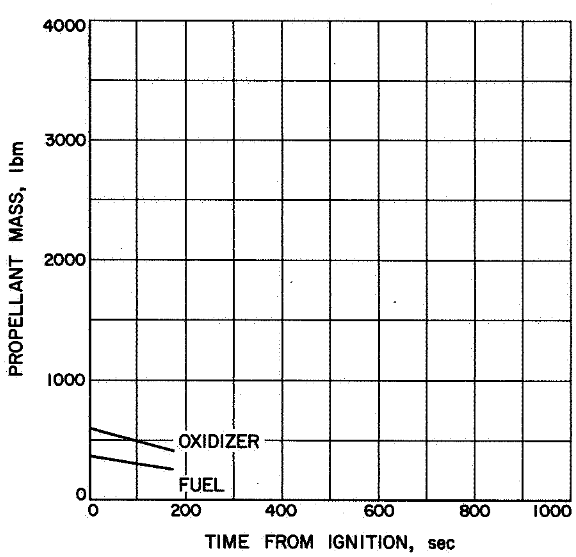


**Fig. B-21. Fuel and oxidizer mass, "maximum" planetary vehicle (orbit trim maneuver)**

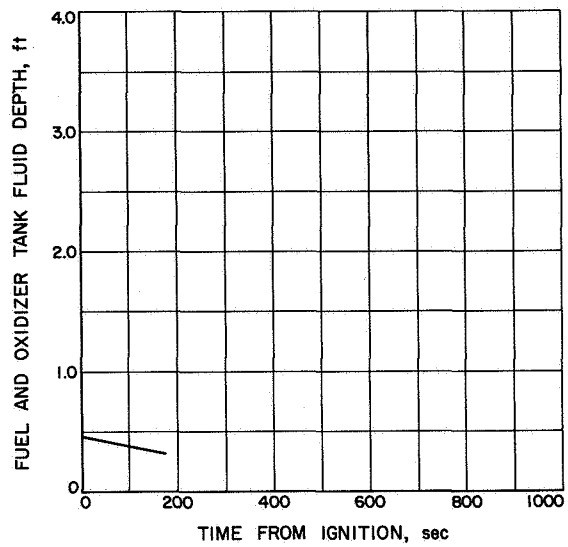


**Fig. B-22. Fuel and oxidizer tank fluid depth, "maximum" planetary vehicle (orbit trim maneuver)**

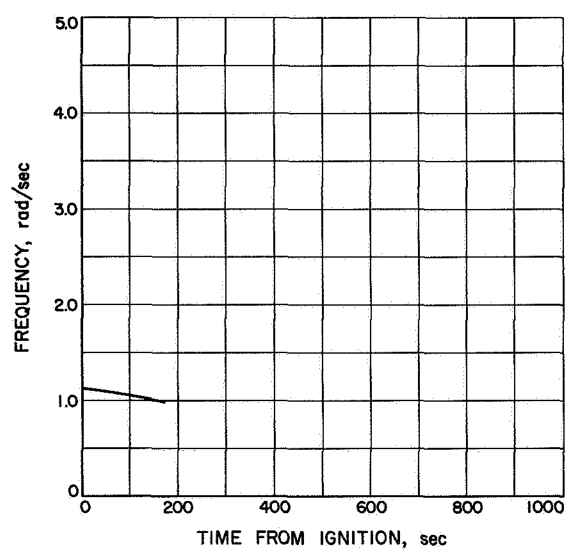


**Fig. B-23. Fuel and oxidizer tank natural slosh frequencies (orbit trim maneuver)**

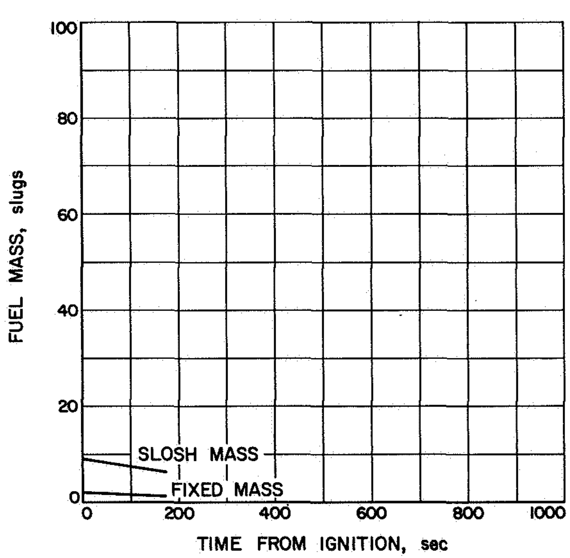


**Fig. B-24. Fixed and slosh fuel mass, "maximum" planetary vehicle (orbit trim maneuver)**

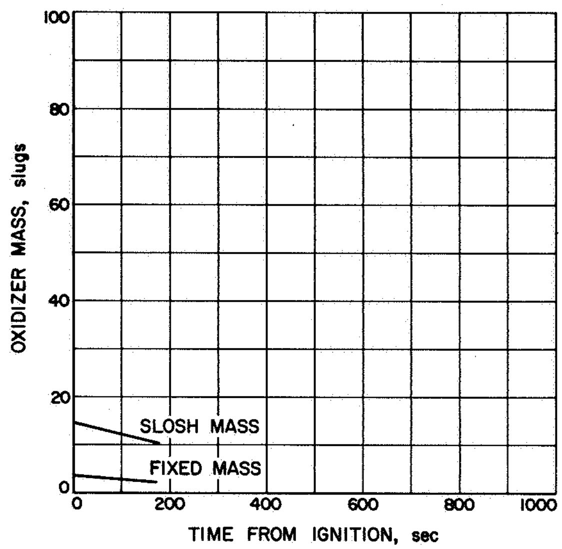


**Fig. B-25. Fixed and slosh oxidizer mass, "maximum" planetary vehicle (orbit trim maneuver)**

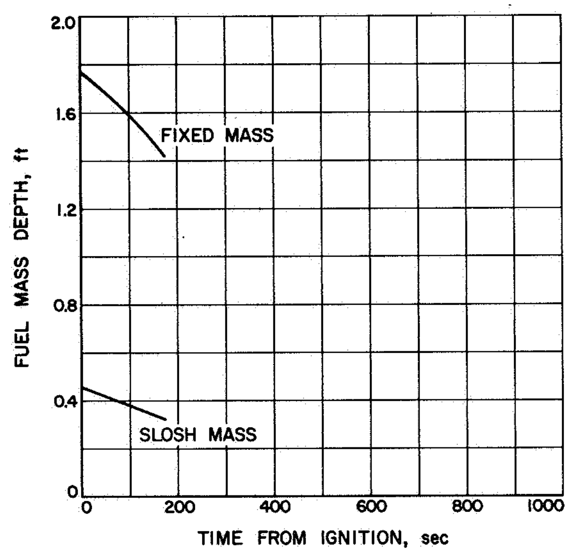


**Fig. B-26. Fixed and slosh fuel mass depth, "maximum" planetary vehicle (orbit trim maneuver)**

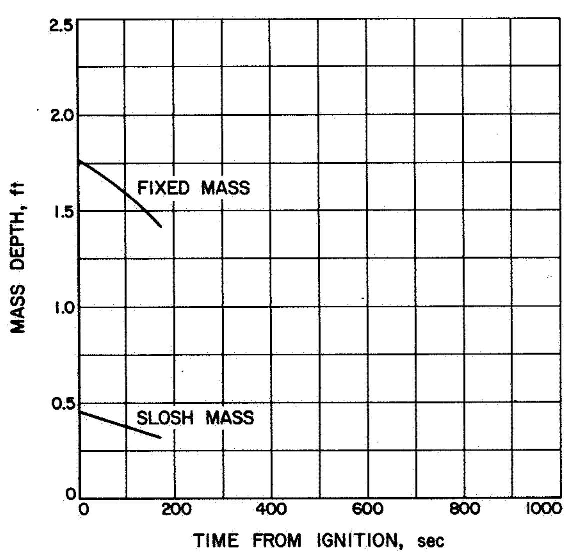


**Fig. B-27. Fixed and slosh oxidizer mass depth, "maximum" planetary vehicle (orbit trim maneuver)**

## II. ORBIT INSERTION WITH CAPSULE SEPARATED

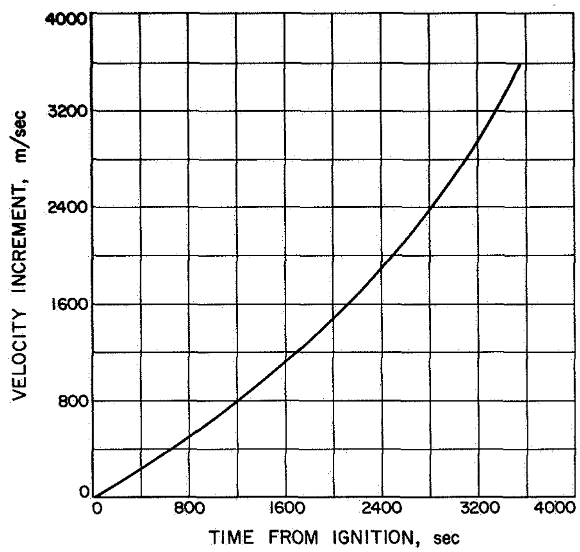


**Fig. B-28. Velocity increment, orbit insertion with capsule separated**

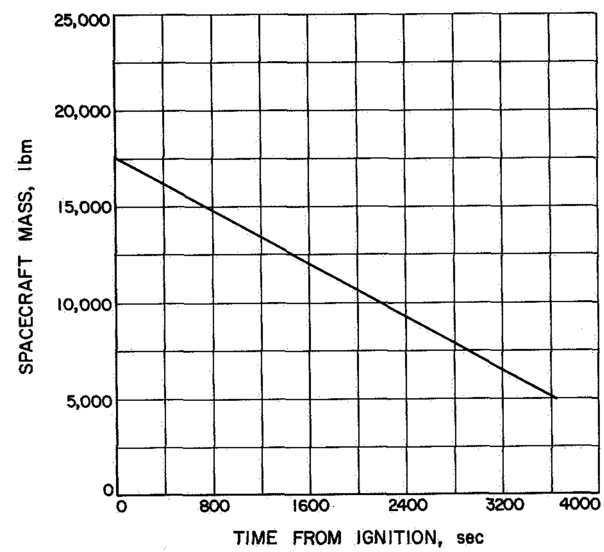


**Fig. B-29. Spacecraft mass, orbit insertion with capsule separated**

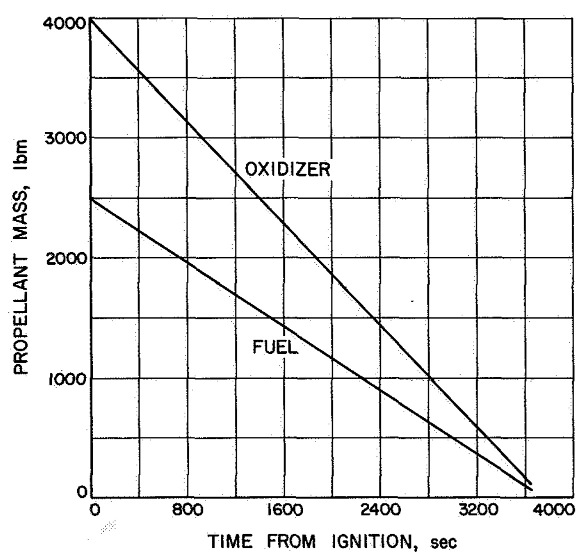


**Fig. B-30. Fuel and oxidizer mass, orbit insertion with capsule separated**

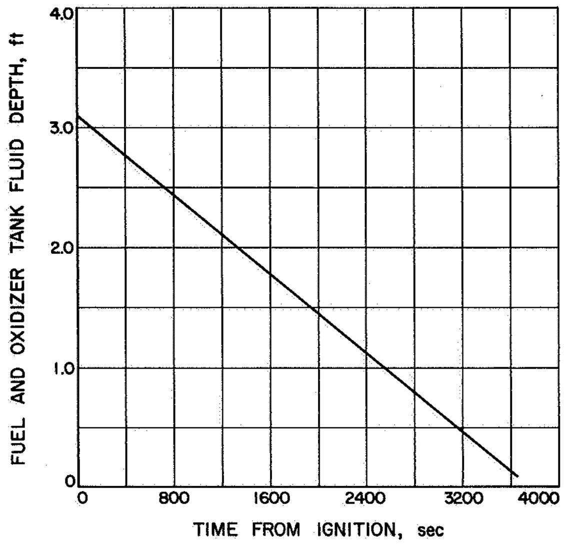


**Fig. B-31. Fuel and oxidizer tank fuel depth, orbit insertion with capsule separated**

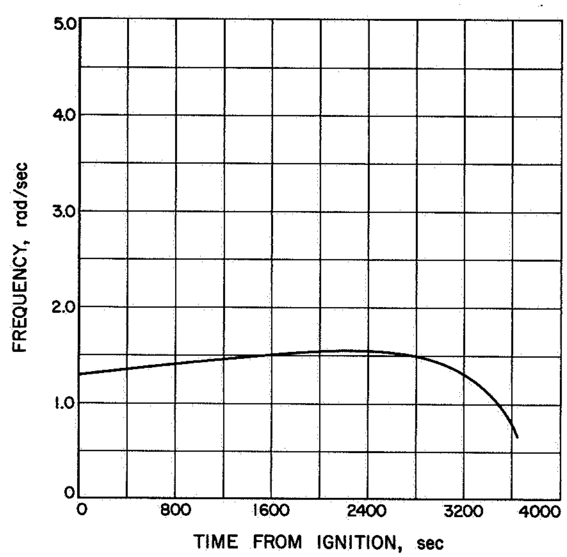


**Fig. B-32. Fuel and oxidizer tank natural slosh frequencies, orbit insertion with capsule separated**

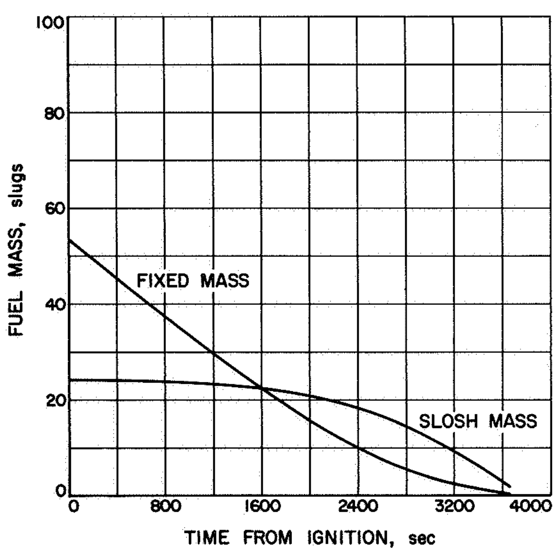


**Fig. B-33. Fixed and slosh fuel mass, orbit insertion with capsule separated**

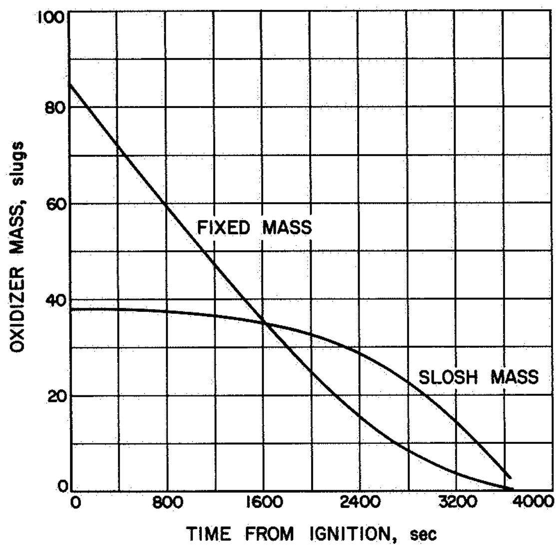


**Fig. B-34. Fixed and slosh oxidizer mass, orbit insertion with capsule separated**

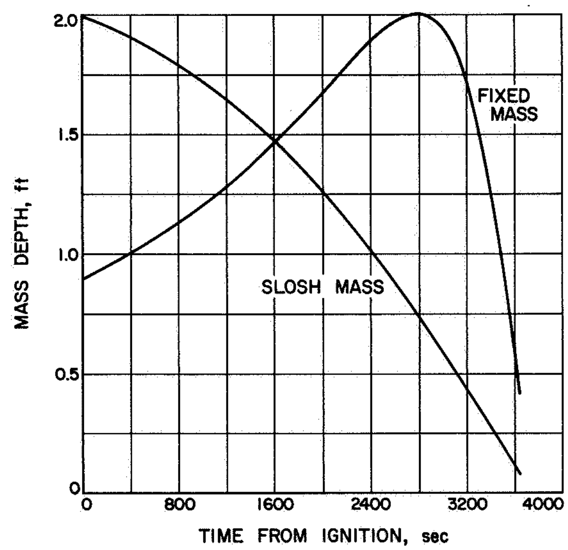


**Fig. B-35. Fixed and slosh fuel mass depth, orbit insertion with capsule separated**

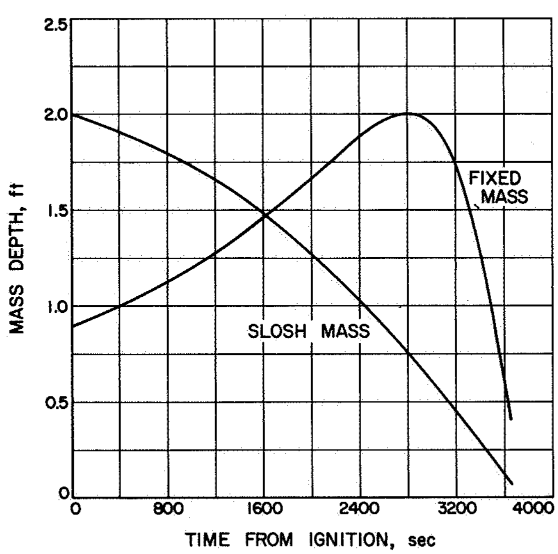


**Fig. B-36. Fixed and slosh oxidizer mass depth, orbit insertion with capsule separated**

## BIBLIOGRAPHY


Abramson, H. N., "Amazing Motions of Liquid Propellants," *Astronautics*, Vol. 6, March 1961, pp. 35–37.

Abramson, H. N., "Liquid Dynamic Behavior in Rocket Propellant Tanks," *Proceedings of the ONR/AIA Symposium on Structural Dynamics of High Speed Flight*, Los Angeles, April 1961, pp. 287–318.

Abramson, H. N., "Discussion on 'Roll Stabilization by Means of Passive Tanks' by Vasta, Giddings, Taplin and Stillwell," *Transactions of the Society of Naval Architects and Marine Engineers*, Vol. 69, 1961, pp. 439–440.

Abramson, H. N., "Dynamic Behavior of Liquid in Moving Container," *Applied Mechanics Reviews*, Vol. 16, No. 7, July 1963, pp. 501–506. Same text and 36 figures appear in "Some Recent Developments Concerned with Liquid Dynamic Behavior in Moving Containers," presented at the ASME Applied Mechanics Conference, Cornell Univ., June 25, 1963.

Abramson, H. N., Chu, W. H., and Garza, L. R., "Liquid Sloshing in Spherical Tanks," *AIAA Journal*, Vol. 1, February 1963, pp. 384–389. Also appears as Technical Report No. 2, Contract NAS8-1555, Southwest Research Institute, San Antonio, Texas, March 1962.

Abramson, H. N., Chu, W. H., and Ransleben, G. E., Jr., "Representation of Fuel Sloshing in Cylindrical Tanks by an Equivalent Mechanical Model," *ARS Journal*, December 1961, pp. 1697–1705.

Abramson, H. N., and Garza, L. R., "Liquid Frequencies and Damping in Compartmented Cylindrical Tanks," *Journal of Spacecraft and Rockets*, Vol. 2, May–June 1965, pp. 453–455.

Abramson, H. N., Garza, L. R., and Kana, D. D., "Some Notes on Liquid Sloshing in Compartmented Cylindrical Tanks," *ARS Journal*, Vol. 32, June 1962, pp. 978–980.

Abramson, H. N., Martin, R. J., and Ransleben, G. E., Jr. "Application of Similitude Theory to the Problem of Fuel Sloshing in Rigid Tanks," Technical Report No. 1 (Contract DA-23-072-ORD-1251), Southwest Research Institute, San Antonio, Texas, May 1958.

Abramson, H. N., and Ransleben, G. E., Jr., "Some Comparisons of Sloshing Behavior in Cylindrical Tanks with Flat and Conical Bottoms," Technical Report No. 4 (Contract DA-23-072-ORD-1251), Southwest Research Institute, San Antonio, Texas, May 1959.

Abramson, H. N., and Ransleben, G. E., Jr. "A Note on Wall Pressure Distributions During Sloshing in Rigid Tanks," Technical Report No. 5 (Contract DA-23-072-ORD-1251), Southwest Research Institute, San Antonio, Texas, June 1959.

Abramson, H. N., and Ransleben, G. E., Jr., *Liquid Sloshing in Rigid Cylindrical Tanks Undergoing Pitching Motion*, Technical Report No. 11 (Contract DA-23-072-ORD-1251), Southwest Research Institute, San Antonio, Texas, May 1961.

Aerojet-General Corp., *Status of Special Problems on F-99 Booster Rocket Program* (Unclassified title), Period Covered: February 15–May 15, 1954, R-838 (Special), June 21, 1954, Confidential.

## BIBLIOGRAPHY (Cont'd)


Armstrong, G. L., and Kachigan, K., "Stability and Control of Carrier Vehicles," *Handbook of Astronautical Engineering*, McGraw-Hill Book Co., New York, 1961.

Bauer, H. F., *The Influence of Fluid in the Tanks on the Moments of Inertia of Jupiter AM7 and AM8*, ABMA SA Memo No. 333, ABMA DA-M-1-58, Army Ballistic Missile Agency, Redstone Arsenal, Ala., March 31, 1958.

Bauer, H. F., *Fluid Oscillations in a Circular Cylindrical Tank*, ABMA R-DA-TR-1-58, Army Ballistic Missile Agency, Redstone Arsenal, Ala., April 1958.

Bauer, H. F., "Fluid Oscillations of a Circular Cylindrical Tank Performing Lissajous Oscillations," ABMA R-DA-TR-2-58, Army Ballistic Missile Agency, Redstone Arsenal, Ala., April 18, 1958.

Bauer, H. F., "The Moment of Inertia of a Liquid in a Circular Cylindrical Tank," ABMA DA-TN-5-58, Army Ballistic Missile Agency, Redstone Arsenal, Ala., April 1958.

Bauer, H. F., "Propellant Sloshing," ABMA R-DA-TR-18-58, Army Ballistic Missile Agency, Redstone Arsenal, Ala., November 5, 1958, Confidential.

Bauer, H. F., "Determination of Approximate First Natural Frequencies of a Fluid in a Spherical Tank," ABMA DA-TN-75-58, Army Ballistic Missile Agency, Redstone Arsenal, Ala., 1958.

Bauer, H. F., *Force and Moment of a Liquid on a Rigidly Fixed Lid on the Free Fluid Surface Due to Translational and Rotational Oscillation of a Tank*, ABMA DA-TN-25-59, Army Ballistic Missile Agency, Redstone Arsenal, Ala., March 20, 1959, Confidential.

Bauer, H. F., *The Effective Moment of Inertia in Roll of Propellant and Roll Damping*, ABMA DA-TM-57-59, Army Ballistic Missile Agency, Redstone Arsenal, Ala., May 26, 1959.

Bauer, H. F., *Theory of the Fluid Oscillations in a Circular Cylindrical Ring Tank Partially Filled with Liquid*, NASA TN-D-557, National Aeronautics and Space Administration, Washington, D. C., December 1960.

Bauer, H. F., "Dynamics of Liquid Propellant Vehicles," *Proceedings of the ONR/AIAA Symposium on Structural Dynamics of High Speed Flight*, Los Angeles, April 1961, pp. 319–355.

Bauer, H. F., "Parametric Study of the Influence of Propellant Sloshing on the Stability of Spacecraft," *Journal of the Aerospace Sciences*, Vol. 28, No. 10, October 1961, pp. 819–820.

Bauer, H. F., *Fluid Oscillations in the Containers of a Space Vehicle and Their Influence Upon Stability*, NASA TR R-187, National Aeronautics and Space Administration, Washington, D. C., February 1964.

Bauer, H. F., *Propellant Oscillations in the Containers of a Roll Oscillating Space Vehicle and Moment of Inertia of Liquid*, AIAA Publication CP-8, AIAA Fifth Annual Structures and Materials Conference, April 1–3, 1964, pp. 184–190.

Benedikt, E. T., *Sloshing in DM-18 Tanks Due to Gusts* (Unclassified title), R-SM-27318, Douglas Aircraft Co., Inc. Santa Monica, Calif., June 27, 1957, Secret.

Benedikt, E. T., *The Effects of Sloshing Upon Pressure Measurements in Missile Tanks: Model DM-18* (Unclassified title) R-SM-27323, Douglas Aircraft Co., Inc., Santa Monica, Calif., July 9, 1957, Secret.

Benjamin, T. B., and Ursell, F., "The Stability of a Plane Free Surface of a Liquid in Vertical Periodic Motion," *Proceedings of the Royal Society (London), Series A*, Vol. 225, 1954, pp. 505–517.

## BIBLIOGRAPHY (Cont'd)


Berlot, R. R., *Representation of Sloshing Modes by the Motions of a Set of Conical Pendulums*, Memo GM-45.3-397, Rev. 1, Ramo-Wooldridge Corp., Los Angeles, Calif., September 19, 1957.

Berlot, R. R., and Freed, L. E., *Comparison of Experimental Sloshing Results with Spherical Pendulum Theory*, Memo GM 42.6-54, Ramo-Wooldridge Corp., Los Angeles, Calif., January 7, 1958.

Bhuta, P. G., and Yeh, G. C. K., "Liquid Sloshing Due to a Time-Dependent Discontinuous Boundary, *Int. J. Mech. Sci.*, Vol. 7, 1965, pp. 475–488.

Binnie, A. M. "Waves in an Open Oscillating Tank," *Engineer*, Vol. 151, 1941, pp. 224–226.

Binnie, A. M., "Self-Excited Oscillations in an Open Circular Water Tank," *Philosophical Magazine*, Vol. 46, 1955, pp. 327–337.

Blackburn, R. R., and St. John, A. D., *Effects and Importance of Penetration and Growth of Lift on Space Vehicle Response*, NASA CR-326, National Aeronautics and Space Administration, Washington, D.C., November 1965.

Bleich, H. H., "Longitudinal Forced Vibrations of Cylindrical Fuel Tanks," *Jet Propulsion*, Vol. 26, No. 2, February 1956, pp. 109–111.

Blokh, E. L., "Horizontal Hydrodynamical Impact of a Sphere in the Presence of a Free Liquid Surface," *PMM*, Vol. 17, No. 5, 1953. English translation by Z. Jakubski appears as STL/TR-61-5110-4, Space Technology Laboratories, Inc., Los Angeles, Calif., January, 1961.

Blokh, E. L., "The Horizontal Impact of an Ellipsoid of Revolution on an Ideal Liquid Having a Free Surface," *PMM*, Vol. 17, No. 6, 1953. English translation by Z. Jakubski appears as STL/TR-61-5110-5, Space Technology Laboratories, Inc., Los Angeles, Calif., January, 1961.

Boeing Airplane Co., *Analysis of the Effect of Airframe Longitudinal Vibration on Propellant Flow of the Bomarc Rocket* (Unclassified title), D-14913, Seattle, Wash., July 1, 1964, Confidential.

Bogoriod, I. B., "On the Solution by a Variational Method of the Problem of the Oscillations of a Liquid Partially Filling a Cavity," *PMM*, Vol. 26, No. 6, 1962, pp. 1122–1127.

Bolotin, V. V., "On the Motion of a Liquid in an Oscillating Container," *PMM*, Vol. 20, No. 2, 1956, pp. 293–294. English translation by K. N. Trirogoff appears as STL-T-Ru-24 (60-5111-104), Space Technology Laboratories Inc., Los Angeles, Calif., August 1960.

Bralitz, H. A., and Heilbron, C. H., "Water Pressures on Dams During Earthquakes," *Transactions of the American Society of Civil Engineers*, Vol. 98, 1933, pp. 452–460.

Budiansky, B., "Sloshing of Liquids In Circular Canals and Spherical Tanks," *Journal of the Aerospace Sciences*, Vol. 27, March 1960, pp. 161–173.

Campbell, I. J., "Wave Motion in an Annular Tank," *Philosophical Magazine*, Series 7, Vol. 44, 1953, pp. 845–854.

Case, K., and Parkinson, W., "On the Motion of a Vessel Containing Fluid with a Free Surface," Memorandum, Space Technology Laboratories, Inc., Los Angeles, Calif., February 8, 1956.

Chu, W. H., "Sloshing of Liquids in Cylindrical Tanks of Elliptic Cross-Section," Technical Report No. 2, Contract DA-23-072-ORD-1251, Southwest Research Institute, San Antonio, Texas, September 26, 1958; also in *ARS Journal*, Vol. 30, April 1960, pp. 360–363.

## BIBLIOGRAPHY (Cont'd)


Chu, W. H., "Free Surface Condition for Sloshing Resulting from Pitching and Some Corrections," *ARS Journal*, Vol. 30, No. 11, November 1960, pp. 1093–1094.

Chu, W. H., "Liquid Sloshing in Spherical Tanks Filled to an Arbitrary Depth," Technical Report No. 4, Contract No. NAS8-1555, Southwest Research Institute, San Antonio, Texas, December 1962.

Clark, C. D., "Natural Frequency Investigation for Propellant in Quarter Tank and Ring Tank Configurations," ABMA DA-TN-86-59, Army Ballistic Missile Agency, Redstone Arsenal, Ala., August 28, 1959.

Clark, C. D., "Effect of Propellant Oscillation on Saturn Roll Moment of Inertia," ABMA DA-TN-87-59, Army Ballistic Missile Agency, Redstone Arsenal, Ala., 1959.

Cockkanoff, O., "Motion of Fluid in a Moving Vessel," *Dissertation Abstracts*, Vol. 24, No. 10, April 1964, 64-3863.

Conrad, A., "Hydrodynamic Forces Induced in Fluid Containers by Earthquakes," Thesis, California Institute of Technology, 1956.

Cooper, R. M., "Dynamics of Liquids in Moving Containers," *ARS Journal*, August 1960, pp. 725–729.

Dodge, F. T., Kana, D. N., and Abramson, H. N., "Liquid Surface Oscillations in Longitudinally Excited Rigid Cylindrical Tanks," AIAA Paper No. 65-83, January 1965; also in *AIAA Journal*, Vol. 3, April 1965, pp. 685–695.

Ehrlich, L. W., Riley, J. D., Strange, W. G., and Troesch, B. A., "Finite-Difference Techniques for a Boundary Problem with an Eigenvalue in a Boundary Condition," *Journal of the Society for Industrial and Applied Mathematics*, Vol. 9, No. 1, March 1961.

Epperson, T. B., Brown R., and Abramson, H. N., "Dynamic Loads Resulting from Fuel Motion in Missile Tanks," *Advances in Missile and Space Technology*, Vol. 2, Pergamon Press, New York, 1961, pp. 313–327.

Eulitz, W. R., "The Sloshing Phenomenon and the Mechanism of a Liquid in Motion in an Oscillating Missile Container," ABMA DS-R-31, Army Ballistic Missile Agency, Redstone Arsenal, Ala., October 1957.

Eulitz, W. R., and Glaser, R. F., *Comparative Experimental and Theoretical Considerations on the Mechanisms of Fluid Oscillations in Cylindrical Containers*, MTP-M-S&M-P-61-11, George C. Marshall Space Flight Center, NASA, Huntsville, Ala., May 29, 1961.

Eulitz, W. R., *Analysis and Control of Liquid Propellant Sloshing During Missile Flight*, MTP-D&VE-P-61-22, George C. Marshall Space Flight Center, NASA, Huntsville, Ala., December 15, 1961.

Ewart, D. G., *Fuel Oscillation in Cylindrical Tanks and the Forces Produced Thereby*, G. W. Technical Note No. 2050, De Havilland Aircraft Co., Ltd., Great Britain, November 1956.

Fickeisen, F. C., *A Simplified Approach to the Problem of Stability for an Airframe and Contained Liquids* (Unclassified title), Document No. D-16135, Contract AF 33(038)19589, Project MX-1599, Boeing Airplane Co., Seattle, Wash., October 25, 1954, Confidential.

Fontenot, L. L., *Model for Longitudinal Vibrations of Liquid Propellant Missiles*, AD-1469, Memo, General Dynamics, January 5, 1962.

Freed, L. E., *Equivalence of Sloshing Wave Height to Pendulum Analogy*, Memo GM 45.3-419, Ramo-Wooldridge Corp., Los Angeles, Calif., October 4, 1957.

## BIBLIOGRAPHY (Cont'd)


Geissler, E. D., "Problems of Attitude Stabilization of Large Guided Missiles," *Aerospace Engineering*, Vol. 19, October 1960, pp. 24–29, 69-71.

Goldsbrough, G. R., "The Tidal Oscillations in an Elliptic Basin of Variable Depth," *Proceedings of the Royal Society (London), Series A*, Vol. 130, 1930, pp. 157–167.

Graham, E. W., *The Forces Produced by Fuel Oscillation in a Rectangular Tank*, SM-13748, Douglas Aircraft Co., Inc., Santa Monica, Calif., April 13, 1950, Revised April 16, 1951.

Graham, E. W., and Rodriguez, A. M., "The Characteristics of Fuel Motion which Affect Airplane Dynamics," SM-14212, Douglas Aircraft Co., Inc., Santa Monica, Calif., November 1951; also *Journal of Applied Mechanics*, Vol. 19, No. 3, September 1952, pp. 381–388.

Hieatt, J. L., and Riley, J. D., *Digital Program for Fluid Sloshing in Tank with Axial Symmetry*, TM-59-0000-00389, Space Technology Laboratories, Inc., Los Angeles, Calif., September 1959.

Hinricks, K., and Kaufman, F. H., *Sloshing Stability Criteria for Vehicles with One Free Surface*, Memo GM 45.3-45, Space Technology Laboratories, Inc., Los Angeles, Calif., July 12, 1956.

Honda, K., and Matsushita, T., "An Investigation of the Oscillations of Tank Water," *Scientific Reports*, First Series, Vol. 2, Tôhoku Imperial Univ., 1913, pp. 131–148.

Hoskins, L. M., and Jacobsen, L. S., "Water Pressure in a Tank Caused by a Simulated Earthquake," *Bulletin of the Seismological Society of America*, Vol. 47, No. 1, January 1957.

Housner, G. W., *Earthquake Pressures on Fluid Containers*, Technical Report No. 8, Earthquake Research Laboratory, California Institute of Technology, ONR Contract N6 ONR-244, August 1954.

Housner, G. W., "Dynamic Pressures on Accelerated Fluid Containers," *Bulletin of the Seismological Society of America*, Vol. 47, 1957, pp. 15–35.

Housner, G. W., "The Dynamic Behavior of Water Tanks," *Bulletin of the Seismological Society of America*, Vol. 53, No. 2, February 1963, pp. 381–387.

Hutton, R. E., and Bhuta, P. G., *Propellant Slosh Loads*, EM 15-10, TRW Systems, Redondo Beach, Calif., June, 1965.

Jacobsen, L. S., "Impulsive Hydrodynamics of Fluid Inside a Cylindrical Tank, and of Fluid Surrounding a Cylindrical Pier," *Bulletin of the Seismological Society of America*, Vol. 39, 1949, pp. 189–204.

Jeffries, H., "Free Oscillations of Water in an Elliptical Lake," *Proceedings of the London Mathematical Society*, Vol. 23, 1924.

Kachigan, K., *Forced Oscillations of a Fluid in a Cylindrical Tank*, ZU-7-046, Convair, San Diego, Calif., October 4, 1955.

Kana, D. D., "Longitudinal Forced Vibration of Partially Filled Tanks," Technical Report No. 6, Contract No. NASw-146, Southwest Research Institute, San Antonio, Texas, February 28, 1963.

Kirchoff, G., and Hausemann, G., "Versuche über Stehened Schwingungen Des Wassers," *Ann. Physik Chem. N. F.*, Vol. 10, No. 246, 1880, pp. 337–347.

Kravtchenko, J., and McNown, J. S., "Seiche in Rectangular Ports," *Quarterly of Applied Mathematics*, Vol. XIII, No. 1, 1955, pp. 19–26.

## BIBLIOGRAPHY (Cont'd)


Kroll R. J., *Longitudinal Vibrations of a Pressurized Cylinder Partially Filled with a Fluid Including the Effect of a Fuel Pipe*, NASA LWP-26 (Langley Working Paper), Langley Research Center, Langley Field, Va., August 14, 1964.

Krzywoblocki, M. Z., "Remarks On 'An Analytical Approach to the Fuel Sloshing and Buffeting Problems of Aircraft,' by Harold Luskin and Ellis Lapin," Readers' Forum, *Journal of the Aerospace Sciences*, Vol. 19, No. 10, October 1952, pp. 715–716.

Lawrence, H. R., Wang, C. J., and Reddy, R. B., "Variational Solution of Fuel Sloshing Modes," *Jet Propulsion*, Vol. 28, No. 11, November 1958, pp. 729–736.

Lee, Y. C., and Miesse, C. C., "On the Forced Vibration of a Tank of Liquid," *Shock and Vibration Bulletin*, No. 22, Naval Research Laboratory, July 1955, pp. 123–133, Confidential.

Leonard, H. W., and Walton, W. C., "An Investigation of the Natural Frequencies and Mode Shapes of Liquids in Oblate Spheroidal Tanks," NASA TN D-904, National Aeronautics and Space Administration, Washington, D. C., June 1961.

Lianis, G., and Fontenot, L., *Analysis of the Free Vibrations of Pressurized Cylindrical Shells Filled With an Ideal Liquid Having a Free Surface*, ERR-AM-194, General Dynamics/Astronautics, San Diego, Calif., September 1962.

Lindholm, U. S., Kana, D. D., Chu, W. H., and Abramson, H. N., *Research in Liquid Dynamics in Missile Fuel Tanks*, Quarterly Programs Report No. 7, Contract No. NASw-2, Southwest Research Institute, San Antonio, Texas, January 1, 1962.

Lorell, J., *Forces Produced by Fuel Oscillation*, Progress Report No. 20-149, Jet Propulsion Laboratory, Pasadena, Calif., 1951.

Lorell, J., *Effect of Fuel Sloshing on the Position of the Center of Rotation of a Missile*, Report No. 20-65, Jet Propulsion Laboratory, Pasadena, Calif., February 14, 1952, Confidential.

Luskin, L., and Lapin, E., "An Analytical Approach to the Fuel Sloshing and Buffeting Problems of Aircraft," *Journal of the Aeronautical Sciences*, Vol. 19, No. 4, April 1952, pp. 217–228.

McNown, J. S., *Sur L'entretien des Oscillations des Eaux Portuaires sous L'action de la Haute Mer*, No. 278, Publications Scientifiques et Techniques du Ministere de L'air, Paris, 1953.

Merton, K. F., and Stephenson, B. H., *Some Dynamic Effects of Fuel Motion in Simplified Model Tip Tanks on Suddenly Excited Bending Oscillations*, NACA TN 2789, National Advisory Committee for Aeronautics, Washington, D. C., 1952.

Moiseyev, N. N., "The Motion of a Rigid Body with a Cavity Partially Filled with an Ideal Heavy Liquid," *Doklady Akadademii Nauk SSSR*, Vol. 85, No. 4, 1952, pp. 719–722. English translation by K. N. Trirogoff appears as STL-T-Ru-26 (60-5111-106), Space Technology Laboratories, Inc., Los Angeles, Calif., August 1960.

Moiseyev, N. N., "On Oscillations of a Heavy Ideal and Incompressible Liquid in a Container," *Doklady Akademii Nauk SSR*, Vol. 85, No. 5, 1952, pp. 963–966.

## BIBLIOGRAPHY (Cont'd)


Moiseyev, N. N., "The Problem of Small Oscillations of an Open Vessel with a Fluid Under the Action of an Elastic Force," *Ukrainian Mat. Zh.*, Vol. 4, 1952, pp. 168–173.

Moiseyev, N. N., "The Problem of the Motion of a Rigid Body Containing Liquid Masses with Free Surfaces," *Matematicheskii Sbornik* (Mathematical Review), Vol. 32 (74), No. 1, 1953.

Moiseyev, N. N., "Some Questions of the Theory of Oscillation of Vessels with a Fluid," *Inzhen. St.*, Vol. 19, 1954, pp. 167–170.

Moiseyev, N. N., "Dynamics of a Ship Having a Liquid Load," *Izvestiya Akademii Nauk SSSR, Otdelenie Tekhnicheskikh Nauk*, No. 7, 1954, pp. 27–45.

Moiseyev, N. N., "On the Theory of Elastic Oscillations of a Fluid-Filled Body," *Doklady Akadademii Nauk SSSR*, Vol. 27, No. 1, 1959, pp. 53–56; English translation in *Soviet Physics–Doklady*, Vol. 4, No. 4, February 1960.

Narimanov, G. S., "On the Motion of a Rigid Body with a Cavity Partially Filled with a Liquid," *PMM*, Vol. 20, No. 1, 1956, pp. 21–38.

Narimanov, G. S., "Concerning the Motion of a Symmetrical Gyroscope, the Cavity of Which is Partially Filled With a Liquid," *PMM*, Vol. 21, 1957, pp. 696–700.

Newell, F. D., *Fuel Sloshing Induced by Lateral Oscillations of the X-1A Airplane*, FRM 185 (AD 39 739), Cornell Aeronautical Laboratories, Inc., Buffalo, N.Y., July 31, 1953.

Okhotsimskii, D. E., "Theory of the Motion of a Body with Cavities Partially Filled With a Liquid," *PMM*, Vol. 20, No. 1, 1956, pp. 3–20. English translation appears as NASA Technical Translation F-33.

Okhotsimskii, D. E., "On the Motion of a Rigid Body with a Cavity Partially Filled With a Liquid," *PMM*, Vol. 21, No. 1, 1956. English translation appears as NASA TT F-33, National Aeronautics and Space Administration, Washington, D. C., May 1960.

Poisson, S. D., "Sur les Petites Oscillations de L'eau Contenue dans un Cylindre," *Annales de Gergonne*, XIX, 1828-9, p. 225; cited by H. Lamb: *Hydrodynamics*, Cambridge University Press, 1932, Dover Publications, New York, 1945, p. 287.

Pode, L., *Linear Equation of Planar Motion of Vehicle Frame Including Effects of Propellant Sloshing*, SGC Report 32R-16, Space-General Corp., El Monte, Calif., June 1961.

Rabinovich, B. I., "On the Equations of Disturbed Motion of a Rigid Body With a Cylindrical Cavity Partially Filled With Liquid," *PMM*, Vol. 20, No. 1, 1956.

Rayleigh, Lord J. W. S., "On Waves," *Philosophical Magazine*, Series 5, Vol. 1, 1876, pp. 257–279. Also in *Scientific Papers*, Vol. 1 (six volumes, 1869–1911), pp. 265.

Reese, J. R., Some Effects of Fluid in Pylon-Mounted Tanks on Flutter, NACA RM L55F10, National Advisory Committee for Aeronautics, Washington, D. C., July 1955, Confidential.

Reese, J. R., and Sewall, J. L., *Effective Moment of Inertia of Fluid in Offset, Inclined, and Swept-Wing Tanks Undergoing Pitching Oscillations*, NACA TN 3353, National Advisory Committee for Aeronautics, Washington, D. C., 1955.

Reismann, H., "Liquid Propellant Inertia and Damping Due to Airframe Roll," *Jet Propulsion*, Vol. 28, No. 11, November 1958, pp. 746–747.

## BIBLIOGRAPHY (Cont'd)


Riley, J. D., and Trembath, N. W., "Sloshing of Liquids in Spherical Tanks," *Journal of the Aerospace Sciences*, Vol. 28, No. 3, March 1961, pp. 245–246.

Rose, R. G., *Dynamics of the Atlas 5-cps Longitudinal Oscillation Following Launch as Related to the Tank Pressure Regulation System*, Report GDA 63-0712 (*Vol. I: Longitudinal Model Development; Vol. II: Pneumatic System Model Development; Vol. III: System Stability Analysis*) General Dynamics/Astronautics, San Diego, Calif., December 31, 1963.

Rumiantsev, V. V., "Equations of Motion of a Rigid Body Having Cavities Partially Filled with a Liquid," *PMM*, Vol. 18, No. 6, 1954, pp. 719–728.

Rumiantsev, V. V., "On Equations of Motion of a Rigid Body with a Cavity Filled with Liquid," *PMM*, Vol. 19, No. 1, 1955, pp. 1–12.

Rumiantsev V. V., "On the Stability of Stationary Motions of Rigid Bodies with Cavities Containing Fluid," *PMM*, Vol. 26, No. 6, 1962, pp. 977–991.

Sanders, D., Rattinger, I., and Gallagher, R., *Investigation and Design Studies for Fuel Storage in High Performance Aircraft, Part I: Basic Report, Part VIII: Fuel Sloshing*, WADC 59-18, Part I, Bell Aircraft Corp., Buffalo, N. Y., May 1959.

Sandorff, P. E., *Principles of Design of Dynamically Similar Models for Large Propellant Tanks*, NASA TN D-99, National Aeronautics and Space Administration, Washington, D. C., January 1960.

Sandorff, P. E., and Petersen, L. K., *Propellant Tank Dynamics*, Report No. 1376, Aerojet-General Corp., Azusa, Calif., December 20, 1957.

Sasaki, S., "On the Oscillations of Water in Circular-Sectorial and Ring-Sectorial Vessels," *Scientific Reports*, First Series, Vol. 3, Tôhoku Imperial Univ., 1914, pp. 257–270.

Schmitt, A. F., *Forced Oscillations of a Fluid in a Cylindrical Tank Undergoing Both Translation and Rotation*, ZU-7-069, Convair, San Diego, Calif., October 16, 1956.

Schmitt, A. F., *Forced Oscillations of a Fluid in a Cylindrical Tank Oscillating in a Carried Acceleration Field — A Correction*, ZU-7-074, Contract No. AF 04(645)-4, Convair, San Diego, Calif., February 4, 1957.

Schultz-Grunow, F., "Stabilität einer Flussigkeits-oberfläche bei periodisch ausgeschalteter Erdschwere," *Forschung auf dem Gebiete des Ingenieurwessens*, Aufl. A. 21, No. 1, 1955, pp. 27-28.

Schy, A. A., *A Theoretical Analysis of the Effects of Fuel Motion on Airplane Dynamics*, NACA Technical Note 2280, National Advisory Committee for Aeronautics, Washington, D. C., January 1951.

Sen, B. M., "Waves in Canals and Basins," *Proceedings of the London Mathematical Society*, Series 2, Vol. 26, 1927, pp. 363–376.

Sewall, J. L., *An Experimental and Theoretical Study of the Effect of Fuel on Pitching-Translation Flutter*, NACA TN 4166, National Advisory Committee for Aeronautics, Washington, D. C., 1957.

Smith, C. C., Jr., *The Effects of Fuel Sloshing on the Lateral Stability of a Free-Flying Airplane Model*, NACA RM L8C16, National Advisory Committee for Aeronautics, Washington, D. C., June 29, 1948.

Smith, K. W., *Fuel Sloshing, Relation Between Surface Movement and Moving Mass Displacement for a Circular Tank*, GW Department File Reference GW/55027/KWS, Royal Aircraft Establishment, Great Britain, August 1947.

## BIBLIOGRAPHY (Cont'd)


Smith, K. W., "Present Work on the Effect of Fuel Sloshing in the Control of Ballistic Missile, Note GW/55027/KWS, Royal Aircraft Establishment, Great Britain, June 1956.

Space Technology Laboratories, Inc., *Advanced Research on Dynamics of Fluids in Motion* (Unclassified title), Quarterly Progress Report, January 1–March 10, 1956, ARL6-17 WDD 56-6532, Contract AF 18(600) 1190, AD 307 845, Confidential.

Sretenskii, L. N., "The Oscillations of Liquid in a Moving Container," *Izvestiya Akademii Nauk SSSR Otdelenie Teknicheskikh Nauk*, 1951, pp. 1483–1494.

Stafford, J., *The Oscillations of Liquid in a Circular Cylindrical Tank with Vertical Axis*, Publication No. TP 183, Saunders-Roe Ltd., Great Britain, June 1956.

Stephens, D. G., *An Analytical and Experimental Investigation of the Stability of an Analogous Liquid Propellant Space Vehicle*, M. S. Thesis, Virginia Polytechnic Institute, Blacksburg, Va., May 1963.

Stephens, D. G., and Leonard, H. W., *The Coupled Dynamic Response of a Tank Partially Filled with a Liquid and Undergoing Free and Forced Planar Oscillations*, NASA TN D-1945, National Aeronautics and Space Administration, Washington, D.C., August 1963.

Stofan, A. J., and Armstead, A. L. *Analytical and Experimental Investigation of Forces and Frequencies Resulting from Liquid Sloshing in a Spherical Tank*, NASA TN D-1281, National Aeronautics and Space Administration, Washington, D.C., July 1962.

Sweitzer, D., *Sloshing of Liquid Propellants*, Astronautics Information Literature Search No. 67, Jet Propulsion Laboratory, Pasadena, Calif., 1960, Confidential.

Tamiya, S., "On the Dynamical Effect of Free Water Surface," *K. Zosen Kyokai*, Vol. 103, 1958, pp. 59–67 (in Japanese).

Taylor, G. I., "The Instability of Liquid Surfaces when Accelerated in a Direction Perpendicular to Their Planes," *Proceedings of the Royal Society (London), Series A*, Vol. 201, 1950, p. 192.

Troesch, B. A., *Pendulum Analogy*, Memo GM 42.4-4, Space Technology Laboratories, Inc., Los Angeles, Calif., November 1957.

Vasta, J., Giddings, A. J., Taplin, A., and Stillwell, J. J. "Roll Stabilization by Means of Passive Tanks," *Transactions of the Society of Naval Architects and Marine Engineers*, Vol. 69, 1961, pp. 411–460.

Ward, B., *Fuel-Sloshing Effects in a Two-Tank Ballistic Missile with Reference to Blue Streak* (Unclassified title), AD 148 238, Royal Aircraft Establishment, Great Britain, August 1957, Secret.

Warner, R. W., and Caldwell, J. T., *Experimental Evaluation of Analytical Models for the Inertias and Natural Frequencies of Fuel Sloshing in Circular Cylindrical Tanks*, NASA TN D-856, National Aeronautics and Space Administration, Washington, D.C., 1961.

Werner, P. W., and Sundquist, K. J., "On Hydrodynamic Earthquake Effects," *Transactions of the American Geophysical Union*, Vol. 30, No. 5, October 1949, pp. 636–657.

Westergaard, H. M., "Water Pressures on Dams During Earthquakes," *Transactions of the American Society of Civil Engineers*, Vol. 98, 1933, pp 418–433.

## BIBLIOGRAPHY (Cont'd)

Widmayer, E., Jr., and Reese, J. R., *Moment of Inertia and Damping of Fluid in Tanks Undergoing Pitching Oscillations*, NACA RM L53E01a, National Advisory Committee for Aeronautics, Washington, D.C., 1953.

Yeh, G. C. K., *Low g Slosh Models*, Memo 65-9713.6-79, TRW Systems, Redondo Beach, Calif., October 26, 1965.

Yeh, G. C. K., "Liquid Sloshing in a Moving Tank with a Time Dependent Discontinuous Boundary," EM 14-12 (6438-6002-RU000) TRW Systems, Redondo Beach, Calif., June 1964.

Zangar, C. N., "Hydrodynamic Pressures on Dams Due to Horizontal Earthquakes," *Proceedings of the Society for Experimental Stress Analysis*, Vol. 10, No. 2, 1953.

Zhukovskii, N. E., "On the Motion of a Rigid Body Having Cavities Filled with Homogeneous Liquid," in *Collected Works*, Vol. 2, Moscow State Publishing House of Technical Literature, 1949.