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MOTION OF AN ARTIFICIAL SATELLITE UNDER COMBINED INFLUENCE OF PLANAR AND KEPLERIAN FORCE FIELDS

by Rowland E. Burns

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DEFINITION OF SYMBOLS

Symbol	Definition
A	Constant measuring force of planar field
a_1^2, a_2^2	Larger roots of $P(\xi)$ and $P(\eta)$, respectively
b_1^2, b_2^2	Smaller roots of $P(\xi)$ and $P(\eta)$, respectively
c_1, c_2, c_3	Integration constants used in solution of Hamilton-Jacobi equation
E	Separation constant in Hamilton-Jacobi equation
E	When used with arguments, denotes an elliptic integral of the second kind
e	Denotes eccentricity of an unperturbed Keplerian orbit
F	Constant measuring force of planar field; $F = mA$
f	Any continuous function of ξ
g	Any continuous function of η
H	Hamiltonian
I_1, I_2, I_3, I_4	Integrals defined by equations (69), (71), (74), and (76)
K_1, K_2	Parameters in elliptic integrals associated, respectively, with ξ and η.
$\vec{L}$	Vectorial angular momentum in an unperturbed Keplerian orbit

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
L	Magnitude of angular momentum in an unperturbed Keplerian orbit
l	Separation constant in the Hamilton-Jacobi equation
m	Mass of satellite
$P(\xi)$	Denotes polynomial in ξ: specifically, as used, a quadratic polynomial
$P(\eta)$	Denotes polynomial in η: specifically, as used, a quadratic polynomial
p_i	Denotes any of the momenta p_ξ, p_η, p_ϕ defined by equations (12), (13), and (14)
q_i	Denotes any of the momenta conjugate to p_ξ, p_η, and p_ϕ
$\vec{r}$	Radius vector from primary to satellite
r	Magnitude of radius vector from primary to satellite
S	Jacobi substitution function
S_1, S_2	Components of S which are functions only of ξ and η, respectively
t	Time, a parameter
U	Potential function
u	Dummy variable used in evaluation of $I_1, \ldots, I_4$; different substitutions are used for each integral

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$\vec{W}$	Laplace integration constant for unperturbed two-body problem
W	Magnitude of integration constant for unperturbed two-body problem
W	Function related to S by equation (24)
x, y, z	Cartesian coordinates describing motion of satellite
x, y	Dummy variables
β	Separation constant in Hamilton-Jacobi equation
η	One of three parabolic coordinates, $\eta = r - z$
θ	Polar angle in spherical coordinates
μ	Gravitational parameter of attracting primary
ξ	One of three parabolic coordinates $\xi = r + z$
ρ	Projection of radius vector in (x, y) plane
ϕ	Angular coordinate from arbitrary zero point in (x, y) plane
ψ	Angle between Laplace vector integration constant and the radius vector

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
Subscripts	
i	Dummy subscript or summation variable
0	Indicates initial point
x, y, z	Indicates Cartesian component of a vector
Other Notation	
$\dot{}$	Indicates derivative with respect to time
$\cdot$	Indicates a scalar product
$\times$	Indicates a vector product

MOTION OF AN ARTIFICIAL SATELLITE UNDER COMBINED INFLUENCE OF PLANAR AND KEPLERIAN FORCE FIELDS

INTRODUCTION

The development of the technology of artificial earth satellites has posed mathematical problems which heretofore were academic. One such problem is the radiation pressure perturbations of satellites with large area-to-mass ratios (i.e., Echo). The standard method of treatment of such a problem is a perturbational analysis of a purely Keplerian orbit which is disturbed by a radiation pressure field.

It will be shown in this report that a generalized form of the two-body problem which more closely approximates the radiation pressure problem can be solved in closed form. Specifically, the problem of the motion of a satellite under the combined influence of Keplerian and planar force fields will be solved in terms of the elliptic functions.

The assumption of a planar force field is still an approximation to the true radiation pressure problem* because the divergence of the field is neglected. It is obvious that neglecting the divergence of the radiation pressure field is far less restrictive than disregarding the entire field. Perturbational treatment with the present solution as a model should converge far more readily than treatments which assume an initially Keplerian model.

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* The treatment here has equal application to the case of the classical Stark effect; that is, the radiation from hydrogen gas in an external planar electric field is (classically) analogous to the radiation pressure problem. See E. Schrodinger, Ann. Physik, 80, 457, 1926, and P. S. Epstein, Phys. Rev., 28, 695, 1926.

The solution may be outlined as follows. The equation of motion in Cartesian coordinates is used to formulate the Hamilton-Jacobi equation of the problem. This equation is then rewritten in terms of parabolic coordinates and separated by means of the usual product of independent functions assumption.* The solution is effected by quadratures of the separated equations in the three-dimensional case. These quadratures are explicitly integrated for the case of two-dimensional motion, which is initially circular. A graphical presentation of the results is given.

FORMULATION OF PROBLEM

The equations of motion of a satellite under the combined influence of an inverse square field and a uniform planar field are

$$\ddot{x} = -\frac{\mu x}{(x^2 + y^2 + z^2)^{3/2}} \quad ,$$

$$\ddot{y} = -\frac{\mu y}{(x^2 + y^2 + z^2)^{3/2}} \quad ,$$

and

$$\ddot{z} = -\frac{\mu z}{(x^2 + y^2 + z^2)^{3/2}} + A \quad .$$

where x, y, z are Cartesian coordinates, μ the gravitational parameter of the attracting center, and A the planar force-to-mass ratio. The force may be derived from a potential U defined by

$$U = -U(x,y,z,A) = -m\left(\frac{\mu}{(x^2 + y^2 + z^2)^{1/2}} + Az\right) \quad .$$

The problem may now be stated as follows. Determine functional relationships which relate x, y, and z to the time, initial conditions, and parameters μ and A. This procedure is quite difficult, if not impossible, in Cartesian coordinates. For this reason, we change from Cartesian coordinates to parabolic coordinates.

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* The separation of the Hamilton-Jacobi equation in these coordinates is suggested in Reference 1. It should be noted that the problem of a planar field superimposed on an inverse square field is a special case of Euler's problem of two fixed centers. See Euler, Mém. de Berlin, 1760, p. 228.

TRANSFORMATION OF COORDINATES AND THE HAMILTON-JACOBI EQUATION

This potential may be rewritten in cylindrical coordinates as

$$U = U(\rho, \phi, z) = \frac{-m\mu}{\sqrt{\rho^2 + z^2}} - mAz \tag{1}$$

where

$$\begin{aligned} x &= \rho \cos \phi \quad , \\ y &= \rho \sin \phi \quad , \\ z &= z \quad . \end{aligned} \tag{2}$$

and

The Lagrangian yielding the basic set of differential equations may be written as

$$\begin{aligned} L &= \frac{m}{2}(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) + \frac{m\mu}{(\rho^2 + z^2)^{\frac{1}{2}}} + mAz \\ &\equiv \frac{m}{2}(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) - U(\rho, \phi, z)^{*} \quad . \end{aligned} \tag{3}$$

The Lagrangian is now transformed to parabolic coordinates ξ, η, ϕ. This will ultimately allow a separation of the Hamilton-Jacobi equation

$$\rho = \sqrt{\xi \eta} \quad , \tag{4}$$

$$z = \frac{\xi - \eta}{2} \quad , \tag{5}$$

and

$$\phi = \phi \quad . \tag{6}$$

The radius vector in spherical coordinates, r, is given by

$$r = \sqrt{\rho^2 + z^2} = \frac{\xi + \eta}{2} \quad . \tag{7}$$

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* The inclusion of m as a multiplicative factor on A is, of course, unnecessary. It simply turns out to be algebraically convenient. It can be removed in subsequent equations by setting A = F/m if desired.

Thus,

$$\dot{\rho} = \frac{\dot{\xi}\eta + \xi\dot{\eta}}{2\sqrt{\xi\eta}} \tag{8}$$

and

$$\dot{z} = \frac{\dot{\xi} - \dot{\eta}}{2} \tag{9}$$

so that

$$L = \frac{m(\xi+\eta)}{8}\left(\frac{\dot{\xi}^2}{\xi} + \frac{\dot{\eta}^2}{\eta}\right) + \frac{m\,\xi\eta\,\dot{\phi}^2}{2} - U(\xi,\eta,\phi)\ . \tag{10}$$

Defining the momenta conjugate to ξ, η, ϕ as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{11}$$

(where q_i is of any of ξ, η, or ϕ and p_i is any of p_ξ, p_η, or p_ϕ), we find that

$$p_\xi = \frac{m(\xi+\eta)}{4\xi}\,\dot{\xi}\quad , \tag{12}$$

$$p_\eta = \frac{m(\xi+\eta)}{4\eta}\,\dot{\eta}\quad , \tag{13}$$

and

$$p_\phi = m\,\xi\,\eta\,\dot{\phi} \tag{14}$$

so that

$$\dot{\xi} = \frac{4\,\xi\,p_\xi}{m(\xi+\eta)}\quad , \tag{15}$$

$$\dot{\eta} = \frac{4\eta\,p_\eta}{m(\xi+\eta)}\quad , \tag{16}$$

and

$$\dot{\phi} = \frac{p_\phi}{m\,\xi\,\eta}\quad . \tag{17}$$

Now

$$L(\xi, \eta, \phi, p_\xi, p_\eta, p_\phi) = \frac{2}{m(\xi+\eta)}\ (\xi\,p_\xi^{\ 2} + \eta\,p_\eta^{\ 2}) + \frac{p_\phi^{\ 2}}{2m\,\xi\,\eta} - U(\xi, \eta, \phi)\ . \tag{18}$$

The Hamiltonian, H, is defined by

$$H = \sum_i \dot{q}_i p_i \Big|_{p_i} - L(q,p) = \frac{2}{m(\xi+\eta)} (\xi p_\xi^2 + \eta p_\eta^2) + \frac{p_\phi^2}{2m\xi\eta}$$

$$+ U(\xi,\eta,\phi) \quad . \tag{19}$$

We now define a function S such that

$$p_\xi = \frac{\partial S}{\partial \xi} \quad , \tag{20}$$

$$p_\eta = \frac{\partial S}{\partial \eta} \quad , \tag{21}$$

and

$$p_\phi = \frac{\partial S}{\partial \phi} \quad . \tag{22}$$

The solutions to the original problem now must be solutions to the Hamilton-Jacobi equation

$$\frac{2}{m(\xi+\eta)} \left[\xi\left(\frac{\partial S}{\partial \xi}\right)^2 + \eta\left(\frac{\partial S}{\partial \eta}\right)^2\right] + \frac{1}{2m\xi\eta} \left(\frac{\partial S}{\partial \phi}\right)^2 + U(\xi,\eta,\phi)$$

$$+ \frac{\partial S}{\partial t} = 0 \quad . \tag{23}$$

SEPARATION OF HAMILTON-JACOBI EQUATION: SOLUTION OF PROBLEM

We shall attempt to find a solution to equation (23) in the form

$$S(\xi,\eta,\phi,t) = W(\xi,\eta,\phi) - Et \tag{24}$$

where E is a constant. Then

$$\frac{2}{m(\xi+\eta)}\left[\xi\left(\frac{\partial W}{\partial \xi}\right)^2+\eta\left(\frac{\partial W}{\partial \eta}\right)^2\right]+\frac{1}{2m\xi\eta}\left(\frac{\partial W}{\partial \phi}\right)^2+U(\xi,\eta,\phi)=E\,. \tag{25}$$

It is not necessary, as yet, to specify the exact form of $U(\xi,\eta,\phi)$. Assume, more generally, that

$$U(\xi,\eta,\phi)=\frac{f(\xi)+g(\eta)}{\xi+\eta} \tag{26}$$

where $f(\xi)$ and $g(\eta)$ are simply well-behaved functions of their arguments which allow the integrals to exist. Inserting (26) into (25) and setting

$$W=l\phi+S_1(\xi)+S_2(\eta) \tag{27}$$

gives

$$\frac{2}{m(\xi+\eta)}\left[\xi\left(\frac{\partial S_1}{\partial \xi}\right)^2+\eta\left(\frac{\partial S_2}{\partial \eta}\right)^2\right]+\frac{l^2}{2m\xi\eta}+\frac{f(\xi)+g(\eta)}{\xi+\eta}=E\,. \tag{28}$$

Multiplying (28) by $m(\xi+\eta)$ and rearranging gives

$$2\xi\left(\frac{\partial S_1}{\partial \xi}\right)^2+mf(\xi)-mE\xi+\frac{l^2}{2\xi}=-2\eta\left(\frac{\partial S_2}{\partial \eta}\right)^2-mg(\eta)$$
$$+mE\eta-\frac{l^2}{2\eta}\,. \tag{29}$$

By the usual argument employed at this state in the solution of a partial differential equation, we note that the left side of (29) depends only upon ξ, while the right side depends only upon η. Since ξ and η are independent coordinates, we can move ξ through an arbitrary sequence of values while η is constant, then reverse the process. But the equality given by (29) must be retained so that both sides must be equal to a constant.

We, thus, have

$$2\xi \left(\frac{dS_1}{d\xi}\right)^2 + m\, f(\xi) - m\, E\, \xi + \frac{l^2}{2\xi} = \beta \tag{30}$$

and

$$2\eta \left(\frac{dS_2}{d\eta}\right)^2 + m\, g(\eta) - m\, E\eta + \frac{l^2}{2\eta} = -\beta \quad . \tag{31}$$

Equations (30) and (31) integrate to yield

$$S_1 = \int_{\xi_0}^{\xi} \sqrt{\frac{m\,E}{2} + \frac{\beta}{2\xi} - \frac{m\,f(\xi)}{2\xi} - \frac{l^2}{4\xi^2}}\; d\xi \tag{32}$$

and

$$S_2 = \int_{\eta_0}^{\eta} \sqrt{\frac{m\,E}{2} - \frac{\beta}{2\eta} - \frac{m\,g(\eta)}{2\eta} - \frac{l^2}{4\eta^2}}\; d\eta \quad . \tag{33}$$

Inserting (32) and (33) into (27) and the resultant equation into (24) gives

$$S(\xi,\eta,\phi,t) = l\phi - Et + \int_{\xi_0}^{\xi} \sqrt{\frac{m\,E}{2} + \frac{\beta}{2\xi} - \frac{m\,f(\xi)}{2\xi} - \frac{l^2}{4\xi^2}}\; d\xi$$

$$+ \int_{\eta_0}^{\eta} \sqrt{\frac{m\,E}{2} - \frac{\beta}{2\eta} - \frac{m\,g(\eta)}{2\eta} - \frac{l^2}{4\eta^2}}\; d\eta \quad . \tag{34}$$

We now specialize $f(\xi)$ and $g(\eta)$ by writing equation (1) in the form

$$\frac{-m\mu}{\sqrt{\rho^2 + z^2}} - mAz = \frac{-2m\mu}{\xi+\eta} - \frac{mA(\xi-\eta)}{2} = \frac{\left(-m\mu - \frac{mA\xi^2}{2}\right) - \left(m\mu - \frac{mA\eta^2}{2}\right)}{\xi + \eta} \quad . \tag{35}$$

Comparison of equations (26) and (35) shows that

$$f(\xi) = -m\mu - \frac{mA\xi^2}{2} \tag{36}$$

and

$$g(\eta) = -m\mu + \frac{mA\eta^2}{2} \tag{37}$$

so that (34) becomes

$$S(\xi,\eta,\phi,t) = l\phi - Et + \int_{\xi_0}^{\xi} \sqrt{\frac{mE}{2} + \frac{(\beta + m^2\mu)}{2\xi} + \frac{m^2A\xi}{4} - \frac{l^2}{4\xi^2}}\, d\xi$$

$$+ \int_{\eta_0}^{\eta} \sqrt{\frac{mE}{2} + \frac{(-\beta + m^2\mu)}{2\eta} - \frac{m^2A\eta}{4} - \frac{l^2}{4\eta^2}}\, d\eta \quad . \tag{38}$$

The arbitrary constants in this equation are l, E, and β. Following the normal procedures of Hamilton-Jacobi mechanics, we now equate the partial derivatives of S with respect to l or E or β to additional constants yielding the three equations

$$c_1 = -t + \frac{m}{2} \int_{\xi_0}^{\xi} \frac{\xi\, d\xi}{\sqrt{+m^2A\xi^3 + 2mE\xi^2 + 2(\beta + m^2\mu)\xi - l^2}}$$

$$+ \frac{m}{2} \int_{\eta_0}^{\eta} \frac{\eta\, d\eta}{\sqrt{-m^2A\eta^3 + 2mE\eta^2 + 2(m^2\mu - \beta)\eta - l^2}} \quad , \tag{39}$$

$$c_2 = \int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{+m^2A\xi^3 + 2mE\xi^2 + 2(\beta + m^2\mu)\xi - l^2}}$$

$$- \int_{\eta_0}^{\eta} \frac{d\eta}{\sqrt{-m^2A\eta^3 + 2mE\eta^2 + 2(m^2\mu - \beta)\eta - l^2}} \quad , \tag{40}$$

and

$$c_3 = \phi - \frac{1}{2}\left\{\int_{\xi_0}^{\xi} \frac{d\xi}{\xi\sqrt{+ m^2 A\xi^3 + 2mE\xi^2 + 2(\beta + m^2\mu)\xi - l^2}} + \int_{\eta_0}^{\eta} \frac{d\eta}{\eta\sqrt{-m^2A\eta^3 + 2mE\eta^2 + 2(m^2\mu-\beta)\eta - l^2}}\right\} . \quad (41)$$

Choosing $\xi = \xi_0$, $\eta = \eta_0$, we find that c_1 is the initial time t_0. Similarly, c_2 vanishes because we initialize motion at a possible orbital point (ξ_0, η_0). Finally, c_3 is the initial value of ϕ. Equation (39) can then be regarded as the relation maintained between ξ and η throughout the motion. If this equation is solved to yield $\eta = \eta(\xi)$, we have the projection of the motion in the ξ-η plane. Then equation (41) is the full three-dimensional description of the spatial motion which relates ϕ to ξ and η. Elimination of η (or ξ) via equation (40) would yield $\phi(\xi)$ (or $\phi(\eta)$). Finally, equation (39) predicts values of the parameter t for known values of ξ and η. We can thus obtain values for any three members of the set $\{\xi, \eta, \phi, t\}$ once any one of the four is specified.

It should be noted that, in a computational sense, we can obtain answers much more easily if a value of ξ or η is originally specified rather than a value of ϕ or t. This observation is not apparent from the material developed thus far but will be indicated by a special case to be treated in detail below. It arises because of the difficulty of inverting the transcendental expressions which result from the integration of the quadratures appearing in equations (39) and (41).

We would expect equation (39) to be complicated because time is usually difficult to treat in celestial mechanics problems. Equation (41) relates all three spatial parameters and, for this reason, may be expected to be more cumbersome than a relationship between two spatial parameters, equation (40).

The next problem of interest is the physical identification of the constants, l, E, and β.

IDENTIFICATION OF CONSTANTS IN TERMS OF INITIAL CONDITIONS

The constant E, first introduced in equation (24), can be most easily identified in Cartesian coordinates. Since we have

$$x = \sqrt{\xi \eta} \cos \phi \qquad , \tag{42}$$

$$y = \sqrt{\xi \eta} \sin \phi \qquad , \tag{43}$$

and

$$z = \frac{\xi - \eta}{2} \qquad , \tag{44}$$

equation (25) becomes

$$E = \frac{2}{m(\xi+\eta)} \left[\xi \, p_\xi^{\;2} + \eta \, p_\eta^{\;2} \right] + \frac{1}{2m\xi\eta} p_\phi^{\;2} + (\xi, \eta, \phi)$$

$$= \frac{m(\xi+\eta)}{8} \left(\frac{\dot{\xi}^2}{\xi} + \frac{\dot{\eta}^2}{\eta} \right) + \frac{m\xi\eta\dot{\phi}^2}{2} + U(\xi, \eta, \phi)$$

$$= \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{m\mu}{\sqrt{x^2+y^2+z^2}} - mAz \quad . \tag{45}$$

Notice that the first two terms following that last equality in equation (45) represent the energy in an unperturbed two-body problem.

For the identification of l we turn to equations (22), (24), and (27) to find that

$$p_\phi = l = m\xi\eta\dot{\phi} \quad . \tag{46}$$

Applying equations (42) and (43) gives

$$l = m(x^2 + y^2)\dot{\phi} = m(x\dot{y} - y\dot{x}) \tag{47}$$

so that l is simply a component of angular momentum.

The constant β requires more work for its identification. Note that from equations (20) and (21)

$$p_\xi^{\;2} = \left(\frac{\partial S}{\partial \xi}\right)^2 = \frac{mE}{2} + \frac{(\beta + m^2\mu)}{2\xi} + \frac{m^2 A\xi}{4} - \frac{l^2}{4\xi^2} \tag{48}$$

and

$$p_\eta^{\;2} = \left(\frac{\partial S}{\partial \eta}\right)^2 = \frac{mE}{2} + \frac{(-\beta + m^2\mu)}{2\eta} - \frac{m^2 A\eta}{4} - \frac{l^2}{4\eta^2} \tag{49}$$

so that

$$p_\eta^{\;2} - p_\xi^{\;2} = -\frac{\beta}{2}\left(\frac{\xi+\eta}{\xi\eta}\right) + \frac{m^2\mu}{2}\left(\frac{\xi-\eta}{\xi\eta}\right) - \frac{m^2 A}{4}(\xi+\eta) - \frac{l^2}{4}\left(\frac{\xi^2-\eta^2}{\xi^2\eta^2}\right) . \tag{50}$$

Solving for β gives

$$\beta = + \left\{\frac{2\xi\eta}{\xi+\eta}\left(p_\xi^{\;2} - p_\eta^{\;2}\right) + m^2\mu\left(\frac{\xi-\eta}{\xi+\eta}\right) - \frac{l^2(\xi-\eta)}{\xi\eta}\right\} - \frac{m^2 A\xi\eta}{2} . \tag{51}$$

This appears much less elegant than the expressions derived for E and l so that a further comment is in order. In Appendix A it is shown that β is a modification of one component of Laplace's vectorial integration constant.

We have now arrived at such a point in the development that no further progress can be made without specializing the general results derived thus far. This is because the integrated forms of equations (39), (40), and (41) cannot be obtained until such time as certain restrictions on the magnitude of A and the initial conditions are spelled out. For any given set of such conditions, the integration can proceed but the conditions must be given.

To illustrate the procedure, we turn our attention to the special case of two-dimensional motion with initially circular orbit conditions.

INTEGRATION OF EQUATIONS OF TWO-DIMENSIONAL MOTION

The choice of initial coordinates was made in such a way that the planar force acts in the direction of increasing z. We can then expect that if motion is to occur in a plane, this plane must include the z axis. It is algebraically simplest to define the plane of two-dimensional motion to be the x-z plane. This, however, is simply a convenience. If such a choice is made, then the coordinate ϕ is identically zero. If $\phi \equiv 0$, then certainly $\dot{\phi} = 0$, and equation (46) shows that $l = 0$. Using this condition, equations (39) and (40) become

$$t - t_0 = \frac{m}{2} \int_{\xi_0}^{\xi} \frac{\sqrt{\xi}\, d\xi}{\sqrt{m^2 A \xi^3 + 2mE\xi + 2(\beta + m^2\mu)}}$$

$$+ \frac{m}{2} \int_{\eta_0}^{\eta} \frac{\sqrt{\eta}\, d\eta}{\sqrt{-m^2 A \eta^2 + 2mE\eta + 2(m^2\mu - \beta)}} \tag{52}$$

and

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi\,[m^2 A \xi^2 + 2mE\xi + 2(\beta + m^2\mu)]}}$$

$$= \int_{\eta_0}^{\eta} \frac{d\eta}{\sqrt{\eta\,[-m^2 A \eta^2 + 2mE\eta + 2(m^2\mu - \beta)]}} \quad . \tag{53}$$

To simplify the algebra, we now make use of the fact that the choice of units for mass, time, and length are at our disposal. Choose these units in such a way that

$$\left\{\begin{aligned} m &= 1 \\ \mu &= 1 \\ r_0 &= 1 \end{aligned}\right. \quad . \tag{54}$$

Having made this choice of units, we further choose initial conditions which ensure an (initially) circular orbit of radius r_0. Then

$$\begin{aligned} x_0 &= 1 \\ y_0 &= 0 \\ z_0 &= 0 \\ \dot{x}_0 &= 0 \\ \dot{y}_0 &= 0 \\ \dot{z}_0 &= \pm 1 \end{aligned} \tag{55}$$

where a subscript of zero indicates an initial condition. Using these conditions in equations (45) and (51)*, we find

$$\left\{\begin{aligned} E &= -\frac{1}{2} \\ \beta &= -\frac{A}{2} \end{aligned}\right. \quad . \tag{56}$$

---

* First, convert initial conditions to ξ_0, η_0 coordinates.

Using equations (54) and (56), the polynomials appearing in the denominator of equations (52) and (53) may be written as

$$P(\xi) = m^2 A\xi^2 + 2mE\xi + 2(\beta + m^2\mu) = A\xi^2 - \xi + 2\left(1 - \frac{A}{2}\right)$$

$$= A(a_1 - \xi)(b_1 - \xi) \tag{57}$$

and

$$P(\eta) = -m^2 A\eta^2 + 2mE\eta + 2(m^2\mu - \beta) = -A\eta^2 - \eta + 2\left(1 + \frac{A}{2}\right)$$

$$= A(a_2 - \eta)(|b_2| + \eta) \tag{58}$$

where a_1, b_1 and a_2, b_2 are the roots of $P(\xi)$ and $P(\eta)$, respectively.

Equations (52) and (53) may now be written as

$$t - t_0 = \frac{1}{2}\left\{ \int_{\xi_0}^{\xi} \frac{\sqrt{\xi}\, d\xi}{\sqrt{(a_1 - \xi)(b_1 - \xi)}} + \int_{\eta_0}^{\eta} \frac{\sqrt{\eta}\, d\eta}{\sqrt{(a_2 - \eta)(|b_2| + \eta)}} \right. \tag{59}$$

and

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(a_1 - \xi)(b_1 - \xi)}} = \int_{\eta_0}^{\eta} \frac{d\eta}{\sqrt{\eta(a_2 - \eta)(|b_2| + \eta)}} \quad . \tag{60}$$

We now examine the possible singularities of the integrands of the last two equations. The integrals will exist only under certain restrictions, and a detailed examination is in order. The equations which define cylindrical coordinates in terms of ξ and η, equations (3) and (4), give

$$\xi = z \pm \sqrt{x^2 + y^2 + z^2}$$

and

$$\eta = -z \pm \sqrt{x^2 + y^2 + z^2} \quad .$$

Either $\xi \le 0$ and $\eta \le 0$ or $\xi \ge 0$ and $\eta \ge 0$ as may be seen from equation (3). Choose, for convenience,

$$\xi = z + \sqrt{x^2 + y^2 + z^2}$$

and

$$\eta = -z + \sqrt{x^2 + y^2 + z^2} \quad . \tag{61}$$

Now, $\xi = 0$ if $x = y = z = 0$ or if $x = y = 0$ and $z < 0$. Similarly, η will vanish for $x = y = z = 0$ or $x = y = 0$ and $z > 0$. This gives $\xi \ge 0$ and $\eta \ge 0$ which establishes a lower bound on ξ and η.

Label the roots of $P(\xi)$ in equation (57) in such a way that $a_1 \le b_1$. With this convention we can now establish upper bounds on both ξ and η. To do this, we notice that from equations (30), (31), (57), and (58) we have

$$\begin{cases} 4\xi \left(\dfrac{dS_1}{d\xi}\right)^2 = P(\xi) \\ 4\eta \left(\dfrac{dS_2}{d\eta}\right)^2 = P(\eta) \end{cases} \quad . \tag{62}$$

Using the non-negativity property of ξ and η, we may conclude that

$$P(\xi) \ge 0 \tag{63}$$

and

$$P(\eta) \ge 0 \tag{64}$$

so that

$$\xi \le a_1$$

or

$$\xi \ge b_1$$

indicating a forbidden range of ξ values. That is, we may expect that the region

$$a_1 < \xi < b_1 \tag{65}$$

is forbidden.

Similarly, the non-negativity of $P(\eta)$ ensures that

$$\eta \leq a_2 \quad .$$

We have thus established the bounds on ξ and η as

$$0 \leq \xi \leq a_1 \tag{66}$$

and

$$0 \leq \eta \leq a_2 \,. \tag{67}$$

It is now apparent that equations (59) and (60) may possess singularities in the integrand. This means that these integrals might necessarily be regarded as improper (Riemann) integrals, however, and the existence of these integrals may be established by standard techniques of elementary calculus.

Next, we shall consider the problem of bounds on A. Let ξ^* and η^* be roots of $P(\xi)$ and $P(\eta)$. Then

$$P(\xi^*) = A\xi^{*2} - \xi^* + 2\left(1 - \frac{A}{2}\right)$$

and

$$P(\eta^*) = -A\eta^{*2} - \eta^* + 2\left(1 + \frac{A}{2}\right) \quad .$$

Then $\xi^* = \xi^*(A)$, $\eta^* = \eta^*(A)$. Specifically,

$$\xi^* = \frac{1 \pm \sqrt{1 - 8A\left(1 - \frac{A}{2}\right)}}{2A}$$

and

$$\eta^* = \frac{1 \pm \sqrt{1 + 8A\left(1 + \frac{A}{2}\right)}}{-2A} \quad .$$

It is not necessary to require that ξ^* and η^* be real. The integrands, however, change character if we allow imaginary values of the roots. For convenience, therefore, we require that ξ^* and η^* be real. Now η^* is real for $A \geq 0$. The values of ξ^* will be imaginary if

$$1 - \frac{\sqrt{3}}{2} < A < 1 + \frac{\sqrt{3}}{2} \quad .$$

Thus, we bound A within the limits

$$0 \leq A \leq 1 - \frac{\sqrt{3}}{2} \quad . \tag{68}$$

For reasons of orientation, we include the following table.

A	a_1	b_1	a_2	b_2
$1 - \frac{\sqrt{3}}{2}$	3.73	3.73	1.73	-9.19
0.1	2.55	7.45	1.78	-11.83
0.01	2.03	97.97	1.97	-101.97
0.001	2.005	998	2.00	-1002
0.0001	2.000	9995	2.00	-10002

(Note $b_2 < 0$ for all allowable values of A.)

To reiterate the results of this section to this point, we now have established the inequalities

$$0 \leq \xi \leq a_1 \quad ,$$

$$0 \leq \eta \leq a_2 \quad ,$$

and

$$0 \leq A \leq 1 - \frac{\sqrt{3}}{2} \quad .$$

In equation (60), we set

$$I_1 = \int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(a_1 - \xi)\,(b_1 - \xi)}} \tag{69}$$

When we introduce a dummy variable u defined by

$$\xi = a_1 \, \mathrm{sn}^2 u$$

with the elliptic parameter K given by

$$K^2 = \frac{a_1}{b_1} ,$$

equation (69) becomes [3]

$$I_1 = \frac{2}{\sqrt{b_1}} \left[\mathrm{sn}^{-1} \left(\sqrt{\frac{\xi}{a_1}} , \sqrt{\frac{a_1}{b_1}} \right) - \mathrm{sn}^{-1} \left(\sqrt{\frac{\xi_0}{a_1}} , \sqrt{\frac{a_1}{b_1}} \right) \right] \quad . \tag{70}$$

Define

$$I_2 = \int_{\eta_0}^{\eta} \frac{d\eta}{\sqrt{\eta(a_2 - \eta)\,(|b_2| + \eta)}} \quad . \tag{71}$$

Set

$$\eta = a_2 \mathrm{cn}^2 u$$

with

$$K^2 = \frac{a_2}{a_2 + |b_2|} .$$

Then, [3]

$$I_2 = \frac{2}{\sqrt{a_2 + |b_2|}} \left[cn^{-1}\left(\sqrt{\frac{\eta_0}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right) - cn^{-1}\left(\sqrt{\frac{\eta}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right)\right] . \qquad (72)$$

Substituting equations (70) and (72) into equation (60) gives the equation of the orbit:

$$\frac{1}{\sqrt{b_1}} \left[sn^{-1}\left(\sqrt{\frac{\xi}{a_1}}, \sqrt{\frac{a_1}{b_1}}\right) - sn^{-1}\left(\sqrt{\frac{\xi_0}{a_1}}, \sqrt{\frac{a_1}{b_1}}\right)\right]$$

$$= \frac{1}{\sqrt{a_2 + |b_2|}} \left[cn^{-1}\left(\sqrt{\frac{\eta_0}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right) - cn^{-1}\left(\sqrt{\frac{\eta}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right)\right] . \qquad (73)$$

Notice that we may not remove the elliptic functions in equation (60) by application of the operators sn or cn due to the multiplicative factors which occur. If $\sqrt{\frac{b_1}{a_2 + |b_2|}}$ happens to be rational, this manipulation could be accomplished, if desired.

Notice that neither transformation used in the derivation of equation (73) possesses a singularity. Equation (73) is a general solution valid for all ξ and η.

Next, set

$$I_3 = \int_{\xi_0}^{\xi} \frac{\sqrt{\xi}\ d\xi}{\sqrt{(a_1 - \xi)\ (b_1 - \xi)}} \quad . \tag{74}$$

Setting

$$\xi = a_1 \, \mathrm{sn}^2 u$$

and

$$K^2 = \frac{a_1}{b_1} \quad ,$$

we obtain [3]

$$\begin{aligned} I_3 = 2\sqrt{b_1} \ \Bigg\{ & \mathrm{sn}^{-1}\left(\sqrt{\frac{\xi}{a_1}}\ ,\ \sqrt{\frac{a_1}{b_1}}\right) - \mathrm{sn}^{-1}\left(\sqrt{\frac{\xi_0}{a}}\ ,\ \sqrt{\frac{a_1}{b_1}}\right) \\ & - E\left[\mathrm{sn}^{-1}\left(\sqrt{\frac{\xi}{a_1}},\ \sqrt{\frac{a_1}{b_1}}\ ,\right)\sqrt{\frac{a_1}{b_1}}\right] \\ & + E\left[\mathrm{sn}^{-1}\left(\sqrt{\frac{\xi_0}{a_1}},\ \sqrt{\frac{a_1}{b_1}}\right),\ \sqrt{\frac{a_1}{b_1}}\right]\Bigg\} \ . \end{aligned} \tag{75}$$

Finally, set

$$I_4 = \int_{\eta_0}^{\eta} \frac{\sqrt{\eta}\ d\eta}{\sqrt{(a_2 - \eta)\ (|b_2| + \eta)}} \quad . \tag{76}$$

Setting

$$\eta = a_2 \, \mathrm{cn}^2 u$$

and

$$K^2 = \frac{a_2}{a_2 + |b_2|}$$

yields [3]

$$I_4 = -2\sqrt{a_2 + |b_2|}\left\{E\left[cn^{-1}\left(\sqrt{\frac{\eta}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right), \sqrt{\frac{a_2}{a_2 + |b_2|}}\right]\right.$$

$$- E\left[cn^{-1}\left(\sqrt{\frac{\eta_0}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right), \sqrt{\frac{a_2}{a_2 + |b_2|}}\right] \tag{77}$$

$$\left. - \frac{|b_2|}{a_2 + |b_2|}\left[cn^{-1}\left(\sqrt{\frac{\eta}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right) - cn^{-1}\left(\sqrt{\frac{\eta}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right)\right]\right\} .$$

Substituting equations (75) and (77) into equation (59) yields

$$t - t_0 = \sqrt{b_1}\left\{sn^{-1}\left(\sqrt{\frac{\xi}{a_1}}, \sqrt{\frac{a_1}{b_1}}\right) - sn^{-1}\left(\sqrt{\frac{\xi_0}{a_1}}, \sqrt{\frac{a_1}{b_1}}\right)\right.$$

$$\left. -E\left[sn^{-1}\left(\sqrt{\frac{\xi}{a_1}}, \sqrt{\frac{a_1}{b_1}}\right), \sqrt{\frac{a_1}{b_1}}\right] + E\left[sn^{-1}\left(\sqrt{\frac{\xi_0}{a_1}}, \sqrt{\frac{a_1}{b_1}}\right), \sqrt{\frac{a_1}{b_1}}\right]\right\}$$

$$- \sqrt{a_2 + |b_2|}\left\{E\left[cn^{-1}\left(\sqrt{\frac{\eta}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right), \sqrt{\frac{a_2}{a_2 + |b_2|}}\right]\right.$$

$$-E\left[cn^{-1}\left(\sqrt{\frac{\eta_0}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right), \sqrt{\frac{a_2}{a_2 + |b_2|}}\right]$$

$$\left. - \frac{|b_2|}{a_2 + |b_2|}\left[cn^{-1}\left(\sqrt{\frac{\eta}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right) - cn^{-1}\left(\sqrt{\frac{\eta}{a_2}}, \sqrt{\frac{a_2}{a_2 + |b_2|}}\right)\right]\right\}. \tag{78}$$

A comparison of equations (73) and (78) indicates that the problem of solving equation (73) for ξ or η is simpler than solving equation (78) for ξ or η. It would thus be easier to write equations of the form

$$t = t(\xi)$$

or

$$t = t(\eta)$$

than to write

$$\xi = \xi(t)$$

or

$$\eta = \eta(t) \quad .$$

The difficulty with specification of ξ or η and, subsequently, solving for t is that we must choose a value of ξ or η which exists for some value of t. In other words, only those values of ξ and η which actually constitute a point on the orbit will yield a value of t.

NUMERICAL RESULTS

The preceding theory has dealt at some length with the solution of a seemingly trivial modification of the Kepler problem. This problem is defined by solution of the following differential equations.

$$\ddot{x} = \frac{-\mu}{(x^2 + y^2 + z^2)^{3/2}} \quad . \tag{79}$$

$$\ddot{y} = \frac{-\mu}{(x^2 + y^2 + z^2)^{3/2}} \quad . \tag{80}$$

$$\ddot{z} = \frac{-\mu}{(x^2 + y^2 + z^2)^{3/2}} + A \quad . \tag{81}$$

The results of a separation of variables of the last three equations is given by equations (39), (40), and (41). For the special case of two-dimensional motion, which is initially a circular orbit, and A restricted to the range

$$0 \leq A \leq 1 - \frac{\sqrt{3}}{2} \quad ,$$

the separated equations were integrated analytically.

The results obtained are fairly complicated, and it is worthwhile to obtain a graphical form of the solution. These graphical results are presented in this section.

The following graphs were prepared by integration of equations (79), (80), and (81).* For illustrative purposes a two-dimensional case was taken with initial conditions given by equation (55). The values of A were chosen to give a representative set of orbits. Values of A were

$$A = \{0,\ 0.001,\ 0.01,\ 0.04,\ 0.05,\ 0.06,\ 0.08,\ 0.1\} .$$

For every value except the first two, two values of z (namely +1 and -1) were chosen. Furthermore, in every case, the results are plotted in both x-z and ξ-η coordinate systems.

The first set of graphs, Figures 1 and 2, is shown for the case of A = 0. In x-y coordinates, the result is a circle but in ξ-η coordinates the result is a straight line. This illustrates the fact that results are strikingly different in the two-coordinate system.

The second set of results, Figures 3 and 4, has an A value of 0.001. The computer was stopped at an arbitrary point in the integration because of excessive running times (approximately 30 minutes on a SDS 930 computer utilizing a Cal Comp plotter). The obvious characteristic of Figure 3 is that the orbit is shifting in the direction of -x, even though the planar force field is acting in the direction of +z. This condition illustrates the gyroscopic effect of orbital motion that is usually less apparent. In Figure 4, as in all other cases, the motion is limited by the roots of the polynomial, as was shown in equations (66) and (67).

A second important point in connection with the numerical integration is illustrated by Figures 5 and 6, for example. The motion, in general, tends

---

* The reason for numerically integrating these equations instead of using equations (73) and (78) was that existing numerical tables for elliptic functions are quite inadequate. A serious study of the behavior of the orbits under discussion would require series generation of the inverse elliptic functions. The above theory was checked by comparison of numerical integration of equations (52) and (53) with numerical integration of equations (79) and (81) (planar case). Equations (73) and (78) were analytically differentiated to establish their validity.

to spiral the point mass toward the attracting center. The integration step size was reduced as the particle approached the center of attraction. The integration was terminated when the constants E or β changed value across an integration step. (E was found to be more sensitive than β to such changes.) A careful examination of the behavior of the motion of the point mass in close proximity with the primary could best be examined via the analytical solution

Without consideration of the remaining graphs (Figs. 7-28) in detail, we shall examine two additional points. The first of these is that in which the combination of forces is such that a backward loop may be attained by the particle. This effect is easily seen in Figure 11 by examining the point of greatest extension of motion in the x direction.

Figures 27 and 28 illustrate a fairly interesting case. For an initially circular orbit, as always, and $\dot{z} = -1$, an extremely close approach (0.000005 unit) occurs during the first orbit. The exact value of A for which a collision occurs during the first orbit could be determined by use of equation (73) if it is of interest.

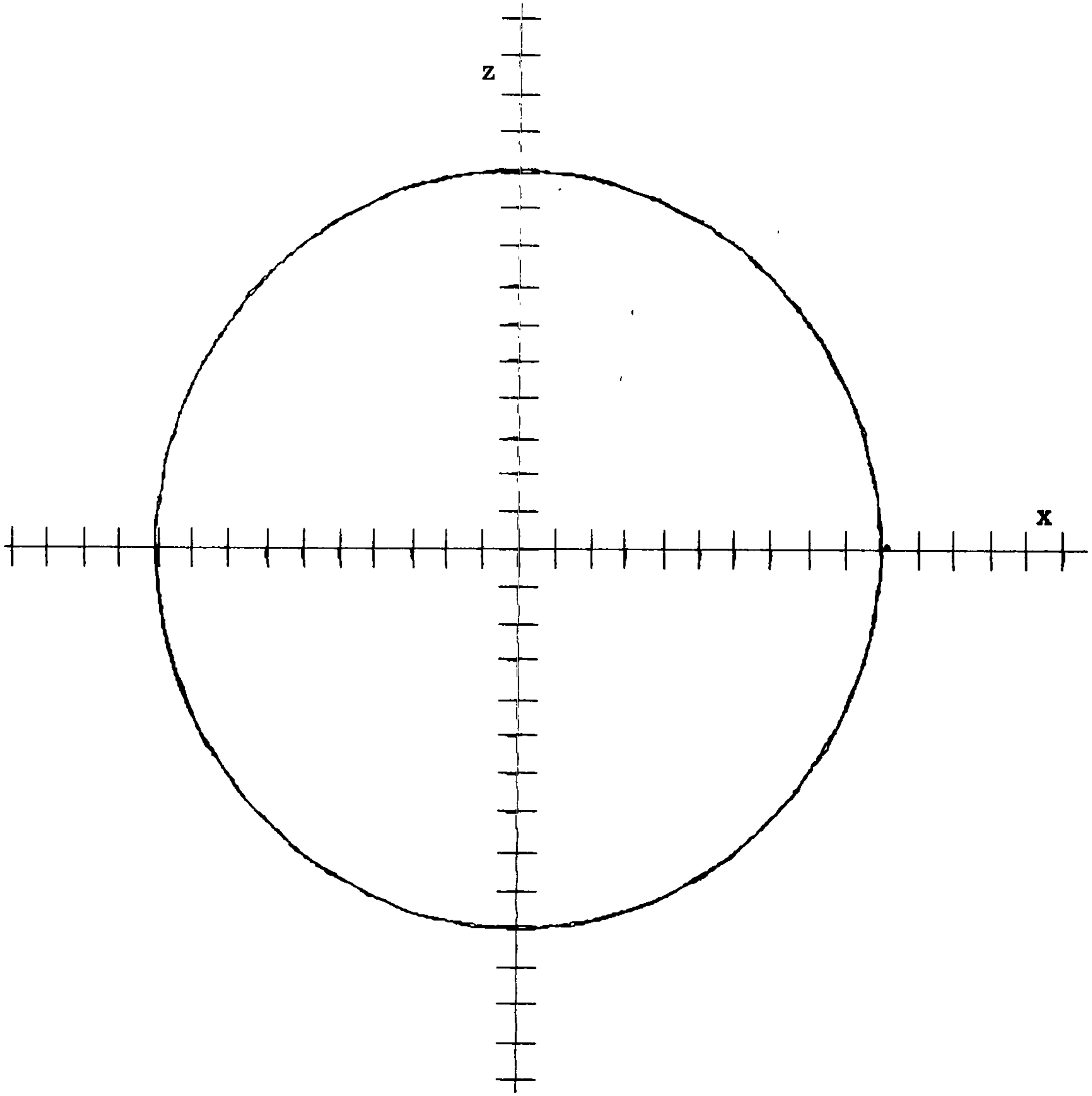

FIGURE 1. $Z_0 = \pm 1$, $A = 0$ CARTESIAN COORDINATES

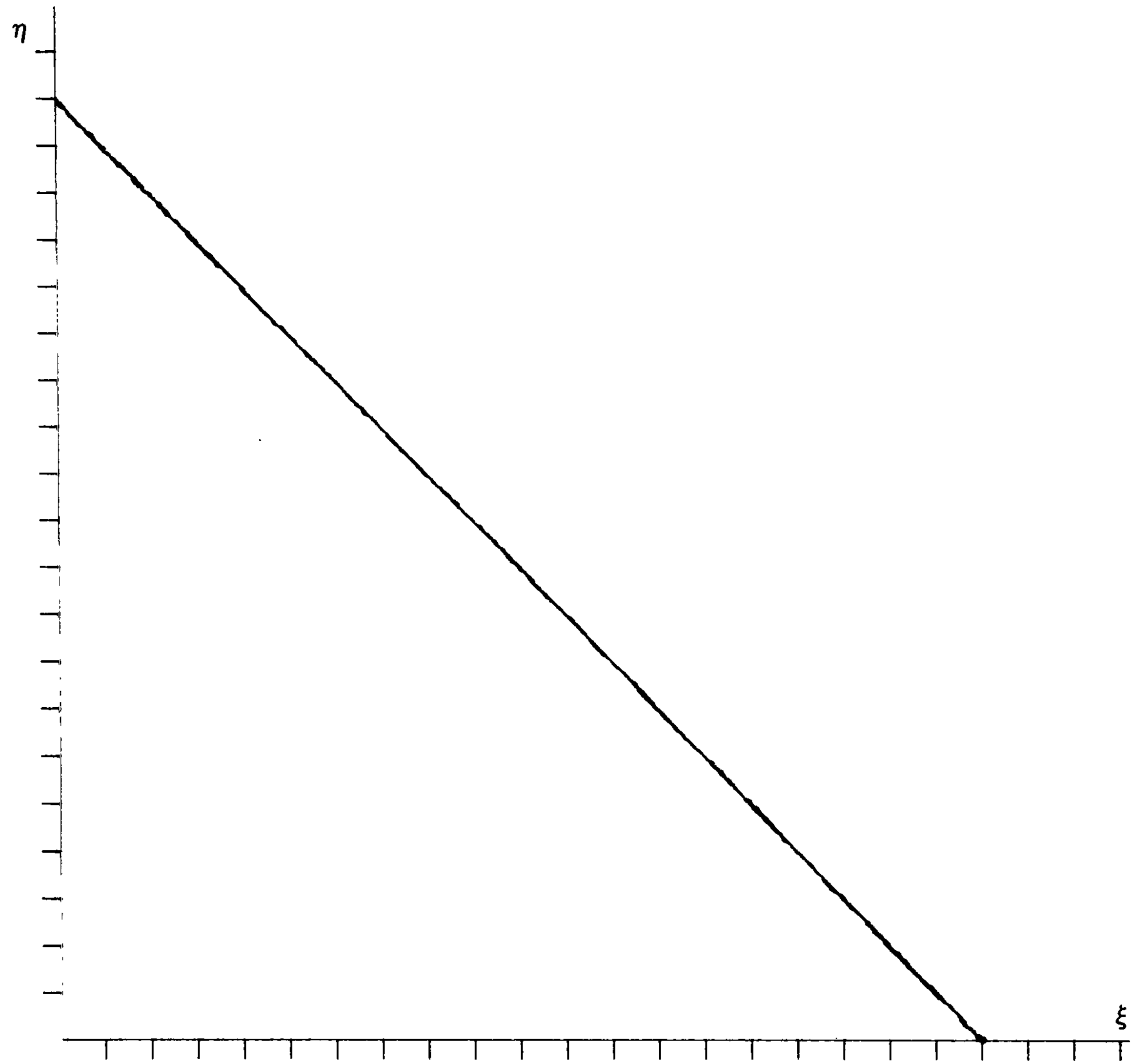

FIGURE 2. $Z_0 = \pm 1$, $A = 0$ PARABOLIC COORDINATES

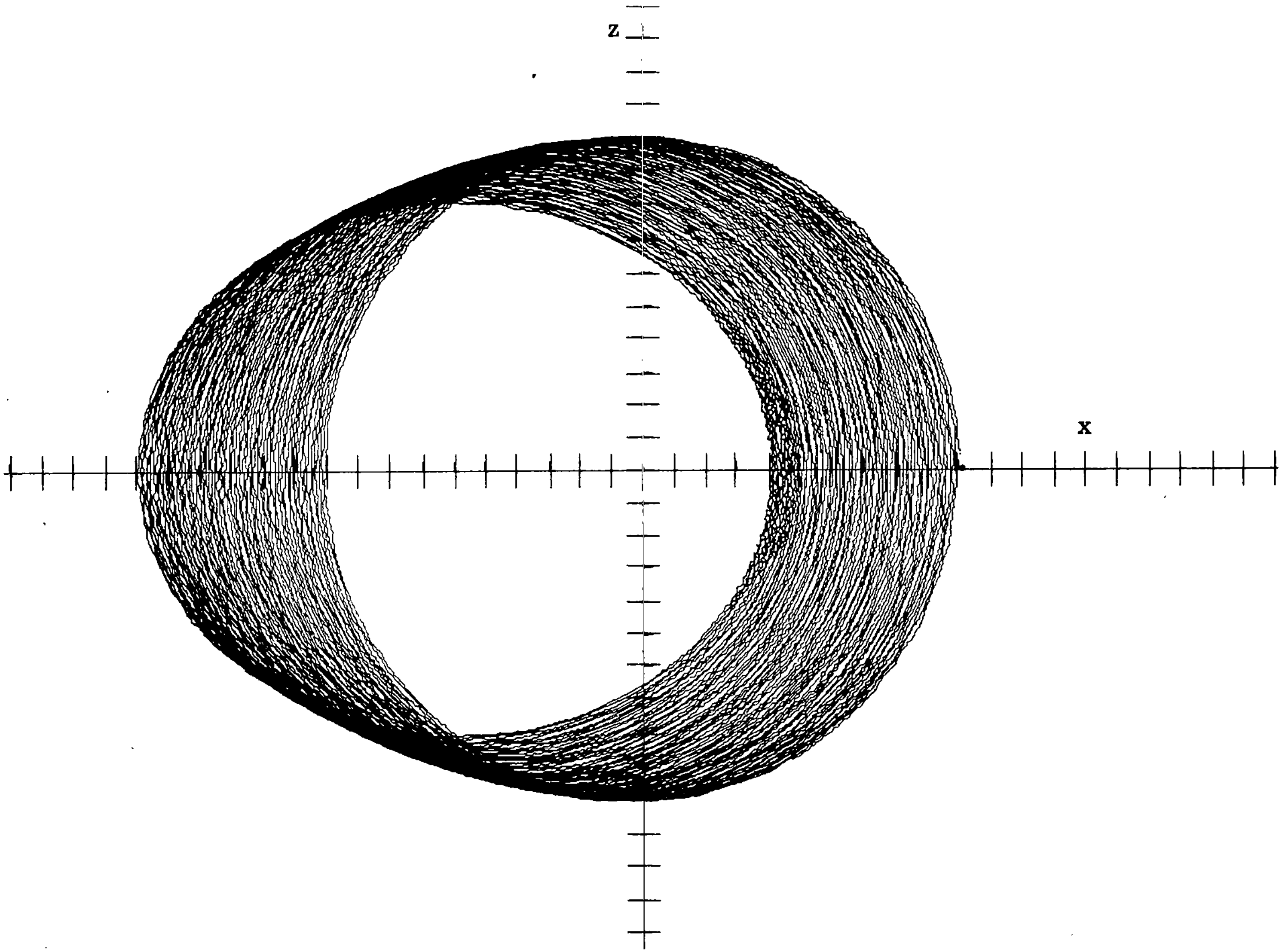

FIGURE 3. $\dot{Z}_0 = 1$, $A = 0.001$ CARTESIAN COORDINATES

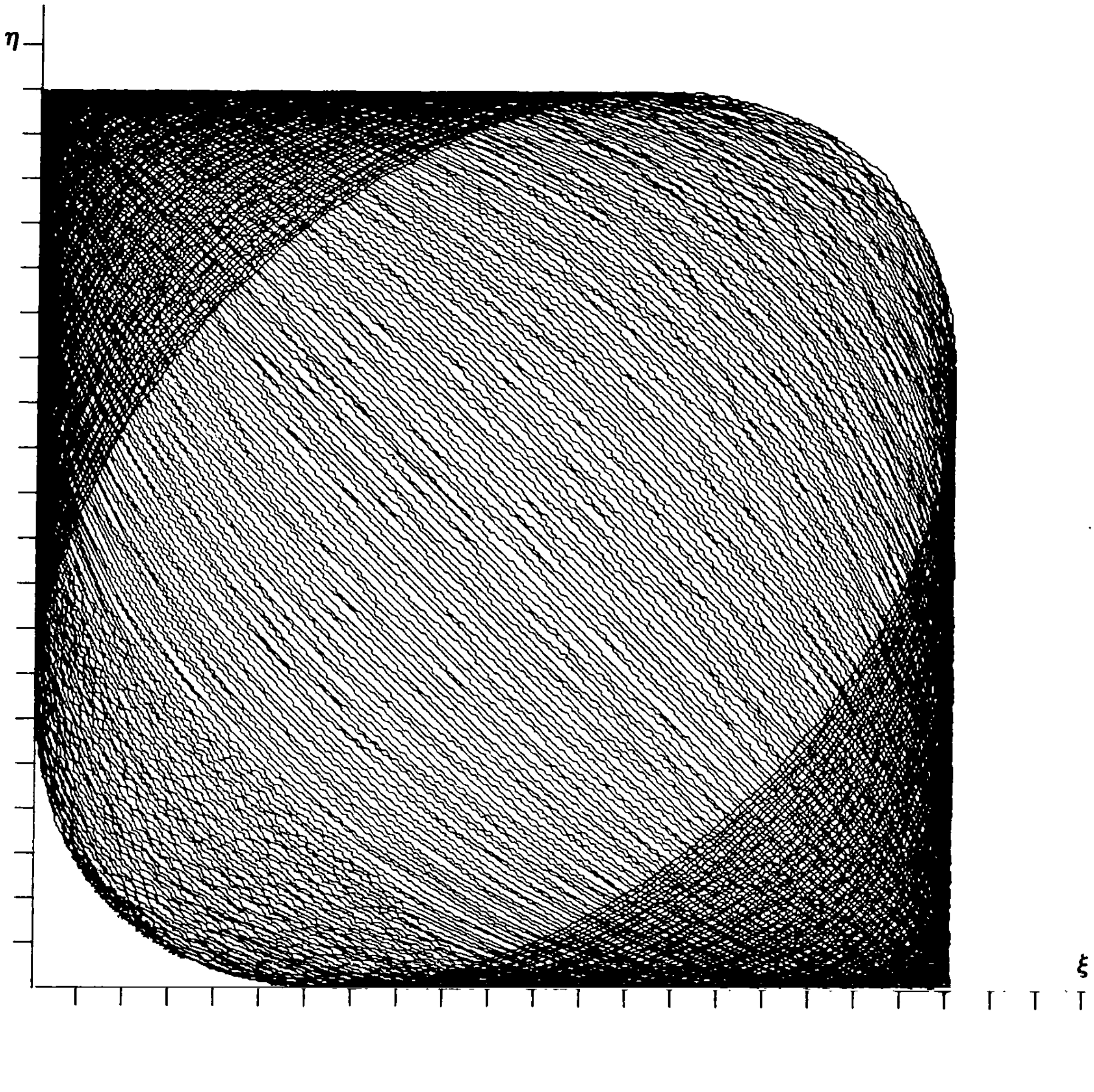

FIGURE 4. $\dot{Z}_0 = 1$, $A = 0.001$ PARABOLIC COORDINATES

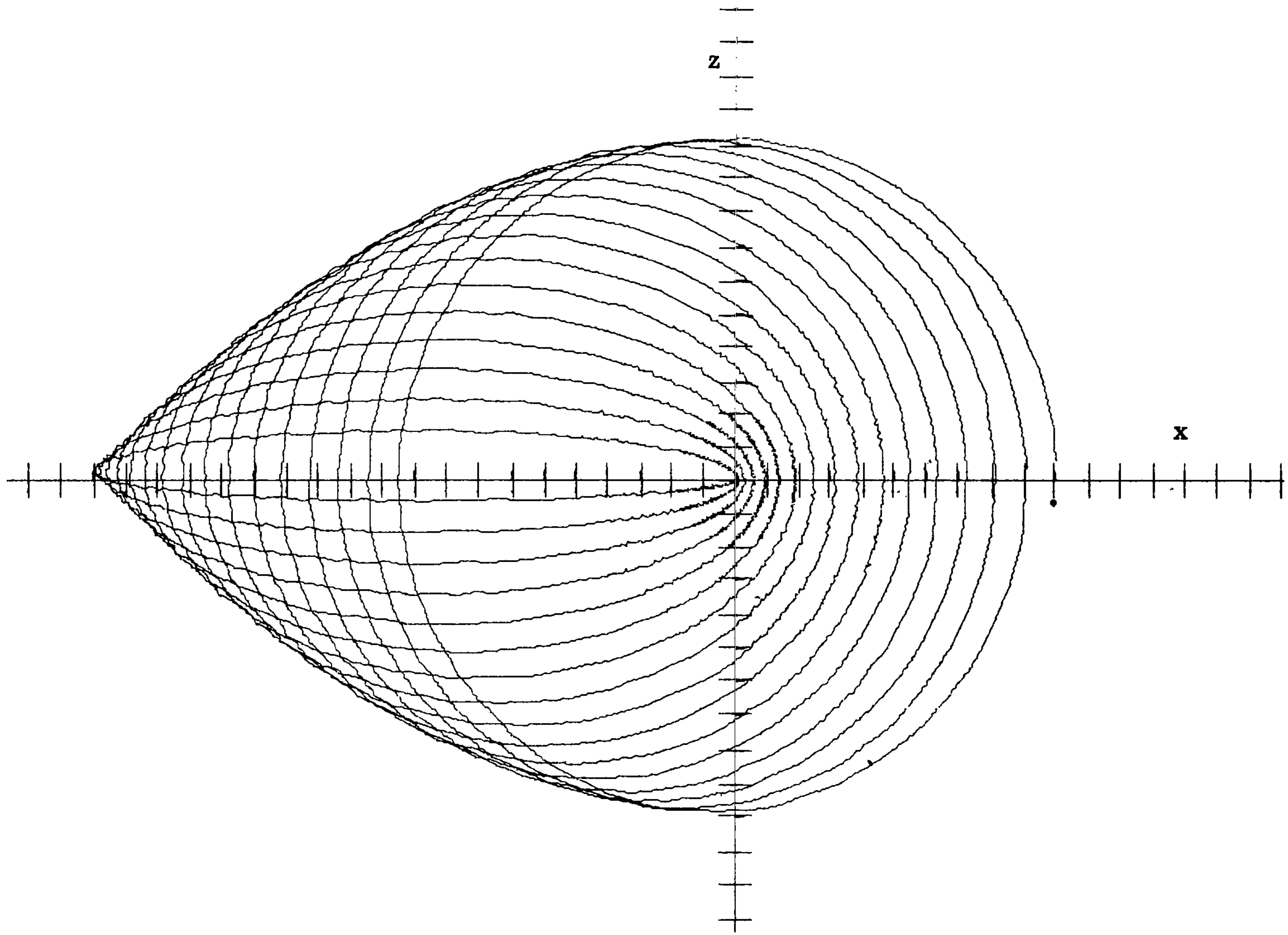

FIGURE 5. $\dot{Z}_0 = 1$, $A = 0.01$ CARTESIAN COORDINATES

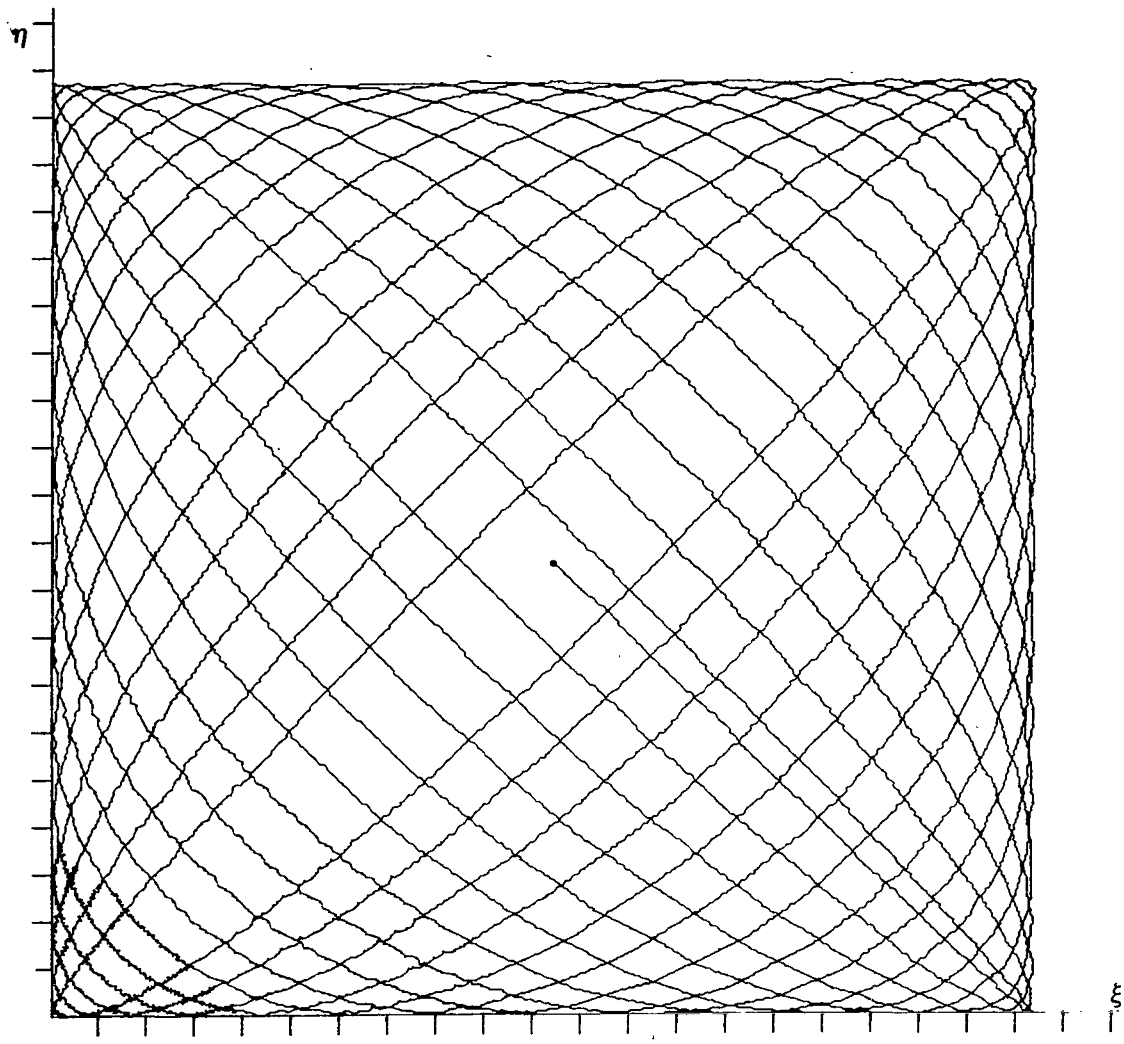

FIGURE 6. $\dot{Z}_0 = 1$, $A = 0.01$ PARABOLIC COORDINATES

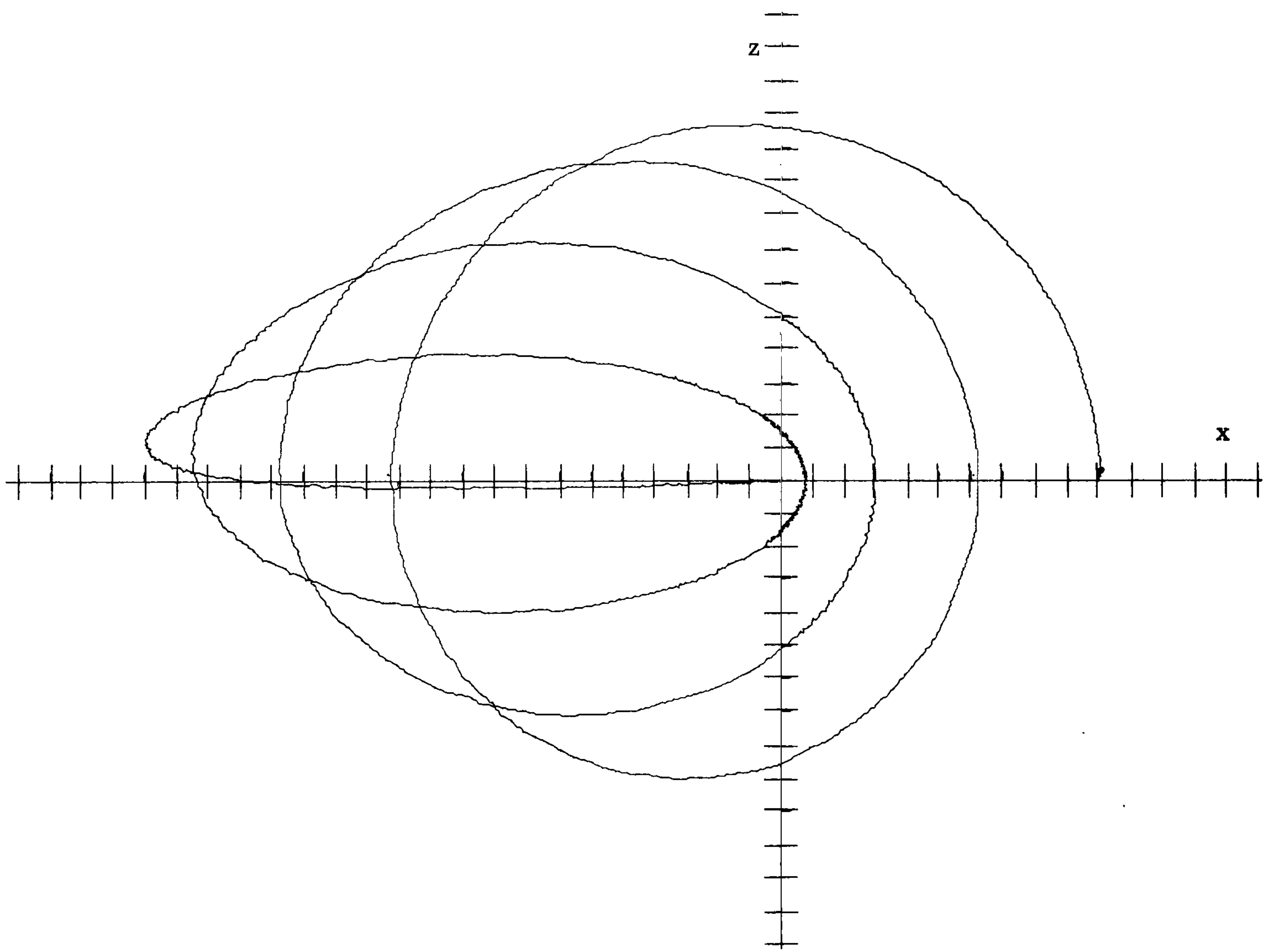

FIGURE 7. $\dot{Z}_0 = 1$, $A = 0.04$ CARTESIAN COORDINATES

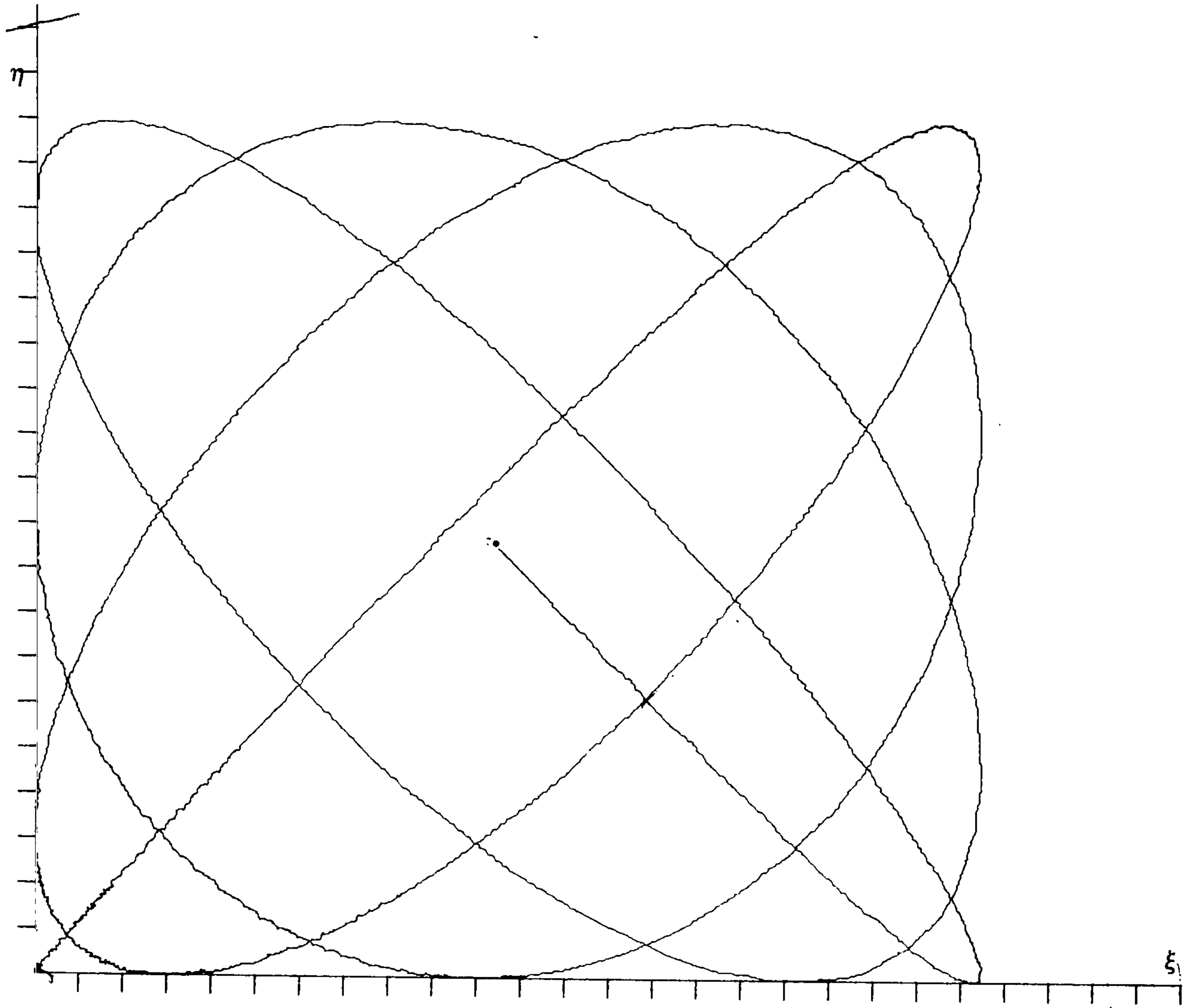

FIGURE 8. $Z_0 = 1$, $A = 0.04$ PARABOLIC COORDINATES

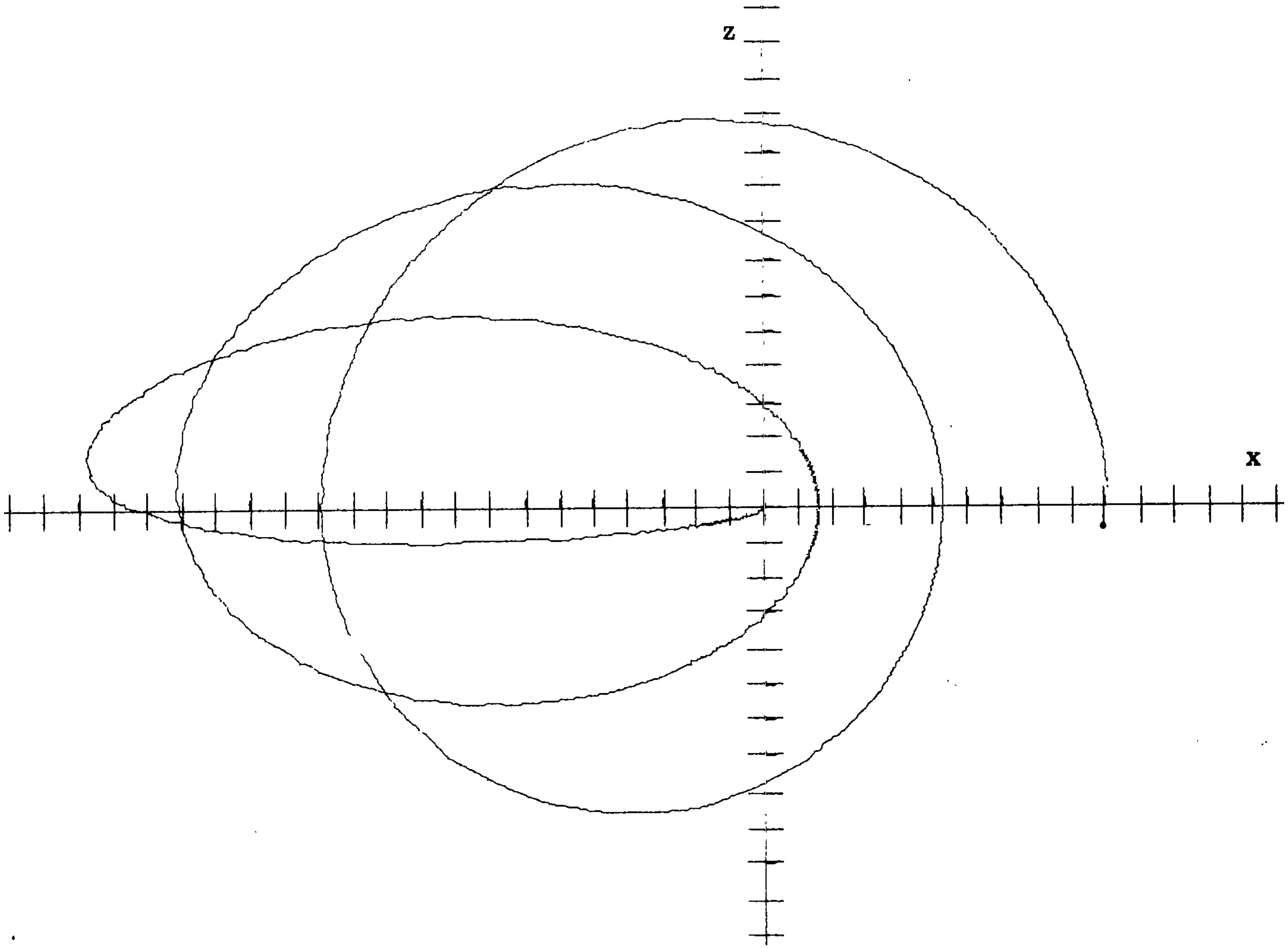

FIGURE 9. $\dot{Z}_0 = 1$, A = 0.05 CARTESIAN COORDINATES

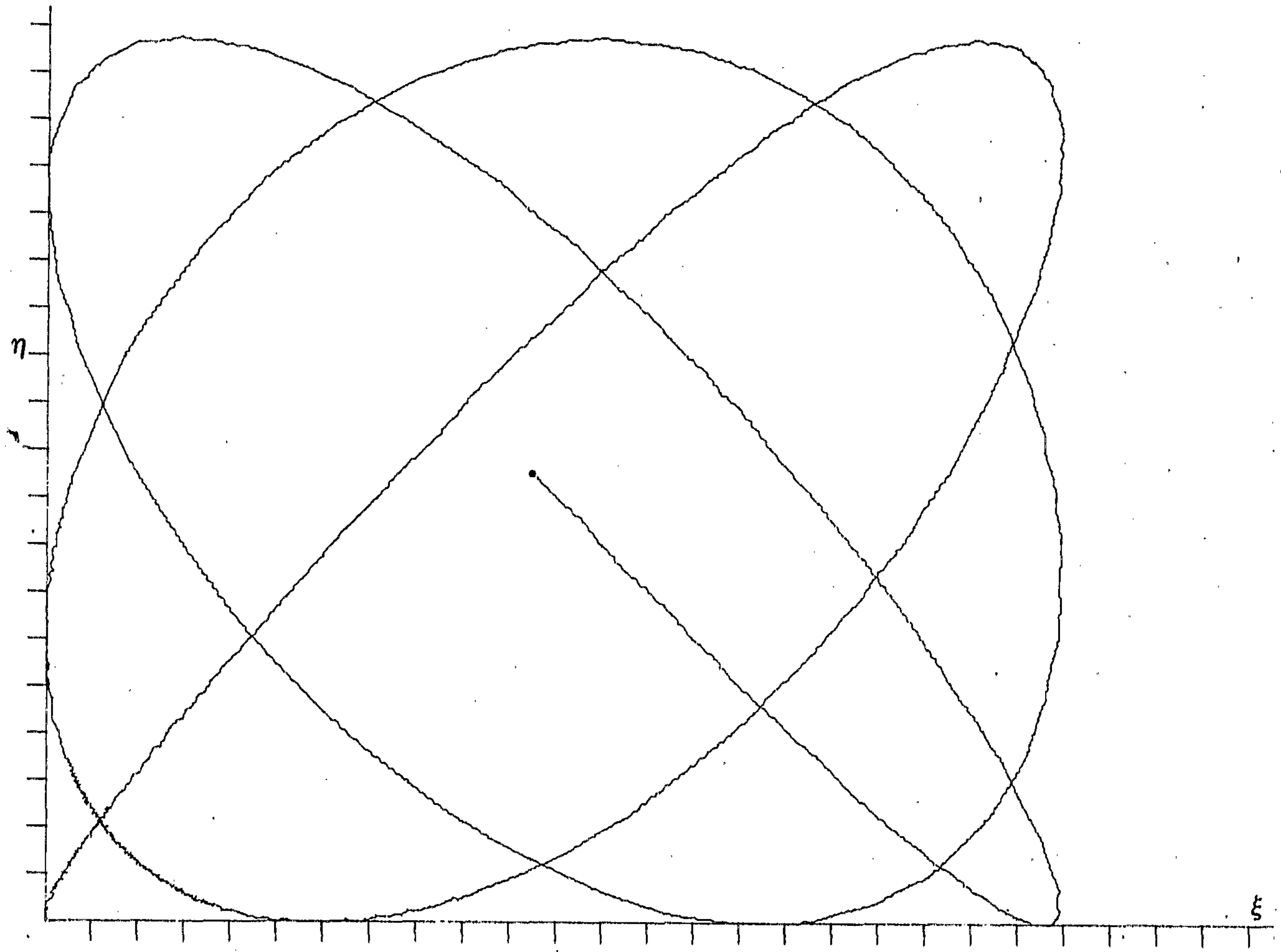

FIGURE 10. $\dot{Z}_0 = 1$, A = 0.05 PARABOLIC COORDINATES

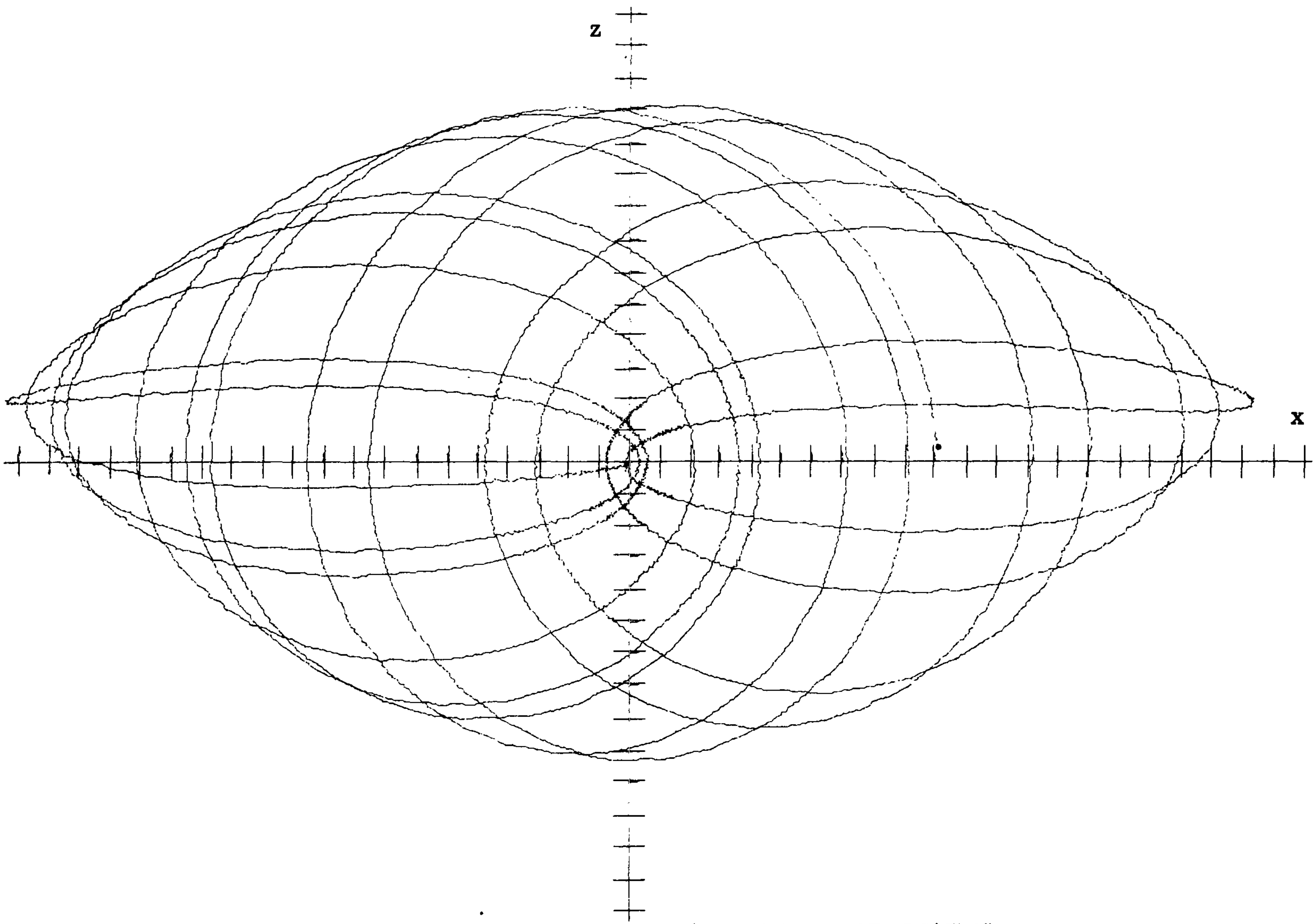

FIGURE 11. $Z_0 = 1$, $A = 0.06$ CARTESIAN COORDINATES

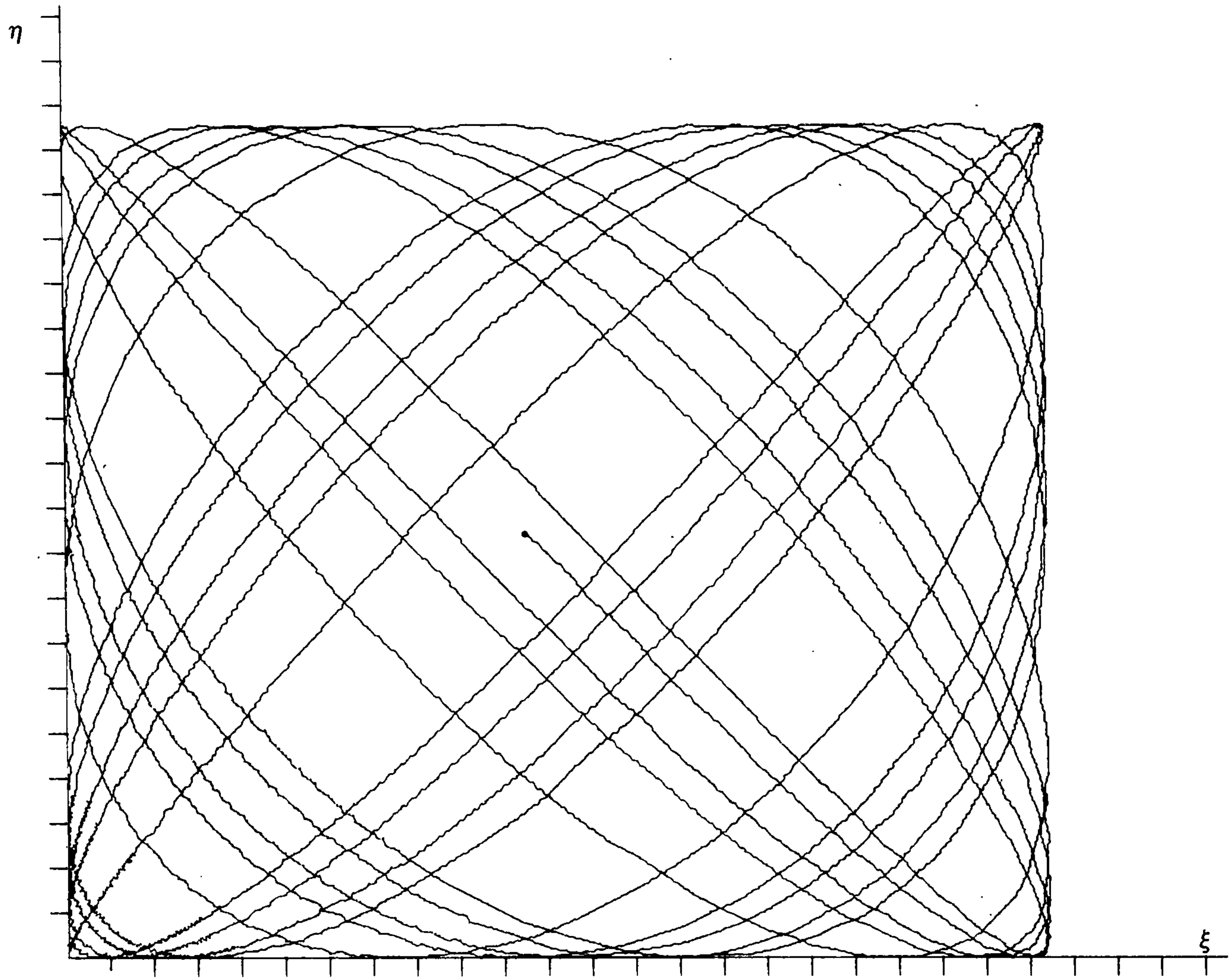

FIGURE 12. $\dot{Z}_0 = 1$, A = 0.06 PARABOLIC COORDINATES

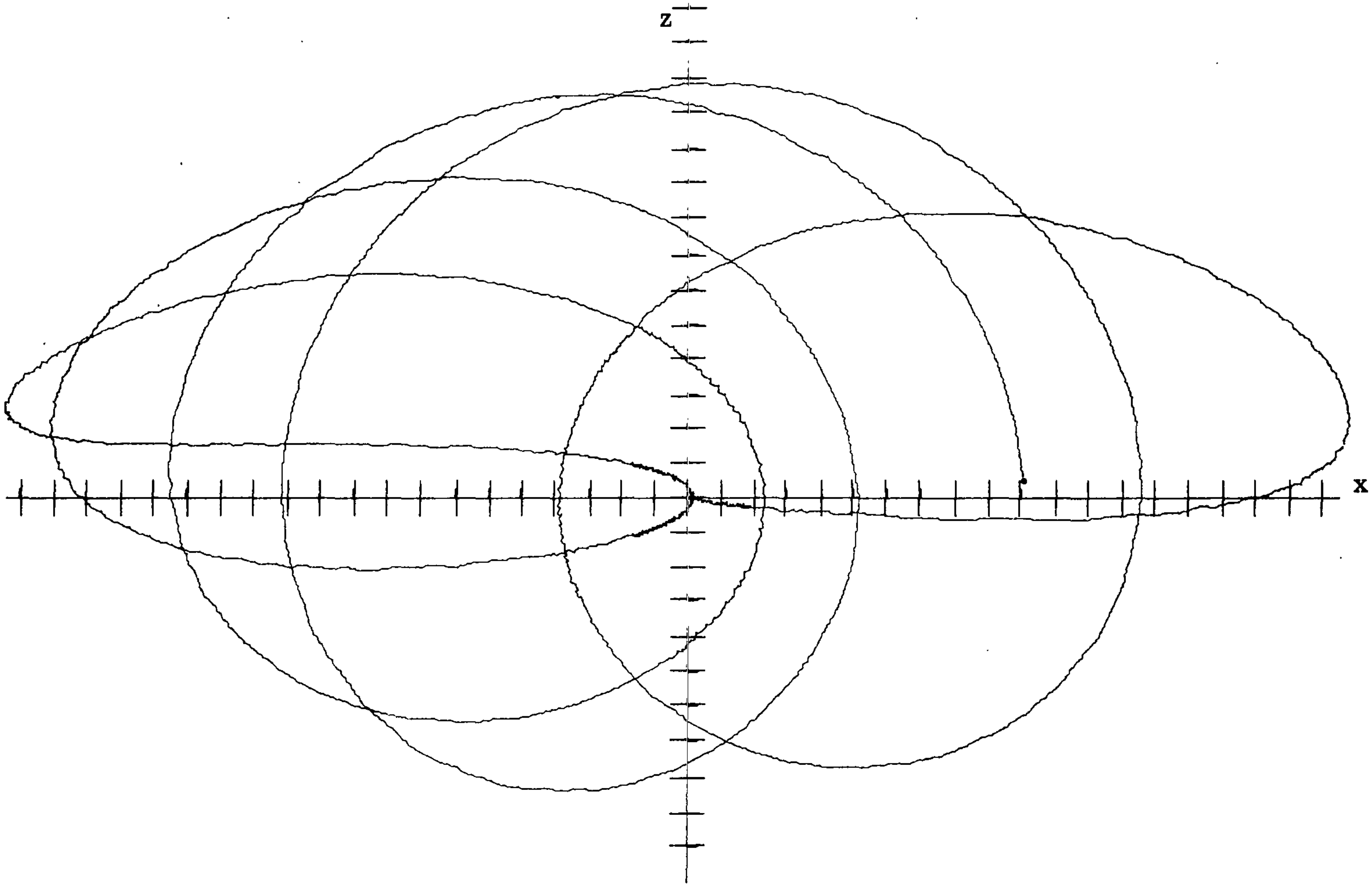

FIGURE 13. $\dot{Z}_0 = 1$, $A = 0.08$ CARTESIAN COORDINATES

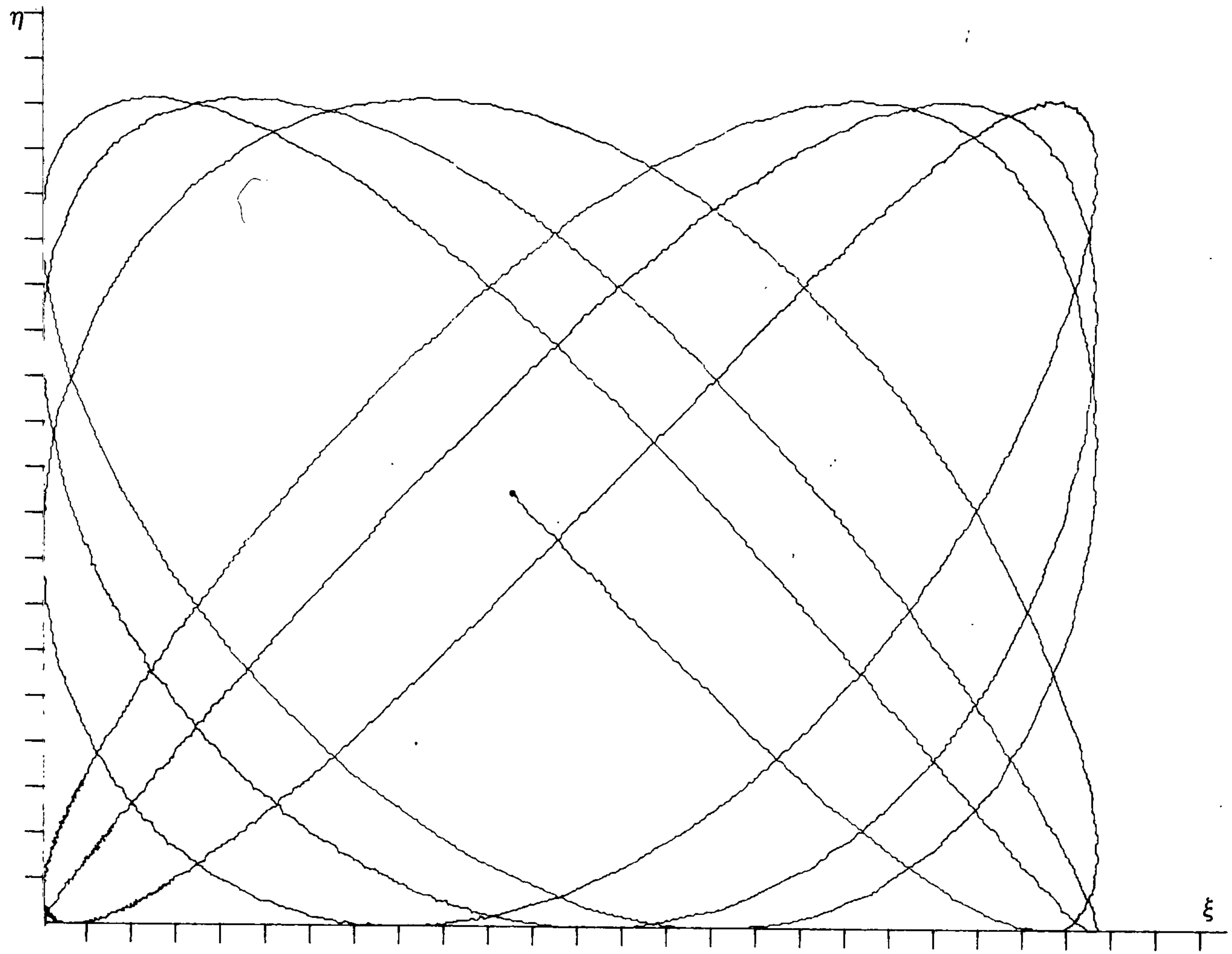

FIGURE 14. $\dot{Z}_0 = 1$, $A = 0.08$ PARABOLIC COORDINATES

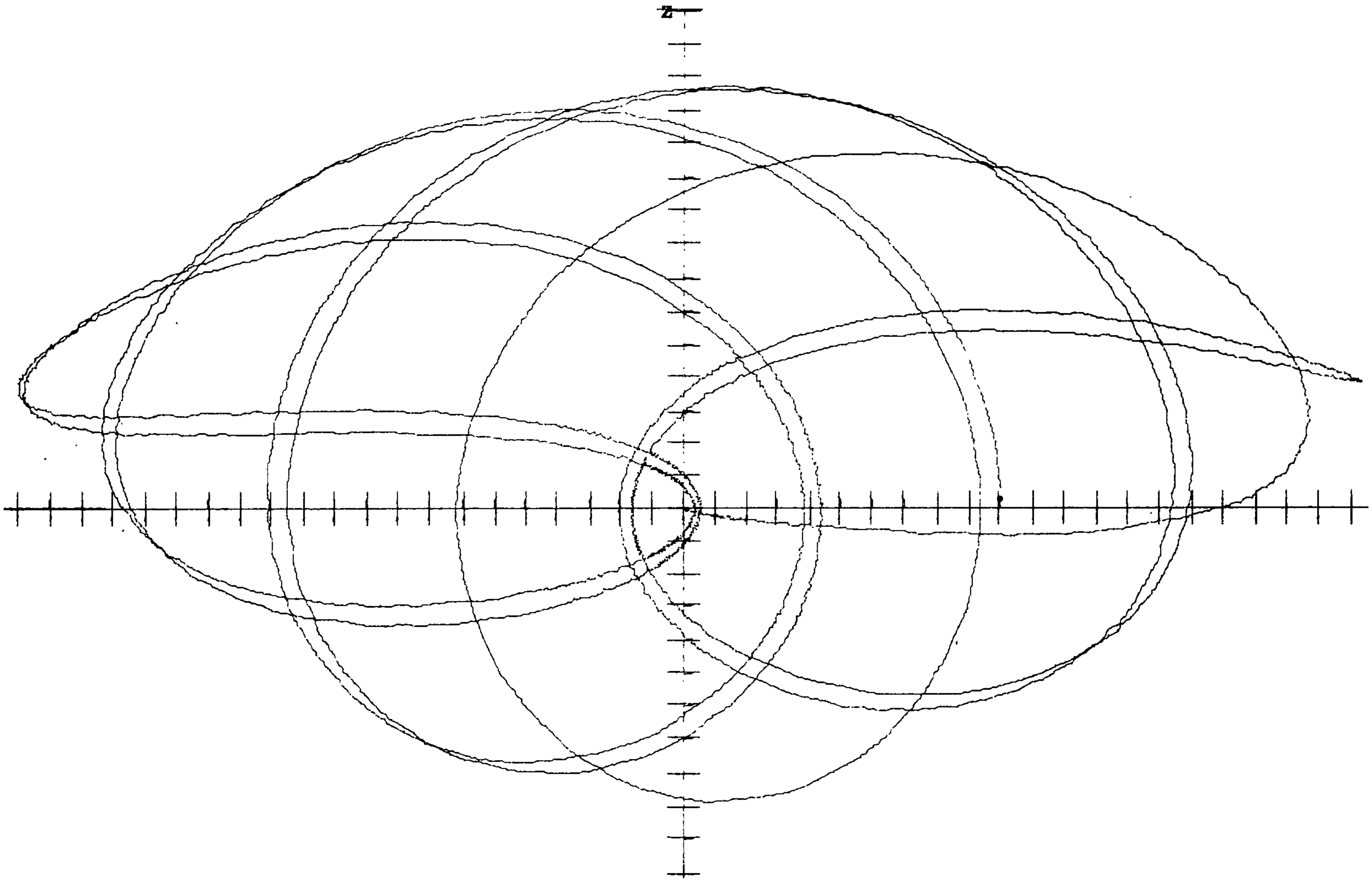

FIGURE 15. $\dot{Z}_0 = 1$, $A = 0.1$ CARTESIAN COORDINATES

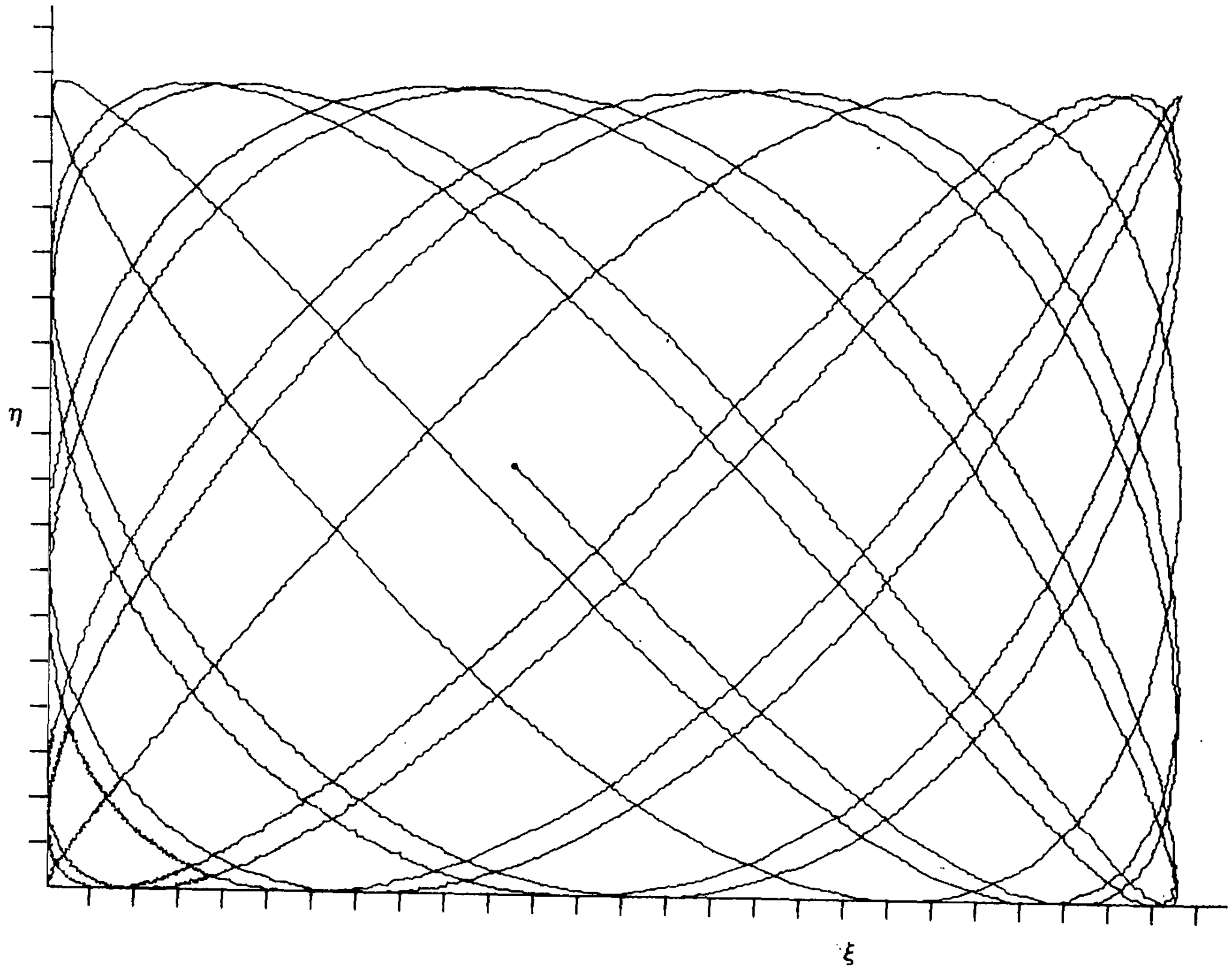

FIGURE 16. $\dot{Z}_0 = 1$, $A = 0.1$ PARABOLIC COORDINATES

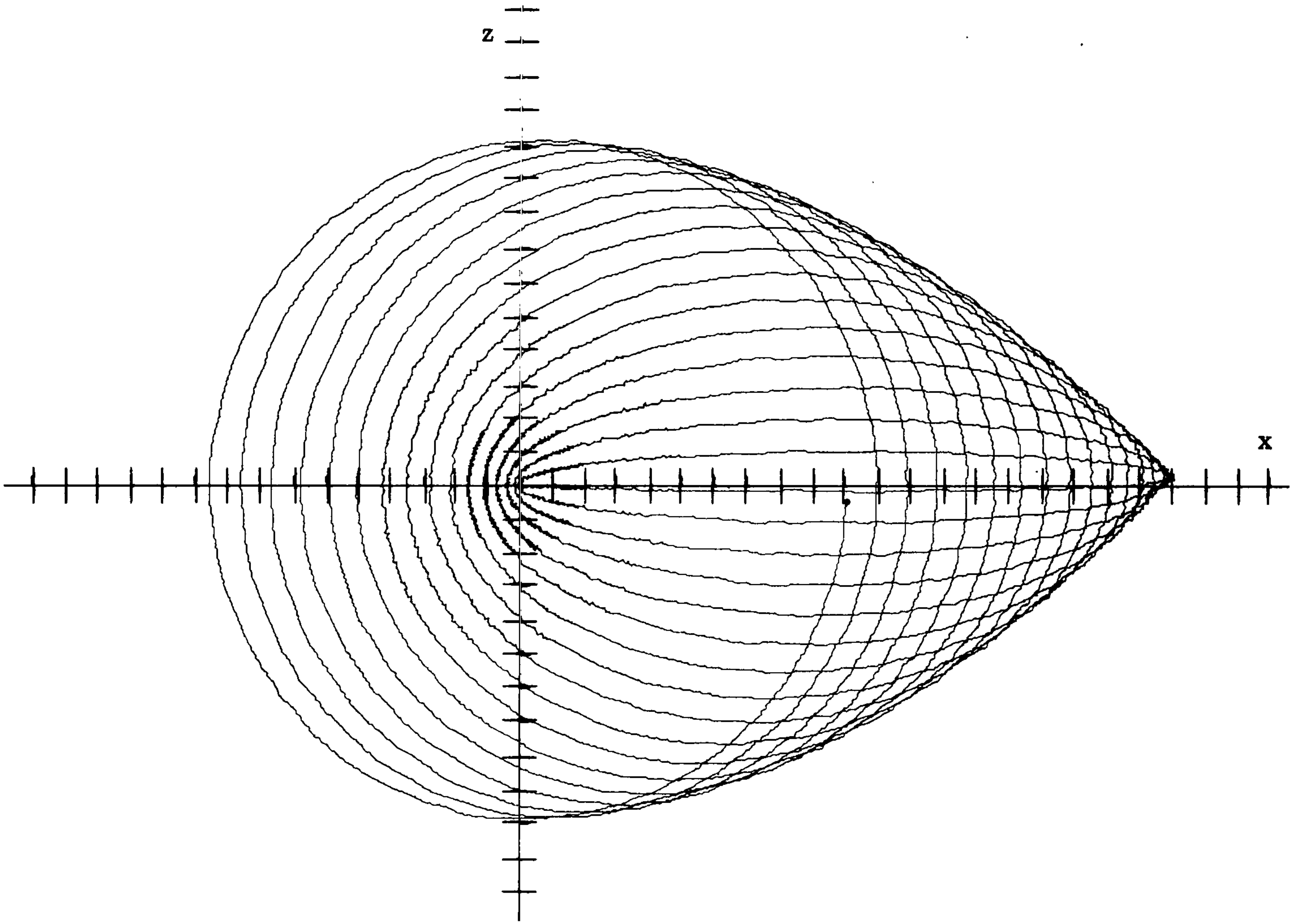

FIGURE 17. $\dot{Z}_0 = -1$, A = 0.01 CARTESIAN COORDINATES

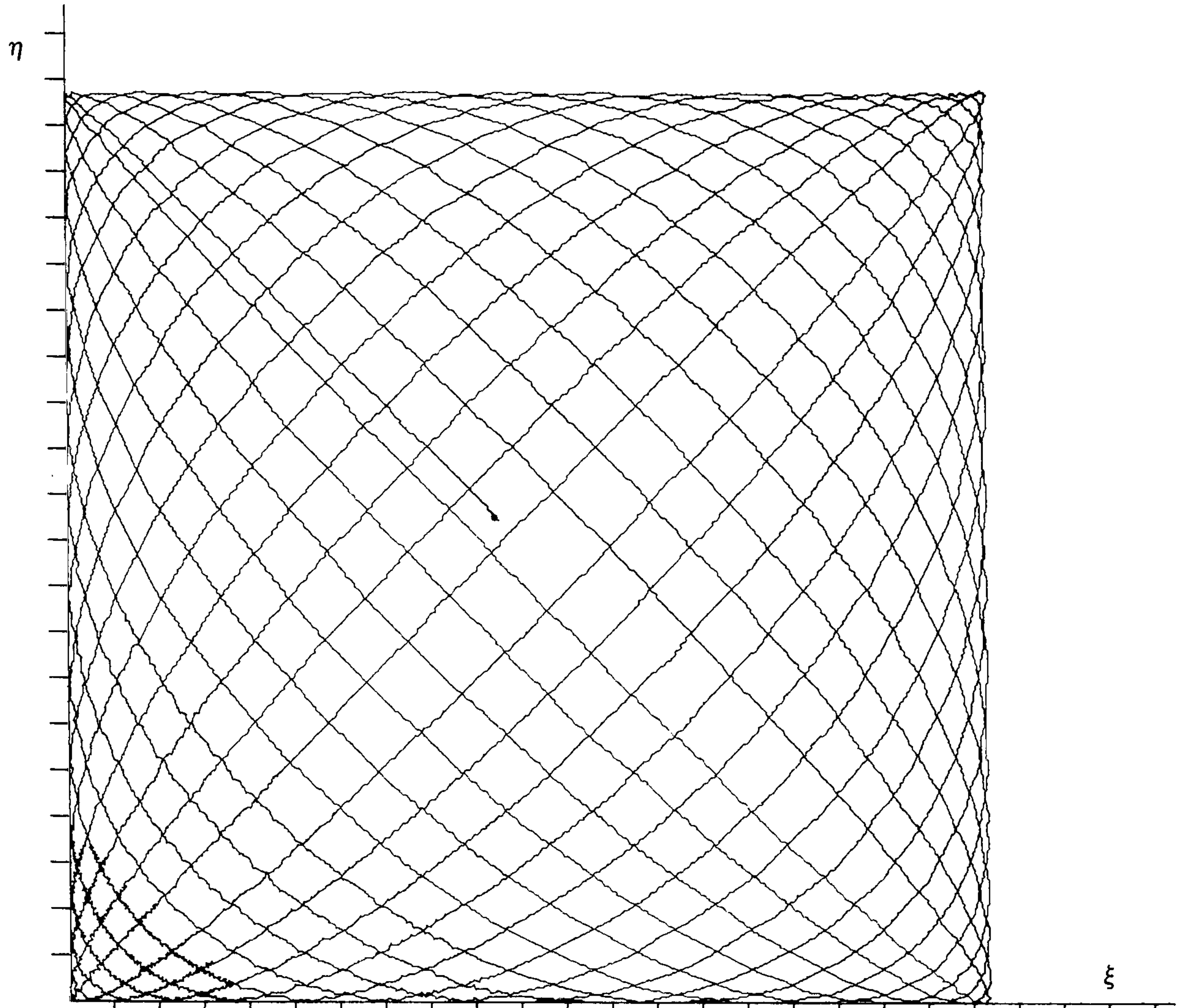

FIGURE 18. $\dot{Z}_0 = -1$, $A = 0.01$ PARABOLIC COORDINATES

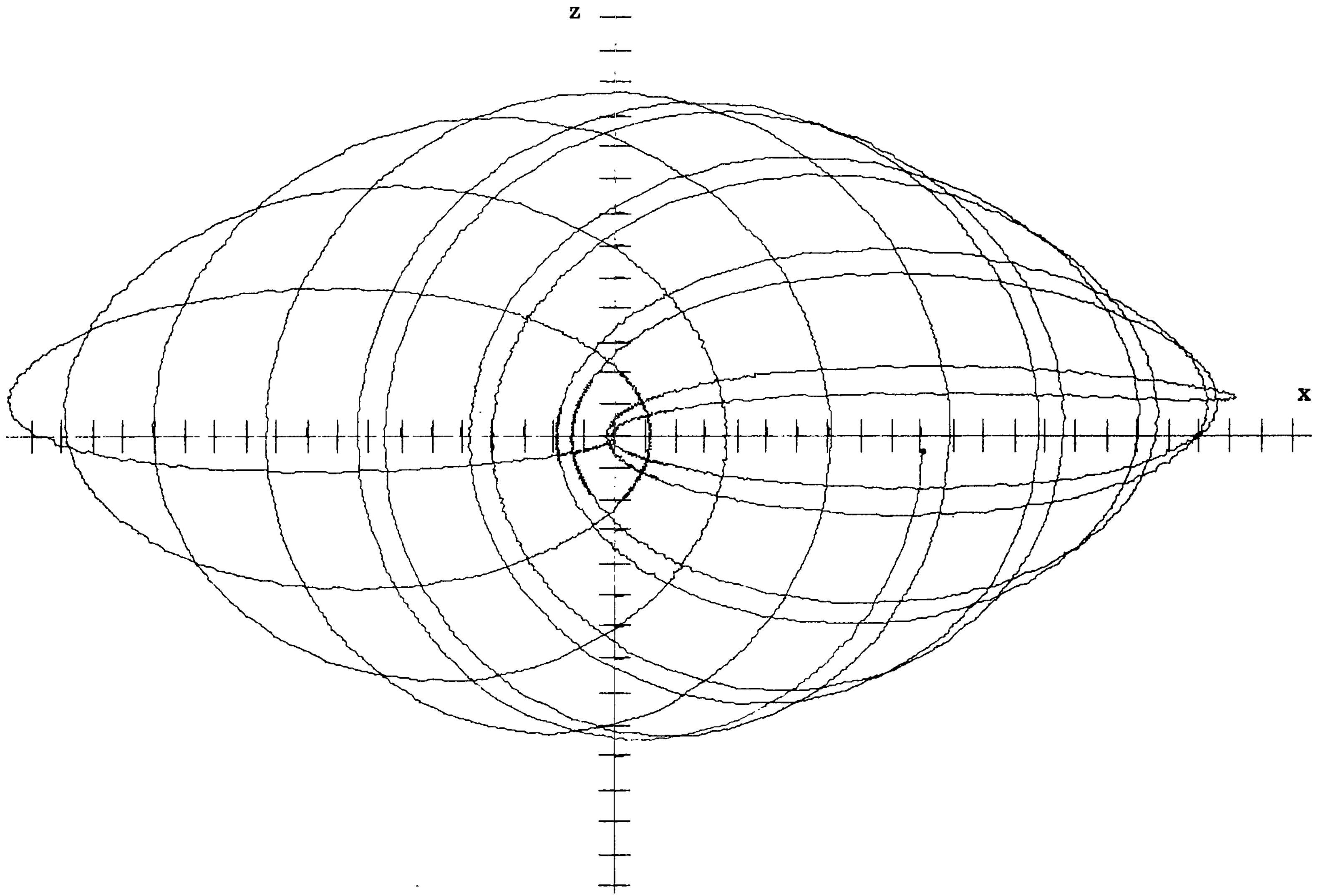

FIGURE 19. $Z_0 = -1$, $A = 0.04$ CARTESIAN COORDINATES

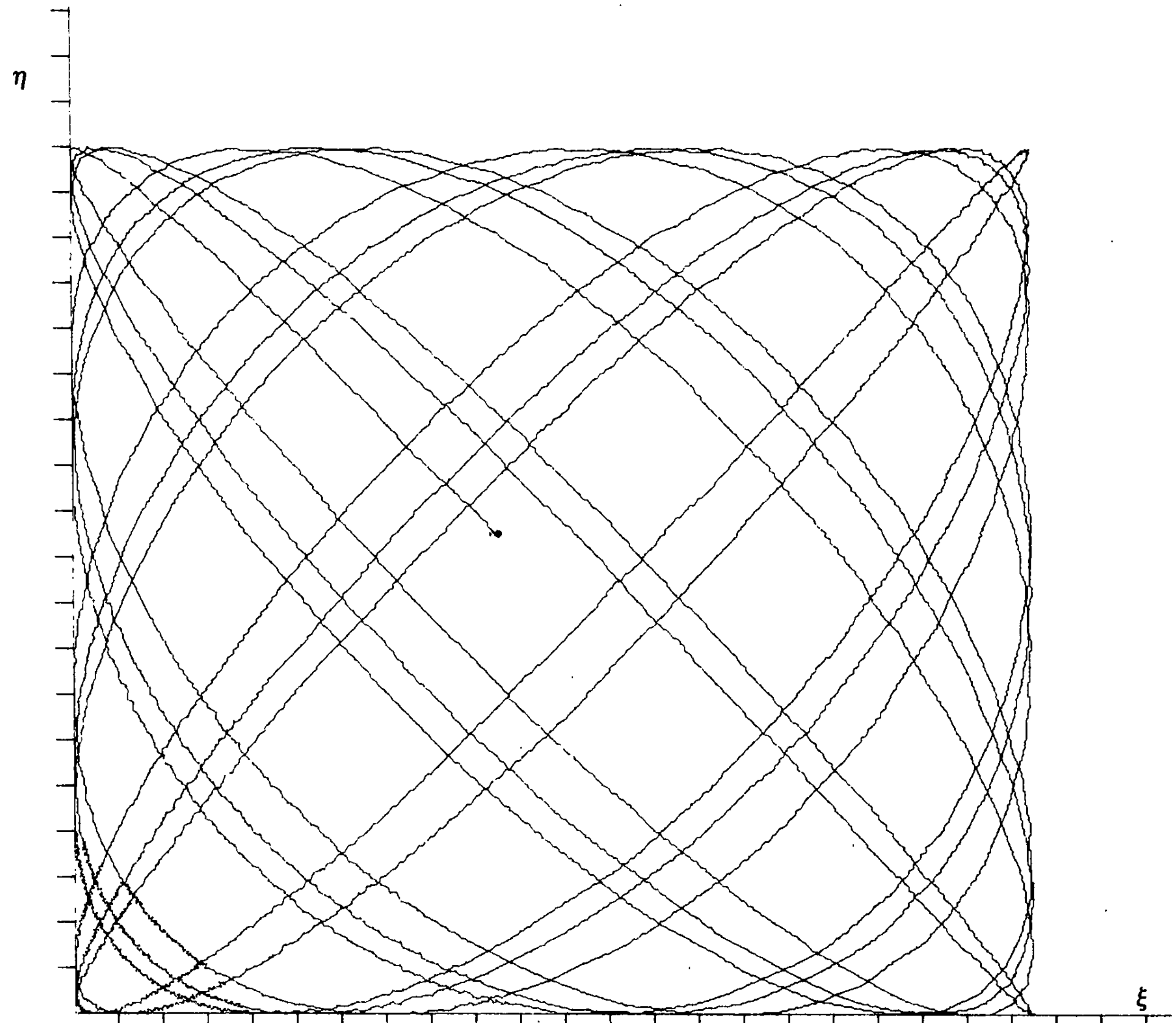

FIGURE 20. $\dot{Z}_0 = -1$, $A = 0.04$ PARABOLIC COORDINATES

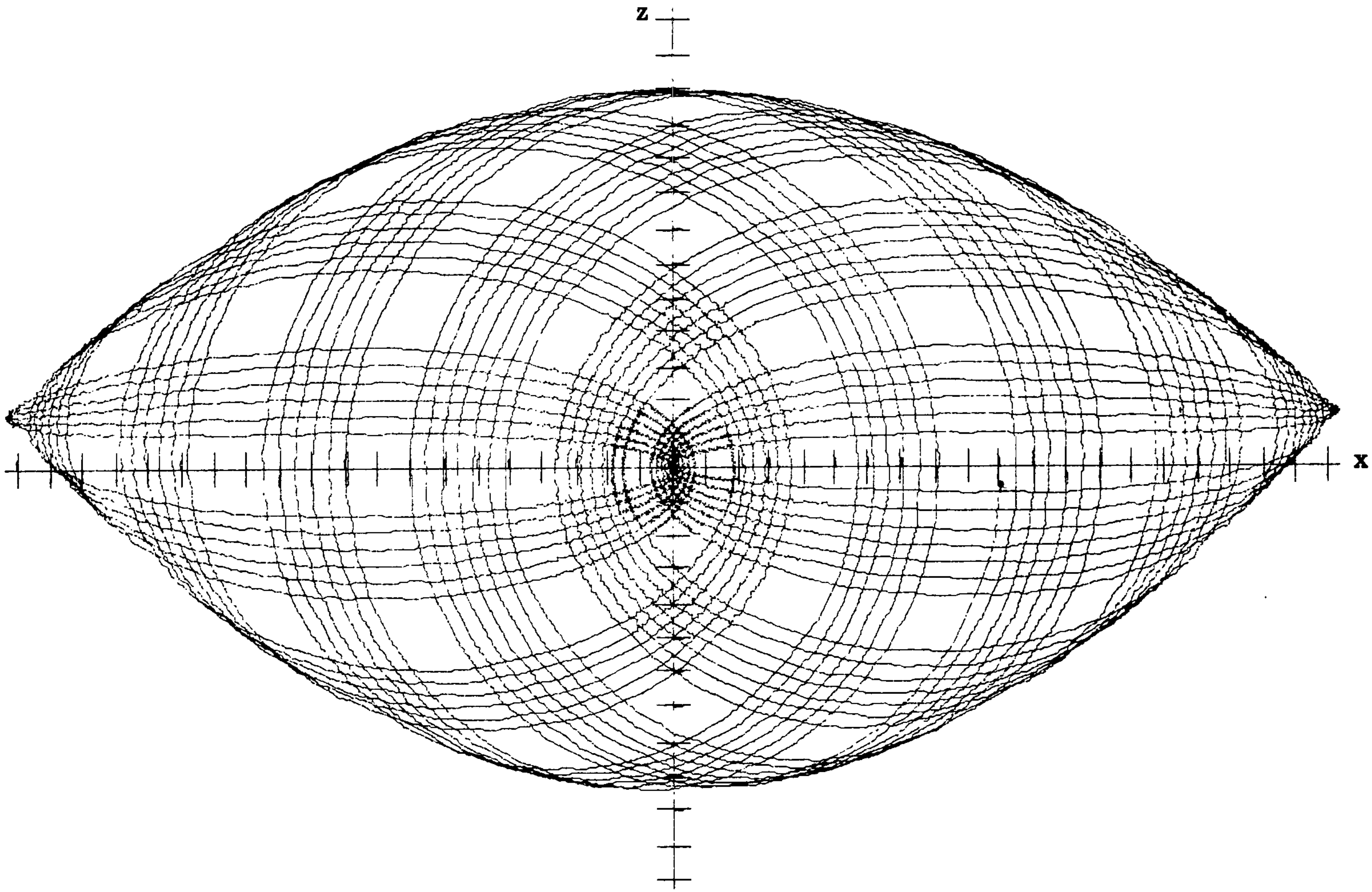

FIGURE 21. $\dot{Z}_0 = -1$, $A = 0.05$ CARTESIAN COORDINATES

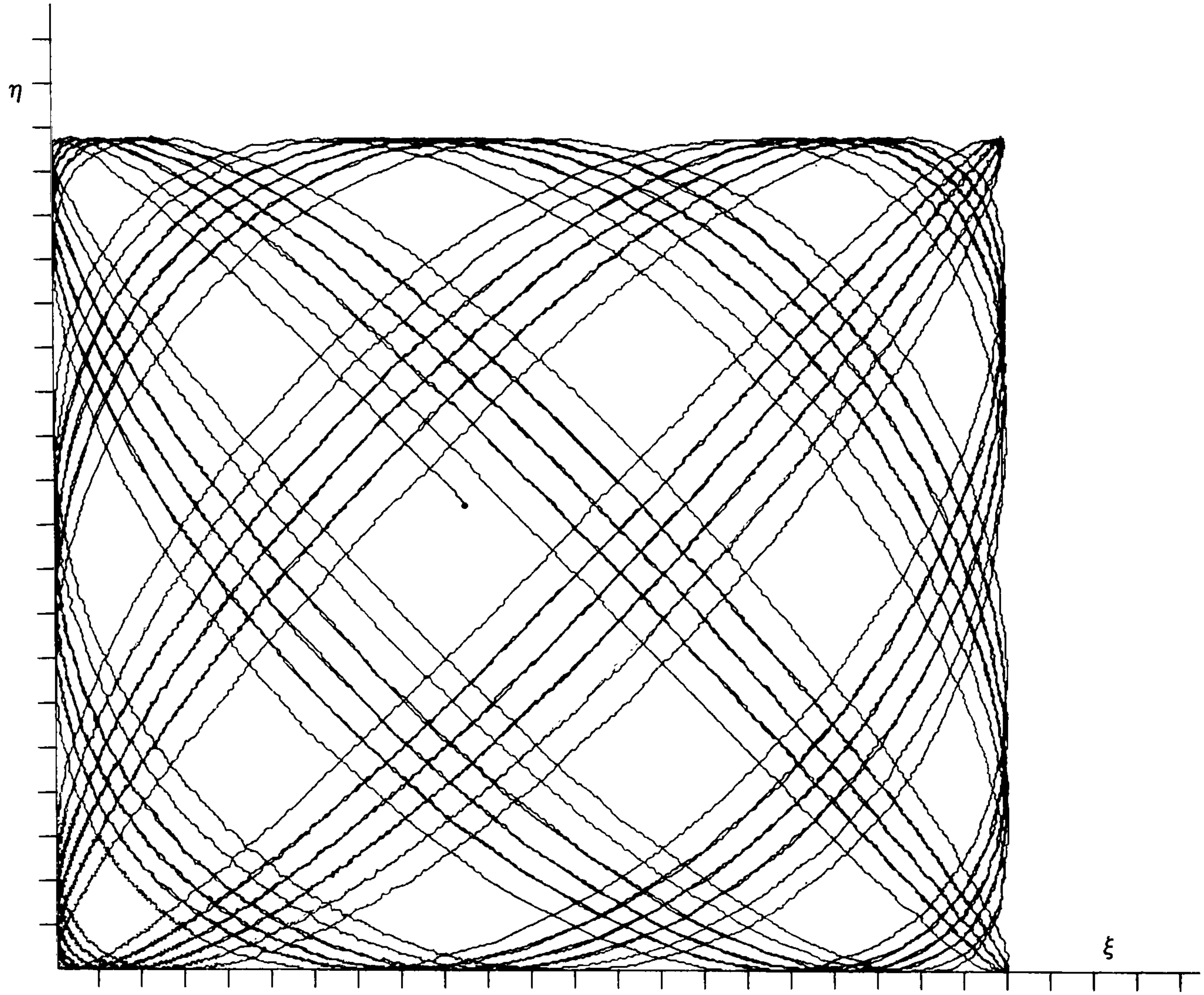

FIGURE 22. $\dot{Z}_0 = -1$, A = 0.05 PARABOLIC COORDINATES

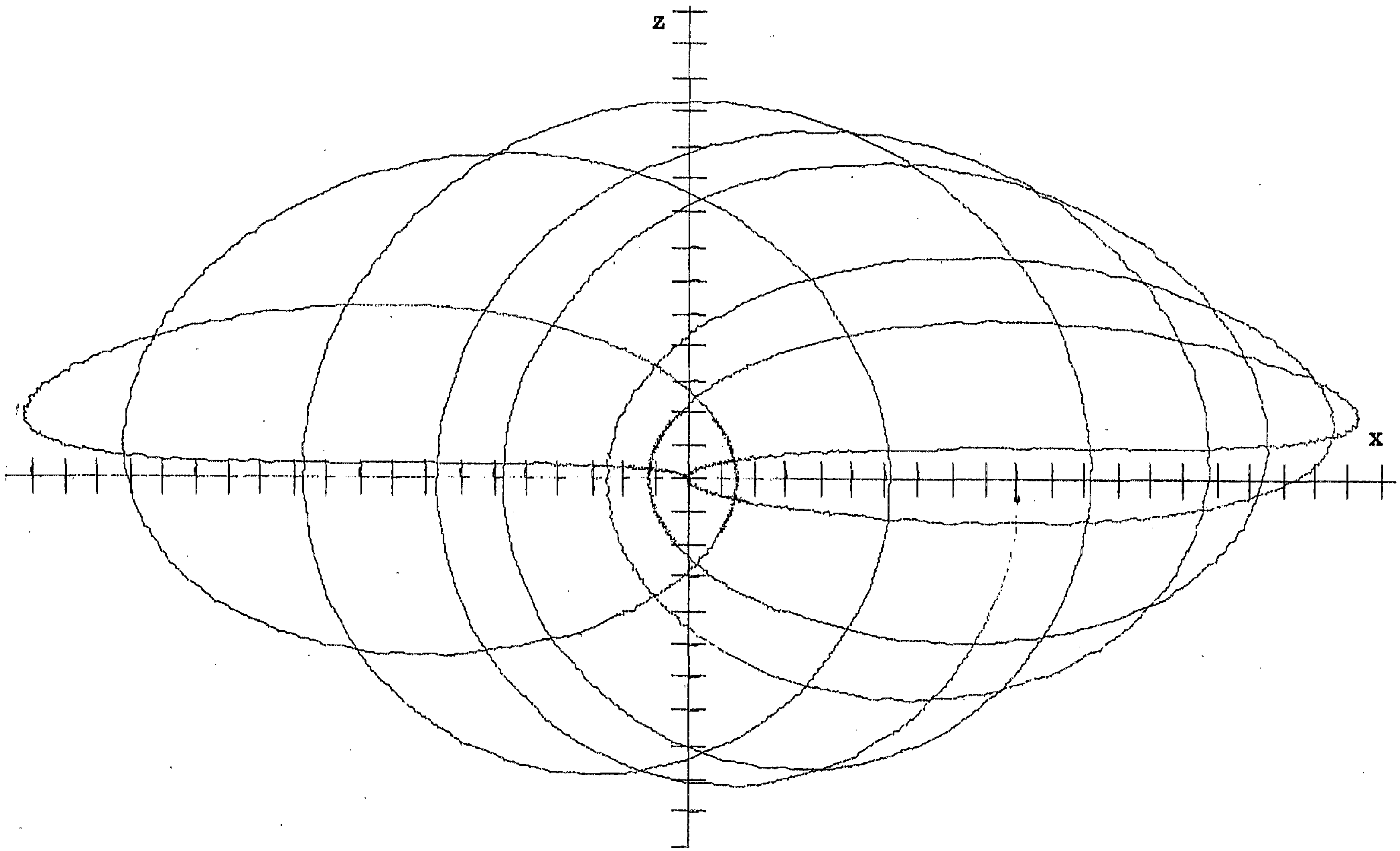

FIGURE 23. $\dot{Z}_0 = -1$, $A = 0.06$ CARTESIAN COORDINATES

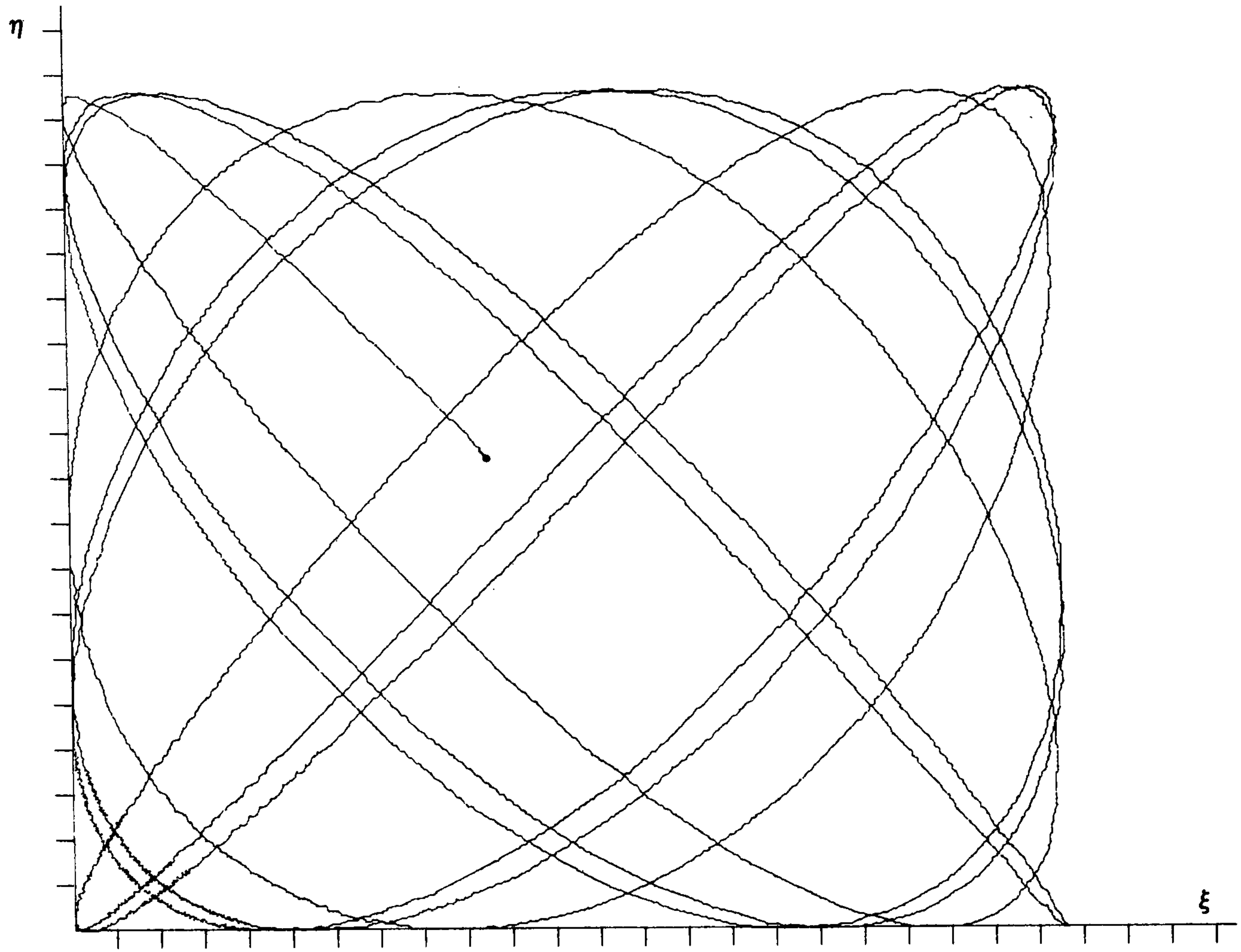

FIGURE 24. $\dot{Z}_0 = -1$, $A = 0.06$ PARABOLIC COORDINATES

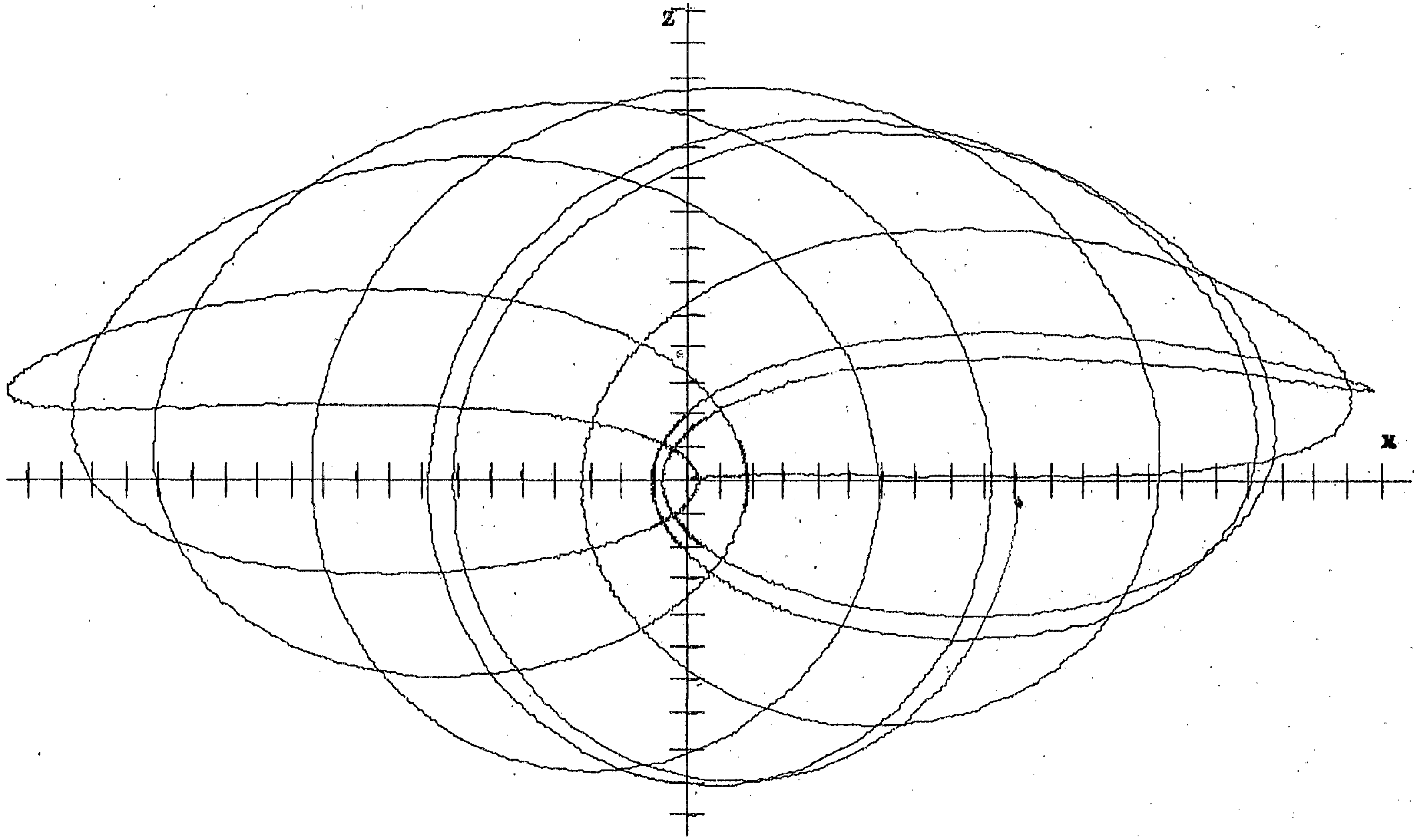

FIGURE 25. $\dot{Z}_0 = -1$, A = 0.08 CARTESIAN COORDINATES

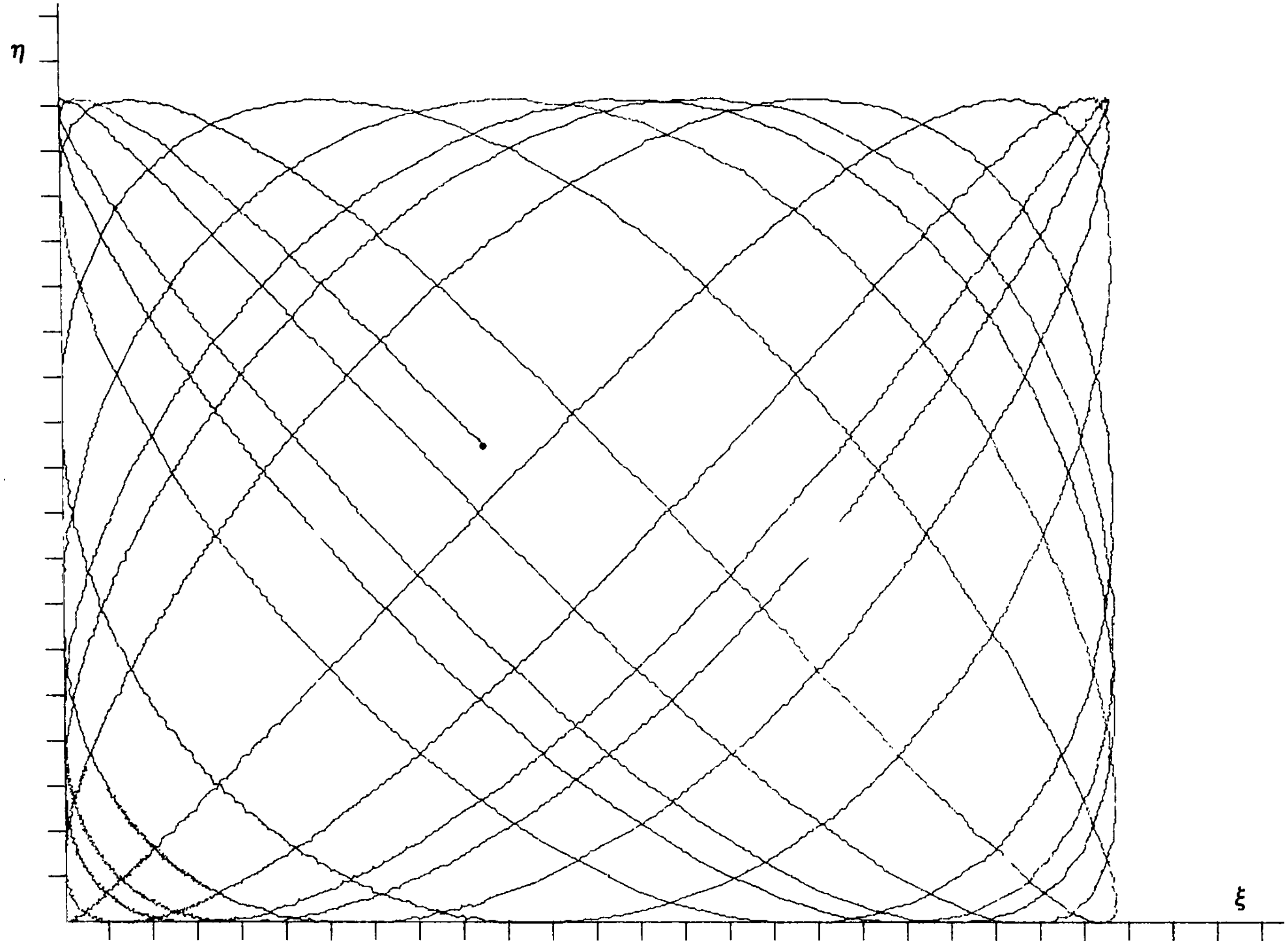

FIGURE 26. $\dot{Z}_0 = -1$, A = 0.08 PARABOLIC COORDINATES

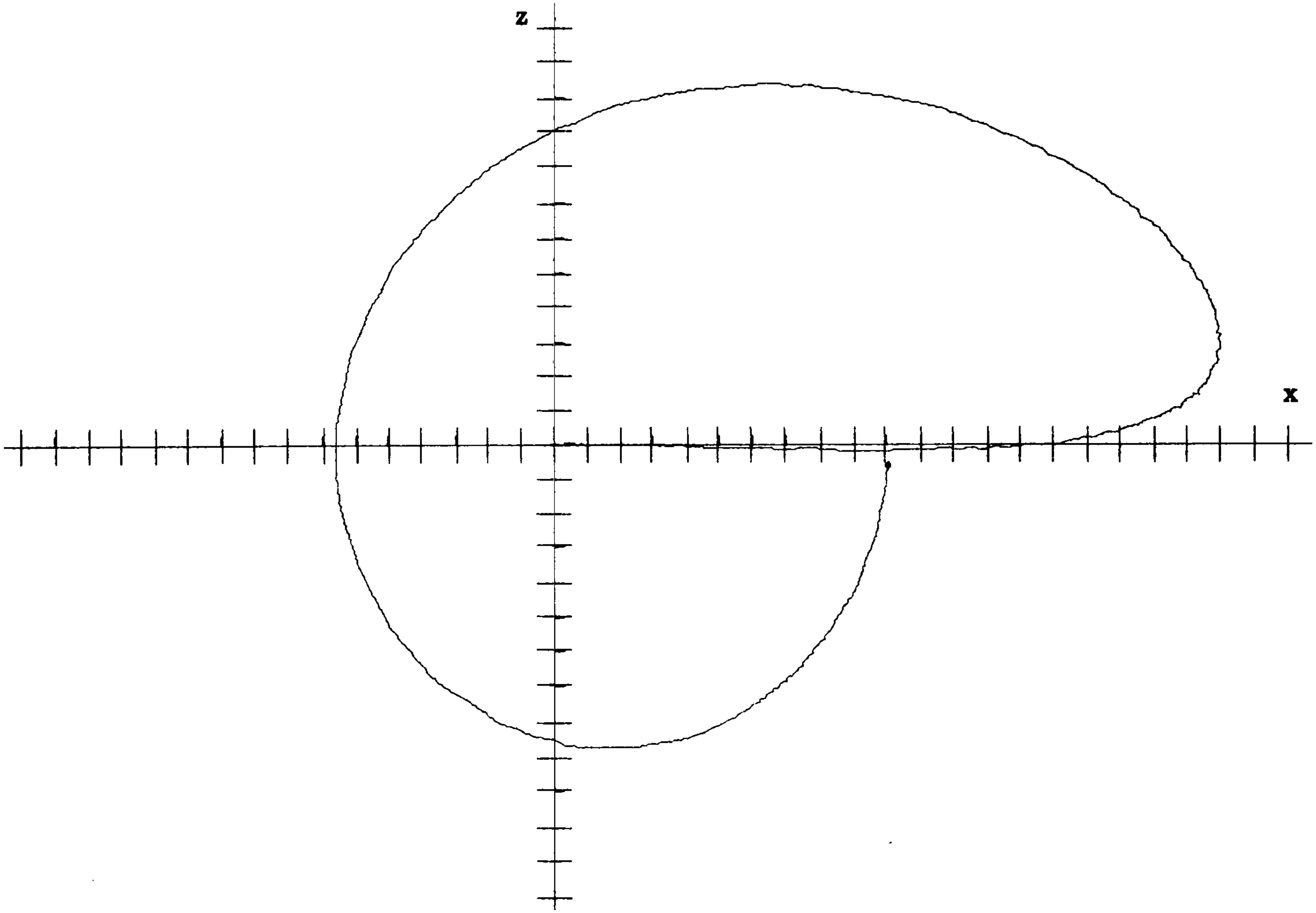

FIGURE 27. $Z_0 = -1$, $A = 0.1$ CARTESIAN COORDINATES

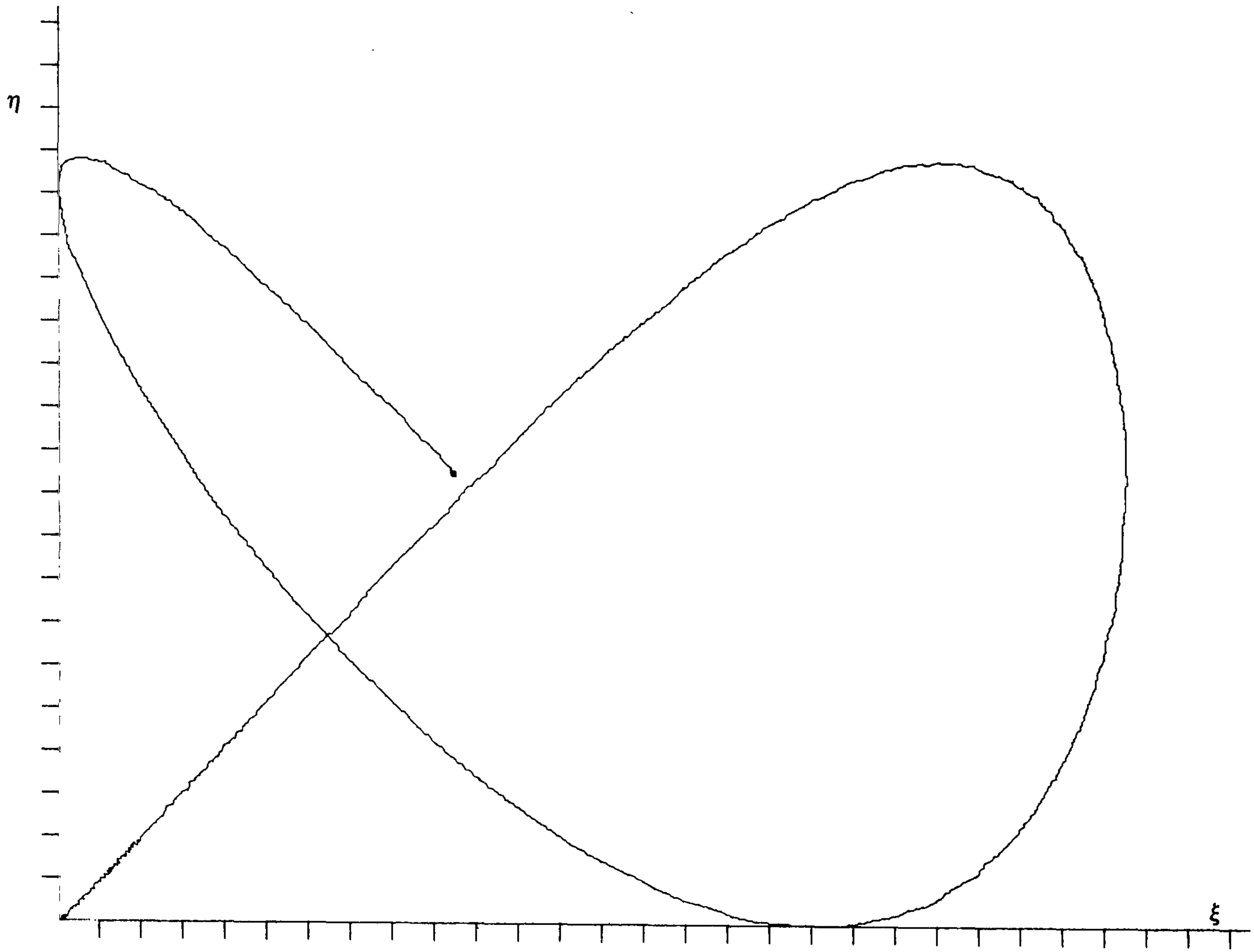

FIGURE 28. $\dot{Z}_0 = 1$, A = 0.01 PARABOLIC COORDINATES

CONCLUSIONS

It has been shown that orbital motion which involves combined Keplerian and planar force fields can be expressed analytically. The results are given in the form of quadratures for general three-dimensional motion.

For the special case of two-dimensional motion and initially circular orbits these integrals are evaluated in terms of elliptic functions. Numerical results for these conditions are presented graphically. The varied results which are due to variations in the ratio of field strengths indicate the generalized form of the analytical results.

The analytical results derived above were not used to generate the graphical results which are presented, due to the inadequacy of published numerical tables in the area of elliptic functions. However, the treatment of unusual behavior such as collisions and long term stability can be approached much more easily via the analytic solution.

The above results are not necessarily directly applicable from the physical point of view. However, they do form a basis for further work in perturbation theory and for cases in which the combination of fields which is assumed in this report is a sufficiently accurate model.

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama, March 19, 1968
125-17-05-00-62

APPENDIX

FURTHER CONSIDERATIONS OF SEPARATION CONSTANT β

Suppose that we consider the unperturbed two-body problem. Let the position be described by a radius vector $\vec{r}$. The equations of motion for this problem are

$$\ddot{\vec{r}} = -\mu \frac{\vec{r}}{r^3} \tag{A-1}$$

so that

$$\vec{r} \times \ddot{\vec{r}} = 0$$

and

$$\vec{r} \times \dot{\vec{r}} = \vec{L} \tag{A-2}$$

where $\vec{L}$ is a vectorial integration constant.

Consider, now, the result,

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{r^2 \dot{\vec{r}} - \vec{r}\left(\vec{r} \cdot \dot{\vec{r}}\right)}{r^3} = \frac{\left(\vec{r} \cdot \vec{r}\right)\dot{\vec{r}} - \vec{r}\left(\vec{r} \cdot \dot{\vec{r}}\right)}{r^3} = \frac{\vec{r} \times \left(\dot{\vec{r}} \cdot \vec{r}\right)}{r^3} . \tag{A-3}$$

Combining equations (A-2) and (A-3) gives

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{-\vec{r} \times \vec{L}}{r^3} . \tag{A-4}$$

Similarly, we compute

$$\frac{d}{dt}\,(\dot{\vec{r}} \times \vec{L}) = \ddot{\vec{r}} \times \vec{L} = -\mu\,\frac{\vec{r} \times \vec{L}}{r^3}. \tag{A-5}$$

Multiplying equation (A-4) by μ and subtracting the result from (A-5) gives

$$\mu\,\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{d}{dt}\,(\dot{\vec{r}} \times \vec{L})$$

yielding

$$\dot{\vec{r}} \times \vec{L} = \mu\left(\frac{\vec{r}}{r}\right) + \vec{W} \tag{A-6}$$

where $\vec{W}$ is a second vectorial integration constant.

It is instructive to identify $\vec{W}$ at this time. From (A-6)

$$\vec{r} \cdot (\dot{\vec{r}} \times \vec{L}) = \mu\, r + \vec{r} \cdot \vec{W}.$$

But

$$\vec{r} \cdot (\dot{\vec{r}} \times \vec{L}) = \vec{L}\,(\vec{r} \times \dot{\vec{r}}) = \vec{L} \cdot \vec{L} = L^2$$

so that

$$\mu\, r + \vec{r} \cdot \vec{W} = L^2. \tag{A-7}$$

Let ψ be the angle between the vectors $\vec{r}$ and $\vec{W}$ so that

$$L^2 = r\,(\mu + w\cos\psi)$$

which gives

$$r = \frac{\dfrac{L^2}{\mu}}{1 + \dfrac{w}{\mu}\cos\psi}\,. \tag{A-8}$$

The expression for the radius in the two-body problem described by spherical coordinate angles ϕ and θ is [2]

$$r = \frac{a(1 - e^2)}{1 + e \cos\theta \cos\phi} \quad . \tag{A-9}$$

Comparing equations (A-8) and (A-9) shows that

$$w = \mu e \tag{A-10}$$

where e is the eccentricity of the orbit. In the standard choice of coordinates where perigee corresponds to $\psi = 0$ we find that $\vec{W}$ is a vector of magnitude μe pointing toward perigee.

Equation (A-6) now yields the z component of the vector $\vec{W}$ which, in parabolic coordinates, ξ, η, ϕ becomes

$$(\vec{W})_z = \frac{2\xi\eta}{m^2(\xi+\eta)} \left(p_\eta^2 - p_\xi^2\right) + \frac{(\xi-\eta)\, p_\phi^2}{2m^2\xi\eta} - L^2 \frac{\xi - \eta}{\xi + \eta} \quad . \tag{A-11}$$

The right side of equation (A-11) can be seen to differ from the portion of equation (51) which is in curly brackets only by a factor of m^2. (Notice that $l = L$ if $A = 0$.) The identification of β is then complete. It is a modified form of one component of Laplace's integration constant just as E and l were, respectively, modified forms of the energy and angular momentum.

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