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ROTATION INVARIANT PROBABILITY DISTRIBUTIONS ON THE SURFACE OF A SPHERE, WITH APPLICATIONS TO GEODESY

BY

JAMES E. POTTER AND ELMER J. FREY

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Abstract

Rotation Invariant Probability Distributions on the Surface of a Sphere, with Applications to Geodesy

by

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Gravity anomalies over the geoid are treated as a stationary stochastic process whose spectrum is shown to be positive. A class of linear transforms defined over the surface of a sphere and invariant under rotations is considered and the spectrum of such transforms is defined. The transform of the random process is shown to have its own spectrum which is expressed as a product of the spectra of the original random process and of the transform. Examples of rotation invariant transforms are the transforms which express undulations in an equipotential surface, or deflections of the vertical, or gravity anomalies at any altitude in terms of the gravity anomalies at the surface. An example is worked out in which it is shown that the convolution expression for the linear transform represented by the Poisson kernel which carries the potential function from one radius to another may be represented by a single integral involving a complete elliptic integral.

1. Introduction

The evaluation of very high harmonics of the earth's gravitational field may be of interest in airborne gravimetry or satellite geodesy, if knowledge is desired, at altitudes above the geoid, of gravity anomalies, deflections of the vertical, or of undulations of the equipotential surface. These latter quantities may be expressed as linear integral transforms of the gravity anomalies at the geoid surface.[1] Without knowledge of the anomalies over the entire surface, the classical Stokes integral formulas cannot provide exact expressions of the desired quantities. However, the statistical information about the anomalies at the surface may be used to provide certain statistical parameters of the desired expressions at altitude.

Kaula[2] has treated gravity anomalies over the geoid as a random process described statistically by a covariance function invariant under translation on the surface, and has derived the spectrum of the random process using the methods of Wiener's generalized harmonic analysis.[3] In communications and control systems theory these methods are widely used in the spectral analysis of random processes with time as the independent variable, and of linear transforms of the random processes. This article is devoted to the description of certain linear transforms defined over the surface of a sphere and to the derivation of the spectra of such linear transforms of stationary random processes on the sphere.

The stationary time series is one whose statistics are invariant under a translation in time and has a particularly simple covariance function, The corresponding definition for a random process defined over the surface of a sphere requires invariance under a translation on the surface, which corresponds to invariance under any rotation about the center. Such statistics are appropriate for the derivation of a common statistical parameter to describe an arbitrary path above the surface. The covariance function of a random process $\chi(t)$ is the statistical average of $\chi(t_1)\chi(t_2)$ and for a stationary process is a function only of the interval $t_2 - t_1$. The covariance function of a stationary random process defined on the sphere depends only on the distance between the two points. The Fourier transform of the covariance function of a stationary time series is a non-negative quantity which represents the power spectral density of the series. The covariance function of the stationary process on the sphere has a non-negative spectrum represented by the coefficients in a Legendre polynomial expansion. The spectrum of the time series is a function of frequency alone and contains no phase information. Similarly the spectrum of the process on a sphere represents only the degree of spherical harmonics and not the order.

The linear operators associated with random processes in control system theory represent linear ordinary differential equations in the time domain, with stochastic inputs. The spectrum of the operator is represented by the transfer function of the differential equation, and the spectrum of the transformed quantity is a product involving the spectra of the random process and of the linear transform. In the case of linear transforms over the sphere, the kernels of certain rotation-invariant transforms are shown to have spectra analogous to those of the constant coefficient linear differential equation operator. The spectrum of the transformed quantity is again a product involving the spectra of the random process and of the transform.

The development involves first the definition of rotation invariance for probability distributions and for linear operators over the surface of a sphere. The spectrum of the random process is shown to be positive, the spectrum of the linear operator is derived, and the spectrum of the transformed function is derived. Finally, an example using the Poisson kernel is chosen and a representation of the convolution with the Poisson kernel is obtained as a single integral involving in the integrand a complete elliptic integral of the second kind.

2. Rotation Invariant Probability Distributions and Operators on the Surface of a Sphere

Let points on the surface S of the unit sphere be represented by unit vectors* $\underline{e}$. Let H denote the Hilbert space $L^2(S)$ with the inner product (f,g) of two functions $f(\underline{e})$ and $g(\underline{e})$ defined as the integral of their product with respect to surface area:

$$(f,g) = \int_S f(\underline{e})g(\underline{e})\, ds(\underline{e}) \tag{1}$$

Let $\chi(\underline{e})$ be a random function on S with covariance function:

$$C^*(\underline{e_1},\underline{e_2}) = \overline{\chi(\underline{e_1})\chi(\underline{e_2})} \tag{2}$$

and let $C^*(\underline{e_1},\underline{e_2})$ be the kernel of a linear "covariance" operator $\tilde{C}$ on H:

$$\tilde{C}f = \int_S C^*(\underline{e_1},\underline{e_2})f(\underline{e_2})\, ds(\underline{e_2}) \tag{3}$$

If f and g are L^2 functions over S, $\xi = (f,\chi)$ and $\eta = (g,\chi)$ are scalar random variables, and:

$$\overline{\xi\eta} = (f,\tilde{C}g) = (\tilde{C}f,g) \tag{4}$$

C is self-adjoint since the kernel C* is symmetric.

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*The following notation is used:

Vectors are represented by underlined lower case letters.

A horizontal line above a symbol represents a statistical average.

The kernel of a linear integral operator which is a function of two unit vectors is represented by an upper case letter with an asterisk superscript; the corresponding operator is represented by the same letter with a tilde, as in Eq (3). When the kernel reduces to a function of one variable, the upper case letter without the asterisk is used to represent the function.

Let $\tilde{L}$ be a linear operator on H, let R be a rotation matrix and let f be L^2. Define the function $f'(\underline{e})$ by the relation:

$$f'(\underline{e}) = f(R\underline{e}) \tag{5}$$

Let $g = \tilde{L}f$ and $g' = \tilde{L}f'$. Then $\tilde{L}$ is defined to be a rotation invariant (RI) operator if:

$$g'(\underline{e}) = g(R\underline{e}) \tag{6}$$

for every R and every f.

If the statistics of the random process $\chi(\underline{e})$ are rotation invariant, $\tilde{C}$ is a RI operator. The Poisson operator which relates potential functions at different radii is an example of a RI operator.

3. Spectrum of RI Operators and Distributions

Let $L^*(\underline{e}_1,\underline{e}_2)$ be the kernel of the integral operator $\tilde{L}$:

$$g(\underline{e}_1) = \tilde{L}f(\underline{e}_1) = \int_S L^*(\underline{e}_1,\underline{e}_2)f(\underline{e}_2)\ ds(\underline{e}_2) \tag{7}$$

For $\tilde{L}$ to be RI it is necessary and sufficient that L^* be a function only of $\underline{e}_1 \cdot \underline{e}_2$, or:

$$L^*(\underline{e}_1,\underline{e}_2) = L(\underline{e}_1 \cdot \underline{e}_2) \tag{8}$$

Let L be expanded in the series of Legendre polynomials:

$$L(x) = \sum_{k=0}^{\infty} \left[\frac{2k+1}{2} \int_{-1}^{1} P_k(u)L(u)\ du \right] P_k(x) \tag{9}$$

and let $\tilde{P}_k$ be the RI operator with kernel $P_k(x)$. Then if $f_e(\underline{e})$ is a spherical harmonic of degree e:

$$\tilde{P}_k f_e = \frac{4\pi}{2e + 1} \delta_{ke} f_e \tag{10}$$

where δ_{ke} is the Kronecker symbol. Equations (9) and (10) show that the eigen functions of $\tilde{L}$ are the spherical harmonic polynomials and the eigenvalues are the coefficients:

$$L_k = 2\pi \int_{-1}^{1} L(x) P_k(x) \, dx \tag{11}$$

so that Eq (9) may be rewritten:

$$L(x) = \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k + 1) L_k P_k(x) \tag{9'}$$

The corresponding Parseval relationship is:

$$\int_{-1}^{1} L^2(x) \, dx = \frac{1}{8\pi^2} \sum_{k=0}^{\infty} (2k + 1) L_k^2 \tag{12}$$

The spectrum of the linear operator $\tilde{L}$, represented by the eigenvalues L_k, corresponds to the transfer function of the linear constant coefficient ordinary differential equation operator in the time domain. The eigenvalues of the covariance operator $\tilde{C}$ will be called the power spectrum of the probability distribution. The next step is to show that the spectrum C_k is positive.

Let f_k be a spherical harmonic of degree k. Then from Eq (4):

$$\overline{(f_k, \chi)^2} = \overline{(f_k, \chi)(f_k, \chi)} = (\tilde{C} f_k, f_k) \tag{13}$$

Since $C(x)$ may be expressed in Legendre polynomials as:

$$C(x) = \sum_{k=0}^{\infty} \frac{(2k + 1)}{4\pi} C_k P_k(x) \tag{14}$$

with coefficients C_k defined as in Eq (9), the operator $\tilde{C}$ may be expressed in the form:

$$\tilde{C} = \sum_{k=0}^{\infty} \frac{(2k + 1)}{4\pi} C_k \tilde{P}_k \tag{15}$$

Application of Eqs (10) and (15) to Eq (13) produces:

$$\overline{(f_k, \chi)^2} = \sum_{e=0}^{\infty} \left(C_e \tilde{P}_e f_k, f_k \right) = C_k \tag{16}$$

if the f_k are normalized. Thus C_k is expressed as an average of the square of a scalar quantity, which must be non-negative, and the spectrum is positive, just as the Fourier transform of the covariance function of a stationary time series is positive.

4. <u>Convolution of Two Kernels</u>

The product of two RI operators is RI. Let:

$$\tilde{N} = \tilde{M}\tilde{L} \tag{17}$$

Then:

$$N^*(\underline{e}_1, \underline{e}_2) = \int_S M^*(\underline{e}_1, \underline{e}_3) L^*(\underline{e}_3, \underline{e}_2) \, ds \, (\underline{e}_3) \tag{18}$$

and:

$$N(x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} M(y) L(z) \cos \phi \, d\theta \, d\phi \tag{19}$$

where:

$$\begin{aligned} y &= x \cos \phi \cos \theta + \sqrt{1 - x^2} \cos \phi \sin \theta \\ z &= \cos \phi \cos \theta \end{aligned} \tag{20}$$

Denote this convolution operation by $N = M*L$. From the preceding section, it follows that $N_k = M_k L_k$.

5. <u>Statistics of Rotation Invariant Transforms</u>

Let $\tilde{L}$ and $\tilde{M}$ be RI operators and let:

$$\psi = \tilde{L}\chi \; , \; \theta = \tilde{M}\chi \tag{21}$$

where $\chi(\underline{e})$ is a RI random process with covariance function $C^*(\underline{e}_1, \underline{e}_2)$.

Now let:

$$P^*(e_1, e\ ) = \overline{\psi(\underline{e}_1)\theta(\underline{e}_2)} \tag{22}$$

be the statistical average of the product $\psi(\underline{e}_1)\theta(\underline{e}_2)$.

Let f and g belong to H and let:

$$\xi = (f,\psi)\ ,\ \eta = (f,\theta) \tag{23}$$

Then:

$$\overline{\xi\eta} = (f,\tilde{P}g) \tag{24}$$

From Eqs (3) and (4):

$$\left.\begin{aligned}(f,\theta) &= (f,\tilde{M}\chi) = (\tilde{M}f,\chi)\\ (g,\psi) &= (g,\tilde{L}\chi) = (\tilde{L}g,\chi)\end{aligned}\right\} \tag{25}$$

and:

$$\overline{\xi\eta} = (\tilde{M}f,\tilde{C}\tilde{L}g) = (f,\tilde{M}\tilde{C}\tilde{L}g) \tag{26}$$

Consequently, the covariance operator $\tilde{P}$, its kernel P, and its spectrum P_k are:

$$\tilde{P} = \tilde{M}\ \tilde{C}\ \tilde{L} \tag{27}$$

$$P = M*C*L \tag{28}$$

$$P_k = M_k C_k L_k \tag{29}$$

Equation (29) shows that the spectrum of an RI transform of an RI stochastic process is also positive, for this is the case when $M_k = L_k$ and P_k consequently has the same sign as C_k. The preceding expressions are also useful in calculating the cross- spectra of two different transforms of the same random process.

6. The Poisson Operator

The kernel of the Poisson operator is:

$$L(x) = \frac{r_o(r^2 - r_o^2)}{4\pi[r^2 + r_o^2 - 2rr_o x]^{3/2}}$$

$$= \frac{1}{4\pi}\left(\frac{r_o}{r}\right)\sum_{k=0}^{\infty} (2k + 1)\left(\frac{r_o}{r}\right)^k P_k(x) \tag{30}$$

so that the spectrum of $\tilde{L}$ is:

$$L_k = \left(\frac{r_o}{r}\right)^{k+1} \tag{31}$$

Let $R(x)$ be the covariance of the potential at radius r_1 with the potential of another point at radius r_2 and with a radius making an angle γ with the first radius ($x = \cos\gamma$). The power spectrum of R is:

$$R_k = C_k\left(\frac{r_o}{r_1}\right)^{k+1}\left(\frac{r_o}{r_2}\right)^{k+1} \tag{32}$$

where C_k is the spectrum of the potential at radius r_o. If $C(x)$ is a covariance function, $C_k \geqq 0$ for all k; furthermore, if the C_k are non-negative and $\sum_{k=0}^{\infty}(2k + 1)C_k$ converges, there exists a Gaussian distribution with covariance function $C(x)$. In particular:

$$C(0) = \overline{\chi^2} = \frac{1}{4\pi}\sum_{k=0}^{\infty} (2k + 1)C_k \tag{33}$$

represents the mean square value of the random function $\chi(\underline{e})$.

7. Representation of Convolution with the Poisson Kernel as a Single Integration

The Poisson kernel for transforming a potential from radius r_o to radius r_1 is:

$$P_\alpha(x) = \frac{1}{4\pi}\frac{\alpha(1 - \alpha^2)}{[1 + \alpha^2 - 2\alpha x]^{3/2}} \tag{34}$$

where:

$$\alpha = \frac{r_o}{r_1} \tag{35}$$

Using this definition

$$P_\alpha * P_\beta = P_\gamma \tag{36}$$

where $\gamma = \alpha\beta$. The calculation of the covariance of the potential at two different radii r_1 and r_2:

$$D\left(\frac{1}{r_1} \underline{r_1}, \frac{1}{r_2} \underline{r_2}\right) = \overline{[(P_\beta \chi)\underline{e_1}][(P_\alpha \chi)\underline{e_2}]} \tag{37}$$

reduces to the convolution:

$$D = P_\gamma * C \tag{38}$$

where C is the covariance function for r_o and

$$\gamma = \frac{r_o^2}{r_1 r_2} \tag{39}$$

The convolution (38) will be expressed as a one-dimensional integral; that is, an $R(x,y)$ will be found such that:

$$D(x) = \int_{-1}^{1} R(x,y)C(y)\, dy \tag{40}$$

Such an expression might be more useful than the Legendre polynomial expansion of $D(x)$ for investigating short correlation distances.

Equation (38) may be rewritten:

$$D(\underline{e_1},\underline{e_2}) = \int_S P_\gamma(\underline{e_1},\underline{e_3})C(\underline{e_3},\underline{e_2})\, ds\, (\underline{e_3}) \tag{38'}$$

Define:

$$\underline{e_1} \cdot \underline{e_2} = \cos \psi \tag{41}$$

Without loss of generality, rotation invariance permits simplifying the equations by assuming:

$$\underline{e_2} = \underline{k}$$

$$\underline{e_1} = \cos\psi\ \underline{k} + \sin\psi\ \underline{i} \tag{42}$$

$$\underline{e_3} = \cos\phi\ \underline{k} + \sin\phi\ (\cos\theta\ \underline{i} + \sin\theta\ \underline{j})$$

where $\underline{i}$, $\underline{j}$, $\underline{k}$ are the usual unit vectors and ϕ and θ represent the colatitude and longitude of the point represented by $\underline{e_3}$. Then:

$$\underline{e_2} \cdot \underline{e_3} = \cos\phi$$

$$\underline{e_1} \cdot \underline{e_3} = \cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta \tag{43}$$

$$ds(\underline{e_3}) = \sin\phi\ d\theta\ d\phi$$

and the convolution expression (40) takes the form:

$$D(\cos\psi) = \int_0^{\pi}\left\{\int_0^{2\pi} P_\gamma(\cos\psi\cos\phi + \sin\psi\sin\phi\cos\theta)\ d\theta\right\}C(\cos\phi)\sin\phi\ d\phi \tag{44}$$

where the expression within the brackets:

$$U(\psi,\phi) = \int_0^{2\pi} P_\gamma(\cos\psi\cos\phi + \sin\psi\sin\phi\cos\theta)\ d\theta \tag{45}$$

replaces $R(x,y)$ Symmetry of $\cos\theta$ reduces (45) to:

$$U(\psi,\phi) = 2\int_0^{\pi} P_\gamma\ (\cos\psi\cos\phi + \sin\psi\sin\phi\cos\theta)\ d\theta \tag{45'}$$

and (44) becomes:

$$D(\cos\psi) = \int_0^{\pi} U(\psi,\phi)C(\cos\phi)\sin\phi\ d\phi \tag{44'}$$

If $\rho(x)$ is defined by

$$\rho(x) = \sqrt{1 + \gamma^2 - 2\gamma x} \tag{46}$$

then:

$$P_\gamma(x) = \frac{\gamma(1 - \gamma^2)}{4\pi\rho^3(x)} \tag{47}$$

Setting: $$x = \cos\psi\cos\phi + \sin\psi\sin\phi\cos\theta \tag{48}$$

$$\rho^2 = 1 + \gamma^2 - 2\gamma(\cos\psi \cos\phi + \sin\psi \sin\phi \cos\theta)$$

$$= 1 + \gamma^2 - 2\gamma\cos(\psi + \phi) - 4\gamma \cos^2\frac{\theta}{2} \sin\psi \sin\phi \qquad (49)$$

or:
$$\rho^2 = \rho_o^2(1 - a^2\cos^2\frac{\theta}{2}) \qquad (50)$$

where:
$$\rho_o = \sqrt{1 + \gamma^2 - 2\gamma\cos(\psi + \phi)} \qquad (51)$$

$$a = \sqrt{\frac{4\gamma\sin\phi \sin\psi}{1 + \gamma^2 - 2\gamma\cos(\psi+\phi)}} \qquad (52)$$

and a is real since both ψ and ϕ range between o and π. From the relationship:

$$1 - a^2 = \frac{1 + \gamma^2 - 2\gamma\cos(\psi-\phi)}{1 + \gamma^2 - 2\gamma\cos(\psi+\phi)} \qquad (53)$$

which is a non-negative quantity, the result follows that:

$$a^2 \leqq 1 \qquad (54)$$

Thus if η is used to replace $\frac{\theta}{2}$, there follows:

$$U(\psi,\phi) = \frac{(1 - \gamma^2)}{\pi\rho_o^3} \int_0^{\frac{\pi}{2}} \frac{d\eta}{[1 - a^2\cos^2\eta]^{3/2}} \qquad (55)$$

This can be simplified into the form of the complete elliptic integral of the second kind:

$$E(a) = \int_0^{\frac{\pi}{2}} \sqrt{1 - a^2\sin^2\eta}\, d\eta \qquad (56)$$

by use of the relationship:

$$\int_0^{\frac{\pi}{2}} \frac{d\eta}{[1 - a^2 \cos^2\eta]^{3/2}} = \frac{E(a)}{1 - a^2} \qquad (57)$$

which is proved below. The final form of Eq (45) thus becomes:

$$U(\psi,\phi) = \frac{\gamma(1-\gamma^2)E(a)}{\pi[1+\gamma^2-2\gamma\cos(\psi-\phi)]\sqrt{1+\gamma^2-2\gamma\cos(\psi+\phi)}} \quad (58)$$

where a and E(a) are defined by (52) and (56).

To prove Eq (57) is valid, use the complete elliptic integral of the first kind:

$$K(a) = \int_0^{\frac{\pi}{2}} \frac{d\eta}{\sqrt{1-a^2\cos^2\eta}} = \frac{1}{a}\int_0^{\frac{\pi}{2}} \frac{d\eta}{\sqrt{1/a^2-\cos^2\eta}} \quad (59)$$

whence:

$$a\frac{dK}{da} = -K + \int_0^{\frac{\pi}{2}} \frac{d\eta}{[1-a^2\cos^2\eta]^{3/2}} \quad (60)$$

However, the elliptic integrals satisfy:

$$a\frac{dK}{da} = -K + \frac{E(a)}{1-a^2} \quad (61)$$

whence Eq (57) follows immediately.

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