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LYAPUNOV FUNCTIONS AND THE EXACT DIFFERENTIAL EQUATION

ABSTRACT

Lyapunov functions are considered from the viewpoint of exactness of the differential equations. It is shown that the search for possible Lyapunov functions can be restated in terms of conditions for exactness and sign-definiteness. This procedure allows a generalization of many well-known techniques that have been described in the literature.

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I. Introduction

If a differential equation is the time derivative of a function of one or more time dependent variables, it can be said to be an exact differential equation; the function is called the first integral of the differential equation [1]. Consider a system described by a set of first order autonomous differential equations

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t)], \tag{1}$$

where $\underline{x}$ and $\underline{f}$ are n dimensional vectors and $f_i = f_i\,(x_1, x_2, \ldots, x_n)$. Assume: the definition of f_i and the existence of its first partial derivatives, unique solutions to eq. (1), and an equalibrium point at $\underline{x}=\underline{0}$.

Defining a new vector $\underline{g}$ such that

$$g_i = -f_1 - f_2 \ldots -f_{i-1} + f_{i+1} + \ldots + f_n,$$

the sum of $g_i \dot{x}_i$ for all i can be shown to be zero, so that eq. (1) can be rewritten as a scalar inner product

$$\langle \underline{g}, \dot{\underline{x}} \rangle = 0. \tag{2}$$

Suppose that there exists a function $\underline{h}(\underline{x})$ such that the addition of $\langle \underline{h}, \dot{\underline{x}} \rangle = \langle \underline{h}, \underline{f} \rangle$ to eq. (2) will result in an exact differential equation

$$\langle \underline{g} + \underline{h}, \dot{\underline{x}} \rangle = \langle \underline{h}, \dot{\underline{x}} \rangle. \tag{3}$$

A first integral, $V(\underline{x})$, can then be found such that

$$\frac{dV(\underline{x})}{dt} = \langle \underline{\nabla V}, \dot{\underline{x}} \rangle = \langle \underline{g} + \underline{h}, \dot{\underline{x}} \rangle. \tag{4}$$

Generalizing a procedure for solving an exact differential equation for n = 2 [1], it can be shown, for example, that

$$V(\underline{x}) = \int \psi(\underline{x})\underline{dx} = \int (g_1+h_1)dx_1$$

$$+ \int [(g_2+h_2) - \frac{\partial}{\partial x_2} \int (g_1+h_1)dx_1]dx_2 + \ldots$$

$$+ \int [(g_n+h_n) - \frac{\partial}{\partial x_n} \int (g_1+h_1)dx_1]dx_n \ . \quad (5)$$

Therefore, if we are successful in finding an $\underline{h}$ which insures:

(a) the exactness of the left hand side of eq. (3),

(b) the negative semidefiniteness of the right hand side of eq. (3), and

(c) the positive definiteness of the first integral, eq. (5),

then $V(\underline{x})$ is a Lyapunov function with respect to eq. (1), by definition [3]. Using an argument similar the one found in [1], it follows that the necessary and sufficient condition for (a) is that:

(a') the matrix

$$\theta = [\underline{\nabla}(g_1+h_1) \quad \underline{\nabla}(g_2+h_2) \quad \ldots \ \underline{\nabla}(g_n+h_n)]$$

be symmetric.

Note that these three conditions represent sufficient statements for the development of a Lyapunov function.

II. Calculation of $\underline{h}$

The problem of finding a Lyapunov function for eq. (1) has been restated as a problem in calculating the components of a vector $\underline{h}$ such that conditions (a) or (a') - (c) are satisfied. A variety of methods can be considered to aid the search for a suitable form for $\underline{h}$. Some of these are summarized below and are readily identified with familiar techniques for developing Lyapunov functions.

i) The simplest case exists when all $h_i = 0$ satisfies (a). This implies that eq. (2) is exact. Condition (b) is satisfied and only (c) needs to be considered. If the application of eq. (5) results in a function that satisfies (c), it is a Lyapunov function. This procedure has been called the method of first integrals [2,3].

ii) If eq. (2) is not exact, all h_i cannot be set to zero, as in (i). Without trying to differentiate between $\underline{g}$ and $\underline{h}$, we could proceed, for example, by trying to select a $\underline{\nabla V}$ which satisfies (a') and (b). If this is accomplished, (c) can then be considered. If (c) is not satisfied, the procedure might be repeated for another $\underline{\nabla V}$ satisfying (a') and (b), etc.

iii) For a given differential equation (2), each g_i is represented by linear and nonlinear terms of the dependent variables, such that we could define

$$\underline{g} = \underline{g}_\ell + \underline{g}_n \tag{6}$$

where $\underline{g}_\ell$ contains only the linear terms. If $\underline{h} = \underline{h}_\ell + \underline{h}_n$ is similarly chosen, the procedure in (ii) could be followed by selecting a linear and nonlinear part of $\underline{\nabla V}$. Equation (5) can then be written as

$$V(\underline{x}) = \int \underline{\psi}_\ell d\underline{x} + \int \underline{\psi}_n d\underline{x} \quad . \tag{7}$$

The first of the two integrals resolves into a quadratic form in $\underline{x}$, and eq. (7) is seen to be the familiar quadratic plus integral form [5]. If the nonlinearity of the system is known analytically, it may also be possible to evaluate the second integral. Otherwise the satisfaction of (c) can be considered, in part, directly through the integral characteristics of the nonlinearity [5].

iv) Rather than effectively solving for g_ℓ in eq. (6) as in (iii), but to improve the likelihood of satisfying (c), we might write, for example,

$$
\begin{aligned}
g_1 + h_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\
&\vdots \\
g_i + h_i &= a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \\
&\vdots \\
g_n + h_n &= a_{n1}x_1 + a_{n2}x_2 + \dots + 2x_n,
\end{aligned}
\tag{8}
$$

where the a_{ij} have a constant part plus a function of $x_1, x_2 \dots, x_{n-1}$, e.g., $a_{ij} = a_{ijk} + a_{ijv}$. The a_{ii} can further be limited such that $a_{iik} > 0$ and $a_{iiv} = a_{iiv}(x_i)$ to facilitate satisfying (c), etc. Proceed by selecting the undetermined constants in eq. (8) to first satisfy (b) and then the remaining terms in the a_{ij} to satisfy (a). Lastly condition (c) would be checked following the use of eq.(5). This corresponds to a variable gradient method [6].

v) Alternately consider $\underline{h}$ in eq. (3) separated into two parts $\underline{h}_1 + \underline{h}_2$, where $\underline{h}_1$ is selected specifically to satisfy (a) only. $\underline{h}_2$ could then be chosen with respect to condition (b), and finally condition (c) would be examined. If (c) cannot be satisfied, an alternate $\underline{h}_2$ may be considered. This procedure is a generalization of a method described in the literature by Infante and by Walker, although their objectives are less apparent [4,7]. In their terminology the "new modified system" would be equivalent to eq. (3), where

$$\langle \underline{g} + \underline{h}, \dot{\underline{x}} \rangle = \langle \underline{g} + \underline{h}_1 + \underline{h}_2, \dot{\underline{x}} \rangle .$$

vi) If one initially selects a function as a first integral such as to satisfy (c) and if the second partial derivatives exist, then (a') is also satisfied, and condition (b) need only be considered. Another first integral can be tried if condition (b) isn't satisfied. While this corresponds to the method of prognostication which so often confuses the neophite, it is also the basis of the more sophisticated method of squaring proposed by Krasovskii [3].

vii) Using a somewhat different technique from those mentioned above, we might begin initially by selecting a function $\phi = \langle \underline{h}, \underline{f} \rangle$, the right side of eq. (3), to satisfy (b) but also to allow a solution for V directly from eq. (4) without appealing to (a) and eq. (5). If this can be done and (c) is satisfied, a Lyapunov function has been found and (a) has been automatically satisfied (assuming sufficient continuity conditions). If (c) cannot be satisfied, a more selective choice of ϕ may be sought. This is the approach proposed by Zubov [3,8]. In effect, one attempts to circumvent a direct consideration of condition (a) and eq. (5) by solving the partial differential equation (4).

III Summary

The construction of Lyapunov functions for the study of the stability properties of nonlinear differential equation solutions has been considered from a unifying perspective. The exact differential equation appears to provide the common base for many of the seemingly unrelated methods suggested in the literature. But in addition it is now also possible to begin investigating new techniques more systematically.

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