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Profile Measurements of Plasma Columns Using Microwave Resonant Cavities*

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ABSTRACT

The density profile of a positive column of a gas discharge is measured by the comparison of two microwave cavity measurements. The two cavity modes used are the TE_{011} and TM_{020}.

INTRODUCTION

The positive column of a low pressure discharge has been diagnosed by various techniques. In particular, the radial density profile has been measured by Langmuir probe and interferometer techniques. No attempt, to our knowledge, has been made to make such measurements using resonant cavities. The Langmuir probe method has been used successfully to measure the profile only for large diameter tubes. Interferometer techniques can be used for small diameter tubes; they have been carried out by many authors and have been summarized in a book.[1] However, it is known that interferometer measurements are in general only 1% as sensitive to plasma density changes as are resonant cavity measurements (with the possible exception of a recently reported experiment using a Fabry-Perot resonator[2]). It is, therefore, desirable that resonant cavities be used to make such measurements.

Resonant cavities have been used extensively to measure the density of plasma in the positive column. In making these measurements, the plasma was always assumed to be uniform, and so an effective number density was actually measured. But, it is noticed that different cavities give different results. This is due to the different electric field configurations associated with the different cavity modes, which interact with the inhomogeneity to give the observed difference in plasma frequencies. By using two or more modes (one or more cavities), this difference can be explored, and allows us to obtain the radial density profile.

From various theories, we can estimate the electron radial density profile. First, one can obtain the well-known Bessel function profile $n = n_o J_o (\sqrt{2C}\ r/r_o)$ based on the balance between the generation of electrons due to ionization collisions and the loss to the wall due to amplipolar diffusion. Second, one can estimate the profile of electrons in the bulk part of plasma through that of the ions because of quasineutrality conditions. It was shown in a recent calculation[3] for the positive column that the ion density has a profile of the form $n = n_o (1 + Cr^2/r_o^2)^{-1+\mu/2(\mu+\nu)}$, where μ and ν are collision and ionozation frequencies, respectively. Since μ is two orders of magnitude less than ν for the mercury plasma,[4] the density profile becomes $n = n_o (1 + Cr^2/r_o^2)^{-1}$. In either case, if we accept these formulas from an asymptotic point of view, we can write

$$n = n_o (1 - Cr^2/r_o^2) \tag{1}$$

where r_o is the tube radius and C is the "profile parameter".[5]

EXPERIMENTAL THEORY

For the chosen general form of the density profile, we consider what cavity modes should be used, and how they should be used. In the literature, Brown[6] suggested using the TE_{011} and TM_{111} modes to obtain information about the radial electron density distribution. However, the three non-zero electric field components of the TM_{111} mode, make calculations virtually impossible. For our purpose the TM_{020} mode is used. This mode has similar desirable features as the TM_{111} except that it has only one non-zero component. The field components of both modes are given below: For the TM_{020} mode:

$$\begin{aligned}
E_r &= 0 \\
E_\varphi &= 0 \\
E_z &= \frac{k^2}{j\omega\epsilon} J_o\,(x_{02}\, r/r_c) \\
H_r &= 0 \\
H_\varphi &= \frac{x_{02}}{r_c}\, J_1\,(x_{02}\, r/r_c) \\
H_z &= 0
\end{aligned} \tag{2}$$

For the TE_{011} mode:

$$E_r = 0$$
$$E_\varphi = \frac{x_{01}'}{r_c} J_1 (x_{01}' r/r_c) \sin\left(\frac{\pi}{d} z\right)$$
$$E_z = 0$$
$$H_r = \frac{x_{01}'}{r_c} \frac{1}{j\omega\mu_0} \frac{\pi}{d} J_1 (x_{01}' r/r_c) \sin\left(\frac{\pi}{d} z\right)$$
$$H_\varphi = 0$$
$$H_z = \frac{x_{01}'^2}{r_c^2} J_o (x_{01}' r/r_c) \sin\left(\frac{\pi}{d} z\right)$$

where r_c is the radius of the cavity, d is the cavity height, and x_{01}', x_{02} are zeros of the Bessel function.

We notice that these modes have two other desirable features in addition to each having only one electric field component. As shown in the diagram (Fig. 1), the electric field intensity of the TM_{020} mode is approximately constant over the plasma region, while that of the TE_{011} varies greatly, being zero at the origin, and growing as r^2. It follows that these two fields interact very differently with the plasma, whose density profile is also shown in the same diagram. It is this difference that allows us to "weigh" the density profile differently so as to determine the "profile parameter" c given in equation (1). The second desirable feature is that in both cases the electric field component is parallel to the plasma-glass boundary. Thus, there are no localized electrostatic modes generated as in the case of Tonks-Dattner resonance.[7] The strongest reflection is from the plasma-glass interface, and there is negligible reflection from the density profile itself. This is a very important point for our experiment and will be made use of in the appendix.

But for plasma tube currents < .1 amps, we can estimate that $E_r \ll E_o$ and therefore shall use the approximation $E_i \approx E_o$, where E_o, E_r, E_i are the electric field components without the plasma column present, that reflected from the plasma-glass boundary, and that inside the plasma, respectively.

The perturbation equation for a cylindrical cavity with a plasma tube symmetrically placed about its center is given by:

$$\frac{\Delta f}{f_o} = \frac{\int_{\Delta v} f_p^2 \, |E_o|^2 \, dv}{2f_o^2 \int_V |E_o|^2 \, dv} \qquad (3)$$

where f_o is the resonant frequency of the particular cavity mode
Δf is the shift in resonant frequency
Δv is the volume occupied by the plasma in the cavity
V is the volume of the cavity.

From equation (1), one writes:

$$f_p^2 = f_{po}^2 \, (1 - cr^2/r_o^2) \qquad (4)$$

where f_{po} is the local plasma frequency at r = 0.

After substitution, the perturbation equation becomes:

$$\frac{1}{f_{po}^2} = \frac{1}{2f_o\,\Delta f}\left\{\frac{\int_{\Delta v} \left|E_o\right|^2 dv}{\int_v \left|E_o\right|^2 dv} - \frac{c}{r_o^2}\,\frac{\int_{\Delta v} r^2 \left|E_o\right|^2 dv}{\int_v \left|E_o\right|^2 dv}\right\} \tag{5}$$

We now evaluate equation (5) for the TM_{020} and TE_{011} modes, where f_o, Δf are superscripted with M or E, depending on whether they are of the TM_{020} or TE_{011} mode, respectively:

TM_{020} Mode:

$$\frac{1}{f_{po}^2} = \frac{1}{2f_o^m\,\Delta f^m}\left\{\frac{\int_0^{r_o} J_o^2\,(x_{02}\,r/r_c)\,r\,dr}{\int_0^{r_c} J_o^2\,(x_{02}\,r/r_c)\,r\,dr} - \frac{c}{r_o^2}\,\frac{\int_0^{r_o} J_o^2\,(x_{02}\,r/r_c)\,r^3\,dr}{\int_0^{r_c} J_o^2\,(x_{02}\,r/r_c)\,r\,dr}\right\} \tag{6}$$

We make the approximation

$$J_o^2\,(x_{02}\,r/r_c) \simeq 1 - \tfrac{1}{2}\,(x_{02}\,r/r_c)^2 \tag{7}$$

where the higher order terms contribute a negligible amount to the integral. Substituting the above, and evaluating the integrals, we obtain:

$$\frac{1}{f_{po}^2} = \frac{1}{2f_o^m \, \Delta f^m} \left\{ \left(\frac{r_o}{r_c} \right)^2 \frac{[J_o^2(u) + J_1^2(u)]^{u=x_{02}\, r_o/r_c}}{[J_o^2(u) + J_1^2(u)]^{u=x_{02}}} \right.$$

$$\left. - \frac{c}{r_o^2} \left(\frac{r_c}{x_{02}} \right)^2 \frac{2/x_{02}^2}{[J_o^2(u) + J_1^2(u)]^{u=x_{02}}} \left[\frac{1}{4} u^4 - \frac{1}{12} u^6 \right]^{u=x_{02}\, r_o/r_c} \right\} \quad (8)$$

The various dimensions for the experiment are listed below:

r_o = .54 cms

r_c = 8.12 cms

x_{02} = 5.52

Substituting these in the perturbations equation (8), one obtains

$$\frac{1}{f_{po}^2} = \frac{1}{2f^m \, \Delta f^m} \left\{ 3.68 - 1.82\, c \right\} \cdot 10^{-2} \quad (9)$$

<u>TE_{011} Mode:</u>

$$\frac{1}{f_{po}^2} = \frac{1}{2f^E \, \Delta f^E} \left\{ \frac{\int_0^{r_o} J_1^2 \, (x'_{01}\, r/r_c)\, r\, dr}{\int_0^{r_c} J_1^2 \, (x'_{01}\, r/r_c)\, r\, dr} \right.$$

$$\left. - \frac{c}{r_o^2} \frac{\int_0^{r_o} J_1^2 \, (x'_{01}\, r/r_c)\, r^3 dr}{\int_0^{r_c} J_1^2 \, (x'_{01}\, r/r_c)\, r\, dr} \right\} \quad (10)$$

Once again we make the reasonable approximation

$$J_1^2\,(x'_{01}\ r/r_c) \simeq \frac{1}{4}\,(x'_{01}\ r/r_c)^2 - \frac{1}{16}\,(x'_{01}\ r/r_c)^4 \qquad (11)$$

and carry out the integration

$$\frac{1}{f_{po}^2} = \frac{1}{2f^E\ \Delta f^E}\left\{ \left(\frac{r_o}{r_c}\right)^2 \frac{[J_1^2(u) - J_o(u)\ J_2(u)]^{u=x'_{01}\ r_o/r_c}}{[J_1^2(u) - J_o(u)\ J_2(u)]^{u=x'_{01}}} \right.$$

$$\left. - \frac{c}{r_o^2}\left(\frac{r_c}{x'_{01}}\right)^2 \frac{\frac{1}{2x'^2_{01}}}{[J_1^2(u) - J_o(u)\ J_2(u)]} \cdot \left[\frac{1}{6}u^6 - \frac{1}{32}u^8\right]^{u=x'_{01}\ r_o/r_c} \right. \qquad (12)$$

where

r_o = .54 as before
r_c = 5.4 cms
x'_{01} = 3.83

Substituting the above in (12), we obtain

$$\frac{1}{f_{po}^2} = \frac{1}{2f^E\ \Delta f^E}\left\{ .113 - .0725c \right\} \cdot 10^{-2}$$

We have evaluated the two perturbation equations for the TM_{020} and TE_{011} modes. The plasma radial inhomogeneity has been included in these equations via the parameter c. By carrying out perturbation experiments such that f_{po}^2 is the same for both modes, we can eliminate it, and hence have an expression of c in terms of the measured quantities f_o^m, f_o^E, Δf^m, Δf^E.

EXPERIMENT AND RESULTS

Two cylindrical cavities, excited in the TM_{020} and TE_{011} modes, respectively, were placed in turn over a chosen region of length 8 cms of the positive column. The cavities were excited by a 2-4 GH_z oscillator, and the tube current read with a calibrated ammeter. After the positive column had warmed up and reached a steady state, measurements of Δf against plasma tube currents for current $\leq$ .12 amps were made. The data is shown below:

Current I (amps)	$\Delta f^m \cdot 10^{-3} GH_z$	$\Delta f^E \cdot 10^{-3} GH_z$
.06	4.4 $\pm$ .1	.10 $\pm$ .005
.04	7.6 $\pm$ .15	.18 $\pm$ .007
.12	11.3 $\pm$ .2	.27 $\pm$ .010

The three points are plotted on the same graph using the same current axis but different units of frequency shift so that both plots may appear on the graph.

As was already stated, it is reasonable to assume that for a given current each cavity sees the same f^2_{po}. Hence, we obtain an expression for C in terms of the known parameters.

$$C = \frac{3.68 \ \Delta f^E/\Delta f^m - .0925}{1.82 \ \Delta f^E/\Delta f^m - .0593}$$

We choose values of Δf^E, Δf^m for currents .070, .075, .080 amps, find C for each, and average to obtain a C corresponding to .75 amps. The values are tabulated below:

	I_{amps}	Δf^E	Δf^m	$\Delta f^E/\Delta f^m$	C
	.07	.127	5.5	.0230	.45
	.075	.140	6.0	.0231	.435
	.08	.152	6.5	.0230	.440
Average	.075	--	--	--	.44

We can evaluate the % difference which, including the radial inhomogeneity in our perturbation equation, is produced in evaluating $(f^2_p)_{ave.}$ instead of f^2_{po}. We use the value of C for 0.075 amps, namely C = .44.

For the TM_{020} mode:

$$\%\ \text{difference} = 100\% \left(1 - \frac{3.68 - 1.82\ c}{3.68}\right)$$

$$\approx 20\%$$

For the TE_{011} mode:

$$\%\ \text{difference} = 100\% \left(1 - \frac{.113 - .0725\ c}{.113}\right)$$

$$\approx 30\%$$

CONCLUSION

The radial electron density distribution of the positive column was measured using resonant cavities. For a current of 0.075 amp, the 'profile parameter' c characterizing this distribution was found to be 0.44 $\pm$ .1. The errors arose from two sources, namely, the measurement of the tube radius (which was made with a travelling microscope after the tube had reached a steady state), and the approximation $E = E_o$ in the perturbation equation. It should be emphasized that some assumption had to be made as to the form of the distribution function. The general method can obviously be applied to other choices of the distribution forms such as trapezoid.[1] Furthermore, one could pick a more complicated form of the distribution function, characterized by say, two 'profile parameters', and by using more than two cavity modes, estimate both of them.

APPENDIX

In equation (3), we have made the approximation that the field inside the plasma tube remains approximately unchanged, i.e., $\overline{E} \sim \overline{E}_o$. In this appendix, the error due to such an approximation is calculated. In the calculation, we shall choose a fixed current of 0.075 amps and write $f_p^2 = (f_p^2)_{ave.}$. This choice is good for the case of no generation of local modes, which is true for our choice of cavity modes.

The exact form for equation (3) is

$$\frac{\Delta f}{f_o} = \frac{\int_{\Delta v} f_p^2 \; \vec{E} \cdot \vec{E}_o \, dv}{2f_o^2 \int_v \left| E_o \right|^2 dv}$$

$$\simeq \frac{(f_p^2)_{ave.} \int_{\Delta v} \vec{E} \cdot \vec{E}_o \, dv}{2f_o^2 \int_v \left| E_o \right|^2 dv} \qquad \text{(A)}$$

where E is the field in the plasma
E_o is the unperturbed field.

The general scheme is to write E in terms of E_o and to compare with the case where E is set equal to E_o. The calculation is carried out for the TM_{020} mode with the remark that the results for the TE_{011} mode are similar.

The incident, reflected, and transmitted fields are given, respectively, as:

$$E_{inc.} \propto J_o\,(kr) \qquad H_{inc.} \propto kJ_1(kr)$$
$$E_{ref.} \propto RJ_o\,(kr) \qquad H_{ref.} \propto kJ_1(kr)$$
$$E_{trans.} \propto TJ_o\,(k'r) \qquad H_{trans.} \propto k'J_1\,(k'r) \tag{B}$$

where

$$k' = nk$$
$$= (1 - (f_p^2)_{ave.}/f_o^2)^{\frac{1}{2}}\; k \tag{C}$$

Matching solutions at the boundary $r = r_o$ for the electric and magnetic components, we obtain

$$(1 + R) = T\,\frac{J_o(nkr_o)}{J_o\,(kr_o)}$$

$$(1 - R) = T\,\frac{nJ_1(nkr_o)}{J_1(kr_o)}$$

$$\therefore\; T = 2\,[\,\frac{J_o(nkr_o)}{J_o\,(kr_o)} + n\,\frac{J_1(nkr_o)}{J_1\,(kr_o)}\,] \tag{D}$$

$$\therefore\; E = TE_o \tag{E}$$

We have here implicitly assumed that because the glass tube itself is thin-walled, there is negligible reflection due to it.

Our perturbation equation becomes:

$$\frac{\Delta f}{f_o} = \frac{(f_p^2)_{ave.}}{2f_o^2} \frac{\int_{\Delta v} TE_o^2 \, dv}{\int_v E_o^2 \, dv} \qquad \text{(F)}$$

Thus, the % error in Δf on making the approximation $E = E_o$ in the numerator of the perturbation equation is:

$(T - 1) \cdot 100\%$ where T is given in equation (D).

For a current of .075 amps, about which our measurements were made

$$n = (1 - (f_p^2)_{ave.}/f_o^2)$$

and if we take $(f_p^2)_{ave.}$ as the mean between f_{po}^2 and its value near the plasma boundary, we obtain from equation (9)

$$n \approx (1 - \frac{2\Delta f^m}{f_o^m} \cdot \frac{10^2}{3} \times .8)$$

But, for the TM_{020} mode,

$$k = .66 \text{ cm}^{-1}$$
$$r_o = .54 \text{ cm}$$

and we find that the % error is:

$$\left(\frac{2}{1 + .94} - 1 \right) \cdot 100\% \approx 3\% .$$

which is small, as desired.

ACKNOWLEDGEMENTS

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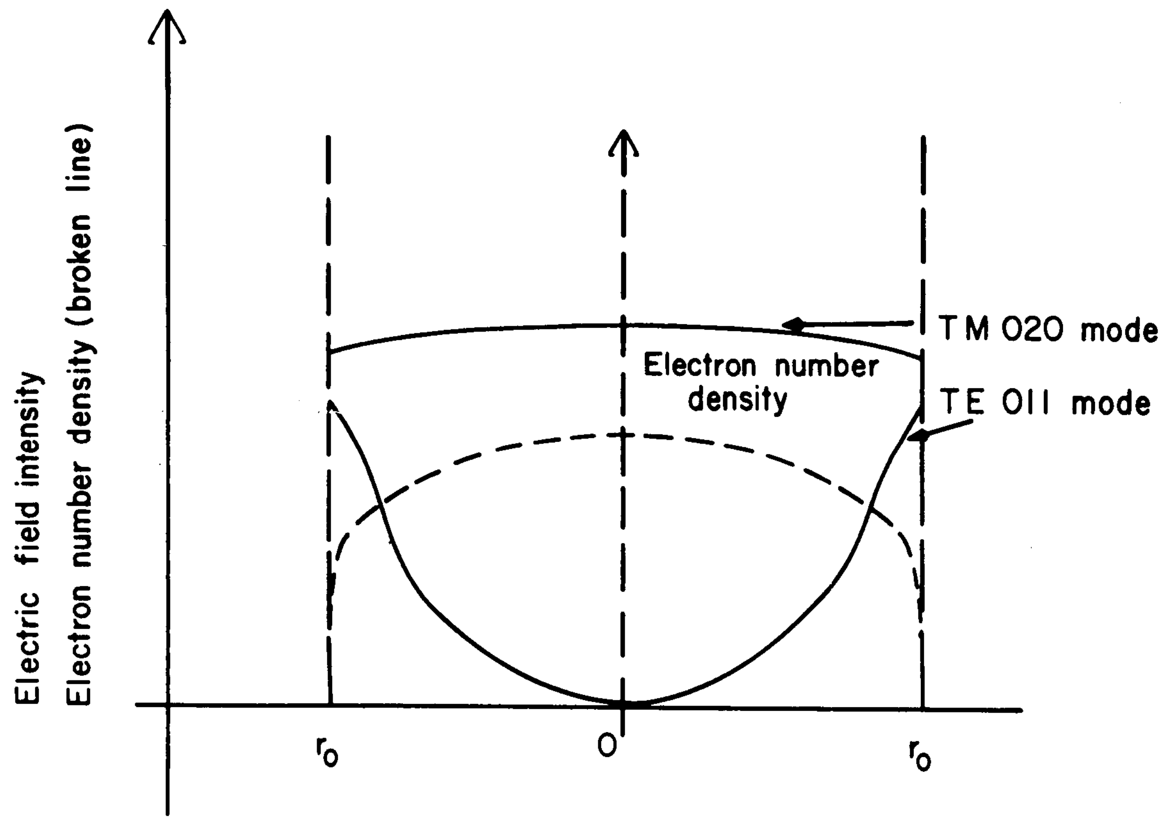

Fig. 1. a. Radial electron number density variation across the positive column.

b. Electric field intensity in the region of the positive column due to the TM 020 and TE 011 modes.

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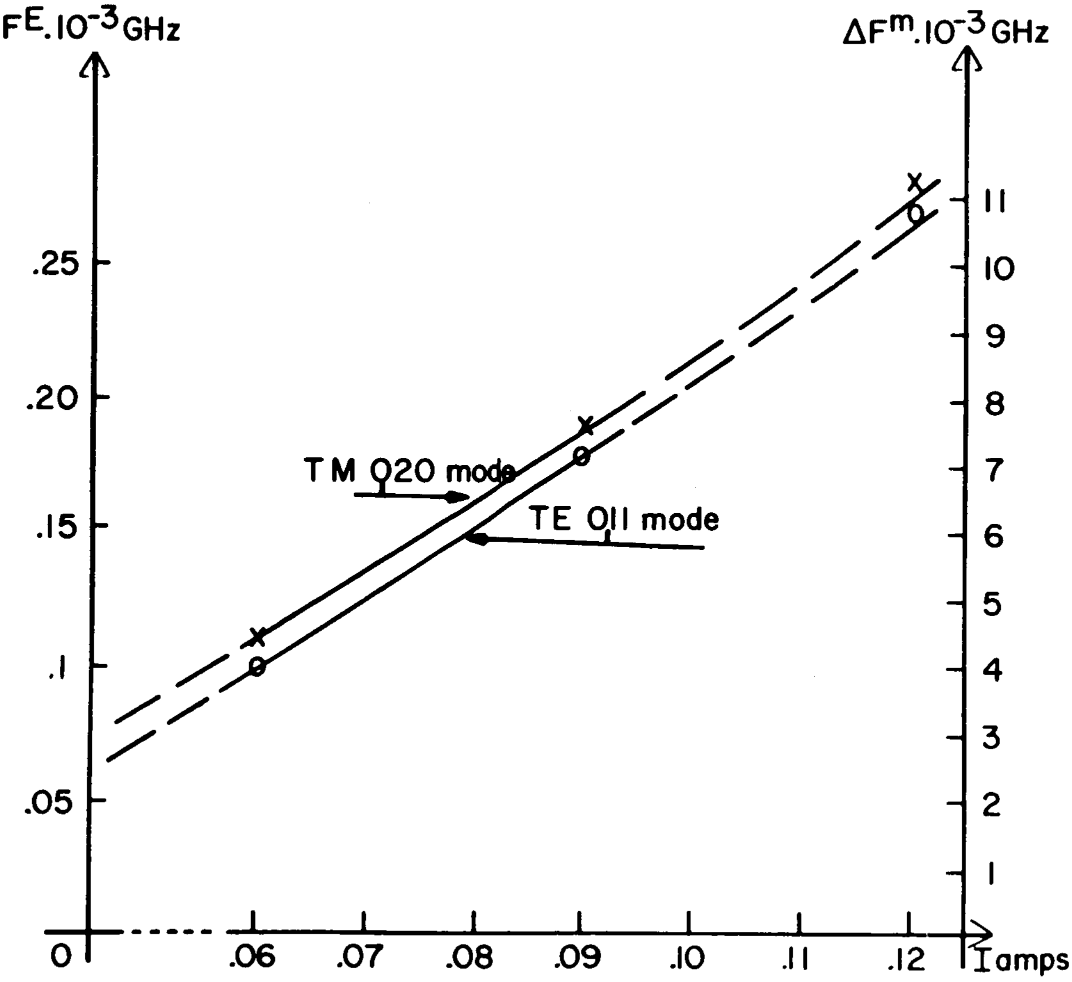

Fig. 2. Frequency shift of the TM 020 and TE 011 cavity modes as a function of current.

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ERRATA

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p. 9, equation 8: Close curly bracket at end of equation.

p. 10, equation 12:
As above.

p. 12, bottom: Last sentence should be followed by the notation (Figure 2).

p. 13, table: Third number under "C" should read .45.

p. 19, Acknowledgement:
The acknowledgement should read: "The authors wish to express their appreciation to . . ."