Technical Report 68-62   February 1968

LOCAL MAPPING RELATIONS
AND
GLOBAL IMPLICIT FUNCTION THEOREMS

by

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This work was in part supported by the National Aeronautics and Space Administration under Grant NsG 398 and by the U.S. Army Research Office (Durham) under Grant DA-AROD31-124-G676.

Abstract

This report is concerned with the problem of determining global existence theorems for solutions y of operator equations $F(x,y) = z$ for fixed x, z once local solvability conditions are known. The problem is studied in the following abstract setting: Given a relation Φ between two topological spaces such that Φ behaves locally like a continuous mapping, determine information about the "global" behavior of Φ. The central condition assumed for Φ is a general continuation property derived from the so-called continuation method used in the solvability theory of operator equations. The abstract theory when applied to equations $Fy = x$, leads to a number of new global existence results for such equations including, as a special case, the wellknown Hadamard-Levy theorem and one of its recent generalizations. For the case of implicit equations $F(x,y) = 0$ in Banachspaces, the theory covers and extends the results obtained by Ficken for the continuation method.

LOCAL MAPPING RELATIONS AND GLOBAL IMPLICIT FUNCTION THEOREMS[1)]

by Werner C. Rheinboldt[2)]

1. Introduction

Consider the problem of solving a nonlinear equation

(1) $$F(x,y) = z$$

with respect to y for given x and z. If, for example, x,y,z are elements of some Banachspaces, then under appropriate conditions about F, the wellknown implicit function theorem assures the "local" solvability of (1). The question arises when such a local result leads to "global" existence theorems for (1). Several authors, as, for example, Cesari [1], Ehrmann [4], Hildebrandt and Graves [7], and Levy [8], have already obtained results along this line. But these results were all closely tied to the classical implicit function theorem, and no general theory appears to exist which permits the deduction of global solvability results once a local one is known.

The problem has considerable interest in numerical analysis. In fact, when an operator equation

(2) $$Fy = x$$

is to be solved iteratively for y, the process converges usually

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1) This work was in part supported by the National Aeronautics and Space Administration under Grant NsG 398 and by the U.S. Army Research Office (Durham) under Grant DA-AROD31-124-G676.
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only in a neighborhood of a solution. If for some other operator F_0 the equation $F_0 y = x$ has been solved, then one may try to "connect" F and F_0 by an operator homotopy $H(t,y)$ such that $H(0,y) \equiv F_0 y$ and $H(1,y) \equiv Fy$, and to "move" along the solution "curve" $y(t)$, $0 \leq t \leq 1$ of $H(t,y) = x$ from the known solution $y(0)$ to the unknown $y(1)$. This can be done, for instance, with the help of a locally convergent iterative process by solving $H(t_k,y) = x$ with $y(t_{k-1})$ as initial approximation, where $0 = t_0 < t, <\ldots< t_m = 1$ is a suitable partition of the interval $J = [0,1]$.

This is the so-called "continuation method". For a review of earlier work about this method see, for example, the introduction of Ficken [5] who then proceeds to develop certain results about the global solvability of $H(t,y) = x$ by using the local solvability provided by the implicit function theorem. Other attacks on this problem are due to Davidenko [2]; see also Yakovlev [12] and for some recent results, Davis [3] and Meyer [9].

In this paper we shall consider the problem of finding global existence theorems for (1) in the following setting. For fixed z the problem depends only on the (multivalued) relation between x and y given by (1). We therefore consider abstract relations Φ between elements of a topological space X and another such space Y. The assumed validity of a local implicit function theorem for (1) leads to the condition that Φ behaves locally like a continuous mapping. The problem then is to obtain from this "local knowledge" information about the global behavior of Φ. The central concept is a general "continuation property" which in turn is equivalent to a so-called "path-lifting property". The latter assures that for a continuous path p in X a path q in Y can be found such that p and q are point-wise related under Φ. The resulting theory resembles somewhat (and was in part influenced by) the theory of covering mappings in algebraic topology.

After introducing some notations in Section 2, Section 3 presents the main theoretical results; then, in Section 4, these results are applied to the case of the equation (2); and in Section 5, to that of (1). The results of Section 4 contain a number of new global existence theorems for (2) and also include the wellknown Hadamard-Levy theorem (see [6] and [8]) as well as a recent generalization by Meyer [9]. Section 5 essentially covers and extends the results of Ficken [5].

Although some of the specific applications in Sections 4 and 5 use the implicit function theorem, the main results do not depend on the way the local invertibility is assured, and hence, corresponding results based on other local solvability theorems can conceivably be phrased.

At this point, I would like to express my heartfelt thanks to my good friend and colleague Professor James M. Ortega of the University of Maryland for his invaluable critical comments and many helpful discussions during the preparation of this paper.

2. Notations

Unless otherwise noted, X and Y shall always denote Hausdorff topological spaces. As usual, a path in the set $Q \subset X$ shall be a continuous mapping $p\colon J = [0,1] \subset R^1 \to Q$, and two paths p and q in Q are called equal, in symbols $p \cong q$, if there exists a "parameter transformation" τ, i.e., a continuous, strongly isotone mapping $\tau\colon J \to J$ with $\tau(0) = 0$, $\tau(1) = 1$, such that $p(\tau(t)) = q(t)$ for $t \in J$. For any path p in Q the reverse path $\bar{p}$ is defined by $\bar{p}(t) = p(1-t)$, $t \in J$, and for any α, β with $0 \leq \alpha < \beta \leq 1$ the path q given by $q(t) = p(\alpha+t(\beta-\alpha))$, $t \in J$, is said to be a segment of p. Finally, if p and q are paths in Q with $p(1) = q(0)$, then the concatenation $\widehat{pq}$ is defined by $\widehat{pq}(t) = p(2t)$ for $0 \leq t \leq 1/2$ and $\widehat{pq}(t) = q(2t-1)$ for $1/2 \leq t \leq 1$.

The set of all possible paths in a set $Q \subset X$ under the above equality definition will be denoted by $\mathcal{P}(Q)$. A subset $\mathcal{P}_0 \subset \mathcal{P}(Q)$ is called admissible if it is closed under reversal, segmentation and finite concatenation of paths. For example, the set of all piecewise continuously differentiable paths in a Banachspace X is admissible. For any $\mathcal{P}_0 \subset \mathcal{P}(Q)$ a set $Q \subset X$ is $\mathcal{P}_0$-path-connected if any two points $x_1, x_2 \in Q$ can be connected by a path from $\mathcal{P}_0$.

A $\mathcal{P}_0$-path-component of Q is a maximal $\mathcal{P}_0$-path-connected subset of Q. If $\mathcal{P}_0$ is admissible, then Q can be partitioned into disjoint $\mathcal{P}_0$-path-components. A set Q is said to be locally $\mathcal{P}_0$-path-connected if for each $x \in Q$ and any (relative) neighborhood $U(x)$ of x in Q there exists a (relative) neighborhood $U_0(x) \subset U(x)$ of x in Q which is $\mathcal{P}_0$-path-connected.

For given $\mathcal{P}_0 \subset \mathcal{P}(Q)$ a $\mathcal{P}_0$-path-homotopy on a set $Q \subset X$ is defined as a continuous mapping $g: J \times J \subset R^2 \to Q$ with the property that for fixed $s_0, t_0 \in J$, $g(s_0,.)$ and $g(.,t_0)$ are paths of $\mathcal{P}_0$. As before, two $\mathcal{P}_0$-path-homotopies g_1 and g_2 in Q are said to be equal if $g_1(\tau_1(s),\tau_2(s)) = g_2(s,t)$ for $s,t \in J$ and certain parameter transformations τ_1 and τ_2. We denote this again by $g_1 \cong g_2$.

Finally, given a relation $\Phi \subset X \times Y$ from X to Y, we shall use the notations $(x,y) \in \Phi$ and $x \Phi y$ interchangeably. The set $D(\Phi) = \{x \in X \mid x \Phi y \text{ for some } y \in Y\}$ is the domain of Φ and for any $Q \subset X$ we define $\Phi[Q] = \{y \in Y \mid x \Phi y \text{ for some } x \in Q \cap D(\Phi)\}$. Then, $\Phi[X] = \Phi[D(\Phi)] = R(\Phi)$ is the range of Φ. If $Q = \{x\}$ we shall also write $\Phi[x]$ instead of $\Phi[\{x\}]$. As usual, the relation $\Phi^{-1} = \{(y,x) \in Y \times X \mid (x,y) \in \Phi\}$ is called the inverse of Φ.

3. Local Mapping Relations

The following definition introduces the local solvability property; as mentioned in the introduction, this property represents an abstract version of the assumption that a "local" implicit function theorem is available for the solution of (1).

3.1 - Definition: A relation $\Phi \subset X \times Y$ is said to be a local mapping relation, or to have the local mapping property, if $D(\Phi) = X$, and if for each $(x_0, y_0) \in \Phi$ there exists an open neighborhood $U(x_0)$ of x_0 in X and a relatively open neighborhood $V(y_0)$ of y_0 in $R(\Phi)$ such that the restriction $\varphi = \Phi \cap (U(x_0) \times V(y_0))$ is a continuous mapping from $U(x_0)$ into $V(y_0)$.

In order to shorten the notation we shall call the neighborhoods $U(x_0), V(y_0)$ a pair of canonical neighborhoods of $(x_0, y_0) \in \Phi$ and the mapping $\varphi: U(x_0) \to V(y_0)$ the corresponding canonical mapping. It may be noted that the condition $D(\Phi) = X$ is not particularly essential and was mainly introduced for the sake of simplicity.

If $F: Y \to X$ is a continuous map from Y onto X, then the graph of F,

$$(3) \qquad \Gamma_F = \{(y,x) \in Y \times X \mid x = Fy\},$$

is clearly a local mapping relation in $Y \times X$, and for any $(y, Fy) \in \Gamma_F$ the sets Y and X are a pair of canonical neighborhoods.

Considerably more interesting is the question when the inverse graph Γ_F^{-1} is a local mapping relation, i.e., when the equation $Fy = x$ has locally a continuous inverse. This will be discussed in Section 4. More general is the case of the relation defined by (1) between x and y for fixed z; this will be the topic of Section 5.

Generally, we are concerned with the question when the existence of local solutions guaranteed by the local mapping property assures the existence of "global" solutions. The tool for studying this problem will be the following "continuation" concept. Given a local mapping relation $\Phi \subset X \times Y$ and some path $p \in \mathcal{P}(X)$, we are interested in finding "solutions" $y \in Y$ of $p(t) \Phi y$, $t \in J = [0,1]$. Suppose that $y_0 \in Y$ is known with $p(0) \Phi y_0$ and that φ is the canonical mapping corresponding to $(p(0), y_0)$. Then $q(t) = \varphi(p(t))$ is uniquely and continuously defined for some interval $0 \leq t < \varepsilon$. Since $q(0) = y_0$, we can call q a "continuation" of the initial solution y_0 of the problem $p(t) \Phi y$, $t \in J$ for small values of t. If q is still defined for $t_1 > 0$, then the process can be repeated with $y_1 = q(t_1)$ in place of y_0. The question now arises whether it is possible to "continue" the solution for all t in J, i.e., whether there exists a path $q \in \mathcal{P}(Y)$ such that $p(t) \Phi q(t)$ for $t \in J$ and $q(0) = y_0$. Without additional assumptions about Φ, this is, of course, not the case. But it turns out that when a path q of the described type does exist for certain paths p and for all corresponding initial solutions, then a number of interesting results can be proved about the <u>global</u> behavior of Φ. Accordingly, we introduce the following condition:

3.2 - <u>Definition</u>: A local mapping relation $\Phi \subset X \times Y$ is said to have the <u>continuation property</u> for a given set $\mathcal{P}_0 \subset \mathcal{P}(X)$ if for any $p \in \mathcal{P}_0$ and any continuous function $q: [0,\hat{t}) \subset J \to Y$ with $p(t)\ \Phi\ q(t)$ for $t \in [0,\hat{t})$ there exists a sequence $\{t_k\} \subset [0,\hat{t})$ with $\lim_{k\to\infty} t_k = \hat{t}$ such that $\lim_{k\to\infty} q(t_k) = \hat{y}$ and $p(\hat{t})\ \Phi\ \hat{y}$.

Before showing that this condition indeed allows the continuation process to be carried out until $t = 1$, it is useful to introduce the following terminology:

3.3 - <u>Definition</u>: A local mapping relation $\Phi \subset X \times Y$ is said to have the <u>path-lifting property</u> for a set $\mathcal{P}_0 \subset \mathcal{P}(X)$ if for any $p \in \mathcal{P}_0$ and any $y_0 \in \Phi\,[p(0)]$ there exists a path $q \in \mathcal{P}(Y)$ such that $q(0) = y_0$ and $p(\tau)\ \Phi\ q(\tau(t))$ for $t \in J$ and some parameter transformation τ. We call q a lifting of p through y_0 and write $p\ \Phi\ q$.

This definition represents a simple modification of the lifting concept used in the theory of covering maps in algebraic topology. The announced result can now be phrased as follows.

3.4 - <u>Theorem</u>: Let $\Phi \subset X \times Y$ be a local mapping relation. Then Φ has the path-lifting property for $\mathcal{P}_0 \subset \mathcal{P}(X)$ if and only if Φ has the continuation property for $\mathcal{P}_0$.

<u>Proof</u>: Clearly, the path-lifting property implies the continuation property. Suppose, therefore, that Φ has the continuation property for $\mathcal{P}_0$. If $p \in \mathcal{P}_0$ and $y_0 \in Y$ with $p(0)\ \Phi\ y_0$ are given, the earlier described continuation process assures the existence of a continuous function $q: [0,\hat{t}) \subset J \to Y$ with $q(0) = y_0$, $p(t)\ \Phi\ q(t)$ for $t \in [0,\hat{t})$ and some $\hat{t} \in (0,1]$. Let $\hat{t}$ be the maximal value with this property in J. By assumption, a sequence $\{t_k\} \subset [0,\hat{t})$ exists with $\lim_{k\to\infty} t_k = \hat{t}$, $\lim_{k\to\infty} q(t_k) = \hat{y}$ and $p(\hat{t})\ \Phi\ \hat{y}$. If $U(p(\hat{t}))$ and $V(\hat{y})$ is a pair of canonical neighborhoods and φ the corresponding canonical map, then $p(t) \in U(p(\hat{t}))$ for $t \in [\hat{t}-\delta,\hat{t}]$ and $q(t_k) \in V(\hat{y})$ for large k. Hence, by construction and continuity of q it follows that $q(t) = \varphi(p(t)) \in V(\hat{y})$ for $t \in [\hat{t}-\delta,\hat{t})$ and therefore, because of the continuity of φ, $\lim_{t\to\hat{t}} q(t) = \lim_{t\to\hat{t}} \varphi(p(t)) = \varphi(p(\hat{t})) = \hat{y}$. Thus, by setting $q(\hat{t}) = \hat{y}$, q is continuously defined for $0 \le t \le \hat{t}$. If $\hat{t} < 1$, then also $p(t)$

$\in U(p(\hat{t}))$ for $t \in [\hat{t},\hat{t}+\delta')$, $\delta' > 0$, and, therefore, $q(t) = \varphi(p(t))$ is defined up to some larger t than $\hat{t}$; and this contradicts the maximality of $\hat{t}$ in J. Therefore, we must have $\hat{t} = 1$; and this proves that Φ has the path-lifting property for $\mathcal{P}_0$.

We turn now to the consequences of the path-lifting property. A first result states that a lifting of a path q is uniquely determined by the initial point y_0. This fact is hardly surprising in view of the continuation idea behind the path-lifting concept.

3.5 - <u>Theorem</u>: Let $\Phi \subset X \times Y$ be a local mapping relation with the path-lifting property for $\mathcal{P}_0 \subset \mathcal{P}(X)$. If $q_1, q_2 \in \mathcal{P}(Y)$ are two liftings of $p \in \mathcal{P}_0$ with $q_1(0) = q_2(0) = y_0$, then $q_1 \cong q_2$.

<u>Proof</u>: By assumption we have $p(t)\ \Phi\ q_i(\tau_i(t))$, $t \in J$, $i=1,2$, where τ_1, τ_2 are certain parameter transformations. Introduce the set $J_0 = \{t \in J \mid q_1(\tau_1(t)) = q_2(\tau_2(t))\}$, then $0 \in J_0$ implies that $\hat{t} = \sup\{t \mid t \in J_0\}$ is well-defined and by continuity we have $\hat{t} \in J_0$. Suppose that $\hat{t} < 1$, and let $U(\hat{x})$, $V(\hat{y}_1)$ be a pair of canonical neighborhoods of $\hat{x} = p(\hat{t})$, $\hat{y}_1 = q_1(\tau_1(\hat{t}))$ with the corresponding canonical map φ. Then $p(t) \in V(\hat{x})$ for $t \in [\hat{t},\hat{t}+\delta]$ with $\delta > 0$ and hence $q_1(\tau_1(t)) = \varphi(p(t)) = q_2(\tau_2(t))$

for these t. This contradicts the construction of $\hat{t}$ and hence we have $\hat{t} = 1$ and therefore $q_1 \cong q_2$.

As a first application we obtain the following result:

3.6 - Theorem: Let $\Phi \subset X \times Y$ be a local mapping relation with the path-lifting property for the admissible set $\mathcal{P}_0 \subset \mathcal{P}(X)$. Assume further that X is locally $\mathcal{P}_0$-path-connected and let $X = \bigcup_{\mu \in M} X_\mu$ be the decomposition of X into its $\mathcal{P}_0$-path-components. Then, each one of the partial relations $\Phi_\mu = \Phi \cap (X_\mu \times Y)$ is a local mapping relation with the path-lifting property for $\mathcal{P}_\mu = \mathcal{P}_0 \cap \mathcal{P}(X_\mu)$. Moreover, for fixed μ the cardinality of the set $\Phi_\mu[x]$ is independent of the choice of $x \in X_\mu$.

Proof: In order to show that $D(\Phi_\mu) = X_\mu$ let $(x_0, y_0) \in \Phi_\mu$ and consider any point $x \in X_\mu$. Then there exists a path $p \in \mathcal{P}_\mu$ connecting x_0 and x and hence also a path $q \in \mathcal{P}(Y)$ with $q(0) = y_0$ and $p \Phi q$. Hence, $x \Phi q(1)$ and therefore $x \in D(\Phi_\mu)$. Let now $U(x_0), V(y_0)$ be a pair of canonical neighborhoods under Φ and φ the corresponding canonical map. Then there exists an open $\mathcal{P}_0$-path-connected neighborhood $U_0(x_0) \subset U(x_0)$ of x_0 in X. But then $U_0(x_0) \subset X_\mu$ is open in X_μ and $U_0(x_0), V(y_0)$ constitutes a pair of canonical neighborhoods of $(x_0, y_0) \in \Phi_\mu$ with the restriction $\varphi \mid U_0(x_0)$ of φ as corresponding canonical map. Thus,

Φ_μ is again a local mapping relation and clearly Φ_μ inherits from Φ the path-lifting property.

To prove the last part of the statement, let $x_0, x_1, \in X_\mu$ be any two points. Then there is a path $p \in \mathcal{P}_\mu$ connecting them and hence for arbitrary $y_0 \in \Phi_\mu[x_0]$ there exists a unique lifting $q \in \mathcal{P}(Y)$ of p through y_0. Thus, $q(1) \in \Phi_\mu[x_1]$ and by Theorem <u>3.5</u> the correspondence $y_0 \to q(1)$ is for fixed p a mapping π from $\Phi_\mu[x_0]$ into $\Phi_\mu[x_1]$. This mapping is injective; in fact, if $q_i \in \mathcal{P}(Y)$, $i = 1,2$ are liftings of p with $q_1(1) = q_2(1)$, then the reverse paths $\bar{q}_i$ are liftings of the reverse path $\bar{p}$ and hence by <u>3.5</u> it follows that $\bar{q}_1 \cong \bar{q}_2$ and therefore $q_1(0) = q_2(0)$. If the reverse path $\bar{p}$ is used to define in the same manner a mapping $\bar{\pi}$ from $\Phi_\mu[x_1]$ into $\Phi_\mu[x_0]$, <u>3.5</u> implies again that $\bar{\pi} = \pi^{-1}$. Hence, altogether, π is bijective and $\Phi_\mu[x_0]$ and $\Phi_\mu[x_1]$ have the same cardinality. This completes the proof.

One of the central results of our theory is the fact that local mapping relations with the path-lifting property for $\mathcal{P}_0$ also lift $\mathcal{P}_0$-path homotopies.

The proof of the following theorem is an adaptation of a proof by Schubert [10] for covering mappings.

3.7 - <u>Theorem</u>: Let $\Phi \subset X \times Y$ be a local mapping relation with the path-lifting property for $\mathcal{P}_0 \subset \mathcal{P}(X)$. For a $\mathcal{P}_0$-path homotopy h in X and a path $q^0 \in \mathcal{P}(Y)$ with $g(.,0)\ \Phi\ q^0$ there exists a

$\mathcal{P}(Y)$-path homotopy h in Y and parameter transformations τ_1, τ_2 such that $g(s,t) \Phi h(\tau_1(s), \tau_2(t))$, $s,t \in J$, and $h(.,0) \cong q^0$. Moreover, if $\hat{h}$ is any other $\mathcal{P}(Y)$-path homotopy in Y with the same properties, then $h \cong \hat{h}$.

Proof: For fixed $s \in J$, $p_s(t) = g(s,t)$, $t \in J$ defines a path of $\mathcal{P}_0$ and hence there exists a unique path $q_s \in \mathcal{P}(Y)$ with $p_s \Phi q_s$ and $q_s(0) = q^0(s)$. Let h be any $\mathcal{P}(Y)$-path homotopy in Y with the stated properties. Then, after a suitable parameter transformation, we have $h(s,0) = q^0(s)$ for any $s \in J$ and hence by 3.5, $h(s,.) \cong q_s$, $s \in J$. This shows that h is uniquely determined up to parameter transformations. Therefore, if the mapping $h: J \times J \subset R^2 \to Y$ is defined by $h(s,t) = q_s(t)$, $s,t \in J$, then all that remains is to prove the continuity of h on $J \times J$.

Suppose g is discontinuous at $(s_0,t_1) \in J \times J$ and that t_0 is the infimum of all those $t \in J$ for which g is discontinuous at (s_0,t). Set $x_0 = g(s_0,t_0)$ and $y_0 = h(s_0,t_0)$, and let $U(x_0)$, $V(y_0)$ be a pair of canonical neighborhoods and φ the corresponding canonical map. Since h is continuous on $J \times J$, there exist open neighborhoods $J(s_0)$ and $J(t_0)$ of s_0 and t_0 in J such that $g(J(s_0),J(t_0)) \subset U(x_0)$. We distinguish now two cases:

Case 1: $t_0 = 0$. Then, $y_0 = h(s_0,0) = q^0(s_0)$ and because of the continuity of q^0 there exists an open neighborhood $J'(s_0)$ of s_0 in J such that $q^0(J'(s_0)) \subset V(x_0)$. It is no restriction to assume that $J'(s_0) = J(s_0)$. This means that $\varphi(g(s,0)) = q^0(s) = h(s,0)$ for $(s,0) \in J(s_0) \times J(0)$ and therefore $\varphi(g(s,t)) = \varphi(p_s(t)) = q_s(t) = h(s,t)$ for $(s,t) \in J(s_0) \times J(0)$. With φg also h is continuous at all points of $J(s_0) \times J(0)$ and this contradicts the construction of $t_0 = 0$.

Case 2: $t_0 > 0$. Since $p_{s_0}(.)$ is continuous in t we can choose $t' < t_0$ such that $t' \in J(t_0)$ and that $h(s_0,t') = q_{s_0}(t') \in V(y_0)$. By assumption h is continuous in both variables at (s_0,t') and hence there exists a neighborhood $J'(s_0)$ of s_0 in J, which we can assume to be equal to $J(s_0)$, such that $h(J(s_0),t') \subset V(y_0)$. But then 3.5 implies again that $\varphi(g(s,t)) = \varphi(p_s(t)) = q_s(t) = h(s,t)$ for each $s \in J(s_0)$ and all $t \in J(t_0)$. This means again that h is continuous in an entire neighborhood of (s_0,t_0) in contradiction to the construction of t_0. This completes the proof.

This theorem permits us to prove the following "global" result for local mapping relations. We call the space Y $\mathcal{P}_0$-simply-connected if it is path-connected under $\mathcal{P}_0$ and if any two paths of $\mathcal{P}_0$ with the same endpoints are $\mathcal{P}_0$-homotopic.

3.8 - Theorem: Let $\Phi \subset X \times Y$ be a local mapping relation with the path-lifting property for the admissible set $\mathcal{P}_0 \subset \mathcal{P}(X)$ and suppose that X is $\mathcal{P}_0$-simply-connected and locally $\mathcal{P}_0$-path-connected. Then there exists a family of continuous mappings $F_\mu : X \to R(\Phi)$ with $F_\mu(X) = Q_\mu$, $\mu \in M$ such that $R(\Phi) = \bigcup_{\mu \in M} Q_\mu$ and $x \, \Phi \, F_\mu x$, $\Phi[x] = \{F_\mu x\}_{\mu \in M}$ for each $x \in X$. The index set M has the same cardinality as the sets $\Phi[x]$ (see 3.6).

Proof: For given $x_0, x \in X$ we generated in the proof of 3.6 a bijection π between $\Phi[x_0]$ and $\Phi[x]$ as follows: Let p be a fixed path connecting x_0 and x and $y_0 \in \Phi[x_0]$ any point; then there exists a unique lifting q of p with initial point y_0, and $\pi(y_0) = q(1)$ defines the desired bijection from $\Phi[x_0]$ onto $\Phi[x]$. We show that when X is $\mathcal{P}_0$-simply connected, then for given $x_0, x \in X$ the bijection $\pi = \pi_{x_0 x}$ from $\Phi[x_0]$ onto $\Phi[x]$ does not depend on the choice of p. Let p' be any other path connecting x_0 with x and q' the lifting of p' with initial point $y_0 \in \Phi[x_0]$. There exists a homotopy y in x with $g(0,t) = p(t)$, $g(1,t) = p'(t)$, $(t \in J)$, $g(s,0) = x_0$, $g(s,1) = x$, $(s \in J)$. Hence, 3.7 implies the existence of a unique path homotopy h in Y for which--after a suitable parameter transformation--$g(s,t) \; \Phi \; h(s,t)$ and $h(s,0) = y_0$, $(s,t \in J)$. Therefore, it follows from 3.5 that $h(0,.) \cong q$, $h(1,.) \cong q'$ and hence that $y = q(1) = h(0,1)$ and

$y' = q'(1) = h(1,1)$. But then, $x = g(s,1)\ \Phi\ h(s,1)$ for $s \in J$, together with the continuity of $h(.,1)$, and the local mapping property of Φ necessarily implies that $y = y'$.

Now let $x_0 \in X$ be a fixed point and $\Phi[x_0] = \{y_\mu\}_{\mu \in M}$. By 3.6 the cardinality of the index set M is invariant under changes of x_0. For given $y_\mu \in \Phi[x_0]$ there exists for each $x \in X$ a unique $y = \pi(y_\mu) \in \Phi[x]$ where $\pi = \pi_{x_0 x}$ is the bijection constructed above. Let $Q_\mu = \{y \in R(\Phi) \mid y = \pi(y_\mu)\}$, then, because of the bijectivity of π, it follows immediately that every $y \in R(\Phi)$ must be contained in at least one Q_μ, i.e., that $\bigcup_{\mu \in M} Q_\mu = R(\Phi)$.

Clearly, the correspondence $x \in X \to y = \pi_{x_0 x}(y_\mu) \in Q$ defines a mapping $F_\mu : X \to R(\Phi)$ with $F_\mu(X) = Q_\mu$. In view of the properties of $\pi_{x_0 x}$ we then have $x\ \Phi\ F_\mu x$ and $\Phi[x] = \{\pi_{x_0 x}(y_\mu)\}_{\mu \in M}$. It remains to be shown that F_μ is continuous as a mapping from x into $R(\Phi)$. For this we prove first that locally F_μ coincides with the canonical mappings. Let $\hat{x} \in X$ and $\hat{y} = F_\mu \hat{x}$ be given and let $U(\hat{x}), V(\hat{y})$ be a pair of canonical neighborhoods and φ the corresponding canonical map. Since X is locally $\mathcal{P}_0$-path connected, it is no restriction to assume from the outset that $U(\hat{x})$ is $\mathcal{P}_0$-path connected. Let $x \in U(\hat{x})$ be any point and $y = \varphi(x) \in V(\hat{y})$. There exists a path $p'' \in \mathcal{P}_0$ from $\hat{x}$ to x in $U(\hat{x})$ and hence, if p' is any path from x_0 to $\hat{x}$, the concatenation $p' \frown p''$ defines a path p from

x_0 to x. Now let q' be any lifting of p' with initial point y_μ, then $q'(1) = F_\mu x = \hat{y}$ and, since $q''(t) = \varphi(p''(t))$ $(t \in J)$ defines a lifting of p'' with initial point $\hat{y}$, it follows that the concatenation $\widehat{q'q''}$ provides us with a lifting q of p with initial point y_μ. But then, by definition of F_μ, we have $F_\mu x = q(1) = y = \varphi(x)$ as stated. The continuity of F_μ is now an easy consequence. In fact, if $\tilde{V}(\hat{y})$ is any open neighborhood of $\hat{y}$ in R_Φ, then so is $V_0(\hat{y}) = V(\hat{y}) \cap \tilde{V}(\hat{y})$ and hence, by continuity of φ, $\varphi^{(-1)}(V_0(\hat{y}))$ is open. Since $F^{(-1)}(\tilde{V}(y)) \supset F^{(-1)}(V_0(\hat{y})) \supset \varphi^{(-1)}(V_0(\hat{y})) \supset \hat{x}$, this shows that F_μ is indeed continuous at $\hat{x}$.

Note that in general the Q_μ are not disjoint as the simple example shows: $X = [0,2] \subset R^1$, $Y = [0,3] \subset R^1$ and $\Phi[x] = \{2-x, 3-x\}$, $0 \leq x \leq 2$. Here we have $M = \{1,2\}$, $Q_1 = [0,2]$, $Q_2 = [1,3]$ and $F_1 x = 2-x$, $F_2 y = 3-x$.

There is one important case when the Q_μ are disjoint and in fact when they are the $\mathcal{P}_0$-path components of Y. We shall discuss this case in the next Section.

We conclude this Section with two somewhat different results. The first provides a sufficient condition for a relation to be a local mapping relation with the path-lifting property, and the second uses the path-lifting property to determine the domain of a local mapping relation.

The following definition represents an extension of the definition of a covering map; accordingly a corresponding name is used.

3.9 - Definition: A relation $\Phi \subset X \times Y$ with $D(\Phi) = X$ is called a covering relation if for each $x \in X$ there exists an open neighborhood $U(x)$ such that $\Phi[U(x)] = \bigcup_{\mu \in M} V_\mu$ where the V_μ are disjoint (relatively) open sets in $R(\Phi)$ and for each $\mu \in M$ the restriction $\varphi_\mu = \Phi \cap (U(y) \times V_\mu)$ is a continuous mapping from $U(x)$ into V_μ. $U(x)$ is called an admissible neighborhood of x.

3.10 - Theorem: A covering relation $\Phi \subset X \times Y$ is a local mapping relation with the path-lifting property for $\mathcal{P}(X)$.

Proof: Let $(x_0, y_0) \in \Phi$, $U(x_0)$ an admissible neighborhood of x_0 and $\Phi[U(x_0)] = \bigcup_{\mu \in M} V_\mu$, where the V_μ are as stated in 3.9. Then, $y_0 \in V_{\mu_0}$ for exactly one $\mu_0 \in M$ and $U(y_0), V_\mu$ is evidently a pair of canonical neighborhoods of $(x_0, y_0) \in \Phi$, i.e., Φ is a local mapping relation.

Let now $p \in \mathcal{P}(X)$ be any path. Since $p(J)$ is compact, there exists a partition $0 = t_0 < t_1 < \ldots < t_{k+1} = 1$ such that $p(t) \in U_i$ for $t_i \leq t \leq t_{i+1}$ where $U_i = U(p(t_i))$, $i = 0,1,\ldots,k$ are admissible neighborhoods of $p(t_i)$. For given $y_0 \in \Phi[p(0)]$ we use induction to construct the lifting q of p through y_0. Suppose q has already been defined for $0 \leq t \leq t_i$, $i \geq 0$. Then, $q(t_i) \in \Phi[p(t_i)]$

$\subset \Phi[U_i] = \bigcup_{\mu \in M} V_\mu$ and $q(t_i)$ belongs to exactly one V_{μ_0}. Since $\varphi_\mu = \Phi \cap (U_i \times V_{\mu_0})$ is a continuous map from U_i into V_{μ_0} and since $p(t) \in U_i$ for $t_i \leq t \leq t_{i+1}$, it follows that $q(t) = \varphi_\mu(p(t))$ $(t_i \leq t \leq t_{i+1})$ represents a well-defined continuous extension of q, i.e., q is now defined for $0 \leq t \leq t_{i+1}$. This completes the induction, and, therefore, Φ has the path-lifting property.

3.11 - Theorem: Let $\Phi \subset Y_0 \times X$ be a local mapping relation on $X_0 \times Y$, where $X_0 \subset X$ is open in X, and X is $\mathcal{P}_0$-path connected for a set $\mathcal{P}_0 \subset \mathcal{P}(X)$ which is closed under segmentation. Suppose further that Φ has the continuation property for $\mathcal{P}_0 \cap \mathcal{P}(X_0)$. Then $X_0 = X$.

Proof: Let $(x_0, y_0) \in \Phi$, $x \in X$ be any point, and $p \in \mathcal{P}_0$ a path between x_0 and x. Then, $J_0 = \{t \in J \mid p(t) \in X_0\}$ is not empty and since X_0 is open in X, J_0 is open in J. If J_0 is a proper subset of J, there exists a $\hat{t} \in J$ such that $[0,\hat{t}) \subset J_0$ but $\hat{t} \notin J_0$. For any $t' \in [0,\hat{t})$ the closed segment of p from $p(0)$ to $p(t')$ is a path p' in $\mathcal{P}_0 \cap \mathcal{P}(X_0)$ and hence p' can be lifted into Y. Since t' was arbitrary, this implies the existence of a continuous function $q\colon [0,\hat{t}) \subset J \to Y$ with $p(t) \,\Phi\, q(t)$ for $0 \leq t < \hat{t}$. Therefore, by the continuation property, there exists a sequence $\{t_k\} \subset [0,\hat{t})$ such that $\lim_{k\to\infty} t_k = \hat{t}$, $\lim_{k\to\infty} q(t_k) = \hat{y}$

and $p(\hat{t}) \Phi \hat{y}$. But this implies that $\hat{t} \in J_0$, against assumption, and hence we have $J_0 = J$, and therefore $x \in X_0$, i.e., $X_0 = X$.

4. Applications to Equations $Fy = x$

Throughout this section we consider a map $F: Y \to X$ from Y onto X and the corresponding inverse graph

$$\Gamma_F^{-1} = \{(x,y) \in X \times Y \mid x = Fy\}$$

The following theorem provides an answer to the question when Γ_F^{-1} is a local mapping relation.

4.1 - Theorem: Given a mapping $F: Y \to X$ from Y onto X, the inverse graph Γ_F^{-1} is a local mapping relation if and only if F is open and locally one-to-one.

Proof: Let Γ_F^{-1} be a local mapping relation and consider for some $(x_0,y_0) \in \Gamma_F^{-1}$ a pair of canonical neighborhoods $U(x_0), V(y_0)$ and the corresponding canonical map φ. Then for any $x \in U(x_0)$ there is exactly one $y \in V(y_0)$ such that $Fy = x$, i.e., $F|V(y_0)$ is injective. If $Q \subset Y$ is open and $y_0 \in Q$, then also $V_0(y_0) = V(y_0) \cap Q$ is open in Y and $U_0(x_0) = \varphi^{-1}(V_0(y_0)) \subset U(x_0)$ is open in $U(x_0)$ and thus in X. Therefore, $F(V_0(y_0)) = (F|V_0(y_0))V_0(y_0) = (F|V_0(y_0))\varphi(U_0(x_0)) = U_0(x_0)$ and hence $F(Q)$ is open in X.

Conversely, let F be open and locally one-to-one. Then, for any $(y_0,x_0) \in \Gamma_F^{-1}$ there exists an open neighborhood $V(y_0)$ of y_0 in Y such that $F|V(y_0)$ is one-to-one. Hence, $FV(y_0) = U(x_0)$

with $x_0 = Fy_0$ is open in X and $(F|V(y_0))^{-1}$ is an injective and continuous mapping from $U(x_0)$ onto $V(y_0)$.

For the following we shall assume that F not only satisfies the conditions of this theorem but is even a local homeomorphism in the usual sense, i.e., that for each $y \in Y$ there exist open neighborhoods $U(Fy)$, $V(y)$ of Fy and y in X and Y, respectively, such that $F|V(y)$ is a homeomorphism from $V(y)$ onto $U(Fy)$. For local homeomorphisms, theorem 3.8 can be strengthened; more specifically, it turns out that in this case the sets Q_μ occurring in that theorem are the path components of Y.

4.2 - Theorem: Let $F: Y \to X$ be a local homeomorphism from Y onto X and $\mathcal{P}_Y \subset \mathcal{P}(Y)$, $\mathcal{P}_X \subset \mathcal{P}(X)$ admissible sets such that $F\mathcal{P}_Y \subset \mathcal{P}_X$ and that X is $\mathcal{P}_X$-simply connected and locally $\mathcal{P}_X$-path connected. Suppose that $\Phi \equiv \Gamma_F^{-1}$ has the path-lifting property for $\mathcal{P}_X$ and that every Φ-lifting of a path from $\mathcal{P}_X$ is contained in $\mathcal{P}_Y$. Then, for each $\mathcal{P}_Y$-path component Y_μ of Y, $F|Y_\mu$ is a homeomorphism from Y_μ onto X.

Proof: In comparison to 3.8 this theorem states mainly that $Q_\mu = Y_\mu$. Yet it is easier to prove 4.2 directly without reference to 3.8.

Given a $\mathcal{P}_Y$-path component Y_μ let $y_0 \in Y_\mu$ and $x_0 = Fy_0$ be fixed, and $x \in X$ an arbitrary point. If $p \in \mathcal{P}_X$ is a path connecting x_0 and x and $q \in \mathcal{P}(Y)$ a Φ-lifting of p with $q(0) = y_0$, then $q \in \mathcal{P}_Y$ and hence necessarily $q \in \mathcal{P}_Y \cap \mathcal{P}(Y_\mu)$. Therefore, $Fq(1) = p(1) = x$ and $q(1) \in Y_\mu$ implies that $x \in FY_\mu$ and thus that $FY_\mu = X$. In order to show that $F|Y_\mu$ is injective, suppose that $Fy' = Fy'' = x$ for $y' \neq y''$ in Y_μ and some $x \in X$. Then, there are paths $q', q'' \in \mathcal{P}_Y \cap \mathcal{P}(Y)$ connecting y_0 with y', and y_0 with y'' respectively, and hence $p'(t) = Fq'(t), p''(t) = Fq''(t)$, $t \in J$ defines two paths $p', p'' \in F\mathcal{P}_Y \subset \mathcal{P}_X$, both of which connect x_0 and x. Therefore, there exists a $\mathcal{P}_X$-path homotopy g in X with $g(0,t) = p'(t), g(1,t) = p''(t)$, $g(s,0) = x_0, g(s,1) = x$, $(t,s \in J)$ and 3.7 implies the existence of a path homotopy h in Y for which--after suitable parameter transformations--$Fh(s,t) = g(s,t), h(s,0) = y_0$, $s,t \in J$. Hence, $h(0,.) \cong q', h(1,.) \cong q''$ and thus $h(0,1) = x' \neq x'' = h(1,1)$, while on the other hand $Fh(.,1) = x$. This contradicts the local homeomorphism property of F, and thus $F|Y_\mu$ is injective. Finally, to show that $(F|Y_\mu)^{-1}$ is continuous, consider $y \in Y_\mu$ and $x = Fy$ and a pair of canonical neighborhoods $U(x), V(y)$ of Φ. Then, there exists a $\mathcal{P}_X$-path-connected neighborhood $U_0(x) \subset U(x)$, i.e., $V_0(y) = (F|U(y))^{-1}U_0(x) \subset V(y)$ is $\mathcal{P}_Y$-path connected and hence contained in Y_μ. Thus, $F|Y_\mu$ is a local homeomorphism from Y_μ

onto X and hence $F|Y_\mu$ is open. This completes the proof.

In many applications the following corollary will be useful:

4.3 - Corollary: Let $F: Y \to X$ be a local homeomorphism and suppose that Y is $\mathcal{P}(Y)$-path connected and X is $\mathcal{P}(X)$-simply connected and locally $\mathcal{P}(X)$-path connected. Then F is a homeomorphism from Y onto X if and only if Γ_F^{-1} has the $\mathcal{P}(X)$-continuation property.

The necessity of the continuation property added in this result is self-evident, and the sufficiency follows directly from 4.2 since $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ satisfy the conditions placed there on $\mathcal{P}_X$ and $\mathcal{P}_Y$, respectively.

For the case of mappings between normed linear spaces we can combine this result with Theorem 3.11 and obtain the following interesting result.

4.4 - Theorem: Let $F: D \subset W \to V$ be a local homeomorphism from an open, $\mathcal{P}(W)$-path-connected subset D of the normed linear space W into another such space V. Then F is a homeomorphism from D onto V if and only if Γ_F^{-1} has the $\mathcal{P}(F(D))$-continuation property.

The necessity is again evident. For the proof of the sufficiency set $Y = D$ and $X_0 = F(D)$. Then, X_0 is open and $\mathcal{P}(X)$-path connected in $X = V$. Because of the $\mathcal{P}(X_0)$-continuation property of Γ_F^{-1}, 3.11 therefore implies that $X_0 = X$, and thus the remainder of the statement is a direct consequence of 4.3.

The continuation property is an operational condition that may be difficult to verify in concrete situations. For compact mappings of a normed linear space into itself there exists a fairly simple condition which assures the continuation property for Γ_F^{-1}. This condition is a modification of the so-called coerciveness condition used in the theory of monotone mappings.

4.5 - Definition: A mapping $F: D \subset V \to V$ with domain and range in a normed linear space V is called norm-coercive if for any $\gamma \geq 0$ there exists a closed, bounded set $D_\gamma \subset D$ such that $\|Fx\| > \gamma$ for all $x \in D \sim D_\gamma$.

Note that for $D = V$, F is norm-coercive if and only if $\|Fx\| \to \infty$ as $\|x\| \to \infty$.

4.6 - Theorem: Suppose that $G: D \subset V \to V$ is sequentially compact on the open, $\mathcal{P}(V)$-path-connected set D in the normed linear space V, and that $F = I - G$ is a norm-coercive local homeomorphism. Then, F is a homeomorphism from D onto V.

Proof: In view of 4.4 it suffices to show that Γ_F^{-1} has the $\mathcal{P}(F(D))$-continuation property. Let $p \in \mathcal{P}(F(D))$ be given and

$q\colon [0,\hat{t}) \subset J \to D$ a continuous function with $Fq(t) = p(t)$ for $0 \leq t < \hat{t}$. Since $p(J)$ is compact, $\gamma = \max_{t\in J} p(t)$ exists and by norm-coerciveness there is a closed bounded set $D_\gamma \subset D$ such that $\|Fx\| > \gamma$ for all $x \in D \sim D_\gamma$. But then necessarily $q(t) \in D_\gamma$ for $t \in [0,\hat{t})$. Hence if $\{t_k\} \subset [0,\hat{t})$ is any sequence with $\lim_{k\to\infty} t_k = \hat{t}$, then $\{Gq(t_k)\} \subset GD_\gamma$ has a convergent subsequence. Suppose this subsequence is again denoted by $\{Gq(t_k)\}$ and that $\lim_{k\to\infty} Gq(t_k) = \hat{y}'$. If $\lim_{k\to\infty} Fq(t_k) = \lim_{k\to\infty} p(t_k) = p(\hat{t}) = \hat{y}$, then

$$\begin{aligned}\|q(t_k) - (\hat{y} + \hat{y}')\| &< \|q(t_k) - Gq(t_k) - \hat{y}\| + \|Gq(t_k) - \hat{y}'\| \\ &= \|Fq(t_k) - \hat{y}\| + \|Gq(t_k) - \hat{y}'\|\end{aligned}$$

and hence $\lim_{k\to\infty} q(t_k) = \hat{y} + \hat{y}'$. Because $\{q(t_k)\} \subset D_\gamma$ and D_γ is closed, we have $\hat{y} + \hat{y}' \in D_\gamma \subset D$ and hence by continuity $F(y + y') = p(\hat{t})$ which proves that Γ_F^{-1} indeed has the $\mathcal{P}(F(D))$ continuation property.

For finite dimensional V this result can, of course, be simplified considerably.

4.7 - <u>Theorem</u>: Let V be a finite dimensional normed linear space and $F\colon D \subset V \to V$ a norm-coercive local homeomorphism on the open, $\mathcal{P}(V)$-path-connected set $D \subset V$. Then, F is a homeomorphism from D onto V.

The proof is a simple adaptation of that of <u>4.6</u>. In private discussions, J. Ortega noted that in this case the norm-coerciveness is also necessary. In fact, if $F: D \subset V \to V$ is a homeomorphism from D onto V and $S = \bar{S}(0,\gamma) = \{x \in V \mid \|x\| \leq \gamma\}$ for some $\gamma \geq 0$, then $D_\gamma = F^{-1}S$ is closed and bounded, and $y \in V \sim D_\gamma$ implies that $Fy \in S$, or $\|Fy\| > \gamma$, i.e., F is norm-coercive.

The local homeomorphism condition in the last theorems can, of course, be replaced by assumptions guaranteeing the validity of some local inverse function theorem. For instance, let us consider the following version of a wellknown theorem of Hildebrandt and Graves [7]:

For the mapping $F: D \subset W \to V$ between the Banachspaces W,V suppose there exists a linear operator $A: W \to V$ with bounded inverse $A^{-1} \in L(V,W)$, such that

$$\|Fy_2 - Fy_1 - A(y_2-y_1)\| \leq \alpha\|y_2-y_1\| < \|A^{-1}\|^{-1}\|y_2-y_1\| \tag{2}$$

for all y_1, y_2 from some ball $\bar{S}(y_0,\gamma) \subset D$. Then, for any $x \in \bar{S}(Fy_0,\sigma)$, where $\sigma = (\|A^{-1}\|^{-1} - \alpha)\,\gamma$, the equation $Fy = x$ has a solution $y_x \in \bar{S}(y_0,\gamma)$ which is unique in that ball and which varies continuously with x.

The proof is wellknown.

In order to apply our previous theorems we need to establish conditions which assure the continuation property for Γ_F^{-1}. A rather simple approach along this line leads to the following theorem essentially due to Ehrmann [4]:

4.8 - Theorem: Let $F: D \subset W \to V$ be a continuous mapping from the open, $\mathcal{P}(W)$-path-connected subset D of the Banachspace W into the Banachspace V. Suppose that for each $y \in D$ there exists a linear operator $A_y: W \to V$ with bounded inverse $A_y^{-1} \in L(V,W)$ such that (2) holds for all y_1, y_2 from some closed ball $\bar{S}(y, \gamma_y) \in D$ (with A and α replaced by A_y and α_y, respectively). If $\sigma_y = (\|A_y^{-1}\|^{-1} - \alpha_y)\gamma_y \geq \sigma > 0$ for $y \in D$, then F is a homeomorphism from D onto V.

From the inverse function theorem it follows that F is a local homeomorphism, and since $\sigma > 0$ is independent of $y \in D$, it is easily verified that Γ_F^{-1} is a covering relation. Thus, by 3.10 Γ_F^{-1} has the path-lifting property and hence the result is a direct consequence of 4.4.

Instead of the assumption that σ is independent of y, we can, of course, also use the norm-coerciveness as a tool for assuring the continuation property. For example, by using the inverse function theorem in its conventional form for continuously (Frechet)-differentiable operators, (i.e., by setting $A = F'(y)$ in the above version) we obtain the following corollary of 4.7.

4.9 - Corollary: Let V be a finite dimensional normed linear space and suppose the mapping $F: D \subset V \to V$ has a continuous, nonsingular derivative on the open set $D \subset V$. If F is norm-coercive on D, then for each $\mathcal{P}(V)$-path component D_μ of D, $F|D_\mu$ is a homeomorphism from D_μ onto V.

We conclude this section by showing that also the following wellknown theorem of Hadamard [6] and Levy [8] (see, e.g., Schwartz [11]) can be proved by means of our results.

4.10 - <u>Theorem</u>: Let $F: W \to V$ be a continuously (Frechet)-differentiable map from the Banachspace W into the Banachspace V, and suppose that for all $y \in W$, $F'(y)$ has a bounded inverse $(F'(y))^{-1} \in L(V,W)$ such that $\|(F'(y))^{-1}\| \leq \gamma$. Then, F is a homeomorphism from W onto V.

<u>Proof</u>: By the standard inverse function theorem we know that F is a local homeomorphism from W onto the open subset FW of V. Moreover, F^{-1} is again continuously differentiable and we have for all $y \in W$

(3) $$(F^{-1}(Fy))' = (F'(y))^{-1}$$

Let $\mathcal{P}_W' \subset \mathcal{P}(W)$ and $\mathcal{P}_V' \subset \mathcal{P}(V)$ be the classes of all piecewise continuously differentiable paths in W and V, respectively, and $\hat{\mathcal{P}}_V' = \mathcal{P}_V \cap \mathcal{P}(FW)$. Clearly then, $F\mathcal{P}_W' \subset \hat{\mathcal{P}}_V'$. Conversely, if $Fq(t) = p(t)$ for $t \in J$, i.e., if $q \in \mathcal{P}(W)$ is a Γ_F^{-1}-lifting of $p \in \hat{\mathcal{P}}_V'$, then clearly also q is again piecewise continuously differentiable and because of (3) we have in each continuity interval of p

$$q'(t) = (F'(q(t))^{-1}p'(t) \tag{4}$$

Hence, we have $q \in \mathcal{P}'_W$ and, therefore, the admissible sets of $\mathcal{P}'_W$ and $\hat{\mathcal{P}}'_V$ satisfy the conditions of Theorem 4.2. In order to show that Γ_F^{-1} has the continuation property for $\hat{\mathcal{P}}'_V$ let $p \in \hat{\mathcal{P}}'_V$ and $q: [0,\hat{t}) \subset J \to W$ be a continuous function with $Fq(t) = p(t)$ for $0 \leq t < \hat{t}$. Then, by the same argument as above, q is piecewise continuously differentiable on $[0,\hat{t})$ and (4) holds on the continuity intervals of p. Let $\{t_k\} \subset [0,\hat{t})$ be any sequence with $\lim_{k\to\infty} t_k = \hat{t}$. It is no restriction to assume that $\{t_k\}$ is increasing, then we have for $k < j$

$$\|q(t_j) - q(t_k)\| \leq \int_{t_k}^{t_j} \|q'(s)\|\, ds \leq \tag{5}$$

$$\int_{t_k}^{t_j} F'(q,s))^{-1}\| \, \|p'(s)\|\, ds \leq \gamma \max_{s\in J}\|p'(s)\| \, |t_j - t_k|$$

which shows that $\{q(t_k)\}$ is a Cauchy sequence and hence convergent. Thus, because of the continuity of F on W it follows immediately that Γ_F^{-1} has the continuation property for $\hat{\mathcal{P}}'_V$. Since FW is open in V, 3.11 therefore implies that $FW = V$, and now the result follows directly from 4.2.

Recently, Meyer [9] extended this theorem to the case where $\|(F'(y))^{-1}\|$ is allowed to go to infinity at most linearly in $\|y\|$. The following theorem is a modification of Meyer's result.

4.11 - Theorem: Let $G: V \to V$ be compact and continuously differentiable on the Banachspace V. Set $F = I - G$ and suppose that $F'(y)$ has a bounded inverse $(F'(y))^{-1} \in L(V,V)$ such that $\|(F'(y))^{-1}\| \leq \alpha\|y\| + \beta$ for all $y \in V$. Then, F is a homeomorphism from V onto itself.

Proof: We use the same notation as in the proof of 4.10, of course, with $W = V$. Let $p \in \hat{\mathcal{P}}^1_V$ and let $q: [0,\hat{t}) \subset J \to V$ be a continuous function such that $Fq(t) = p(t)$ for $0 \leq t < \hat{t}$. Then, as in the case of (5), we find

$$\|q(t) - q(0)\| \leq \int_0^t \|q'(s)\|\,ds \leq \int_0^t \|(F'(q(s)))^{-1}\|\,\|p'(s)\|\,ds$$

$$\leq \hat{\beta}t + \int_0^t \hat{\alpha}\,\|q(s) - q(0)\|\,ds$$

where

$$\hat{\beta} = (\alpha\|q(0)\| + \beta) \max_{s\in J} \|p'(s)\| , \quad \hat{\alpha} = \alpha \max_{s\in J} \|p'(s)\|$$

The wellknown Gronwall inequality now implies that

$$\|q(t) - q(0)\| \leq \frac{\hat{\beta}}{\hat{\alpha}} (e^{\hat{\alpha}\hat{t}}-1) , \quad t \in [0,\hat{t}),$$

i.e., for any sequence $\{t_k\} \subset [0,\hat{t})$ with $\lim_{k\to\infty} t_k = \hat{t}$ it follows that $\{q(t_k)\}$ is bounded. Because of the compactness of G, there exists a convergent subsequence of $\{Gq(t_k)\}$. If this subsequence is again denoted by $\{Gq(t_k)\}$ and if $\lim_{k\to\infty} Gq(t_k) = \hat{y}'$, then with $\lim_{k\to\infty} Fq(t_k) = \lim_{k\to\infty} p(t_k) = \hat{y}$ it follows that

$$\|q(t_k) - (\hat{y}+\hat{y})\| \leq \|q(t_k) - Gq(t_k) - \hat{y}\| + \|Gq(t_k) - \hat{y}'\|$$

which proves that Γ_F^{-1} has the continuation property for $\hat{\mathcal{P}}'_V$. The remainder of the proof is identical to that of <u>4.10</u>.

5. Applications to Equations $F(x,y) = 0$

In this section we shall discuss a few possible results about equations of the form $F(x,y) = 0$ where $F\colon D \subset X \times Y \to Z$ and X,Y,Z are Banachspaces. Let

(6a) $$\Phi_0 = \{(x,y) \in D \mid F(x,y) = 0\} \quad ,$$

then the relation to be considered is the restriction

(6b) $$\Phi = \Phi_0 \cap (D(\Phi_0) \times R(\Phi_0))$$

There are various possibilities of assuring that Φ has the local mapping property. We shall restrict ourselves to those derivable from the classical implicit function theorem, and use that theorem in the following form:

> Let $F\colon D \subset X \times Y \to Z$ be continuous on D and $F(x_0,y_0) = 0$ at $(x_0,y_0) \in D$ with $\bar{S}(x_0,\sigma) \times \bar{S}(y_0,\rho_0) \subset D$. Suppose $A\colon Y \to Z$ is a linear operator with bounded inverse $A^{-1} \in L(Z,Y)$ such that
>
> $$\|F(x,y_2) - F(x,y_1) - A(y_2-y_1)\| \le \alpha_0 \|y_2-y_1\| < \|A^{-1}\|^{-1} \|y_2-y_1\|$$
>
> for $x \in \bar{S}(x_0,\sigma)$ and $y_1,y_2 \in \bar{S}(y_0,\rho_0)$. Finally, let $\sigma_0 \le \sigma$ be such that $\|F(x,y_0)\| \le (\|A^{-1}\|^{-1} - \alpha_0)\rho_0$ for $x \in S(x_0,\sigma_0)$. Then for each $x \in S(x_0,\sigma_0)$ the equation $F(x,y) = 0$ has a solution $y_x \in S(y_0,\rho_0)$ which is unique in that ball and depends continuously on x.

The proof is standard and is omitted here (see, e.g., Hildebrandt and Graves [7], and Cesari [1]).

Our formulation does not represent the weakest possible result. In fact, in going through the proof it is readily seen that the continuity assumption about F can be reduced to separate continuity in each variable. Also, the conditions on A can be weakened, as Cesari has shown. However, the conventional implicit function theorem for continuously differentiable operators is contained in this particular version.

Based on this implicit function theorem, we obtain the following result assuring that Φ is a local mapping relation:

5.1 - Lemma: Let $F: D \subset X \times Y \to Z$ be continuous on the open set D in $X \times Y$, and suppose that for each $(x,y) \in D$ there exists a linear operator $A(x,y): Y \to Z$ with bounded inverse $A^{-1}(x,y) \in L(Z,Y)$ such that A depends continuously on (x,y) in the strong operator topology. Finally, assume there exists an $\alpha > 0$ such that for each $(x,y) \in D$ a $\rho = \rho(x,y) > 0$ can be found with

$$(7) \quad \|F(x,y_2) - F(x,y_1) - A(x,y)(y_2-y_1)\| \le \alpha\|y_2-y_1\| < \|A^{-1}(x,y)\|^{-1}\ \|y_2-y_1\|$$

for $y_1, y_2 \in \bar{S}(y,\rho)$. Then Φ is a local mapping relation and $D(\Phi) = D(\Phi_0)$ is open in X.

Proof: For $(x_0,y_0) \in \Phi$ we can find $\sigma > 0$, $\rho > 0$ such that

$\bar{S}(x_0,\sigma) \times \bar{S}(y_0,\rho) \subset D$. Set $A \equiv A(x_0,y_0)$ and let $\epsilon > 0$ be such that $\alpha_0 = \alpha + \epsilon \|A^{-1}\|^{-1}$. Then, $\sigma_0 \in (0,\sigma]$, $\rho_0 \in (0,\rho(x_0,y_0)]$ can be chosen such that $\|A(x,y) - A(x_0,y_0)\| \leq \epsilon$ and $\|F(x,y_0)\| < (\|A^{-1}\|^{-1} - \alpha_0)\rho_0$ for $x \in \bar{S}(x_0,\sigma_0)$ and $y \in \bar{S}(y_0,\rho_0)$. Then for any $x \in \bar{S}(x_0,\sigma_0)$ and $y_1,y_2 \in \bar{S}(y_0,\rho_0)$

$$\begin{aligned}
&\|F(x,y_2) - F(x,y_1) - A(x_0,y_0)(y_2-y_1)\| \\
&\leq \|F(x,y_2) - F(x,y_1) - A(x,y_1)(y_2-y_2)\| \\
&+ \|A(x,y_1) - A(x_0,y_0)\| \, \| y_2 - y_1\| \\
&\leq (\alpha + \epsilon) \|y_2-y_1\|
\end{aligned}$$

Thus, all conditions of the implicit function theorem are satisfied and hence the restriction $\varphi = \Phi \cap (S(x_0,\sigma_0) \times S(y_0,\rho_0))$ is a continuous map from $S(x_0,\sigma_0)$ into $S(y_0,\rho_0)$. Since $(x_0,y_0) \in \Phi$ was arbitrary, this completes the proof.

Condition (7) is evidently satisfied if F has a continuous partial derivative $\partial_2 F(x,y)$ in D with a bounded inverse $(\partial_2 F(x,y))^{-1} \in L(Z,Y)$.

The question now arises when Φ has the path-lifting property. As in Section 4, a possible approach is here to add conditions to 5.1 which assure that Φ is a covering relation. Loosely speaking, these conditions have to guarantee that the radius σ_0 does not depend on $(x_0,y_0) \in \Phi$. This means, of course, that we have to make uniformity assumptions about the continuity of A, F, etc. One possible result along this line can be stated as follows:

5.2 - Theorem: Let $F: D \subset X \times Y \to Z$ be uniformly continuous on the open set D in $X \times Y$, and suppose that for each $(x,y) \in D$ a linear operator $A(x,y): Y \to Z$ with bounded inverse $A^{-1}(x,y) \in L(Z,Y)$ is defined such that $\|A^{-1}(x,y)\| \leq 1/\mu > 0$, for all $(x,y) \in D$. Assume further that A is uniformly continuous for $(x,y) \in D$. Finally, let $\alpha > 0$ and $\rho > 0$ be such that for each $(x,y) \in D$ the inequality (7) holds with $\alpha < \mu$ for all $y_1, y_2 \in \bar{S}(y,\rho)$. Then, Φ is a covering relation.

Proof: By going through the proof of 5.1 again, we see that both ρ_0 and σ_0 can now be chosen independently of $(x_0, y_0) \in \Phi$. Hence, the restriction φ is a continuous mapping from $S(x_0,\sigma_0)$ into $S(y_0,\rho_0)$ and for every $x \in S(y_0,\sigma_0)$, $\varphi(x) \in S(y_0,\rho_0)$ is the only solution of $F(x,y) = 0$ in that ball. Clearly, then, $\Phi[S(x_0,\sigma_0)] \subset \bigcup_{y_0 \in \Phi[x_0]} (S(y_0,\rho_0) \cap R(\Phi))$ and the sets on the right are disjoint and open in $R(\Phi)$. This proves that Φ is indeed a covering relation.

Theorem 5.2 essentially generalizes the first main theorem of Ficken [5] who considers an operator $F: J \times D_Y \subset R^1 \times Y \to Z$ and uses instead of the operator function A the partial derivative $\partial_2 F(s,y)$, yet who places slightly less stringent continuity assumptions upon F. As indicated earlier, our continuity assumptions for F in the basic implicit function theorem, and therefore also in 5.2, can also be weakened to correspond to those of Ficken.

As in Section 4 we can also assure the path-lifting property for Φ by making assumptions about $R(\Phi)$. Following is a simple result along this line:

5.3 - Theorem: Suppose that $F: D \subset X \times Y \to Z$ is continuous on D, and that Φ has the local mapping property. If $R(\Phi)$ is compact, then Φ has the $\mathcal{P}(D(\Phi))$ continuation property.

Proof: Let $p \in \mathcal{P}(D(\Phi))$ and $q: [0,\hat{t}) \subset J \to R(\Phi)$ be a continuous map such that $F(p(t),q(t)) = 0$ for $0 \leq t < \hat{t}$. If $\{t_k\} \subset [0,\hat{t})$ is any sequence with $\lim_{k\to\infty} t_k = \hat{t}$, then $\{q(t_k)\} \subset R(\Phi)$ and hence there exists a convergent subsequence. For the sake of simplicity assume that $\lim_{k\to\infty} q(t_k) = \hat{y} \in R(\Phi)$. Since $p(\hat{t}) \in D(\Phi)$, we have $(p(\hat{t}),\hat{y}) \in D(\Phi) \times R(\Phi) \subset D$ and, therefore, by the continuity of F, $F(p(\hat{t}),\hat{y}) = 0$, i.e., Φ has the continuation property for $\mathcal{P}(D(\Phi))$.

This result, together with 5.1, represents essentially an extension of the second main result of Ficken [5], except for the form of the continuity assumption about F. In this case this continuity assumption can again be weakened if the local mapping property is derived from conditions of the type used in 5.2. The compactness assumption about $R(\Phi)$ can be guaranteed by suitable compactness conditions about F. For both these points we refer to Ficken [5].

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