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ADDITIONAL STUDIES OF QUASI-OPTIMUM FEEDBACK CONTROL TECHNIQUES

*by B. Friedland, F. E. Thau, S. Welt,
C. K. Ling, and M. Schilder*

Prepared by
GENERAL PRECISION SYSTEMS INC.
Little Falls, N. J.
for Ames Research Center

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PREFACE

This report contains further results on the theory and applications of a quasi-optimum control technique obtained in "Study of Quasi-Optimum Feedback Control Techniques" under Contract NAS 2-3636 with the Ames Research Center, National Aeronautics and Space Administration. The results of an earlier study, performed under Contract NAS 2-2648 with the same agency, are contained in NASA Contractor Report CR-527, "Study of Quasi-Optimum Feedback Control Techniques" to which this report can be regarded as a sequel.

The principal investigator was Dr. Bernard Friedland; contributors included Dr. Frederick E. Thau and Messrs. Sanford Welt and Chong K. Ling, all of the Controls Department, Aerospace Research Center, Kearfott Group, General Precision Systems Inc. Dr. Michael Schilder, of the same department assisted with Section 2.1 . Dr. Elwood C. Stewart, of the NASA Ames Research Center, served as Contract Technical Monitor.

CONTENTS

INTRODUCTION AND SUMMARY

The principal impediment to widespread application of optimum control theory has been the lack of practically feasible techniques for implementing the required computation. In particular, the implementation requires a very rapid solution of a two-point boundary value problem in ordinary differential equations. Since there is rarely a need for absolute optimality, and in fact the performance criterion is often quite subjective, a moderate sacrifice in performance is generally an acceptable price to pay for simplicity of implementation.

In 1965, Friedland [A1] presented a quasi-optimum control technique which offered the possibility of achieving nearly optimum performance by means of a control system which can be readily implemented. When the technique was first introduced, only the rudiments of the theory and a simple example to demonstrate feasibility were given. Further development of the theory and its application to realistic guidance and control problems were undertaken in 1965 under NASA Contract NAS 2-2648 and reported in NASA Contractor Report CR-527 [A1] and in several technical papers [A2 - A8] . This study was continued in 1966-67 under Contract NAS 2-3636; this report gives the results achieved under the latter contract.

The basis of the quasi-optimum control technique under investigation is the well-established engineering practice of approximating a complicated dynamic process by a simpler process, designing a control system for the latter, and then amending the design (if necessary) to account for the difference between the original process and the approximation used. A systematic application of this design approach, within the framework of modern optimum control theory is the essence of our quasi-optimum control technique. In the application of this technique it is necessary that the "simplified process", in addition to being a reasonably faithful representation of the true process, must be such that the solution of the two-point boundary-value problem governing its optimum control law can be reduced to manageable proportions. The correction to the optimum control law then requires the evaluation of a correction matrix by the solution of a matrix Riccati equation. The solution matrix of this Riccati equation is used to correct the solution to the simplified process.

The process for which the quasi-optimum control law is sought is represented by the system of first-order differential equations

$$\dot{x} = f(x, u) \tag{1}$$

where $x = \{x_0, x_1, \ldots, x_n\}$ is the state vector, $u = \{u_1, u_2, \ldots, u_r\}$ is the control vector, and $f = \{f_0, f_1, f_2, \ldots, f_n\}$ is a vector-valued function. The component x_0 of x is a measure of the performance. A feedback control law $u = u(x)$ is to be determined which takes the process from some current* state $x(t)$ to a final state $x(T)$, such that the performance index $x_0(T)$ is a minimum, and the remaining n states satisfy the boundary conditions

$$\varphi(x(T)) = 0 \tag{2}$$

where $\varphi = \{\varphi_1, \varphi_2, \ldots, \varphi_s\}$, $s \leq n$. The terminal time T may be either free or specified. In addition, the control u may be required to be a member of a closed, bounded set Ω.

The structure of the optimum controller can be determined by the maximum principle of Pontryagin [B1]. Define the Hamiltonian function:

$$h(p, x, u) = p' f(x, u) \tag{3}$$

where $p = \{p_0, p_1, \ldots, p_n\}$ and ($'$) denotes transposition, and where p satisfies the adjoint equation

$$\dot{p} = -\text{grad}_x h = -h_x \tag{4}$$

It is seen from (1) that

$$\dot{x} = \text{grad}_p h = h_p \tag{5}$$

* The current time is denoted by the variable t, terminal time by T; time when it is used as an independent variable is denoted by τ, e.g., $t < \tau < T$.

Necessary conditions for the existence of an optimum control u^* are:

(i) h is maximum with respect to $u \in \Omega$, that is,

$$h(x, u^*, p) = \max_{u \in \Omega} h(x, u, p) \tag{6a}$$

(ii) $$h(x, u^*, p) = \text{const} \tag{6b}$$

(iii) The adjoint vector satisfies the "transversality conditions"

$$p(T) = \left[\begin{array}{c} -1 \\ \hline \Phi' \lambda \end{array}\right] \tag{7}$$

where λ is a vector of s constants

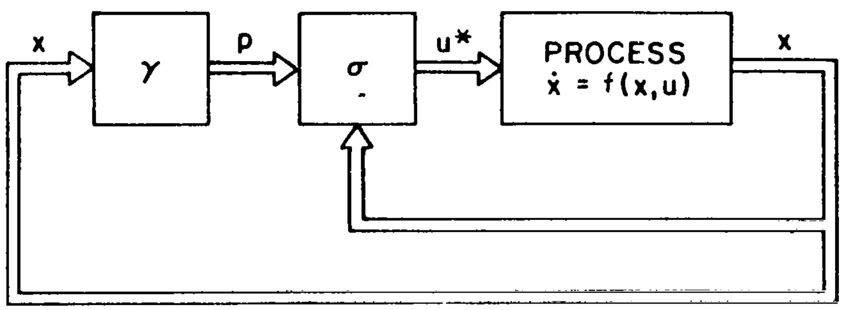
and

$$\Phi = \left[\frac{\partial \varphi_i}{\partial x_j}\right] \qquad i = 1, 2, \ldots, s; \quad j = 0, 1, \ldots, n$$

The optimum control system may thus be conceived as having the structure shown in Figure 1. The transformation σ of the process state vector x and the adjoint vector p into the control

$$u^* = \sigma(p, x) \tag{8}$$

is defined by (6a), and is determined by maximizing the Hamiltonian (3) with $u \in \Omega$. Equations (4), (5) and (8), together with boundary conditions (2) and (7), define a two-point boundary-value problem. Given the current state $x(t)$ (if a solution of the boundary-value problem exists), then the adjoint $p(t)$ may be determined as the solution to the two-point boundary-value problem. Thus, (2), (4), (5), (7) and (8) define a transformation γ of the current state $x(t)$ into the adjoint $p(t)$. For most applications, the transformation implicit in the solution of the two-point boundary-value problem cannot be obtained by any practical method of computation, and hence an approximate solution to the two-point boundary-value problem is needed.



STRUCTURE OF OPTIMUM CONTROL SYSTEM
FIGURE I

Suppose the state x can be regarded as the sum of two terms

$$x = X + \xi \tag{9}$$

where X is the state of the "simplified process". Then (1) can be written

$$\dot{X} + \dot{\xi} = f(X + \xi, u)$$

Furthermore, assume that ξ is small. Then the original system can be approximated by the system

$$\dot{X} = \lim_{\xi \to 0} f(X + \xi, u) = F(X, u) \tag{10}$$

where $\varphi(X(T)) = 0$. By defining a "simplified Hamiltonian" $H = P'F(X, u)$, a corresponding two-point boundary-value for the simplified system can be derived, i.e.,

$$\dot{X} = H_P \qquad ; \qquad \dot{P} = -H_X \tag{11}$$

and

$$P(T) = \left[\frac{1}{\Phi' \Lambda} \right]$$

where Λ is an s-dimensional vector of "slack"variables.

The "simplified adjoint" vector P, which, by assumption, can be solved for in terms of X, may be regarded as an approximate solution for p of the exact problem. For nonzero ξ, however, this approximation may be inadequate. Consequently, it is desirable to include the effects of the state "error" ξ more exactly. For this purpose suppose that a change ψ in the adjoint vector results because of the error ξ, i.e.,

$$p = P + \psi \tag{12}$$

Since p can be expressed as a function of x, i.e. $p(x) = p(X + \xi)$, by expanding about the state X, and retaining only the first two terms, we obtain

$$p(x) = p(X) + \left[\frac{\partial p_j}{\partial x_i} \right]_{x=X} \xi$$

By (12), the first term $p(X)$ is the adjoint vector P of the simplified problem; the second term is the vector ξ premultiplied by a gain matrix

$$M(X) = \left[\frac{\partial p_j}{\partial x_i} \right]_{x=X}$$

Thus (12) can be written

$$p(x) = P(X) + M(X)\xi \tag{13}$$

and consequently

$$\psi(t) = M(t)\,\xi(t) \tag{14}$$

The structure of the quasi-optimum control system based on this approximation is shown in Figure 2. The suboptimum controller comprises three units: the σ - unit which is the same as determined for Figure 1 by maximizing h with respect to $u \in \Omega$, the unit Γ which transforms X into P, and the gain unit $M(X)$ by which ξ is multiplied to yield a correction to P.

To obtain M, differentiate (13) with respect to time:

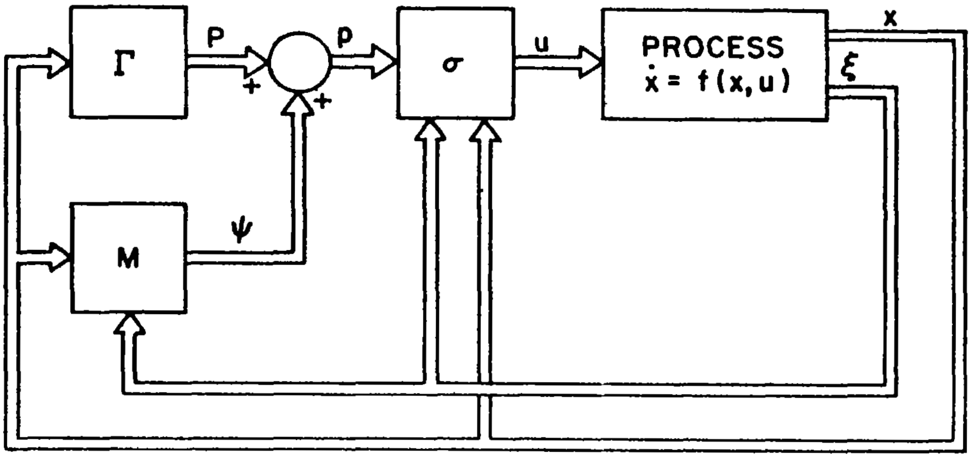
$$\dot{p} = \dot{P} + M\dot{\xi} + \dot{M}\xi \tag{15}$$

Likewise

$$\dot{x} = \dot{X} + \dot{\xi}$$

Substituting these relations into the canonical equations (3) and (4) and expanding about the state and the adjoint for the simplified process gives

$$\begin{aligned} \dot{X} + \dot{\xi} &= h_p = h_P + (H_{XP} + H_{PP}M)\,\xi + O(\xi^2) \\ \dot{P} + \dot{M}\xi + M\dot{\xi} &= -h_x = -h_X - (H_{XX} + H_{PX}M)\xi + O(\xi^2) \end{aligned} \tag{16}$$



STRUCTURE OF
QUASI-OPTIMUM CONTROL SYSTEM
FIGURE 2

where,

$$H_{XP} = \left[\frac{\partial^2 h}{\partial x_j \partial p_i}\right]_{x=X} \qquad H_{PX} = \left[\frac{\partial^2 h}{\partial p_j \partial x_i}\right]_{x=X} = H'_{XP}$$

$$H_{PP} = \left[\frac{\partial^2 h}{\partial p_j \partial p_i}\right]_{x=X} \qquad H_{XX} = \left[\frac{\partial^2 h}{\partial x_j \partial x_i}\right]_{x=X}$$

Upon use of (11), and after dropping terms of $O(\xi^2)$, (16) reduces to

$$\dot{\xi} = (H_{XP} + H_{PP}M)\xi \tag{17}$$

$$M\dot{\xi} + \dot{M}\xi = -(H_{XX} + H_{PX}M)\xi \tag{18}$$

Substitution of (17) into (18) gives:

$$(\dot{M} + MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX})\xi = 0$$

If this relationship is to hold for all ξ, the matrix M must satisfy the matrix Riccati equation:

$$-\dot{M} = MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX} \tag{19}$$

One method of solving the matrix Riccati equation is to observe that it corresponds to the auxiliary equations

$$\begin{aligned} \dot{\xi} &= H_{XP}\xi + H_{PP}\psi \\ \dot{\psi} &= -H_{XX}\xi - H_{PX}\psi \end{aligned} \tag{20}$$

which is equivalent to (16) when the higher-order terms are dropped. This is a linear system whose solution can be expressed as

$$\begin{aligned} \xi(T) &= \Phi_{11}(T, t)\xi(t) + \Phi_{12}(T, t)\psi(t) \\ \psi(T) &= \Phi_{21}(T, t)\xi(t) + \Phi_{22}(T, t)\psi(t) \end{aligned} \tag{21}$$

where

$$\Phi(T, t) = \begin{bmatrix} \Phi_{11}(T, t) & \Phi_{12}(T, t) \\ \Phi_{21}(T, t) & \Phi_{22}(T, t) \end{bmatrix} \tag{22}$$

is the "transition matrix" corresponding to:

$$\mathcal{A} = \begin{bmatrix} H_{XP} & H_{PP} \\ -H_{XX} & -H_{PX} \end{bmatrix}$$

Equations (21) are actually $2(n + 1)$ equations in $n + 1$ unknowns. To solve we need $(n + 1)$ relations in addition to (21). These relations come from the boundary conditions. Suppose that for the exact problem the boundary conditions at $\tau = T$ are given by (2) and (7). If in the simplified process the boundary conditions are satisfied at time T, then in the exact problem these conditions must be satisfied at $T + dT$. By expanding the exact state and adjoint about the time T and dropping second-order infinitesimals, we obtain

$$\begin{aligned} x(T + dT) &= x(T) + \dot{x}(T)dT \\ &= X(T) + \xi(T) + \dot{X}(T)dT \end{aligned} \tag{23a}$$

$$\begin{aligned} p(T + dT) &= p(T) + \dot{p}(T)dT \\ &= P(T) + \psi(T) + \dot{P}(T)dT \end{aligned} \tag{23b}$$

Substituting (23a) in (2) and expanding about the state of the simplified process gives

$$\varphi(X(T)) + \Phi\xi(T) + \Phi\dot{X}(T)dT = 0 \tag{24a}$$

Similarly, for the adjoint we have

$$P(T) + \psi(T) + P(T)dT = \begin{bmatrix} -1 \\ \hline \Phi'\lambda \end{bmatrix} \tag{24b}$$

Since the simplified problem has been assumed to satisfy the boundary conditions of the same form, i.e., $\varphi(X(T)) = 0$ and $P(T) = \left[\dfrac{-1}{\Phi' \Lambda}\right]$, then (24a) and (24b) reduce to the $n + 1$ independent equations

$$\Phi[\xi(T) + \dot{X}(T)dT = 0 \tag{25a}$$

$$\psi(T) + \dot{P}(T)dT = \Phi' \eta \tag{25b}$$

where

$$\eta = \lambda - \Lambda$$

Finally, we must have

$$\begin{aligned} dH &= \xi' \frac{\partial H}{\partial X} + \psi' \frac{\partial H}{\partial P} \\ &= -\dot{P}'\xi + \dot{X}'\psi = 0 \end{aligned} \tag{26}$$

Equations (25a), (25b) and (26) give a total of $n + 2$ relations. Since dT is an additional variable, there are just enough equations needed to solve (21) for $\varphi(t)$ as a function of $\xi(t)$ and thereby obtain $M(t)$. In most cases, the linear differential equations (20) have time-varying coefficients and as a result cannot be solved analytically, Hence, it becomes necessary either to approximate the solution to the Riccati equation or to integrate (19) numerically.

Numerical integration of the Riccati equation requires that boundary conditions (25a) and (25b) be translated into conditions on $M(T)$. Consequently, (19) must be integrated backwards in time starting at $\tau = T$. Part of the complexity of this problem arises because the matrix $M(\tau)$ may not exist at $\tau = T$, hence, the boundary conditions cannot be translated directly into conditions on $M(T)$. This problem may be circumvented by expressing $M(t)$ in the form

$$M(t) = S(t) - R(t)Q^{-1}(t)R'(t) \tag{27}$$

integrating systems of differential equations for S, Q and R for a small time Δ backwards from T and using the results to compute $M(T - \Delta)$. It was shown in [A1] and [A7] that the matrix S satisfies (19) with $S(T) = 0$, and R and Q satisfy

$$-\dot{R} = (A' + SB)R \tag{28}$$

$$-\dot{Q} = R'BR \tag{29}$$

with boundary conditions

$$R(T) = [\,\Phi' \mid -\dot{P}(T)\,] \tag{30}$$

$$Q(T) = \left[\begin{array}{c|c} 0 & \Phi X(T) \\ \hline \dot{X}'(T)\Phi' & -\dot{X}'(T)\dot{P}(T) \end{array}\right] \tag{31}$$

where

$$\Phi = \left[\frac{\partial \varphi_i}{\partial x_j}\right] \tag{32}$$

is the Jacobian matrix of the terminal constraint vector $\varphi(x(T)) = 0$.

Although the solution of (27) - (31) is well-suited to numerical integration by means of a high-speed digital computer, we have found, in several examples, that it is practical to further simplify the determination of M by assuming $\dot{M} \approx 0$, and hence to solve the algebraic system

$$MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX} = 0 \tag{33}$$

It was anticipated at the beginning of our investigation (under Contract NAS 2-2648) that the quasi-optimum control technique would be limited to practical problems in which ξ is so small that the simplified control law gives "passable" performance. We were pleased to discover that the quasi-optimum control technique works even when the simplified control law is patently unacceptable.

One of the examples considered [A1, A4] was minimum time rendezvous in free space. The simplified problem was obtained by assuming negligible angular velocity in a relative coordinate system and thereby reducing the problem to a one-dimensional (second-order) case for which the explicit control law is well-known. The application of this control law to the true process results in purely radial acceleration of the controlled vehicle relative to target and causes the vehible to orbit the target in order to conserve angular momentum, and is completely unacceptable even when the initial angular momentum is quite small. When the quasi-optimum control law is used, however, the rendezvous is actually achieved, and the rendezvous time and trajectory compares favorably with the exact optimum even when the initial relative velocity is purely tangential.

Another application which verified that the quasi-optimum control law may work even when the simplified control law doesn't, was in the flight control of a flexible booster [A1, A5]. In this case, the simplified problem was obtained by assuming negligible bending. When the control law obtained for the rigid vehicle was used for the flexible vehicle, excessive bending moments were produced and led to vehicle failure. The quasi-optimum control law, which corrected the rigid body control law to account for bending, however, gave good performance for a vehicle of moderate flexibility.

In other applications, considered in 1966-67 under Contract NAS 2-3636, and described in detail below, we found that the quasi-optimum control law gave visible improvement over that obtained with the simplified control law, but, because the simplified control law gave passable performance, the improvement is not as striking.

One of the general theoretical questions concerning the quasi-optimum control technique is the estimation of the degradation of performance resulting from the use of the quasi-optimum control law. This problem has received attention in 1966-67. In particular, in one approach, we considered the "mildly-nonlinear" process

$$\dot{x} = Ax + \mu f(x) + Bu \tag{34}$$

with a performance criterion

$$V = \frac{1}{2} \int_{t}^{T} (x'Rx + u'Qu)\, d\tau \tag{35}$$

to be minimized, where T is fixed, Q is a positive-definite matrix, μ is a small parameter, and $f(x)$ is a nonlinear function which is twice differentiable with respect to all its arguments. Earlier [A1] we showed that the quasi-optimum control law for this process, in accordance with the theory summarized above, is

$$u = Q^{-1}B' (K(T, \tau)x + \underline{m}_{\mu x} (x, \tau)\mu) \tag{36}$$

where $K(\tau, t)$ is the solution to the matrix Riccati equation for the simplified process, i.e.

$$-\dot{K} = KA + A'K + KBQ^{-1}B'K - R \tag{37}$$

with $K(\tau, T) = 0$, and $\underline{m}_{\mu x}$ is the solution to

$$-\dot{\underline{m}}_{\mu x} = (A' + KBQ^{-1}K')\, \underline{m}_{\mu x} + Kf(x) + \frac{\partial f}{\partial x} Kx \tag{38}$$

with

$$\underline{m}_{\mu x}(T) = 0$$

In the present investigation we have demonstrated that, for sufficiently small μ, the quasi-optimum control law (36) is indeed better then the simplified control law

$$u = Q^{-1}B'K(\tau, t)x \tag{39}$$

Specifically the quasi-optimum control law results in a performance V_q which is smaller than the performance V_s obtained by use of the simplified control law (39) by a positive quantity times μ^2. The details of this calculation, which it would appear can be extended to a more general class of problems, are given below. Another approach which was considered was to expand the solution of the differential equations obtained by use of the optimum, the quasi-optimum, and the simplified control laws on the actual process about the solution to the simplified process. A linear, nonhomogeneous differential equation for the difference between these solutions is obtained. The properties of this differential equation can be used to compare the performance of the various cases. The above approaches yield some results on the problem of performance; a considerable amount of work, however, still remains to be done on this problem.

As an alternative to determining the performance of the quasi-optimum system for processes with performance functionals of the form

$$V = \frac{1}{2} x'(t)Mx(t) = \int_t^\infty [q(x) + h(u)]\, d\tau$$

it is reasonable to examine whether the quasi-optimum control law optimizes anything, and if so, what is optimized. This question has led to a study of the general inverse optimum control problem: what is optimized by a control law of specified form?

We have found that performance indices in the above form which are minimized by a given control law

$$u = \varphi(x)$$

for the system

$$\dot{x} = f(x) + Gu$$

must satisfy

$$\varphi(x) = \eta^{-1}(-G'Mx)$$

and

$$q(x) = -h(\varphi(x)) - x'M[f(x) + G\varphi(x)]$$

for all x, where $\eta = dh/du$. Furthermore, we found for linear and nonlinear single-input systems that if the optimum performance is required to be a positive-definite quadratic form in the state variables, then the optimum control must be a function of a linear combination of (at least) those state variables which are directly affected by the control. Details of these calculations are contained in Appendix 1 and in [A8].

One of the important topics in optimum control theory is the stochastic optimum control problem, in which it is desired to minimize

$$V(x,t) = E\{\int_t^T L(x(s))ds \mid x(t) = x\} \tag{40}$$

for the stochastic process

$$\dot{x} = f(x,u(x)) + Gv \tag{41}$$

where v is Gaussian white noise with spectral density matrix Σ. Determination of the stochastic optimum control law necessitates the solution of the "stochastic Hamilton-Jacobi" equation, which is a second-order partial differential equation. The use of the quasi-optimum control technique appears, as is shown below, to offer an effective method of obtaining an approximate solution for the quasi-optimum control law when the disturbance v is small (i.e. Σ is small) and the solution to the noise-free problem is known exactly. This application of this technique has been worked out for a simple example and a Monte-Carlo simulation has been performed which shows that the quasi-optimum control law is superior to the control law for the noise-free process. The amount of improvement, however, is only modest; it remains to be determined under which circumstances the quasi-optimum control law is worth the additional complexity.

PART I. THEORETICAL STUDIES

1.1 PERFORMANCE OF QUASI-OPTIMUM CONTROL LAW.

Since the implementation of quasi-optimum control law necessarily entails the use of a system of greater complexity than required by the control law for the simplified process, it is worth having an estimate of the improvement which can be achieved by use of the quasi-optimum control law. The general problem of estimating performance has not yet been solved, but a definite answer has been obtained for the mildly-nonlinear process (34), with the performance index (35). For convenience, we assume that the upper limit on the integral in (35) is ∞ ; this results in no real loss in generality.

Consider any control law $u_\alpha(x)$ and the corresponding value of the performance index $V_\alpha(x)$. From the integral definition of $V = V_\alpha$, we have

$$\left(\frac{dV_\alpha}{dt}\right) = - \frac{1}{2}(x'Rx + u'_\alpha Qu_\alpha)$$

But, in general, if V_α is not a function of the present time t, then

$$\frac{dV_\alpha}{dt} = \left(\frac{\partial V}{\partial x}\right)' \dot{x}_\alpha = \left(\frac{\partial V_\alpha}{\partial x}\right)' (Ax + Bu_\alpha(x) + \mu f(x))$$

Where $\partial V/\partial x$ denotes the gradient of V with respect to x, and ($'$) denotes transposition. Thus, upon equating the above expressions for dV_α/dt, the following partial differential equation for V_α is obtained:

$$\left(\frac{\partial V_\alpha}{\partial x}\right)' (Ax + Bu_\alpha(x) + \mu f(x)) + \frac{1}{2}(x'Rx + u'_\alpha Qu_\alpha) = 0 \tag{42}$$

We now consider the following 3 control laws:

(i) Simplified Control law $\quad u_\alpha(x) = u_s(x) = Q^{-1}B'Kx \quad$ (43s)

(ii) Quasi-optimum control law $\quad u_\alpha(x) = u_q(x) = Q^{-1}B'(Kx + \underline{m}_{\mu x}\mu) \quad$ (43q)

(iii) Exact-optimum control law $\quad u_\alpha(x) = u_o(x) = Q^{-1}B'(Kx + \underline{m}_{\mu x}\mu + O(\mu^2)) \quad$ (43o) where K is the optimum gain matrix for the simplified process, i.e. the solution to (37) with $\dot{K} \equiv 0$. Note that the adjoint vector for the exact process is

$$p(x, \mu) = Kx + \underline{m}_{\mu x} \mu + O(\mu^2) \tag{44}$$

and

$$\underline{m}_{\mu x} = \frac{\partial p}{\partial x} \Big|_{\mu = 0} \tag{45}$$

which makes it permissible to express $u_o(x)$ as given by (43o), since the maximum principle asserts that

$$u_o(x) = Q^{-1} B' p(x) \, .$$

For each of the control laws of (43) we assume that the solution $V_\alpha(x)$ exists* and is at least twice differentiable in the parameter μ. Then we can expand V_α in a series in μ up to μ^2, i.e.,

$$V_\alpha(x) = V_{\alpha 0}(x) + V_{\alpha 1}(x)\mu + V_{\alpha 2}(x)\mu^2 + O(\mu^3), \quad \alpha = s, q, o \tag{46}$$

Substitution of (46) and (43s) into (42) results in the following partial differential equation for $V_s(x)$, the performance attained by use of the simplified control law:

$$\left[\frac{\partial V_{so}}{\partial x} + \mu \frac{\partial V_{s1}}{\partial x} + \mu^2 \frac{\partial V_{s2}}{\partial x} + O(\mu^3) \right]' \quad [(A + B'Q^{-1}BK)x + \mu f(x)]$$
$$+ \frac{1}{2} [x'(R + KBQ^{-1}B'K)x] = 0 \tag{47s}$$

* An implication of this assumption is that the control law $\mu_\alpha(x)$ results in an asymptotically stable system, unless the integrand of (35) vanishes identically along any trajectory, since then $dV_\alpha/dt < 0$ and V_α is positive definite which implies asymptotic stability.

Equating the coefficients of μ^0, μ, and μ^2 of (47) results in

$$\left(\frac{\partial V_{so}}{\partial x}\right)' \hat{A}x + \frac{1}{2}[x'(R + KBQ^{-1}K)x] = 0 \tag{48s}$$

$$\left(\frac{\partial V_{s1}}{\partial x}\right)' \hat{A}x + \left(\frac{\partial V_{so}}{\partial x}\right)' f(x) = 0 \tag{49s}$$

$$\left(\frac{\partial V_{s2}}{\partial x}\right)' \hat{A}x + \left(\frac{\partial V_{s1}}{\partial x}\right)' f(x) = 0 \tag{50s}$$

where

$$\hat{A} = A + BQ^{-1}B'K$$

Similarly, substitution of (43q) and (46) into (42) gives

$$\left(\frac{\partial V_{qo}}{\partial x} + \mu\frac{\partial V_{q1}}{\partial x} + \mu^2\frac{\partial V_{q2}}{\partial x} + O(\mu^3)\right)'[\hat{A}x + \mu(f(x) + BQ^{-1}B'\underline{m}_{\mu x})]$$
$$+ \frac{1}{2}[x'Rx + (x'K + \mu\underline{m}'_{\mu x})BQ^{-1}B'(Kx + \mu\underline{m}_{\mu x})] = 0 \tag{47q}$$

and equating coefficients of μ^0, μ, and μ^2 results in

$$\left(\frac{\partial V_{qo}}{\partial x}\right)' \hat{A}x + \frac{1}{2}x'(R + KBQ^{-1}B'K)x = 0 \tag{48q}$$

$$\left(\frac{\partial V_{q1}}{\partial x}\right)' \hat{A}x + \left(\frac{\partial V_{qo}}{\partial x}\right)'[f(x) + BQ^{-1}B'\underline{m}_{\mu x}] + \underline{m}'_{\mu x}BQ^{-1}B'Kx = 0 \tag{49q}$$

$$\left(\frac{\partial V_{q2}}{\partial x}\right)' \hat{A}x + \left(\frac{\partial V_{q1}}{\partial x}\right)'[f(x) + BQ^{-1}B'\underline{m}_{\mu x}] + \frac{1}{2}\underline{m}_{\mu x}BQ^{-1}B'\underline{m}_{\mu x} = 0 \tag{50q}$$

Finally, substitution of (43o) and (46) into (42) gives

$$\left[\frac{\partial V_{oo}}{\partial x} + \mu\frac{\partial V_{o1}}{\partial x} + \mu^2\frac{\partial V_{o2}}{\partial x} + O(\mu^3)\right]'[\hat{A}x + \mu(f(x) + BQ^{-1}B'\underline{m}_{\mu x}) + O(\mu^2)]$$
$$+ \frac{1}{2}[x'Rx + (x'K + \mu\underline{m}'_{\mu x} + O(\mu^2))BQ^{-1}B'(Kx + \mu\underline{m}_{\mu x} + O(\mu^2))] = 0 \tag{47o}$$

Again equating coefficients of μ^0, μ, and μ^2 gives

$$\left(\frac{\partial V_{oo}}{\partial V}\right)' \hat{A}x + \frac{1}{2}x'(R + KBQ^{-1}B'K)x = 0 \tag{48o}$$

$$\left(\frac{\partial V_{o1}}{\partial x}\right)' \hat{A}x + \left(\frac{\partial V_{oo}}{\partial x}\right)' [f(x) + BQ^{-1}B'\underline{m}_{\mu x}] + \underline{m}_{\mu x} BQ^{-1}B'Kx = 0 \tag{49o}$$

$$\left(\frac{\partial V_{o2}}{\partial x}\right)' \hat{A}x + \left(\frac{\partial V_{o1}}{\partial x}\right)' [f(x) + B'Q^{-1}B'\underline{m}_{\mu x}] + \frac{1}{2}\underline{m}'_{\mu x} BQ^{-1}B'\underline{m}_{\mu x} + O(1) = 0 \tag{50o}$$

Comparison of (47s), (47q), and (47o) reveals that V_{so}, V_{qo}, and V_{oo} satisfy the same differential equation. Since each must satisfy the same condition $V_{\alpha o}(0) = 0$, they are all equal, and given by

$$V_{so} = V_{qo} = V_{oo} = -\frac{1}{2}x'Kx \qquad (K = K') \tag{51}$$

This is verified by noting that if (50) is the solution then

$$\frac{\partial V_{so}}{\partial x} = \frac{\partial V_{qo}}{\partial x} = \frac{\partial V_{oo}}{\partial x} = -Kx \tag{52}$$

and hence (48s) becomes

$$-x'K\hat{A}x + \frac{1}{2}x'(R + KBQ^{-1}BK)x = 0$$

or

$$-\frac{1}{2}x'[K(A + BQ^{-1}B'K) + (A' + KBQ^{-1}B')K - R - KBQ^{-1}BK]x = 0$$

The matrix of the quadratic form in the above is

$$KA + A'K + KBQ^{-1}B'K - R$$

But by (37) this matrix is zero. Consequently (51) is the desired solution for the zero order term, and (52) is its gradient.

Substitution of (51) and (52) into (49s), (49q), and (49o) gives

$$\left(\frac{\partial V_{s1}}{\partial x}\right)' \hat{A}x - x'Kf(x) = 0 \tag{53}$$

$$\left(\frac{\partial V_{q1}}{\partial x}\right)' \hat{A}x - x'K[f(x) + BQ^{-1}B'\underline{m}_{\mu x}] + m_{\mu x} BQ^{-1}B'Kx = 0$$

$$\left(\frac{\partial V_{o1}}{\partial x}\right)' \hat{A}x - x'K[f(x) + BQ^{-1}B'm_{\mu x}] + m_{\mu x} BQ^{-1}Kx = 0$$

Clearly,

$$V_{s1} = V_{q1} = V_{o1} = V_1$$

Since they all satisfy (53). Thus the first-order as well as the zero order terms using all three control laws are equal. This is an expected result, since if $V_{q1} \neq V_{o1}$ or $V_{s1} \neq V_{o1}$, it would be possible to find a value of μ so that $V_{q1} < V_{o1}$ or $V_{s1} < V_{o1}$ which is impossible if V_o is optimum.

The differences between V_s, V_q, and V_o are thus in the second order terms. To evaluate this difference, we make use of (45). In particular, since

$$p(x,\mu) = -\frac{\partial V_o}{\partial x} = -\frac{\partial}{\partial x}(V_{oo} + \mu V_{o1} + \mu^2 V_{o2} + O(\mu^3))$$

$$= -\frac{\partial V_{oo}}{\partial x} - \mu \frac{\partial V_{o1}}{\partial x} - \mu^2 \frac{\partial V_{o2}}{\partial x}$$

it follows that

$$-\underline{m}_{\mu x} = \frac{\partial V_{o1}}{\partial x} \quad \left(= \frac{\partial V_{q1}}{\partial x} = \frac{\partial V_{s1}}{\partial x} \right)$$

(This also follows from (38).) As a consequence, the second-order terms in each of the three cases, from (50s), (50q), and (50o), satisfy

$$\left(\frac{\partial V_{s2}}{\partial x}\right)' \hat{A}x - \underline{m}'_{\mu x} f(x) = 0 \tag{54s}$$

$$\left(\frac{\partial V_{q2}}{\partial x}\right)' \hat{A}x - \underline{m}'_{\mu x} f(x) - \frac{1}{2}\underline{m}'_{\mu x} BQ^{-1}B'\underline{m}_{\mu x} = 0 \tag{54q}$$

$$\left(\frac{\partial V_{o2}}{\partial x}\right)' \hat{A}x - \underline{m}'_{\mu x} f(x) - \frac{1}{2}\underline{m}'_{\mu x} BQ^{-1}B'\underline{m}_{\mu x} + O(1) = 0 \tag{54o}$$

Now let

$$W = V_{s2} - V_{q2}$$

be the difference between the second-order terms of the simplified and the quasi-optimum performance values. Then W satisfies

$$\left(\frac{\partial W}{\partial x}\right)' \hat{A}x + \frac{1}{2}\underline{m}'_{\mu x} BQ^{-1}B'\underline{m}_{\mu x} = 0 \quad , \quad W(0) = 0 \tag{55}$$

The solution of this equation, by characteristics is

$$W = \frac{1}{2} \int_{o}^{\infty} \underline{m}_{\mu x} (\xi (\tau)) BQ^{-1} B' \underline{m}_{\mu x} (\xi (\tau)) d\tau \tag{56}$$

where

$$\xi (\tau) = e^{\hat{A}\tau} x$$

It is evident that W is positive-definite and hence, we have established that $V_{s2} > V_{q2}$. Since $V_{qo} = V_{so}$ and $V_{q1} = V_{s1}$, we have the principal result that

$$V_{s2} > V_{q2} \tag{57}$$

for sufficiently small values of μ.

An alternative approach to the question of estimating the difference between the performance indices of the optimal, quasi-optimal, simplified controls was also considered.

Using the theory developed in the introduction, rewrite (1),

$$\dot{x} = f(x, u(x)) \qquad x(T) = x$$

in the form

$$\dot{x} = \overline{f}(x, p(x)) \qquad x(T) = x \tag{58}$$

where $p(x)$ is an adjoint vector,

$$\overline{f}(x, p(x)) = f(x, \sigma (p(x), x))$$

and where $\sigma (p, x)$ is defined by (6a). The additional assumption is made that the simplified state X is in a lower dimensional subspace L^* of the state space than the actual state space L and that if the solution of (58) starts out in this lower dimensional subspace, it stays in L^* for all time. If the initial state is in this subspace, the exact control problem can be explictly solved, in accordance with the basic assumption of this method of quasi-optimum control.

Let Q be the linear projection operator from L onto L^*. Thus $Q^2 = Q$, Q is the identity on L^* and maps all vectors not in L^* onto the zero vector. Let $p_o(x)$,

$x_o(\tau)$; $p_q(x)$, $x_q(\tau)$; and $p_s(x)$, $x_s(\tau)$ be the adjoint vectors and paths followed by the process using the optimum control, the quasi-optimum control, and the simplified control, respectively. Also let $X(\tau)$ be the path followed by the process using the optimum control but with initial value Qx.

It follows from the above definitions and (59) that

$$\dot{x}_\alpha = \overline{f}(x_\alpha, p_\alpha(x_\alpha)) \qquad x_\alpha(t) = x \qquad \alpha = o, q, s \tag{59}$$

and that

$$\dot{X} = \overline{f}(X, p_o(X)) \qquad X(t) = Qx \tag{60}$$

A possible Taylor series expansion for $p_o(x)$ is

$$p_o(x) = p_o(Qx + (I - Q)x) = p_o(Qx) + \frac{\partial}{\partial Qx} p_o(Qx)(I - Q)x + O((I - Q)x)^2 \tag{61}$$

Assume now that $Qx = X\ (I - Q)x = \xi$; in the new notation

$$\frac{\partial}{\partial Qx} p_o(Qx) = M(X)$$

From (12) and (13)

$$p_s(x) = p_o(Qx) \tag{62}$$

$$p_q(x) = p_o(Qx) + \frac{\partial}{\partial Qx} p_o(Qx)(I - Q)x \tag{63}$$

Now compare the solutions of (59) to that of (60). By assumption, (60) is soluable in closed form and thus all of its properties can be readily determined. If it can be shown that the solutions to (59) either converge to the solution of (60) or stay very close to it, then it follows that the solutions to (59) have the same properties. In particular, this method will sometimes allow comparison of the zeroth index of the various solutions of (59), which gives a way of estimating the performance indices of the various processes. Also by this method, in some cases, one can deduce that the quasi-optimal is (asymptotically) stable. In particular, let

$$x_q = X + \beta_q$$

where x_q is defined by (59) and X is defined by (60). Thus β_q is the difference between (60) and (59). We will derive a differential equation for β. From (59) and (63),

$$\dot{X} + \dot{\beta}_q = \bar{f}(X + \beta_q \ , p_o(Q(X + \beta_q)) + \frac{\partial}{\partial x} p_o(Q(X + \beta_q)(I - Q)(X + \beta_q))$$

$$X(t) + \beta_q(t) = Qx + (I - Q)x$$

By definition, X satisfies (60), and thus by the assumption above, since X starts in L^* , it stays in L^* for all subsequent time. Thus

$$Q(X(\tau)) = X(\tau) \qquad t \le \tau$$

Therefore $(I - Q)X(\tau) = (I - Q)QX(\tau) = (Q - Q^2)X(\tau) = 0$ for $t \le \tau$. We may rewrite (64) as

$$\dot{X} + \dot{\beta}_q = \bar{f}(X + \beta_q \ , p_o(Q(X + \beta_q)) + \frac{\partial}{\partial x} p_o(Q(X + \beta_q))(I - Q)\beta_q) \tag{65}$$

Now expand the right hand side of (65) in a Taylor series about $X(\tau)$ in powers of $\beta_q(\tau)$, to obtain

$$\dot{X} + \dot{\beta}_q = \bar{f}(x, p_o(QX) + \frac{\partial}{\partial x}\bar{f}(X, p_o(QX))\beta_q + \frac{\partial}{\partial p}\bar{f}(X, p(QX))[\frac{\partial}{\partial x} p_o(QX)\beta_q$$
$$+ \frac{\partial}{\partial x} p_o(Q(X)(I - Q)\beta_q] + O(\beta_q^2) \tag{66}$$

We can rewrite (66) as

$$\dot{X} + \beta_q = \bar{f}(X, p_o(QX + [\frac{\partial}{\partial x}\bar{f}(X, p_o(QX) + \frac{\partial}{\partial p}\bar{f}(X, p_o(X, p_o(QX)\frac{\partial}{\partial x} p_o(QX)]\beta_q + O(\beta_q^2) \tag{67}$$

Since $QX = X$ by assumption, and $X = f(X, p_o(X))$ with initial conditions $X(t) = Qx(t)$,

$$\dot{\beta}_q = A(\tau)\beta_q + O(\beta_q^2) \tag{68}$$

with initial conditions $\beta(t) = (\mathcal{I} - Q)x(t)$, and $A(\tau)$ is the matrix

$$A(\tau) = \frac{\partial}{\partial x}\bar{f}(X(\tau), p_o(QX(\tau))) + \frac{\partial}{\partial p}\bar{f}(X(\tau), p_o(QX(\tau))\frac{\partial}{\partial x}p_o(QX(\tau)))$$

Using the definitions of H_{XP} , H_{PP} and M given in section 1, it is seen that

$$A = H_{XP} + H_{PP}M$$

The required differential equation for β_q is given by (68). Sometimes the first-order properties of a differential equation determine the stability of the solution. In particular, if $A(\tau)$ is constant and has all its eigenvalues negative, β_q is asymptotically stable about zero [B2] , if $A(\tau)$ is constant, and has a positive eigenvalue, then the solution to (68) is not stable.

If we write $x_o(\tau) = X(\tau) + \beta_o(\tau)$ and suppose $x_o(\tau)$ to satisfy equation (59) with initial values $x_o(t) = x = Qx + (\mathcal{I} - Q)x$, then using a similar procedure as above, it is found that

$$\begin{aligned} \dot{\beta}_o(\tau) &= A(\tau)\beta_o(\tau) + O(\beta_o^2) \\ \beta_o(t) &= (\mathcal{I} - Q)x \end{aligned} \tag{69}$$

Thus the differential equations satisfied by β_o and β_q are the same, to the first order.

It should be noted that the possible initial conditions of (68) and (69) are not the whole space but only the complement $L - L^*$ of the subspace L^* . More effort is needed to determine if the first order matrices which arise in control differential equations such as (58) have properties which ensure that the solutions of (69) have some type of stability conditioned on the fact that solutions start out in $L - L^*$.

To estimate the difference γ between x_o and x_q we note that this difference satisfies

$$\dot{x}_o - \dot{x}_q = \dot{\beta}_o - \dot{\beta}_q = A(\tau)[x_o - x_q] + O(\beta_o^2) + O(\beta_q^2) \tag{70}$$

i.e.

$$\dot{\gamma} = A(\tau)\gamma + O(\beta_o^2) + O(\beta_q^2) \quad , \quad t \le \tau \le T \tag{71}$$

with the initial condition

$$\gamma(t) = 0 \tag{72}$$

The solution to (71) thus depends on the forcing terms $O(\beta_o^2)$ and $O(\beta_q^2)$. In particular

$$\gamma(\tau) = \int_t^T \Phi(\tau, s)[O(\beta_o^2(s)) + O(\beta_q^2(s))]\, ds$$

where Φ is the fundamental matrix corresponding to $A(\tau)$. An estimate of the forcing term and knowledge of Φ would permit the estimation of γ.

If we write $x_s(\tau) = X(\tau) + \beta_s(\tau)$ and suppose $x_s(t) = Qx + (I - Q)x$, and use the same procedure as used in analyzing x_q and x_o , we arrive at the differential equation

$$\dot{\beta}_s(\tau) = B(\tau)\beta_s(\tau) + O(\beta_s^2)$$

$$\beta_s(t) = (I - Q)x$$

where $B(\tau)$ is the matrix

$$B(\tau) = \frac{\partial}{\partial x}\bar{f}(X(T),\ p_o(QX(\tau))) + \frac{\partial}{\partial p}\bar{f}(X(\tau), p_o(Q(X(\tau)))Q$$

It is seen that the differential equations satisfied by β_s and β_q are different even if one only considers the first term since the matrices $A(\tau)$ and $B(\tau)$ are in general different. It is therefore perfectly possible that β_o and β_q will be stable about zero, and that β_s will converge to infinity. This fact will be brought out in the following example.

Consider the control problem governed by

$$\dot{x}_0 = \frac{1}{2}(x_1^2(\tau) + u^2(x(\tau)))$$

$$\dot{x}_1 = -x_2(\tau)x_1(\tau) + u(x(\tau))$$

$$\dot{x}_2 = 0 \tag{73}$$

$$\dot{x}_3 = 1$$

with initial conditions $x_0(t) = 0$, $x_1(t) = \omega$, $x_2(t) = a$, $x_3(t) = t$.

We wish to compute $u(x(\tau))$ $(x(\tau) = \{x_0(\tau) , x_1(\tau) , x_2(\tau) , x_3(\tau)\})$ in such a way that $x_0(T) = \frac{1}{2}\int_t^T (x_1^2(\tau) + u^2(x(\tau))d\tau$ is minimized. Proceeding as in the introduction, we write (see(3)),

$$h(p , x, u) = \frac{p_0}{2}(x_1^2 + u^2) - p_1x_2x_1 + p_1u + p_3$$

h is maximized with respect to u if

$$u = -\frac{p_1}{p_0}$$

Thus $\bar{f}(x , p(x))$ (see(58)) is

$$\bar{f}_0(x , p(x)) = \frac{1}{2}\left[x_1^2 + \left(\frac{p_1(x)}{p_0(x)}\right)^2\right]$$

$$\bar{f}_1(x , p(x)) = -x_2x_1 - \frac{p_1(x)}{p_0(x)}$$

$$\bar{f}_2(x , p(x)) = 0 \tag{74}$$

$$\bar{f}_3(x , p(x)) = 1$$

Using the Pontryagin maximum principle, we arrive at the two point boundary value problem

$$p_{s1}(x) = x_1 \left[\frac{\exp[x_3 - T] - \exp[T - x_3]}{\exp[x_3 - T] + \exp[T - x_3]} \right]$$

$$p_{q0}(x) = -1 + m_{02}x_2$$

and that

$$p_{q1}(x) = x_1 \left[\frac{\exp[x_3 - T] - \exp[T - x_3]}{\exp[x_3 - T] + \exp[T - x_3]} \right] + m_{12}x_2$$

where m is defined by (19). After some calculation, we find that

$$m_{02} = 0 \qquad \text{and} \qquad m_{12} = x_1 \left[\frac{(\exp[x_3 - T] - \exp[T - x_3])^2}{(\exp[x_3 - T] - \exp[T - x_3])^2} \right]$$

Thus

$$\bar{f}_0(x_s , p_s(x_s)) = \dot{x}_{s0} = \frac{1}{2} \left(x_{s1}^2 + x_{s1}^2 \left[\frac{\exp[x_{s3} - T] - \exp[T - x_{s3}]}{\exp[x_{s3} - T] - \exp[T - x_{s3}]} \right]^2 \right)$$

$$\bar{f}_1(x_s , p_s(x_s)) = \dot{x}_{s1} = - x_{s2}x_{s1} + x_{s1} \left[\frac{\exp[x_{s3} - T] - \exp[T - x_{s3}]}{\exp[x_{s3} - T] - \exp[T - x_{s3}]} \right]$$

$$\bar{f}_2(x_s , p_s(x_s)) = \dot{x}_{s2} = 0$$

$$\bar{f}_3(x_s , p_s(x_s)) = \dot{x}_{s3} = 1$$

and

$$\bar{f}_0(x_q , p_q(x_q)) = x_{q0} = \frac{1}{2} (x_{q1}^2 + x_{q1}^2 \left(\left[\frac{\exp[x_3 - T] - \exp[T - x_3]}{\exp[x_3 - T] - \exp[T - x_3]} \right] \right.$$

$$\left. + x_{q2} \left[\frac{\exp[x_3 - T] - \exp[T - x_3]}{\exp[x_3 - T] - \exp[T - x_3]} \right]^2 \right)^2$$

$$\bar{f}_1(x_q, p_q(x_q)) = \dot{x}_{q1} = -x_{q2}x_{q1} + x_{q1}\left(\left[\frac{\exp[x_{q3} - T] - \exp[T - x_{q3}]}{\exp[x_{q3} - T] + \exp[T - x_{q3}]}\right] + \left[\frac{\exp[x_{q3} - T] - \exp[T - x_{q3}]}{\exp[x_{q3} - T] + \exp[T - x_{q3}]}\right]^2 x_{q2}\right)$$

$$\bar{f}_2(x_q, p_q(x_q)) = \dot{x}_{q2} = 0$$

$$\bar{f}_3(x_q, p_q(x_q)) = \dot{x}_{q3} = 1$$

with initial conditions

$$x_{s0} = x_{q0} = 0$$

$$x_{s1} = x_{q1} = \omega$$

$$x_{s2} = x_{q2} = a$$

$$x_{s3} = x_{q3} = t$$

Solving these equations we find that

$$x_{1s}(\tau) = \left[\frac{\exp|T - \tau| + \exp|\tau - T|}{\exp|t - T| + \exp|T - t|}\right] \exp|-a(\tau - t|\,\omega$$

$$x_{2s}(\tau) = a$$

$$x_{3s}(\tau) = \tau$$

$$V_s = \frac{\omega^2}{2}\left(\frac{1}{\exp[t - T] + \exp[T - t]}\right)^2 \left[\frac{2a\exp[-2a(T - t)] + (1 - a)\exp[2(T - t)] - (1 + a)\exp[2(t - T)]}{1 - a^2}\right]$$

and that

$$x_{1q} = \omega \exp\left[\frac{2a}{1+\exp[2(\tau - T)]} - \frac{2a}{1+\exp[2(t - T)]}\right]\left(\frac{\exp[t-\tau] + \exp[\tau - T]}{\exp[t - T] + \exp[T - t]}\right)$$

$$x_{2q} = a$$

$$x_{3q} = \tau$$

$$V_q = \frac{\omega^2}{2(\exp[t-T] + \exp[T-t])^2}\int_t^T \exp\left[\frac{4a}{1+\exp[2(\tau - T)]} - \frac{4a}{1+\exp[2(t-T)]}\right]$$

$$\left[(\exp[T-\tau] + \exp[\tau - T])^2 + (\exp[T-\tau] - \exp[\tau - T])^2 + 2a\,\frac{(\exp[\tau - T] - \exp[T-\tau])^3}{\exp[T-\tau] + \exp[\tau - T]}\right] d\tau$$

If we let $T \to \infty$, then the preceding equations simplify considerably. We have

$$V_o = \frac{\omega^2}{2}\left[\sqrt{1+a^2} - a\right]$$

$$x_{o1} = \exp\left[-\sqrt{a^2+1}\;(\tau - t)\right]\omega$$

$$x_{o2} = a$$

$$x_{o3} = \tau$$

$$V_s = \frac{\omega^2}{2}\left(\frac{1}{a+1}\right) \qquad (a > -1)$$

$$x_{1s} = \exp[-(a+1)(\tau - t)]\,\omega$$

$$x_{2s} = a$$

$$x_{3s} = \tau$$

and

$$V_q = \frac{\omega^2}{2}(1 - a + \frac{a^2}{2})$$

$$x_{1q} = \exp\{-(\tau - t)\}\,\omega$$

$$x_{2q} = a$$

$$x_{3q} = \tau$$

It can also be seen that $X(\tau)$ is

$$X_1(\tau) = \exp\{-(\tau - t)\}\,\omega$$

$$X_2(\tau) = 0$$

$$X_3(\tau) = \tau$$

The matrix $A(\tau)$ is

$$A(\tau) = \begin{bmatrix} 0 & 2\,\omega \exp[t - \tau] & -\omega^2 \exp[2(t - \tau)] & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is seen that the solution to

$$\dot{\beta}_\alpha(\tau) = A(\tau)\beta_\alpha(\tau) \qquad \alpha = o\,,\, q$$

$$\beta_\alpha(t) = (I - Q)x(t) = \{0\,,\,0\,,\,a\,,\,0\}$$

is

$$\beta_\alpha(\tau) = \begin{bmatrix} \frac{1}{2}\int_t^\tau \omega^2 a \exp[t-s]\,ds \\ 0 \\ a \\ 0 \end{bmatrix}$$

The matrix $B(\tau)$ is

$$B(\tau) = \begin{bmatrix} 0 & 2\omega\exp[t-\tau] & 0 & 0 \\ 0 & -1 & -\exp[t-\tau]\,\omega & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution to

$$\dot{\beta}_s(\tau) = B(\tau)\beta_s(\tau)$$

$$\beta_s(t) = (I-Q)x(t) = \{0, 0, a, 0\}$$

is

$$\beta_s(\tau) = \begin{bmatrix} 2\omega^2 a\int_t^\tau (t-s)\exp[t-s]\,ds \\ \omega a(t-\tau)\exp[t-\tau] \\ a \\ 0 \end{bmatrix}$$

If $x_o(\tau)$, $x_q(\tau)$ and $x_s(\tau)$, which we computed previously, are expanded out in powers of a, then it is seen that the zeroth order term for each is $X(\tau)$ and the first order terms are $\beta_o(\tau)$, $\beta_q(\tau)$ and $\beta_s(\tau)$ respectively. The first order correction for the performance index at ∞ is $-(\omega^2/2)a$ for all three, i.e.

$$\beta_{o0}(\infty) = \beta_{q0}(\infty) = \beta_{s0}(\infty) = -\frac{\omega^2}{2}a$$

1.2 STOCHASTIC OPTIMUM CONTROL

Consider the problem of finding an optimum control law for the process

$$\dot{x} = f(x, u) + v \tag{75}$$

with performance criterion

$$V(t, x, u) = E\{\int_t^T L(x(\tau), u(\tau))d\tau + g[x(T)] \mid x(t) = y\} \tag{76}$$

where $E\{\cdot\}$ denotes expectation, $x = \{x_1, \ldots, x_n\}$ is the state vector, $u = \{u_1, \ldots, u_m\}$ is the control vector, $f = \{f_1, \ldots, f_n\}$ is a vector-valued function, and $\{v\}$ is a zero-mean, vector white-noise distrubance process with $E\{v(t)v'(\tau)\} = \Sigma\delta(t - \tau)$. The state is required to satisfy the boundary conditions

$$\varphi(x(T)) = 0 \tag{77}$$

where $\varphi = \{\varphi_1, \ldots, \varphi_s\}$, $s \le n$, and the control u may be required to be a member of a closed, bounded set Ω.

The optimum control which minimizes (76) for the process (75) satisfies the stochastic Hamilton-Jacobi equation [B3]

$$-\frac{\partial V}{\partial t} = \operatorname*{Min}_{u\in\Omega} \{L(y, u) + \mathcal{L}[V]\} \tag{78}$$

Subjecr to the terminal condition

$$V(T, x, u) = g(x) \tag{79}$$

where $\mathcal{L}$ is the differential generator for (75),

$$\mathcal{L}[V] = \sum_{i=1}^{n} f_i(y, u)\frac{\partial V}{\partial y_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1} D_{ij}\frac{\partial^2 V}{\partial y_i \partial y_j} \tag{80}$$

and

$$D = [D_{ij}] = G\Sigma G' \tag{81}$$

An approximate solution to (78) will be obtained under the assumption that the noise acting on the process is "small" (i.e., that Σ is a "small" matrix). Suppose Σ is a diagonal matrix,

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \cdots & \cdots & 0 \\ \vdots & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma_n^2 \end{bmatrix} \tag{82}$$

Then (78) becomes

$$-\frac{\partial V}{\partial t} = \underset{u \in \Omega}{\text{Min}} \left\{ L + \sum_{i=1}^{n} f_i \frac{\partial V}{\partial y_i} + \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2 \frac{\partial^2 V}{\partial y_i^2} \right\} \tag{83}$$

Stratonovich [B4] has suggested the following iterative technique for solving (83): first obtain the noise-free solution $V^{(0)}$ by setting all $\sigma_i^2 \equiv 0$ in (83). Then use the following recursive scheme to obtain an approximate solution:

$$-\frac{\partial V^{(k+1)}}{\partial t} = \underset{u \in \Omega}{\text{Min}} \left\{ L + \sum_{i=1}^{n} f_i \frac{\partial V^{(k+1)}}{\partial y_i} + \sum_{i=1}^{n} \left(\frac{\sigma_i^2}{2}\right) \frac{\partial^2 V^{(k)}}{\partial y_i^2} \right\} \tag{84}$$

Thus, once $V^{(0)}$ is determined one must solve a sequence of nonlinear, first-order partial differential equations for the controls $u^{(k)}$ and performance $V^{(k)}$. Unfortunately, even for systems (75) with relatively simple structure there appears to be no way of finding an exact solution to the first equation for $V^{(1)}$,

$$-\frac{\partial V^{(1)}}{\partial t} = \underset{u \in \Omega}{\text{Min}} \left\{ L + \sum_{i=1}^{n} f_i \frac{\partial V^{(1)}}{\partial y_i} + \sum_{i=1}^{n} \frac{\sigma_i^2}{2} \frac{\partial^2 V^{(0)}}{\partial y_i^2} \right\} \tag{85}$$

Quasi-Optimum Control - An approximate solution for $V^{(1)}$ will be obtained by using the quasi-optimum control technique developed by Friedland [A1]. Note that if the constants $\sigma_i^2/2$ are defined as additional state variables then (85) corresponds to the following set of canonical equations

$$\dot{x} = h_p = \begin{bmatrix} -L(y, u^*(x,p)) \\ 1 \\ f(y, u^*(x,p)) \\ 0 \end{bmatrix} \qquad \dot{p} = -h_x = -\begin{bmatrix} 0 \\ 0 \\ (\frac{\partial f}{\partial y})' p - (\frac{\partial F}{\partial y})' \frac{\sigma^2}{2} \\ -F(y) \end{bmatrix} \tag{86}$$

where $x = \{x_0, \tau, y, \frac{\sigma^2}{2}\}$, $p = \{p_0, p_\tau, p_y, p_\sigma\}$, $\sigma^2/2 = \{\sigma_1^2/2, \ldots, \sigma_n^2/2\}$,
$F = \{\partial^2 V^{(0)}/\partial y_1^2, \ldots, \partial^2 V^{(0)}/\partial y_n^2\}$,

$$p_0 = -\partial V/\partial x_0, \qquad p_\tau = -\partial V/\partial \tau, \qquad p_y = -\partial V/\partial y, \qquad p_\sigma = -\partial V/\partial(\frac{\sigma^2}{2}) \tag{87}$$

and for notational simplicity we have used $V = V^{(1)}$. The Hamiltonian function h is given by

$$h = p_\tau + p_0 L + p_y' f - (\frac{\sigma^2}{2})' F \tag{88}$$

and

$$h(x, p, u^*) = \max_{u \in \Omega} h(x, p, u) \tag{89}$$

The following boundary conditions apply at the terminal time T

$$\begin{aligned} x_0(T) &= \text{minimum} & p_0(T) &= -1 \\ \tau(T) &= T & p_\tau(T) &= \begin{cases} \text{free for } T \text{ fixed} \\ 0 \text{ for } T \text{ free} \end{cases} \\ \varphi(y(T)) &= 0 & p_y(T) &= \Phi'\lambda \\ \sigma^2/2 & & p_\sigma(T) &= 0 \end{aligned} \tag{90}$$

where

$$\Phi = [\partial\varphi_i/\partial y_j] \qquad i = 1, 2, \ldots, s; \; j = 1, \ldots, n \tag{91}$$

and λ is an arbitrary s - vector.

If the optimum control law for (85) can be expressed as

$$u^* = f(x, p(x)) \tag{92}$$

the approximate solution is written as

$$u = f(x, P(X) + M(X)\xi) \tag{93}$$

where X is the state of the noise-free system, $x = X + \xi$, and the correction matrix M can be obtained with the aid of the auxiliary system

$$\begin{aligned} \dot{\xi} &= H_{XP}\xi + H_{PP}\psi \\ \dot{\psi} &= -H_{XX}\xi - H_{PX}\psi \end{aligned} \tag{94}$$

where

$$\psi = M\xi \tag{95}$$

and where the coefficient matrices $H_{XP}, \ldots, H_{XX}$ are matrices of second partial derivatives of the Hamiltonian of the exact problem evaluated at $x = X$. The boundary conditions for (94) are

$$\begin{aligned} \Phi[\xi(T) + \dot{X}(T)dT] &= 0 \\ \psi(T) + \dot{P}(T)dT &= \Phi'\eta \\ X'\psi &= P'\xi \end{aligned} \tag{96}$$

Derivations of (94), (95), (96) are given in [A1].

This approach will be applied to the second-order process

$$\begin{aligned} \dot{x}_1 &= u_1 + v_1 \\ \dot{x}_2 &= u_2 + v_2 \end{aligned} \tag{97}$$

where the controls u_1, u_2 are subject to the constraint

$$u_1^2 + u_2^2 = 1 \tag{98}$$

and the random disturbances v_1 and v_2 are zero-mean white noise processes with variance

$E\{v_i(t)v_i(\tau)\} = \sigma_i^2\,\delta(t-\tau)$, $i = 1,2$. The problem is to choose u_1 and u_2 to minimize the expected time to hit a circle centered at the origin. Thus the performance criterion is the conditional expectation

$$V = E\{\int_t^T d\lambda \mid x_1(t) = y_1,\ x_2(t) = y_2\} \tag{99}$$

where t denotes the current time and T is the first time that the random process $\{x_1, x_2\}$ hits the circle S,

$$S = \{x_1, x_2;\ x_1^2 + x_2^2 = R^2\} \tag{100}$$

The stochastic Hamilton-Jacobi equation for this problem

$$0 = \min_{u_1^2+u_2^2=1} \{1 + u_1\frac{\partial V}{\partial y_1} + u_2\frac{\partial V}{\partial y_2} + \frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial y_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial y_2^2}\} \tag{101}$$

Thus the "optimum" control is given by

$$u_1 = -\frac{\partial V/\partial y_1}{[(\frac{\partial V}{\partial y_1})^2 + (\frac{\partial V}{\partial y_2})^2]^{\frac{1}{2}}}, \qquad u_2 = -\frac{\partial V/\partial y_2}{[(\frac{\partial V}{\partial y_1})^2 + (\frac{\partial V}{\partial y_2})^2]^{\frac{1}{2}}} \tag{102}$$

and the "optimum" performance satisfies

$$0 = 1 - [(\frac{\partial V}{\partial y_1})^2 + (\frac{\partial V}{\partial y_2})^2]^{\frac{1}{2}} + \frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial y_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial y_2^2} \tag{103}$$

subject to the boundary condition

$$V(y_1, y_2) = 0 \qquad \text{for} \qquad y_1^2 + y_2^2 = R^2 \tag{104}$$

The canonical equations corresponding to (103) are

$$\begin{aligned} \dot{y}_1 &= p_1[p_1^2 + p_2^2]^{-\frac{1}{2}} \\ \dot{y}_2 &= p_2[p_1^2 + p_2^2]^{-\frac{1}{2}} \\ \tfrac{1}{2}(\dot{\sigma}_1^2) &= 0 \\ \tfrac{1}{2}(\dot{\sigma}_2^2) &= 0 \end{aligned} \tag{105}$$

and

$$\dot{p}_1 = -\frac{\partial h}{\partial y_1} = -\frac{1}{2}[\sigma_1^2 \frac{\partial F_1}{\partial y_1} + \sigma_2^2 \frac{\partial F_2}{\partial y_1}]$$

$$\dot{p}_2 = -\frac{\partial h}{\partial y_2} = -\frac{1}{2}[\sigma_1^2 \frac{\partial F_1}{\partial y_2} + \sigma_2^2 \frac{\partial F_2}{\partial y_2}]$$

$$\dot{\lambda}_1 = -\frac{\partial h}{\partial\left(\frac{\sigma_1^2}{2}\right)} = -F_1 \tag{106}$$

$$\dot{\lambda}_2 = -\frac{\partial h}{\partial\left(\frac{\sigma_2^2}{2}\right)} = -F_2$$

where

$$p_i = \partial V/\partial y_i \quad , \quad \lambda_i = \partial V/\partial(\sigma_i^2/2) \quad , \quad i = 1, 2 \tag{107}$$

The solution $\bar{V}(y_1, y_2)$ to the "noise-free" problem

$$0 = 1 - [(\partial\bar{V}/\partial y_1)^2 + (\partial\bar{V}/\partial y_2)^2]^{\frac{1}{2}} \tag{108}$$

is well-known and is given by

$$\bar{V} = [(y_1^2 + y_2^2)^{\frac{1}{2}} - R] \tag{109}$$

Thus, $P_i = +\partial\bar{V}/\partial y_i$ are given by

$$P_1 = y_1(y_1^2 + y_2^2)^{-\frac{1}{2}}$$

$$P_2 = y_2(y_1^2 + y_2^2)^{-\frac{1}{2}} \tag{110}$$

From (106) it is seen that P_1 and P_2 are constant in time and thus from (105),

$$X_1(\tau) = y_1 - [y_1(y_1^2 + y_2^2)^{-\frac{1}{2}}](\tau - t)$$

$$X_2(\tau) = y_2 - [y_2(y_1^2 + y_2^2)^{-\frac{1}{2}}](\tau - t) \tag{111}$$

and

$$X_1^2(\tau) + X_2^2(\tau) = [(y_1^2 + y_2^2)^{\frac{1}{2}} - (\tau - t)]^2 \tag{112}$$

The quasi-optimum control law is found by letting

$$\frac{\partial V}{\partial y_1} = P_1 + m_{13}\left(\frac{\sigma_1^2}{2}\right) + m_{14}\left(\frac{\sigma_2^2}{2}\right)$$
$$\frac{\partial V}{\partial y_2} = P_2 + m_{23}\left(\frac{\sigma_1^2}{2}\right) + m_{24}\left(\frac{\sigma_2^2}{2}\right) \tag{113}$$

where the correction terms $m_{i,j}$ are determined from the solution to

$$\begin{aligned}
\dot{\xi}_1 &= (1 + X_1^2/r^2)\psi_1 + (X_1X_2/r^2)\psi_2 \\
\dot{\xi}_2 &= (X_1X_2/r^2)\psi_1 + (1 + X_2^2/r^2)\psi_2 \\
\dot{\xi}_3 &= 0 \ , \ \dot{\xi}_4 = 0 \\
\dot{\psi}_1 &= (3X_1X_2^2/r^5)\xi_3 - (X_1(2X_2^2 - X_1^2)/r^5)\xi_4 \\
\dot{\psi}_2 &= -(X_2(2X_1^2 - X_2^2)/r^5)\xi_3 + (3X_2X_1^2/r^5)\xi_4 \\
\dot{\psi}_3 &= (3X_1X_2^2/r^5)\xi_1 - (X_2(2X_1^2 - X_2^2)/r^5)\xi_2 \\
\dot{\psi}_4 &= -(X_1(2X_2^2 - X_1^2)/r^5)\xi_1 + (3X_2X_1^2/r^5)\xi_2
\end{aligned} \tag{114}$$

with boundary conditions

$$\begin{aligned}
\psi_1(T) &= 2R\eta y_1/r \quad , \quad \psi_2(T) = 2R\eta y_2/r \\
\psi_3(T) &= -dT(1 - y_1^2/R^2)/R \\
\psi_4(T) &= -dT(1 - y_2^2/R^2)/R \\
0 &= [y_1\xi_1(T) + y_2\xi_2(T)]/r + dT \\
0 &= [y_1\psi_1(t) + y_2\psi_2(t)]/r - (y_2^2/r^2)\xi_3 - (y_1^2/r^2)\xi_4
\end{aligned} \tag{115}$$

where $r^2 = y_1^2 + y_2^2$. Since ξ_3 and ξ_4 are constant, it is a simple matter to integrate (114) and to apply (115) to obtain

$$
\begin{aligned}
m_{13} &= \frac{y_1 y_2^2}{r^3}\left[\frac{3}{r} - \frac{2}{R}\right] \\
m_{14} &= -\frac{2y_1 y_2^2}{r^3}\left[\frac{1}{r} - \frac{1}{R}\right] + \frac{y_1^3}{r^4} \\
m_{23} &= -\frac{2y_2 y_1^2}{r^3}\left[\frac{1}{r} - \frac{1}{R}\right] + \frac{y_2^3}{r^4} \\
m_{24} &= \frac{y_2 y_1^2}{r^3}\left[\frac{3}{r} - \frac{2}{R}\right]
\end{aligned}
\tag{116}
$$

Thus, the quasi-optimum control law is given by (102) where

$$
\begin{aligned}
\frac{\partial V}{\partial y_2} &= \frac{y_1}{r} + \frac{\sigma_1^2}{2}\left[\frac{y_1 y_2^2}{r^3}\left(\frac{3}{r} - \frac{2}{R}\right)\right] + \frac{\sigma_2^2}{2}\left[-\frac{2y_1 y_2^2}{r^3}\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{y_1^3}{r^4}\right] \\
\frac{\partial V}{\partial y_2} &= \frac{y_2}{r} + \frac{\sigma_1^2}{2}\left[-\frac{2y_2 y_1^2}{r^3}\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{y_2^3}{r^4}\right] + \frac{\sigma_2^2}{2}\left[\frac{y_2 y_1^2}{r^3}\left(\frac{3}{r} - \frac{2}{R}\right)\right]
\end{aligned}
\tag{117}
$$

It is interesting to note that in the symmetric special case, $\sigma_1^2 \equiv \sigma_2^2$ the above control law reduces to the exact optimum control law which can be derived directly from the solution of (103) with $\sigma_1^2 \equiv \sigma_2^2$.

By using the quasi-optimum solution as the input to next stage of the iteration (84) (i.e., by compounding the approximation) a sequence $V^{(k)}$, $u^{(k)}$ is generated. The question of convergence of this sequence to the optimum solution has not yet been thoroughly investigated. However, in practical problems one would expect that the first iteration should yield a satisfactory control law.

We have obtained a Monte-Carlo simulation of the performance of the above second-order process using both the quasi-optimum and simplified control laws. For all cases considered $\sigma_1^2 = 0$, the initial state was (5,5), and the average time to reach a circle of radius 1.0 was recorded for a 10-member "ensemble". Table 1 contains the computer results for various values of σ_2^2.

TABLE 1

AVERAGE TIME TO REACH UNIT CIRCLE

σ_2^2	Quasi-Optimum	Simplified	% Improvement
.1	5.679	5.682	.053
1	6.162	6.208	.75
2	7.206	7.235	.40
2.25	7.117	7.622	7.1
3	8.629	8.953	3.8
4	9.316	9.414	1.1
4.5	9.141	9.204	.68
4.8	6.977	6.611	-5.2
5	-	-	-

The fact that the expected hitting time for both the quasi-optimum and simplified systems does not increase monotonically with increasing σ_2^2 indicates that a larger ensemble should probably be used in the performance evaluation. The quasi-optimum control seems to provide an improvement which has a maximum at a value of σ_2^2 between 2 and 3. At $\sigma_2^2 = 4.8$ the simplified control law provides better performance than does the quasi-optimum control law, and for $\sigma_2^2 \doteq 5.0$ neither system can reach the unit circle. Since the improvement in performance is not striking in this example further study of stochastic quasi-optimum control is required.

PART II. APPLICATIONS

In order to verify the validity of the quasi-optimum control technique and to obtain some qualitative insight into some of the difficulties and limitations of the method, a number of "practical" problems to which the technique appears to be applicable were studied. The word "practical" is enclosed in quotation marks here to emphasize that even the equations (1) for the exact model entailed a considerable simplification of the actual physical behavior of the process; the simplified model (10) is a still further simplification. In all cases considered the further simplification led to a lower-order system of differential equations. In these applications, one of the devices employed as a basis for using the quasi-optimum control technique is to represent parameters which are small by additional state variables x_i with the differential equations $\dot{x}_i = 0$. The simplified problem is then the same order as the original problem before introduction of the additional state variables, but is simple enough to permit an analytic solution.

No theoretical difficulties were encountered in any of the examples studied; the algebraic calculations, however, although straightforward, were quite tedious and involved. Consequently progress was slow and calculations had to be checked frequently.

The following problems were considered.

1. Three-Axis Attitude Control of a Space Vehicle
2. Minimum-Time, Bounded Acceleration Rendezvous in a Central Force Field
3. Aircraft Landing Problem

In the sections below the equations for each problem are numbered according to the format $(i - 1)$, $(i - 2)$, . . . , where $i = 1, 2,$ or 3 according to which of the above problems is under discussion.

2.1 THREE-AXIS ATTITUDE CONTROL OF A SPACE VEHICLE

The first application study is that of controlling the attitude of a space vehicle in which the gyroscopic coupling torques are small but not negligible.

Problem Formulation - The equations governing the components of angular velocity along principal axes of the vehicle are

$$\dot{\omega}_i = [(I_j - I_k)\,\omega_j \omega_k + c_i f_i]/I_i \qquad \begin{array}{l} i, j, k = 1,2,3 \text{ in cyclic order} \\ i \neq j \neq k \end{array} \tag{1-1}$$

where I_i represents the momont of inertia about the $i\underline{^{th}}$ principal axis, ω_i represents the component of angular velocity along the $i\underline{^{th}}$ principal axis, c_i represents the moment arm of jet control, and f_i represents the thrust of jet control,

$$|f_i(t)| \leq M_i \qquad i = 1, 2, 3 \tag{1-2}$$

Three additional coordinates required to completely describe the vehicle attitude are the Euler angles $[\theta_1, \theta_2, \theta_3]$ defined as in [B5] :

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\cos\theta_3}{\cos\theta_2} & -\dfrac{\sin\theta_3}{\cos\theta_2} & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ -\tan\theta_2\cos\theta_3 & \tan\theta_2\sin\theta_3 & 1 \end{bmatrix} \omega \tag{1-3}$$

Thus (1-1) and (1-3) describe the vehicle attitude motion.

We will assume that the angles θ_2 and θ_3 are sufficiently small throughout the control interval so that the matrix in (1-3) becomes the identity matrix. Then the equations of motion reduce to

$$\begin{aligned} \dot{\theta}_i &= \omega_i \qquad & i, j, k = 1, 2, 3 \\ & & i \neq j \neq k \\ \dot{\omega}_i &= [(I_j - I_k)\,\omega_j\omega_k + c_i f_i]/I_i \end{aligned} \tag{1-4}$$

We will define the state variables

$$x_0 = t$$

$$x_i = I_i \theta_i \qquad i = 1, 2, 3 \qquad (1\text{-}5)$$

$$x_{i+3} = I_i \omega_i$$

The cross-axis inertia ratios $(I_j - I_k)/I_j I_k$ are assumed to be small but nonzero, and are represented by additional state variables

$$x_{6+i} = (I_j - I_k)/I_j I_k \qquad \begin{array}{l} i, j, k = 1, 2, 3 \\ i \neq j \neq k \end{array} \qquad (1\text{-}6)$$

Hence the state equations can be written as

$$\begin{aligned}
\dot{x}_0 &= 1 \\
\dot{x}_1 &= x_4 \\
\dot{x}_2 &= x_5 \\
\dot{x}_3 &= x_6 \\
\dot{x}_4 &= x_7 x_5 x_6 + k_1 u_1(t) \\
\dot{x}_5 &= x_8 x_6 x_4 + k_2 u_2(t) \\
\dot{x}_6 &= x_9 x_4 x_5 + k_3 u_3(t) \\
\dot{x}_7 &= \dot{x}_8 = \dot{x}_9 = 0
\end{aligned} \qquad (1\text{-}7)$$

where $k_i = c_i M_i$ and

$$|u_i(t)| \leq 1 \qquad i = 1, 2, 3 \qquad (1\text{-}8)$$

To simplify notation we will assume that $k_1 = k_2 = k_3 = 1$.

The problem is to minimize the time required to reduce $x_1, \ldots x_6$, to zero, i.e., to minimize $x_0(T)$ subject to the constraint (1-8), where $x_i(T) = 0$, $i = 1, 2, \ldots, 6$. The Hamiltonian for this problem is

$$h = p_0 + p_1 x_4 + p_2 x_5 + p_3 x_6 + p_4 x_7 x_5 x_6 + p_4 u_1$$
$$+ p_5 x_8 x_6 x_4 + p_5 u_2 + p_6 x_9 x_4 x_5 + p_6 u_3 \tag{1-9}$$

Maximization of h with respect to u_1, u_2 and u_3 results in the optimum control law

$$u_i = \mathrm{sgn}\,(p_{i+3}) \qquad i = 1, 2, 3 \tag{1-10}$$

where the adjoint variables p_i, $i = 0, 1, \ldots, 9$ satisfy

$$\begin{aligned}
\dot{p}_0 &= \dot{p}_1 = \dot{p}_2 = \dot{p}_3 = 0 \\
\dot{p}_4 &= -p_1 - p_5 x_8 x_6 - p_6 x_9 x_5 \\
\dot{p}_5 &= -p_2 - p_4 x_7 x_6 - p_6 x_9 x_4 \\
\dot{p}_6 &= -p_3 - p_4 x_7 x_5 - p_5 x_8 x_4 \\
\dot{p}_7 &= -p_4 x_5 x_6 \\
\dot{p}_8 &= -p_5 x_6 x_4 \\
\dot{p}_9 &= -p_6 x_4 x_5
\end{aligned} \tag{1-11}$$

Simplified System - Suppose the cross-axis inertia ratios are zero $(x_7 \equiv x_8 \equiv x_9 \equiv 0)$. Then from (1-7) the three axes are uncoupled, and the optimum control law for each axis can be obtained from the well-known solution to the Bushaw problem [A2]. Thus we select

$$X = [x_0, x_1, \ldots, x_6, 0, 0, 0] \tag{1-12}$$

as the state of the simplified problem.

The Hamiltonian for the simplified problem is

$$H = P_0 + \sum_{i=1}^{3} P_i x_{i+3} + \sum_{i=1}^{3} u_i P_{i+3} \tag{1-13}$$

where $P = [P_0, P_1, \ldots, P_6, 0, 0, 0]$ is the adjoint vector for the simplified problem. The maximum principle applied to (1-13) yields the optimum control for the simplified problem

$$u_i = \mathrm{sgn}\,(P_{i+3}) \qquad i = 1, 2, 3 \qquad (1\text{-}14)$$

where the adjoint varables satisfy

$$\begin{aligned} \dot{P}_0 &= 0 \\ \dot{P}_i &= 0 \qquad i = 1, 2, 3 \\ \dot{P}_{i+3} &= -P_i \end{aligned} \qquad (1\text{-}15)$$

By integrating (1-15) we find that

$$\begin{aligned} u_i &= \mathrm{sgn}\,[P_{(i+3)0} - P_{i0}t] \\ &= \begin{cases} U_i\,, & t < t_i = P_{(i+3)0}/P_{i0} \\ U_i\,, & t > t_i \end{cases} \qquad i = 1, 2, 3 \end{aligned} \qquad (1\text{-}16)$$

where

$$U_i = \mathrm{sgn}\,(P_{(i+3)0}) = \pm 1$$

Substituting (1-16) into (1-7) with $x_7 = x_8 = x_9 = 0$, and integrating to the terminal time T_i, results in expressions for $x_i(T_i)$ and $x_{i+3}(T_i)$ as functions of T_i, t_i, and the initial conditions. Solving simultaneously for T_i and t_i yields

$$t_i = -\frac{X_{(i+3)0}}{U_i} + \left[\frac{X^2_{(i+3)0}}{2} - U_i X_{i0}\right]^{\frac{1}{2}} \qquad i = 1, 2, 3 \qquad (1\text{-}17a)$$

$$T_i = -\frac{X_{(i+3)0}}{U_i} + 2\left[\frac{X^2_{(i+3)0}}{2} - U_i X_{i0}\right]^{\frac{1}{2}} \qquad (1\text{-}17b)$$

By using (1-13) and (1-17a) we find the following expressions for the initial adjoint variables in terms of the initial state variables:

$$P_{i0} = \frac{U_i}{\left[\frac{X^2_{(i+3)0}}{2} - U_i X_{i0}\right]^{\frac{1}{2}}} \tag{1-18a}$$

and $\qquad i = 1, 2, 3$

$$P_{(i+3)0} = \frac{1}{U_i} - \frac{X_{(i+3)0}}{\left[\frac{X^2_{(i+3)0}}{2} - U_i X_{i0}\right]^{\frac{1}{2}}} \tag{1-18b}$$

Since the initial state is arbitrary, dropping the subscript zero in (1-18a) results in the general relations for the adjoint variables in terms of the state variables. Substitution of (1-18b) into (1-14) leads to the well-known control law for the simplified problem.

If the cross-axis inertia ratios are not zero, however, then this control may be unsatisfactory for the original problem. Thus we will modify this simplified control law to take into account small, but non-zero, cross-axis inertia ratios.

Quasi-Optimum Control Law - In accordance with the procedure outlined in the Introduction, the quasi-optimum feedback control law is given by

$$\begin{aligned} u_1 &= \text{sgn}\,(P_4 + m_{47}x_7 + m_{48}x_8 + m_{49}x_9) \\ u_2 &= \text{sgn}\,(P_5 + m_{57}x_7 + m_{58}x_8 + m_{59}x_9) \\ u_3 &= \text{sgn}\,(P_6 + m_{67}x_7 + m_{68}x_8 + m_{69}x_9) \end{aligned} \tag{1-19}$$

where P_4, P_5, P_6 are the adjoint variables of the simplified system defined in the last section and the m_{ij} are components of the correction matrix M. The correction matrix is obtained by finding the fundamental matrix for the auxiliary system where the coefficient matrices are given by

$$
H_{XP} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & X_5X_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_6X_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_4X_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{1-20}
$$

$$
H_{PX} = H'_{XP} \tag{1-21}
$$

$$
H_{XX} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_5X_6 & P_6X_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & P_4X_6 & 0 & P_6X_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & P_4X_5 & P_5X_4 & 0 \\
0 & 0 & 0 & 0 & 0 & P_4X_6 & P_4X_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_5X_6 & 0 & P_5X_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_6X_5 & P_6X_4 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{1-22}
$$

$T_1 < T_2, T_3$	$T_2 < T_1, T_3$	$T_3 < T_1, T_2$
$\xi_1(T_1) = 0$	$\xi_2(T_2) = 0$	$\xi_3(T_3) = 0$
$\xi_4(T_1) = U_1 dT_1$	$\xi_5(T_2) = U_2 dT_2$	$\xi_6(T_3) = U_3 dT_3$
$\psi_2(T_1) = 0$	$\psi_1(T_2) = 0$	$\psi_1(T_3) = 0$
$\psi_3(T_1) = 0$	$\psi_3(T_2) = 0$	$\psi_2(T_3) = 0$
$\psi_5(T_1) = P_2 dT_1$	$\psi_4(T_2) = P_1 dT_2$	$\psi_4(T_3) = P_1 dT_3$
$\psi_6(T_1) = P_3 dT_1$	$\psi_6(T_2) = P_3 dT_2$	$\psi_5(T_3) = P_2 dT_3$
$\psi_7(T_1) = P_4(T_1)X_5(T_1)X_6(T_1)dT_1$	$\psi_7(T_2) = 0$	$\psi_7(T_3) = 0$
$\psi_8(T_1) = 0$	$\psi_8(T_2) = P_5(T_2)X_6(T_2)X_4(T_2)dT_2$	$\psi_8(T_3) = 0$
$\psi_9(T_1) = 0$	$\psi_9(T_2) = 0$	$\psi_9(T_3) = P_6(T_3)X_4(T_3)X_5(T_3)dT_3$

TABLE 1-1

BOUNDARY CONDITIONS FOR

QUASI-OPTIMUM CONTROL LAW

and

$$0 = P_1\xi_4 + P_2\xi_5 + P_3\xi_6 + P_4X_5X_6\xi_7 + P_5X_6X_4\xi_8 + P_6X_4X_5\xi_9 + X_4\psi_1$$
$$+ X_5\psi_2 + X_6\psi_3 + U_1\psi_4 + U_2\psi_5 + U_3\psi_6$$

where the last equation is evaluated at time t and is applicable to each of the three cases. By applying the appropriate boundary conditions defined above we find that

$$-L\xi(t) = N\psi(t)$$

where, for the case $T_1 < T_2, T_3$,

$$L = \begin{bmatrix} 0 & 0 & 0 & \varphi_{14} & 0 & 0 & \varphi_{17} & \varphi_{18} & \varphi_{19} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_2/U_1 & 0 & 0 & \gamma_{57} & -P_2\varphi_{48}/U_1 & \gamma_{59} \\ 0 & 0 & 0 & -P_3/U_1 & 0 & 0 & \gamma_{67} & \gamma_{68} & -P_3\varphi_{49}/U_1 \\ 0 & 0 & 0 & X & X & X & X & X & X \\ 0 & 0 & 0 & X & 0 & X & X & X & X \\ 0 & 0 & 0 & X & X & 0 & X & X & X \\ 0 & 0 & 0 & P_1 & P_2 & P_3 & P_4X_5X_6 & P_5X_6X_4 & P_6X_4X_5 \end{bmatrix}$$

and

$$N = \begin{bmatrix} \varphi_{11} & 0 & 0 & \varphi_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -P_2\varphi_{41}/U_1 & \hat{\varphi}_{52} & 0 & -P_2\varphi_{44}/U_1 & 1 & 0 & 0 & 0 & 0 \\ -P_3\varphi_{41}/U_1 & 0 & \varphi_{63} & -P_3\varphi_{44}/U_1 & 0 & 1 & 0 & 0 & 0 \\ X & X & X & X & X & X & 1 & 0 & 0 \\ X & X & X & X & X & X & 0 & 1 & 0 \\ X & X & X & X & X & X & 0 & 0 & 1 \\ X_4 & X_5 & X_6 & U_1 & U_2 & U_3 & 0 & 0 & 0 \end{bmatrix}$$

where $\gamma_{57} = \beta_{57} - P_2\varphi_{47}/U_1$, $\gamma_{67} = \beta_{67} - P_3\varphi_{47}/U_1$,

$\gamma_{59} = \beta_{59} - P_2\varphi_{49}/U_1$, and $\gamma_{68} = \beta_{68} - P_3\varphi_{48}/U_1$.

Upon solving for ψ_4 , ψ_5 , ψ_6 in terms of ξ_7 , ξ_8 , and ξ_9 we find the required corrections m_{ij} . The terms labeled X are not required in this calculation.

d) To account for all possible permutations of the t_i and T_j, the relationship between the switching times t_1 , t_2 , and t_3 is determined and a transformation $t_i \to t_j$, $T_i \to T_j$, $X_i \to X_j$, $P_i \to P_j$ is made such that the order $t_1 < t_2 < t_3$ is always maintained. Based on the above ordering, the gains m_{ij} are applied to the corresponding x_j. Then the inverse transformation is used to re-order the resulting controls u_i to obtain the input u_i which is applied to the dynamical system.

The gains m_{ij} are given as follows:

Whenever $T_1 < T_2, T_3$ we have

$$m_{57} = \frac{P_2}{U_1}(\varphi_{47} - \frac{\varphi_{17}}{T_1 - t_1}) - \beta_{57} \qquad m_{67} = \frac{P_3}{U_1}(\varphi_{47} - \frac{\varphi_{17}}{T_1 - t_1}) - \beta_{67}$$

$$m_{58} = 0.0 \qquad m_{68} = -\beta_{68}$$

$$m_{59} = -\beta_{59} \qquad m_{69} = 0.0$$

$$m_{47} = \frac{t_1}{U_1 t_1 + X_4}[X_4 \frac{\varphi_{17}}{\varphi_{11}} - (\varphi_{47} - \frac{\varphi_{17}}{T_1 - t_1})$$

$$[P_2 U_2 + P_3 U_3]/U_1 + \beta_{57} U_2 + \beta_{67} U_3 - P_4 X_5 X_6]$$

$$m_{48} = \frac{t_1}{U_1 t_1 + X_4}[X_4 \frac{\varphi_{18}}{\varphi_{11}} + U_3 \beta_{68} - P_5 X_6 X_4]$$

$$m_{49} = \frac{t_1}{U_1 t_1 + X_4}[X_4 \frac{\varphi_{19}}{\varphi_{11}} + U_2 \beta_{59} - P_6 X_4 X_5]$$

Whenever $T_2 < T_1, T_3$ we have

$$m_{47} = 0.0 \qquad m_{67} = -\beta_{67}$$

$$m_{48} = \frac{P_1}{U_2}(\varphi_{58} - \frac{\varphi_{28}}{T_2 - t_2}) - \beta_{48} \qquad m_{68} = \frac{P_3}{U_2}(\varphi_{58} - \frac{\varphi_{28}}{T_2 - t_2}) - \beta_{68}$$

$$m_{49} = -\beta_{49} \qquad m_{69} = 0.0$$

$$m_{57} = \frac{t_2}{U_2 t_2 + X_5}[X_5 \frac{\varphi_{27}}{\varphi_{27}} + U_3 \beta_{67} - P_4 X_5 X_6]$$

$$m_{58} = \frac{t_2}{U_2 t_2 + X_5} [X_5 \frac{\varphi_{28}}{\varphi_{22}} - (\varphi_{58} - \frac{\varphi_{28}}{T_2 - t_2}$$

$$[P_1 U_1 + P_3 U_3]/U_2 + \beta_{48} U_1 + \beta_{68} U_3 - P_5 X_6 X_4]$$

$$m_{59} = \frac{t_2}{U_2 t_2 + X_5} [X_5 \frac{\varphi_{29}}{\varphi_{22}} + U_1 \beta_{49} - P_6 X_4 X_5$$

For $T_3 < T_1 , T_2$ we have

$$m_{47} = 0.0 \qquad m_{57} = -\beta_{67}$$

$$m_{48} = -\beta_{48} \qquad m_{58} = 0.0$$

$$m_{49} = \frac{P_1}{U_3} (\varphi_{69} - \frac{\varphi_{39}}{T_3 - t_3}) - \beta_{49} \qquad m_{59} = \frac{P_2}{U_3} (\varphi_{69} - \frac{\varphi_{39}}{T_3 - t_3}) - \beta_{59}$$

$$m_{67} = \frac{t_3}{U_3 t_3 + X_6} [X_6 \frac{\varphi_{37}}{\varphi_{33}} + U_2 \beta_{57} - P_4 X_5 X_6]$$

$$m_{68} = \frac{t_3}{U_3 t_3 + X_6} [X_6 \frac{\varphi_{38}}{\varphi_{33}} + U_1 \beta_{48} - P_5 X_6 X_4]$$

$$m_{69} = \frac{t_3}{U_3 t_3 + X_6} [X_6 \frac{\varphi_{39}}{\varphi_{33}} - (\varphi_{69} - \frac{\varphi_{39}}{T_3 - t_3})$$

$$[U_1 P_1 + U_2 P_2]/U_3 + U_2 \beta_{59} + U_1 \beta_{49} - P_6 X_4 X_5]$$

The terms φ_{ij} , β_{ij} are given as follows:

Case 1: $t_1 < T_1 < t_2$, $T_1 < T_2 , T_3$

$$\varphi_{11} = -2U_1 t_1 (T_1 - t_1)/P_1$$

$$\varphi_{17} = U_{23} T_1^4/12 + (X_6 U_2 + X_5 U_3) T_1^3/6 + X_{56} T_1^2/2$$

$$\varphi_{47} = U_{23} T_1^3/3 + (X_6 U_2 + X_5 U_3) T_1^2/2 + X_{56} T_1$$

$$\varphi_{18} = 2\frac{U_1}{P_1}(T_1 - t_1)[\frac{1}{3}U_3P_2t_1^3 - \frac{1}{2}(U_3P_5 - X_6P_2)t_1^2 - P_5X_6t_1]$$

$$\varphi_{19} = 2\frac{U_1}{P_1}(T_1 - t_1)[\frac{1}{3}U_2P_3t_1^3 - \frac{1}{2}(U_2P_6 - X_5P_3)t_1^2 - P_6X_5t_1]$$

$$\varphi_{57} = P_1U_3[T_1^3/3 - T_1^2t_1/2] + P_1X_6[T_1^2/2 - T_1t_1]$$

$$\beta_{67} = P_1U_2[T_1^3/3 - T_1^2t_1/2] + P_1X_5[T_1^2/2 - T_1t_1]$$

$$\beta_{59} = U_1P_3(-\frac{T_1^3}{3} + T_1^2t_1 - \frac{t_1^3}{3} + t_1^2t_3 - 2t_1T_1t_3 + \frac{t_3}{2}T_1^2) + X_4P_3(T_1^2/2 - T_1t_3)$$

$$\beta_{68} = P_2U_1(-T_1^3/3 + T_1^2t_1 - \frac{t_1^3}{3} + t_1^2t_2 - 2t_1t_2T_1 + \frac{t_2}{2}T_1^2) + P_2X_4(T_1^2/2 - t_2T_1)$$

Case 2: $t_2 < T_1 < t_3 ,\ T_1 < T_2 , T_3$

$$\varphi_{11} = -2U_1t_1(T_1 - t_1)/P_1$$

$$\varphi_{17} = U_{23}[-T_1^4/12 + T_1^3t_2/3 - T_1t_2^3/3 + t_2^4/6] + X_5U_3T_1^3/6$$
$$+ X_6U_2[-T_1^3/6 + T_1^2t_2 - T_1t_2^2 + t_2^3/3] + X_{56}T_1^2/2$$

$$\varphi_{47} = U_{23}[-T_1^3/3 + T_1^2t_2 - t_2^3/3] + U_2X_6[-T_1^2/2 - t_2^2 + 2t_2T_1] + X_5U_3T_1^2/2$$
$$+ X_5X_6T_1$$

$$\varphi_{18} = 2\frac{U_1}{P_1}(T_1 - t_1)[U_3P_2(\frac{t_1^3}{3} - \frac{1}{2}t_2t_1^2) + P_2X_6(\frac{t_1^2}{2} - t_1t_2)]$$

$$\varphi_{19} = 2\frac{U_1}{P_1}(T_1 - t_1)[U_2P_3(\frac{t_1^3}{3} - \frac{1}{2}t_3t_1^2) + P_3X_5(\frac{t_1^2}{2} - t_1t_3)]$$

$$\beta_{57} = P_1U_3(\frac{T_1^3}{3} - \frac{t_1T_1^2}{2}) + P_1X_6(\frac{T_1^2}{2} - t_1T_1)$$

$$\beta_{67} = U_2P_1[-\frac{1}{3}T_1^3 + T_1^2t_2 - \frac{t_2^3}{3} + \frac{1}{2}t_1T_1^2 - 2t_1t_2T_1 + t_1t_2^2] + P_1X_5[\frac{T_1^2}{2} - T_1t_1]$$

$$\beta_{59} = U_1P_3[-\frac{1}{3}T_1^3 + t_3\frac{T_1^2}{2} + t_1T_1^2 - 2t_1t_3T_1 - \frac{1}{3}t_1^3 + t_1^2t_3] + P_3X_4[\frac{T_1^2}{2} - T_1t_3]$$

$$\beta_{68} = P_2U_1[-\frac{1}{3}T_1^3 - \frac{t_1^3}{3} + \frac{1}{2}T_1^2t_2 + t_1T_1^2 - 2T_1t_2t_1 + t_1^2t_2] + P_2X_4[\frac{T_1^2}{2} - T_1t_2]$$

Case 3: $t_2 < T_2 < t_3$, $T_2 < T_1$, T_3

$$\varphi_{22} = -2U_2t_2(T_2 - t_2)/P_2$$

$$\varphi_{28} = U_{13}[-T_2^4/12 + t_1T_2^3/3 - T_2t_1^3/3 + t_1^4/6] + X_6U_1[-T_2^3/6 + t_1T_2^2 - T_2t_1^2 + t_1^3/3]$$

$$+ X_4U_3T_2^3/6 + X_{46}T_2^2/2$$

$$\varphi_{58} = U_{13}[-T_2^3/3 + t_1T_2^2 - t_1^3/3] + X_6U_1[-T_2^2/2 + 2t_2T_2 - t_1^2] + X_4U_3T_2^2/2 + X_{46}T_2$$

$$\varphi_{29} = 2\frac{U_2}{P_2}(T_2 - t_2)[\frac{1}{3}U_1P_3((t_2 - t_1)^3 + t_1^3) - \frac{1}{2}(U_1P_6 - X_4P_3)$$

$$[(t_2 - t_1)^2 + t_1^2] - P_6X_4t_2]$$

$$\varphi_{27} = 2\frac{U_2}{P_2}(T_2 - t_2)[U_3P_1(\frac{1}{3}[(t_2 - t_1)^3 + t_1^3] - \frac{1}{2}t_1[(t_2 - t_1)^2 + t_1^2])$$

$$+ X_6P_1(\frac{1}{2}[(t_2 - t_1)^2 + t_1^2] - t_1t_2)]$$

$$\beta_{48} = U_3P_2[\frac{1}{3}T_2^3 - \frac{1}{2}T_2^2t_2] + P_2X_6[T_2^2/2 - t_2T_2]$$

$$\beta_{68} = P_2U_1(-T_2^3/3 + T_2^2t_1 - t_1^3/3 + t_1^2t_2 - 2t_1t_2T_2 + t_2T_2^2/2) + P_2X_4(T_2^2/2 - t_2T_2)$$

$$\beta_{67} = P_1U_2[T_2^3/3 - T_2^2t_1/2] + P_1X_5[T_2^2/2 - T_2t_1]$$

$$\beta_{49} = U_2P_3[-T_2^3/3 - t_2^3/3 + T_2^2t_2 - 2t_2t_3T_2 + t_3t_2^2 + t_3T_2^2/2] + P_3X_5[T_2^2/2 - t_3T_2]$$

Case 4: $t_3 < T_1 < T_2 , T_3$

$$\varphi_{11} = -2U_1t_1(T_1 - t_1)/P_1$$

$$\varphi_{17} = U_{23}[T_1^4/12 - T_1^3(t_3 + t_2)/3 + 2T_1^2t_2t_3 + T_1(t_3^3/3 - 2t_2t_3^2 - t_2^3/3) - t_3^4/6 + 2t_2t_3^3/3 + t_2^4/6] + U_2X_6[-T_1^3/6 + T_1^2t_2 - T_1t_2^2 + t_2^3/3] + U_3X_5[-T_1^3/6 + T_1^2t_3 - T_1t_3^2 + t_3^3/3] + X_{56}T_1^2/2$$

$$\varphi_{47} = U_2(T_1^3/3 - T_1^2(t_3 + t_2) + 4T_1t_3t_2 + t_3^3/3 - 2t_2t_3^2 - t_2^3/3) + X_6U_2(-T_1^2/2 + 2T_1t_2 - t_2^2) + X_5U_3(-T_1^2/2 + 2t_3T_1 - t_3^2) + X_{56}T_1$$

$$\varphi_{18} = 2U_1(T_1 - t_1)[P_2U_3(t_1^3/3 - t_2t_1^2/2) + P_2X_6(t_1^2/2 - t_1t_2)]/P_1$$

$$\varphi_{19} = 2U_1(T_1 - t_1)[P_3U_2(t_1^3/3 - t_3t_1^2/2) + P_3X_5(t_1^2/2 - t_3t_1)]/P_1$$

$$\beta_{57} = P_1U_3(-T_1^3/3 + T_1^2(t_3 + t_1/2) - 2t_1t_3T_1 - t_3^3/3 + t_1t_3^2) + P_1X_6(T_1^2/2 - T_1t_1)$$

$$\beta_{67} = P_1U_2(-T_1^3/3 + T_1^2(t_2 + t_1/2) - 2t_1t_2T_1 - t_2^3/3 + t_1t_2^2) + P_1X_5(T_1^2/2 - T_1t_1)$$

$$\beta_{59} = P_3U_1(-T_1^3/3 + T_1^2(t_1 + t_3/2) - 2t_1t_3T_1 - t_1^3/3 + t_3t_1^2) + P_3X_4(T_1^2/2 - T_1t_3)$$

$$\beta_{68} = P_2U_1(-T_1^3/3 + T_1^2(t_1 + t_2/2) - 2t_1t_2T_1 - t_1^3/3 + t_2t_1^2) + P_2X_4(T_1^2/2 - T_1t_2)$$

Case 5: $t_3 < T_2 < T_1, T_2$

$$\varphi_{22} = -2U_2 t_2 (T_2 - t_2)/P_2$$

$$\varphi_{28} = U_{13}[T_2^4/12 - T_2^3(t_3 + t_1)/3 + 2t_1 t_3 T_2^2 + T_2(t_3^3/3 - 2t_1 t_3^2 - t_1^3/3 - t_3^4/6 + 2t_1 t_3^3/3 + t_1^4/6] + U_1 X_6[-T_2^3/6 + T_2^2 t_1 - T_2 t_1^2 + t_1^3/3] + U_3 X_4[-T_2^3/6 + T_2^2 t_3 - T_2 t_3^2 + t_3^3/3] + X_{46} T_2^2/2$$

$$\varphi_{58} = U_{13}[T_2^3/3 - T_2^2(t_1 + t_3) + 4t_1 t_3 T_2 + t_3^3/3 - 2t_1 t_3^2 - t_1^3/3] + X_6 U_1(-T_2^2/2 + 2t_1 T_2 - t_1^2) + X_4 U_3(-T_2^2/2 + 2t_3 T_2 - t_3^2) + X_{46} T_2$$

$$\varphi_{29} = 2\frac{U_2}{P_2}(T_2 - t_2)[P_3 U_1(-t_2^3/3 - t_1^3/3 + t_1 t_2^2 + t_3 t_2^2/2 - 2t_1 t_2 t_3 + t_3 t_1^2) + P_3 X_4(t_2^2/2 - t_2 t_3)]$$

$$\varphi_{27} = 2\frac{U_2}{P_2}(T_2 - t_2)[P_1 U_3(t_2^3/3 - t_1 t_2^2/2) + P_1 X_6(t_2^2/2 - t_1 t_2)]$$

$$\beta_{48} = P_2 U_3[-T_2^3/3 + T_2^2(t_3 + t_2/2) - 2T_2 t_3 t_2 - t_3^3/3 + t_2 t_3^2] + P_2 X_6[T_2^2/2 - T_2 t_2]$$

$$\beta_{68} = P_2 U_1[-T_2^3/3 + T_2^2(t_1 + t_2/2) - 2t_1 t_2 T_2 - t_1^3/3 + t_2 t_1^2] + P_2 X_4(T_2^2 - T_2 t_2)$$

$$\beta_{67} = P_1 U_2[-T_2^3/3 + T_2^2(t_2 + t_1/2) - 2T_2 t_1 t_2 - t_2^3/3 + t_1 t_2^2] + P_1 X_5[T_2^2/2 - T_2 t_1]$$

$$\beta_{49} = P_3 U_2[-T_2^3/3 + T_2^2(t_2 + t_3/2) - 2T_2 t_2 t_3 - t_2^3/3 + t_3 t_2^2] + P_3 X_5(T_2^2/2 - T_2 t_3)$$

Case 6: $t_3 < T_3 < T_1, T_2$

$$\varphi_{33} = -2U_3 t_3 (T_3 - t_3)/P_3$$

$$\varphi_{39} = U_{12}[T_3^4/12 - T_3^3(t_1 + t_2)/3 + 2t_1t_2T_3^2 + T_3(t_2^3/3 - 2t_1t_2^2 - t_1^3/3) - t_2^4/6$$
$$+ 2t_1t_2^3/3 + t_1^4/6] + U_1X_5(-T_3^3/6 + T_3^2t_1 - T_3t_1^2 + t_1^3/3) + U_2X_4(-T_3^3/6$$
$$+ T_3^2t_2 - T_3t_2^2 + t_2^3/3) + X_{45}T_3^2/2$$

$$\varphi_{69} = U_{12}[T_3^3/3 - T_3^2(t_1 + t_2) + 4t_2t_1T_3 + t_2^3/3 - 2t_1t_2^2 - t_1^3/3] + X_5U_1[-T_3^2/2$$
$$+ 2t_1T_3 - t_1^2] + X_4U_2[-T_3^2/2 + 2T_3t_2 - t_2^2] + X_{45}T_3$$

$$\varphi_{37} = 2\frac{U_3}{P_3}(T_3 - t_3)[P_1U_2(-t_3^3/3 - t_2^3/3 + t_2t_3^2 + t_1t_3^2/2 - 2t_1t_2t_3 + t_1t_2^2)$$
$$+ P_1X_5(t_3^2/2 - t_1t_3)]$$

$$\varphi_{38} = 2\frac{U_3}{P_3}(T_3 - t_3)[P_2U_1(-t_3^3/3 - t_1^3/3 + t_1t_3^2 + t_3^2t_2/2 - 2t_1t_2t_3 + t_2t_1^2)$$
$$+ P_2X_4(t_3^2/2 - t_3t_2)]$$

$$\beta_{48} = P_2U_3(-T_3^3/3 + T_3^2(t_3 + t_2/2) - 2T_3t_2t_3 - t_3^3/3 + t_3^2t_2) + P_2X_6(T_3^2/2 - T_3t_2)$$

$$\beta_{67} = P_1U_2(-T_3^3/3 + T_3^2(t_2 + t_1/2) - 2t_1t_2t_3 - t_2^3/3 + t_1t_2^2) + P_1X_5(T_3^2/2 - t_1T_3)$$

$$\beta_{49} = P_3U_2(-T_3^3/3 + T_3^2(t_2 + t_3/2) - 2T_3t_2t_3 - t_2^3/3 + t_3t_2^2) + P_3X_5(T_3^2/2 - t_3T_3)$$

$$\beta_{59} = P_3U_1(-T_3^3/3 + T_3^2(t_1 + t_3/2) - 2T_3t_1t_3 - t_1^3/3 + t_3t_1^2) + P_3X_4(T_3^2/2 - T_3t_3)$$

It is important to note that when (1-17) and (1-18) are substituted into the above expressions for α_{ij} and β_{ij}, the quasi-optimum control law is an explicit function of the state of the vehicle and is thus a feedback control law.

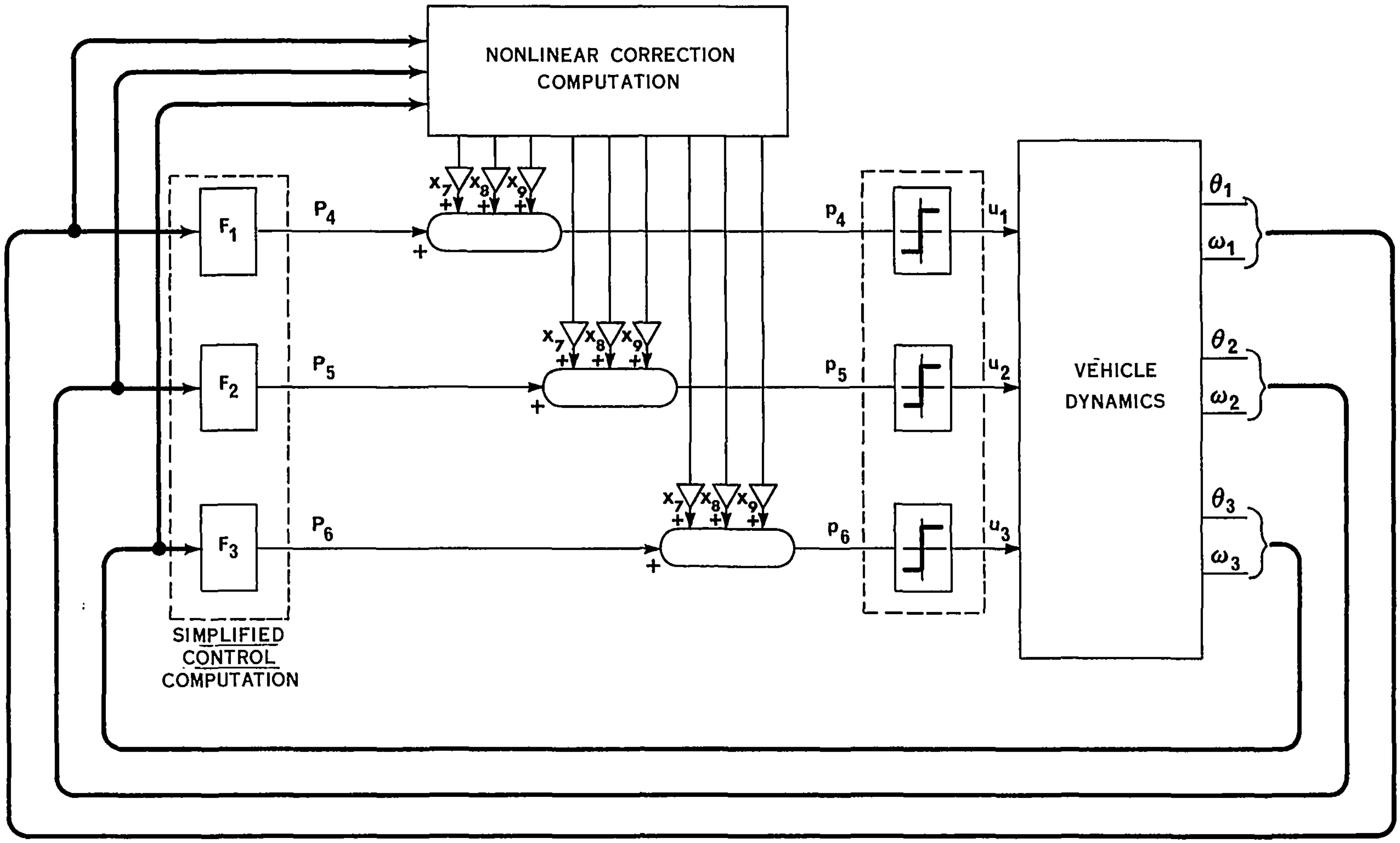


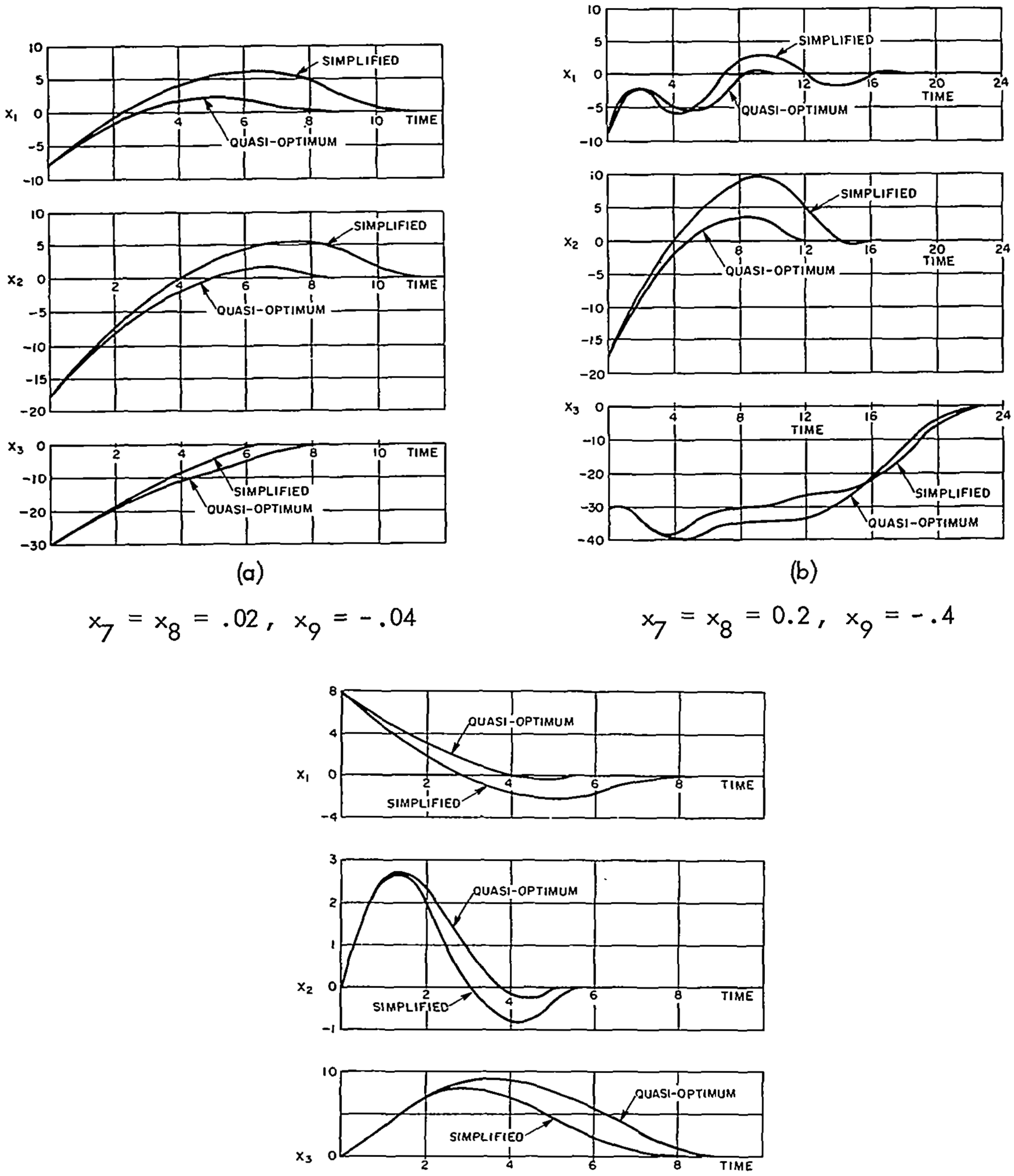
FIGURE 1.1 STRUCTURE OF QUASI-OPTIMUM CONTROL SYSTEM

The resulting feedback control system has the structure shown in Figure 1-1 . Using the simplified control law indicated by transformations $F_1, F_2, F_3,$ the adjoint variables P_4, P_5, P_6 are determined from the measured angular positions θ_i and velocities ω_i . Each nonlinear correction is applied to the appropriate amplifier whose gain depends on the cross-axis coupling. Thus the output of the ideal switch functions provides the physical realization of control law(1-19).

Performance with Quasi-Optimum Control Law - To evaluate the effectiveness of the quasi-optimum closed-loop system compared to the simplified system, the transient response of both systems was obtained by means of a digital computer simulation. Figures 1-2a and 1-2b correspond to the same initial angular positions and velocities but with cross-axis inertia ratios that differ by a factor of 10. It is clear from Figure 1-2a that after 8 units on the time scale the quasi-optimum system has reached the origin in 3-dimensional space whereas the simplified system requires 11 time units to reach the origin. The figure also indicates that the transient response of the quasi-optimum system undergoes smaller overshoot on the x_1 and x_2 axes, but requires more time to null the position and velocity of the third axis.

From Figure 1-2b it is seen that, because of the increase in the cross-axis coupling, both the quasi-optimum and simplified systems require more time to reduce all of the angular position and velocity coordinates to zero. As before the quasi-optimum control law results in a more "damped" transient response than that of the simplified system on the x_1 and x_2 axes. This improvement of performance is again accompanied by an increase in the time required for the quasi-optimum control law to bring the coordinates of the third axis to the origin.

Similar results are obtained for initial conditions shown in Figure 1-2c. Thus it would appear that the quasi-optimum control law provides a generally smoother transient response and reduces the coordinates of two axes to zero in a smaller time interval than does the simplified control law.



(a)

$x_7 = x_8 = .02, \ x_9 = -.04$

(b)

$x_7 = x_8 = 0.2, \ x_9 = -.4$

(c)

$x_7 = x_8 = .2, \ x_9 = -.4$

FIGURE 1-2

COMPARISON OF TRANSIENT RESPONSE
OF CLOSED-LOOP SYSTEM USING
QUASI-OPTIMUM AND SIMPLIFIED CONTROL LAW

2.2 MINIMUM-TIME, BOUNDED ACCELERATION RENDEZVOUS IN A CENTRAL FORCE FIELD

This is an extension of the minimum-time bounded acceleration rendezvous in free space problem which was considered under Contract No. NAS 2-2648, [A1, A4] wherein it was assumed that the gravity field was not significant and hence was omitted. In the present application a central gravitational force is considered.

Exact Problem - It is assumed that the motion of the target vehicle is known, that is $r_t(t)$ and $\Phi_t(t)$ are specified (see Figure 2-1). The motion of the vehicle with respect to the target is defined in a target-referenced polar coordinate system as follows

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\Phi}{dt}\right)^2 = a_r - g_r \tag{2-1}$$

$$r\frac{d^2\Phi}{dt^2} + 2\,\frac{dr}{dt}\,\frac{d\Phi}{dt} = a_t - g_t \tag{2-2}$$

The forcing terms on the right hand side can be expressed as

$$\begin{aligned} a_r &= a\cos\theta \\ a_t &= a\sin\theta \end{aligned} \tag{2-3}$$

$$\begin{bmatrix} g_r \\ \\ g_t \end{bmatrix} = \mu \begin{bmatrix} \cos\Phi & \sin\Phi \\ \\ -\sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} \dfrac{\cos\Phi_v}{r_v^2} - \dfrac{\cos\Phi_t}{r_t^2} \\ \\ \dfrac{\sin\Phi_v}{r_v^2} - \dfrac{\sin\Phi_t}{r_t^2} \end{bmatrix}$$

where a is the (constant) thrust to mass ratio and μ is the gravity constant $(= g_0 R_0^2$, g_0 is the gravitational acceleration at the surface of the attracting body of radius $R_0)$.

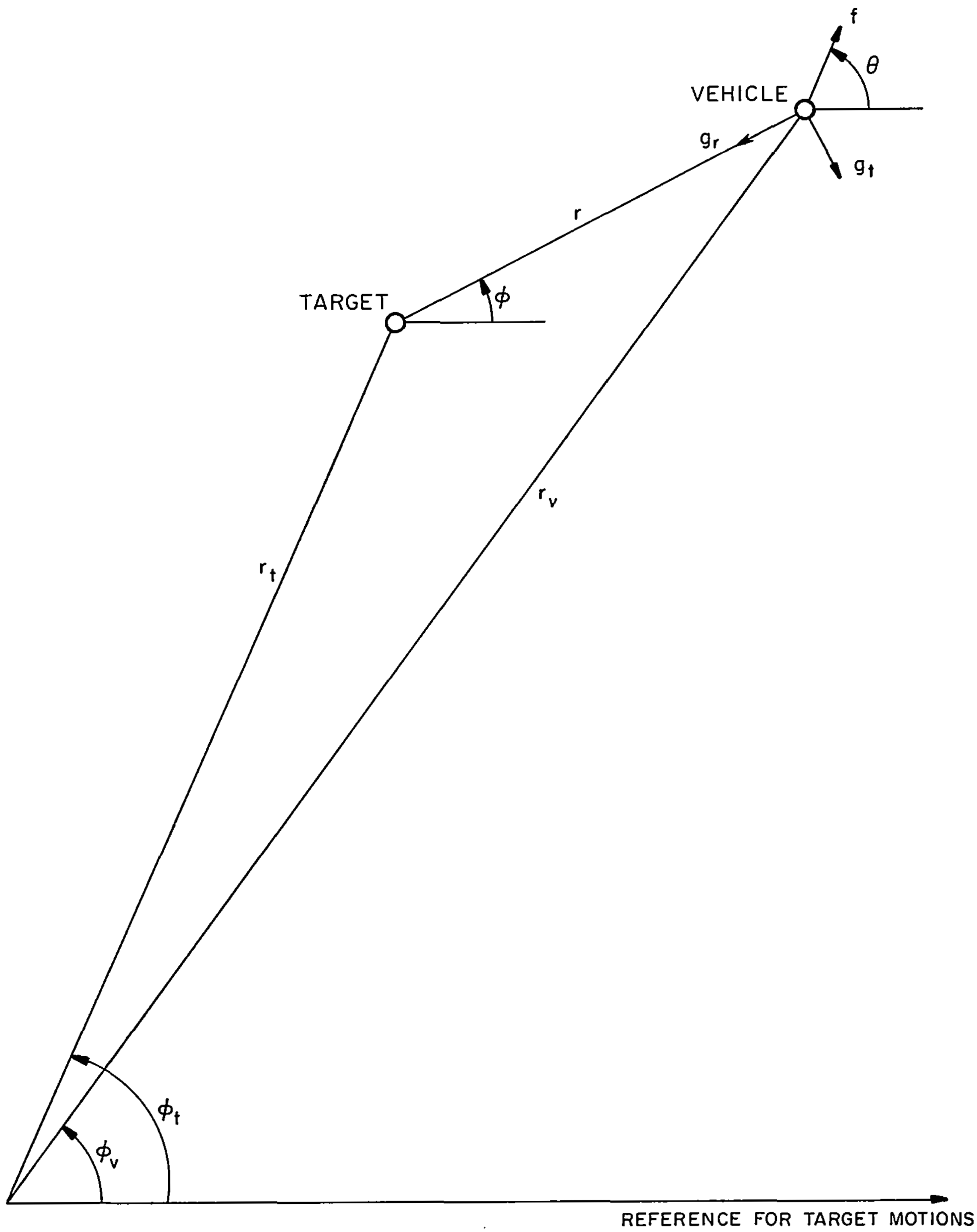


FIGURE 2.1
RELATIVE COORDINATE SYSTEM, FOR RENDEZVOUS

To apply the quasi-optimum control technique a new set of variables is defined as follows:

$$\begin{aligned} x_0 &= t & x_3 &= r\dot{\Phi}/a \\ x_1 &= \dot{r}/a & x_4 &= \Phi \\ x_2 &= r/a & x_5 &= \mu/a \end{aligned} \tag{2-4}$$

The constant μ/a is represented by an additional state variable x_5 which, in the simplified problem is assumed zero, i.e., the simplified problem assumes the absence of a gravitational field.

In terms of these new variables, the equations of motion (2-1) and (2-2) become

$$\begin{aligned} \dot{x}_0 &= 1 \\ \dot{x}_1 &= \frac{x_3^2}{x_2} + u_1 - x_5 h_1 \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= -\frac{x_1 x_3}{x_2} + u_2 - x_5 h_3 \\ \dot{x}_4 &= \frac{x_3}{x_2} \\ \dot{x}_5 &= 0 \end{aligned} \tag{2-5}$$

where $u_1 = \cos\theta$, $u_2 = \sin\theta$,

$$h_1 = \frac{a x_2 + r_t \cos(x_4 - \Phi_t)}{\left[a^2 x_2^2 + 2 a x_2 r_t \cos(x_4 - \Phi_t) + r_t^2\right]^{3/2}} - \frac{\cos(x_4 - \Phi_t)}{r_t^2}$$

$$h_3 = \frac{-r_t \sin(x_4 - \Phi_t)}{\left[a^2 x_2^2 + 2 a x_2 r_t \cos(x_4 - \Phi_t) + r_t^2\right]^{3/2}} + \frac{\sin(x_4 - \Phi_t)}{r_t^2}$$

The problem is then to force the system (2-5) from any initial state to the origin of the state space in the shortest possible time T, i.e., to minimize $x_o(T)$ subject to the constraint

$$u_1^2 + u_2^2 = 1 \tag{2-6}$$

The Hamiltonian function for this problem is given by

$$h = p_0 + p_1\left(\frac{x_3^2}{x_2} + u_1 - x_5 h_1\right) + p_2 x_1 + p_3\left(-\frac{x_1 x_3}{x_2} + u_2 - x_5 h_3\right) + p_4 \frac{x_3}{x_2} \tag{2-7}$$

Maximization of h with respect to u_1 and u_2 subject tothe constraint (2-6) results in the following steering law

$$u_1 = \frac{p_1}{\left(p_1^2 + p_3^2\right)^{1/2}} \qquad u_2 = \frac{p_3}{\left(p_1^2 + p_3^2\right)^{1/2}} \tag{2-8}$$

Using these values of u_1 and u_2, along with the condition that $p_0 \equiv -1$ in the Hamiltonian, yields

$$h = -1 + (p_1^2 + p_3^2)^{\frac{1}{2}} + p_1\left(\frac{x_3^2}{x_2} - x_5 h_1\right) + p_2 x_1 - p_3\left(\frac{x_1 x_3}{x_2} + x_5 h_3\right) + p_4 \frac{x_3}{x_2} \tag{2-9}$$

Simplified Problem - Suppose that the initial tangential velocity of the vehicle with respect to the target and the gravitational field of the attracting body are both zero. Then clearly, the optimum solution is to apply the vehicle thrust along the initial radius vector pointing either toward or away from the origin in accordance with the well known solution for the minimum time double integral plant (Bushaw's Problem).

If both the relative initial tangential velocity and gravitational attraction are suitably small, it is reasonable to use the solution of Bushaw's Problem as the basis for an approximate solution to the exact problem. Thus we select as the state vector of the simplified process

$$X = \{x_0, x_1, x_2, 0, 0, 0\} \tag{2-10}$$

Then $\xi = \{\xi_0, 0, 0, x_3, x_4, x_5\}$ where ξ_0 is the approximate change in the performance index due to the simplification. For the simplified problem, the Hamiltonian is

$$H = P_0 + P_1 U + P_2 X_1 \tag{2-11}$$

where $P = \{P_0, P_1, P_2, 0, 0, 0\}$ is the adjoint vector in the simplified problem. Maximizing the Hamiltonian subject to $U = \pm 1$ gives

$$U = \operatorname{sgn} P_1(t) \tag{2-12}$$

The state and adjoint equations for the simplified problem are

$$\begin{aligned} \dot{X}_0 &= 1 & \dot{P}_0 &= 0 \\ \dot{X}_1 &= U & \dot{P}_1 &= -P_2 \\ \dot{X}_2 &= X_1 & \dot{P}_2 &= 0 \end{aligned} \tag{2-13}$$

Integration of (2-13) yields

$$\begin{aligned} X_0 &= t \\ X_1 &= X_1(0) + Ut \\ X_2 &= X_2(0) + X_1(0)t + Ut^2/2 \end{aligned} \tag{2-14a}$$

$$\begin{aligned} P_0 &= -1 \\ P_1 &= P_1(0) - P_2(0)t \\ P_2 &= P_2(0) = \text{constant} \end{aligned} \tag{2-14b}$$

Eliminating time from

$$X_1(t) = X_1(0) + Ut$$

$$X_2(t) = X_2(0) + X_1(0)t + Ut^2/2$$

gives the switching curve for the simplified problem

$$X_2 + \frac{X_1 |X_1|}{2} = 0 \tag{2-15a}$$

Thus the control is $U = -\operatorname{sgn}\left(X_2 + \dfrac{X_1|X_1|}{2}\right).$ (2-15b)

In the event the argument of the sgn function is zero, it is indicated that the vehicle is on the switching curve and $U = -\,\mathrm{sgn}\,(X_2)$.

Substituting (2-12) into (2-14a) and integrating to the switch time t_s and terminal time T gives

$$\begin{aligned} X_1(t_s) &= X_1(0) + U t_s \\ X_2(t_s) &= X_2(0) + X_1(0) t_s + U t_s^{\,2}/2 \end{aligned} \qquad (2\text{-}16a)$$

$$\begin{aligned} X_1(T) &= X_1(t_s) - U(T - t_s) \\ X_2(T) &= X_2(t_s) + X_1(t_s)(T - t_s) - U(T - t_s)^2/2 \end{aligned} \qquad (2\text{-}16b)$$

Solving simultaneously for T and t_s and utilizing these along with the Hamiltonian (2-11) results in the following expressions for the switching time, terminal time and initial adjoint variables.

$$t_s = \left[-X_1(0) + \frac{1}{P_2(0)} \right] U = \frac{P_1(0)}{P_2(0)} \qquad (2\text{-}17a)$$

$$T = \left[-X_1(0) + \frac{2}{P_2(0)} \right] U \qquad (2\text{-}17b)$$

$$P_1(0) = \left[1 - \frac{X_1(0)U}{\left[\dfrac{X_1(0)^2}{2} - U X_2(0) \right]^{1/2}} \right] U \qquad (2\text{-}17c)$$

$$P_2(0) = \frac{U}{\left[\dfrac{X_1(0)^2}{2} - U X_2(0) \right]^{1/2}} \qquad (2\text{-}17d)$$

Since the initial state is arbitrary, dropping the subscript zero in (2-17) results in the general relations for the switch time, terminal time and adjoint variables.

If the initial tangential velocity and gravitational field are not zero, however, the simplified steering law is unsatisfactory for the exact problem because no tangential acceleration is ever produced. As a result the initial angular momentum is conserved, and as the radial distance decreases the tangential velocity increases until the vehicle either orbits the target or escapes entirely. Satisfactory performance can be achieved only by use of a tangential component of acceleration.

In addition, if the gravitational field is ignored, the vehicle will either miss the target or reduce the range to zero with a non-zero velocity. Either situation is unsatisfactory since the boundary conditions require the range and range rate to be simultaneously reduced to zero.

Quasi-Optimum Control Law - In the quasi-optimum control law the radial and tangential components of the normalized acceleration are given by (2-8), in which the approximate values of p_1 and p_3 are used. The approximations are given by

$$p_i = P_i + \sum_{j=0}^{5} m_{ij} \xi_i \qquad i = 1, 3 \tag{2-18}$$

From (2-10) however, $\xi_1 = \xi_2 = 0$ and $\xi_3 = x_3$, $\xi_4 = x_4$, and $\xi_5 = x_5$; hence (2-18) becomes

$$p_1 = P_1 + m_{10}\xi_0 + m_{13}x_3 + m_{14}x_4 + m_{15}x_5 \tag{2-19}$$

$$p_3 = m_{30}\xi_0 + m_{33}x_3 + m_{34}x_4 + m_{35}x_5 \tag{2-20}$$

Thus only m_{10}, m_{13}, m_{14}, m_{15}, m_{30}, m_{33}, m_{34}, and m_{35} in the matrix M are needed. These are calculated with the aid of the matrix Riccati equation (19). The coefficient matricies H_{XX}, H_{PP}, and H_{XP} appearing therein are found by performing the required partial differentiations on the Hamiltonian for the exact problem given by (2-7), and evaluating the result at $x = X$, i.e., $x_3 = x_4 = x_5 = 0$. The results are

$$H_{XX} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -P_1 \frac{\partial h_1}{\partial X_0} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -P_1 \frac{\partial h_1}{\partial X_2} \\ 0 & 0 & 0 & 2\frac{P_1}{X_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -P_1 \frac{\partial h_1}{\partial X_4} \\ -P_1 \frac{\partial h_1}{\partial X_0} & 0 & -P_1 \frac{\partial h_1}{\partial X_2} & 0 & -P_1 \frac{\partial h_1}{\partial X_4} & 0 \end{bmatrix}$$

$$H_{PP} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\delta(P_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{U}{P_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H_{PX} = H'_{XP} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{X_1}{X_2} & \frac{1}{X_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h_1 & 0 & 0 & -h_3 & 0 \end{bmatrix}$$

The result of substituting these matrices into the auxiliary system (20) expressed in component form, is

$$\dot{\xi}_0 = 0 \tag{2-21a}$$

$$\dot{\xi}_1 = -h_1 \xi_5 + 2\delta(P_1)\psi_1 \tag{2-21b}$$

$$\dot{\xi}_2 = \xi_1 \tag{2-21c}$$

$$\dot{\xi}_3 = -\frac{X_1}{X_2}\xi_3 - h_3\xi_5 + \frac{U}{P_1}\psi_3 \tag{2-21d}$$

$$\dot{\xi}_4 = \frac{1}{X_2}\xi_3 \tag{2-21e}$$

$$\dot{\xi}_5 = 0 \tag{2-21f}$$

$$\dot{\psi}_0 = P_1 \frac{\partial h_1}{\partial x_0}\bigg|_{x = X} \xi_5 \tag{2-22a}$$

$$\dot{\psi}_1 = -\psi_2 \tag{2-22b}$$

$$\dot{\psi}_2 = P_1 \frac{\partial h_1}{\partial x_2}\bigg|_{x = X} \xi_5 \tag{2-22c}$$

$$\dot{\psi}_3 = -2\frac{P_1}{X_2}\xi_3 + \frac{X_1}{X_2}\psi_3 - \frac{1}{X_2}\psi_4 \tag{2-22d}$$

$$\dot{\psi}_4 = P_1 \frac{\partial h_1}{\partial x_4}\Bigg|_{x = X} \xi_5 \tag{2-22e}$$

$$\dot{\psi}_5 = P_1 \frac{\partial h_1}{\partial x_0}\Bigg|_{x = X} \xi_0 + P_1 \frac{\partial h_1}{\partial x_2}\Bigg|_{x=X} \xi_2 + P_1 \frac{\partial h_1}{\partial x_4}\Bigg|_{x=X} \xi_4 + h_1\psi_1 + h_3\psi_3 \tag{2-22f}$$

The boundary equations for these differential equations are

$$\xi_1(T) = -\dot{X}_1(T)\,dT = -\operatorname{sgn}[P_1(T)]\,dT \tag{2-23a}$$

$$\xi_2(T) = -\dot{X}_2(T)\,dT = 0 \tag{2-23b}$$

$$\psi_0(T) = -\dot{P}_0(T)\,dT = 0 \tag{2-23c}$$

$$\psi_3(T) = -\dot{P}_3(T)\,dT = 0 \tag{2-23d}$$

$$\psi_4(T) = -\dot{P}_4(T)\,dT = 0 \tag{2-23e}$$

$$\psi_5(T) = -\dot{P}_5(T)\,dT = P_1(T)h_1(T) = 0 \tag{2-23f}$$

$$-\xi_1(T)P_2(T) + \xi_5(T)P_1(T)h_1(T) = \psi_1(T)\operatorname{sgn}[P_1(T)] \tag{2-23g}$$

Utilizing the auxiliary equations, their boundary conditions and the relationship

$$\psi = M\xi$$

we can determine the following properties of the M matrix.

(1) Equation (2-21f) indicates that $\xi_5(t)$ = constant. Thus the first row and column of the M matrix can be determined form (2-22a) and (2-23c).

$$\dot{\psi}_0(\tau) = P_1(\tau)\frac{\partial h_1(\tau)}{\partial X_0}\xi_5 \tag{2-22a}$$

$$\psi_0(T) = \psi_0(t) + \xi_5\int_t^T P_1(\tau)\frac{\partial h_1(\tau)}{\partial X_0}\,d\tau = 0 \tag{2-24}$$

Thus we have determined that

$$m_{0i} = m_{i0} = 0 \qquad i = 0, 1, 2, 3, 4$$

$$m_{05} = m_{50} = -\int_t^T P_1(\tau)\frac{\partial h_1(\tau)}{\partial x_0}\, d\tau \tag{2-25}$$

(2) In a similar manner, using (2-22e) and (2-23e) it is established that

$$m_{4i} = m_{i4} = 0 \qquad i = 0, 1, 2, 3, 4$$

$$m_{45} = m_{54} = -\int_t^T P_1(\tau)\frac{\partial h_1(\tau)}{\partial X_4}\, d\tau \tag{2-26}$$

(3) Rewriting equations (2-21d) and (2-22d) in matrix form and noting that $\psi_4 = m_{45}\xi_5$ we obtain the system of equations

$$\begin{bmatrix} \dot{\xi}_3 \\ \dot{\psi}_3 \end{bmatrix} = \begin{bmatrix} -\frac{x_1}{x_2} & \frac{U}{P_1} \\ -\frac{2P_1}{x_2} & \frac{x_1}{x_2} \end{bmatrix} \begin{bmatrix} \xi_3 \\ \psi_3 \end{bmatrix} + \begin{bmatrix} -h_3\xi_5 \\ -\frac{m_{45}}{x_2}\xi_5 \end{bmatrix} \tag{2-27}$$

Defining the system fundamental matrix as

$$\Phi(T,\lambda) = \begin{bmatrix} a(T,\lambda) & b(T,\lambda) \\ c(T,\lambda) & d(T,\lambda) \end{bmatrix}$$

$$\begin{bmatrix} \xi_3(T) \\ \psi_3(T) \end{bmatrix} = \Phi(T,\lambda)\begin{bmatrix} \xi_3(t) \\ \psi_3(t) \end{bmatrix} + \int_t^T \Phi(T,\lambda)\begin{bmatrix} -h_3\xi_5 \\ -\frac{m_{45}}{x_2}\xi_5 \end{bmatrix} d\lambda \tag{2-28}$$

it is found that

$$\psi_3(t) = -\frac{c(T,t)}{d(T,t)} - \xi_3(t) + \frac{\int_t^T [c(T,\lambda)\, h_3 - \frac{m_{45}}{x_2} d(T,\lambda)]\, d\lambda}{d(T,t)} \xi_5 \tag{2-29}$$

Equation (2-29) implies that

$$m_{i3} = m_{3i} = 0 \qquad i = 0, 1, 2, 4$$

$$m_{33} = -\frac{c(T,t)}{d(T,t)} \tag{2-30a}$$

$$m_{35} = m_{53} = \frac{\int_t^T [c(T,\lambda) h_3 - \frac{m_{45}}{x_2} d(T,\lambda)]\, d\lambda}{d(T,t)} \tag{2-20b}$$

(4) The remaining elements of the M matrix can be determined by integrating (2-22b), (2-22c), (2-21b), and (2-21c). Special attention must be given to the integration of $\dot{\xi}_1(\tau)$ due to the impulse function that appears at the switch time t_s. In particular,

$$\psi_2(\tau) = \psi_2(t) + \xi_5 I_1(\tau, t) \tag{2-31}$$

$$\psi_1(\tau) = \psi_1(t) - \psi_2(t)(\tau - t) - \xi_5 I_2(\tau, t) \tag{2-32}$$

where

$$I_1(\tau,t) = \int_t^\tau P_1(\lambda) \frac{\partial h_1(\lambda)}{\partial x_2} d\lambda \tag{2-33}$$

$$I_2(\tau,t) = \int_t^\tau \int_t^\lambda P_1(\beta) \frac{\partial h_1(\beta)}{\partial x_2} d\beta\, d\lambda \tag{2-34}$$

and

$$t \le \tau \le T$$

<u>For $\tau < t_s^-$</u>

$$\xi_1(t_s^-) = \xi_1(t) + \xi_5 I_3(t_s^-, t) \tag{2-35}$$

$$\xi_2(t_s^-) = \xi_2(t) + \xi_1(t)(t_s^- - t) - \xi_5 I_4(t_s^-, t) \tag{2-36}$$

where

$$I_3(\tau, t) = \int_t^{\tau} h_1(\lambda)\, d\lambda \tag{2-37}$$

$$I_4(\tau, t) = \int_t^{\tau} \int_t^{\lambda} h_1(\beta)\, d\beta\, d\lambda \tag{2-38}$$

For $\tau = t_s$

$$\xi_1(t_s^+) = \xi_1(t_s^-) + 2\int_{t_s^-}^{t_s^+} \delta[P_1(\lambda)]\, \psi_1(\lambda)\, d\lambda \tag{2-39}$$

Substituting $P_1(\lambda) = P_1(0) - P_2(0)\lambda$ into the integral and making the change of variable

$$X = \frac{P_1(0) - P_2(0)\lambda}{P_2(0)} = t_s - \lambda$$

yields

$$\xi_1(t_s^+) = \xi_1(t_s^-) - 2\int_{t_s^+}^{t_s^-} \delta[P_2(0)X]\, \psi_1(t_s - X)\, dX \tag{2-40}$$

Making the substitution $\delta(at) = \delta(t)/|a|$, (2-40) becomes

$$\xi_1(t_s^+) = \xi_1(t_s^-) + \frac{2\psi_1(t_s)}{|P_2(0)|} \tag{2-41}$$

After integrating from the switch time t_s to the terminal time T the four integrals are

$$\psi_1(T) = \psi_1(t) - \psi_2(t)(T - t) - \xi_5(t)\, I_2(T, t) \tag{2-42a}$$

$$\psi_2(T) = \psi_2(t) + \xi_5(t)\, I_1(T, t) \tag{2-42b}$$

$$\xi_1(T) = \psi_1(t) f_{11} + \psi_2(t) f_{12} + \xi_1(t) + \xi_5(t) g_1 \qquad \text{(2-42c)}$$

$$\xi_2(T) = \psi_1(t) f_{21} + \psi_2(t) f_{22} + \xi_1(t)(T-t) + \xi_2(t) + \xi_5(t) g_2 \qquad \text{(2-42d)}$$

where

$$\begin{aligned}
f_{11} &= \frac{2}{|P_2(0)|} \\
f_{12} &= \frac{-2}{|P_2(0)|}(t_s - t) \\
f_{21} &= \frac{2}{|P_2(0)|}(T - t_s) \\
f_{22} &= \frac{2}{|P_2(0)|}(t_s - t)(T - t_s) \\
g_1 &= -I_3(T, t) - \frac{2}{|P_2(0)|} I_2(t_s, t) \\
g_2 &= \frac{-2}{|P_2(0)|} I_2(t_s, t)(T - t_s) - I_4(T, t)
\end{aligned} \qquad \text{(2-43)}$$

Since there is no boundary condition for $\psi_2(T)$, equation (2-42b) is not required. Applying boundary conditions (2-23a), (2-23b), and (2-23g) to (2-42) and noting that

$$\xi_1(T) = -\operatorname{sgn}[P_1(T)] = U d T$$

$$\xi_2(T) = 0$$

$$\psi_1(T) = \frac{-\xi_1(T) P_2(T) + \xi_5(T) P_1(T) h_1(T)}{\operatorname{sgn}[P_1(T)]} = P_2(T) d T$$

since $h_1(T) = 0$. The remaining equations in matrix form become

$$\begin{bmatrix} P_2(T) d T \\ U d T \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -(T-t) & 0 & 0 & -I_2(T,t) \\ f_{11} & f_{12} & 1 & 0 & g_1 \\ f_{21} & f_{22} & (T-t) & 1 & g_2 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \xi_1(t) \\ \xi_2(t) \\ \xi_5(t) \end{bmatrix} \qquad \text{(2-44)}$$

Elimination of dT and obtaining $\psi_1(t)$ and $\psi_2(t)$ as functions of ξ_1, ξ_2, and ξ_5 results in

$$m_{1j} = m_{j1} = 0 \qquad\qquad j = 0, 3, 4$$

$$m_{2j} = m_{j2} = 0 \qquad\qquad j = 0, 3, 4$$

with

$$\Delta m_{11} = P_2(0)(f_{22} - f_{12}(T-t)) - U(T-t)^2 \tag{2-45}$$

$$\Delta m_{12} = -P_2(0)f_{12} - U(T-t) \tag{2-46}$$

$$\Delta m_{15} = f_{22}(-U\mathcal{I}_2(T,t) - P_2(0)g_1) + g_2(U(T-t) + P_2(0)f_{12}) \tag{2-47}$$

$$\Delta m_{21} = P_2(0)(f_{11}(T-t) - f_{21}) - U(T-t) \tag{2-48}$$

$$\Delta m_{22} = P_2(0)f_{11} - U \tag{2-49}$$

$$\Delta m_{25} = -f_{21}(-U\mathcal{I}_2(T,t) + g_1P_2(0)) - g_2(U - P_2(0)f_{11}) \tag{2-50}$$

where

$$\Delta = (U - P_2(0)f_{11})f_{22} + f_{21}(U(T-t) + P_2(0)f_{12})$$

which simplifies to

$$\Delta = \frac{2}{P_2(0)^3} \tag{2-51}$$

Thus the only nonzero components in (2-19) and (2-20) are m_{15}, m_{33}, and m_{35} and the quasi optimum control law becomes

$$p_1 = P_1 + m_{15}x_5 \tag{2-52a}$$

$$p_3 = m_{33}x_3 + m_{35}x_5 \tag{2-52b}$$

and the M matrix is of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & m_{05} \\ 0 & m_{11} & m_{12} & 0 & 0 & m_{15} \\ 0 & m_{21} & m_{22} & 0 & 0 & m_{25} \\ 0 & 0 & 0 & m_{33} & 0 & m_{35} \\ 0 & 0 & 0 & 0 & 0 & m_{45} \\ m_{50} & m_{51} & m_{52} & m_{53} & m_{54} & m_{55} \end{bmatrix} \tag{2-53}$$

The coefficients $m_{33}(t)$ and $m_{35}(t)$ are given by (2-30). To obtain $a(t)$, $b(t)$, $c(t)$, and $d(t)$ the time-varying second order system (2-21d) and (2-22d) must be solved. We were unable to solve this system and accordingly obtained scalar Riccati equations for m_{33} and m_{35} with the use of (19):

$$-\dot{m}_{33} = \frac{2P_1}{X_2} - 2m_{33}\frac{X_1}{X_2} + \frac{Um_{33}^2}{P_1} \tag{2-54a}$$

$$-\dot{m}_{35} = -m_{33}h_3 - m_{35}\frac{X_1}{X_2} + \frac{m_{45}}{X_1} + \frac{U}{P_1} m_{33}m_{35} \tag{2-54b}$$

$$-\dot{m}_{45} = -P_1 \frac{\partial h_1}{\partial x_4} \tag{2-54c}$$

where (2-54c) is required to solve for m_{35}. Since the first two of these equations are equally intractable, an approximate solution for m_{33} and m_{35} was obtained by assuming $\dot{m}_{35}$ and $\dot{m}_{35} = 0$, i.e.,

$$m_{33} = \frac{P_1 U}{X_2} \left[X_1 - (X_1^2 - 2X_2U)^{1/2}\right] \tag{2-55a}$$

$$m_{35} = \frac{\left[m_{33}h_3 - \dfrac{m_{45}}{X_2}\right]}{\left[\dfrac{Um_{33}}{P_1} - \dfrac{X_1}{X_2}\right]} \tag{2-55b}$$

where

$$m_{45} = -\int_{t}^{T} P_1(\tau) \frac{\partial h_1(\tau)}{\partial x_4} d\tau = I_5(T,t) \qquad (2\text{-}55c)$$

and P_1 and U are given by (2-17c) and (2-15b) respectively, with zero subscript omitted. It should be noted that m_{33} is exactly the same coefficient that was obtained for the tangential velocity correction in the problem without the central force field, which was expected.

After inserting (2-43) and (2-51) into (2-47), noting that $2/|P_2| = 2/UP_2$, and simplifying we obtain

$$\begin{aligned} m_{15} &= I_2(T,t) + (P_2X_1 - 2)I_2(T,t_s) \\ &\quad - UP_2(P_2X_1 - 1)I_3(T,t) + \frac{P_2^3}{2} X_1 I_4(T,t) \end{aligned} \qquad (2\text{-}55d)$$

The remaining task is to evaluate the integrals I_2 thru I_5 (I_1 is not needed) which are functions of h_1 and its partial derivatives with respect to X_2 and X_4. For the case of a target in a circular orbit, evaluation of these integrals is facilitated because the angle $X_4 - \Phi_t$ of the simplified problem is a constant. These integrals must be recalculated for other types of orbits. However, the performance of the quasi-optimum controller should not be effected by the choice of target trajectory.

For the case of a circular target orbit, further simplification may be obtained in h_1 by assuming that the target radius is much larger than the distance between the vehicle and target, i.e.,

$$r_t >> ax_2$$

Employing this simplification yields

$$h_1 \cong \frac{ax_2}{r_t^3} \qquad (2\text{-}56)$$

$$\frac{\partial h_1}{\partial x_2} \cong \frac{a}{r_t^5}\left(-2a^2x_2^2 - 4ax_2r_t\cos(x_4 - \Phi_t) + r_t^2(1 - 3\cos^2(x_4 - \Phi_t))\right) \qquad (2\text{-}57)$$

$$\frac{\partial h_1}{\partial x_4} \cong \frac{a}{r_t^4}\sin(x_4 - \Phi t)(2ax_2^2 + x_2r_t\cos(x_4 - \Phi t)) \qquad (2\text{-}58)$$

Inserting these relations into the integrals yeilds

$$\begin{aligned} I_2(t_s,t) &= \frac{P_1a}{r_t^5}\left[\frac{t_s^6A_4}{30} + \frac{t_s^5A_3}{20} + \frac{t_s^4A_2}{12} + \frac{t_s^3A_1}{6} + \frac{t_s^2A_0}{2}\right] \\ &- \frac{P_2a}{r_t^5}\left[\frac{t_s^7A_4}{42} + \frac{t_s^6A_3}{30} + \frac{t_s^5A_2}{20} + \frac{t_s^4A_1}{12} + \frac{t_s^3A_0}{6}\right] \end{aligned} \qquad (2\text{-}59)$$

$$\begin{aligned} I_2(T,t_s) &= \frac{P_1a}{r_t^5}\left[\frac{(T-t_s)^6A_4}{30} + \frac{(T-t_s)^5A_3'}{20} + \frac{(T-t_s)^4A_2'}{12} + \frac{(T-t_s)^3A_1'}{6} + \frac{(T-t_s)^2A_0'}{2}\right] \\ &- \frac{P_2a}{r_t^5}\left[\frac{(T-t_s)^7A_4}{42} + \frac{(T-t_s)^6A_3'}{30} + \frac{(T-t_s)^5A_2'}{20} + \frac{(T-t_s)^4A_1'}{12} + \frac{(t-t_s)^3A_0'}{6}\right] \end{aligned} \qquad (2\text{-}60)$$

$$I_3(t_s,t) = \frac{a}{r_t^3}\left[\frac{t_s^3U}{6} + \frac{t_s^2X_1}{2} + t_sX_2\right] \qquad (2\text{-}61)$$

$$I_3(T,t_s) = \frac{a}{r_t^3}\left[-\frac{(T-t_s)^3U}{6} + \frac{(T-t_s)^2X_1(t_s)}{2} + (T-t_s)X_2(t_s)\right] \qquad (2\text{-}62)$$

$$I_4(t_s,t) = \frac{a}{r_t^3}\left[\frac{t_s^4U}{24} + \frac{t_s^3X_1}{6} + \frac{t_s^2X_2}{2}\right] \qquad (2\text{-}63)$$

$$I_4(T,t_s) = \frac{a}{r_t^3}\left[-\frac{(T-t_s)^4 U}{24} + \frac{(T-t_s)^3 X_1(t_s)}{6} + \frac{(T-t_s)^2 X_2(t_s)}{2}\right] \tag{2-64}$$

$$\begin{aligned} I_5(T,t) = & -\frac{P_1 a \sin(X_4-\Phi_t)}{r_t^4}\left[\frac{D_4}{5}\{t_s^5 + (T-t_s)^5\} + \frac{1}{4}\{D_3 t_s^4 + D_3'(T-t_s)^4\}\right. \\ & \left. + \frac{1}{3}\{D_2 t_s^3 + D_2'(T-t_s)^3\} + \frac{1}{2}\{D_1 t_s^2 + D_1'(T-t_s)^2\} + D_0 t_s + D_0'(T-t_s)\right] \\ & + \frac{P_2 a \sin(X_4-\Phi_t)}{r_t^4}\left[\frac{D_4}{6}\{t_s^6 + (T-ts)^6\} + \frac{1}{5}\{D_3 t_s^5 + D_3'(T-t_s)^5\}\right. \\ & \left. + \frac{1}{4}\{D_2 t_s^4 + D_2'(T-t_s)^4\} + \frac{1}{3}\{D_1 t_s^3 + D_1'(T-t_s)^3\} + \frac{1}{2}\{D_0 t_s^2 + D_0'(T-t_s)^2\}\right] \end{aligned} \tag{2-65}$$

where t_s and $T-t_s$ are given by (2-17a) and (2-17b) respectively. The coefficients A_i and D_i are as follows:

$$A_0 = -2aX_2\{2r_t\cos(X_4-\Phi_t) + aX_2\} + r_t^2\{1-3\cos^2(X_4-\Phi_t)\}$$

$$A_0' = -2aX_2(t_s)\{2r_t\cos(X_4-\Phi t) + aX_2(t_s)\} + r_t^2\{1-3\cos^2(X_4-\Phi_t)\}$$

$$A_1 = -4aX_1\{aX_2 + r_t\cos(X_4-\Phi_t)\}$$

$$A_1' = -4aX_1(t_s)\{aX_2(t_s) + r_t\cos(X_4-\Phi_t)\}$$

$$A_2 = -2a\{aX_1^2 + aUX + Ur_t\cos(X_4-\Phi_t)\}$$

$$A_2' = -2a\{aX_1(t_s)^2 - aUX_2(t_s) - Ur_t\cos(X_4-\Phi_t)\}$$

$$A_3 = -2a^2UX_1$$

$$A_3' = 2a^2UX_1(t_s)$$

$$A_4 = -a^2/2$$

$$D_0 = X_2\{2aX_2 + r_t\cos(X_4-\Phi_t)\}$$

$$D_0' = X_2(t_s)\{2aX_2(t_s) + r_t\cos(X_4-\Phi_t)\}$$

$$D_1 = X_1\{4aX_2 + r_t\cos(X_4-\Phi_t)\}$$

$$D_1' = X_1(t_s)\{4aX_2(t_s) + r_t\cos(X_4-\Phi_t)\}$$

$$D_2 = 2aX_1^2 + 2aUX_2 + Ur_t\cos(X_4-\Phi_t)/2$$

$$D_2' = 2aX_1^2(t_s) - 2aUX_2(t_s) - Ur_t\cos(X_4-\Phi_t)/2$$

$$D_3 = 2aUX_1$$

$$D_3' = -2aUX_1(t_s)$$

$$D_4 = a/2$$

Hence use of (2-55a), (2-55b) and (2-55c) in (2-19) and (2-20) results in the following quasi optimum control law.

$$u_1 = \frac{P_1}{\left(P_1^2 + P_3^2\right)^{1/2}}$$

$$u_2 = \frac{P_3}{\left(P_1^2 + P_3^2\right)^{1/2}} \qquad (2\text{-}8)$$

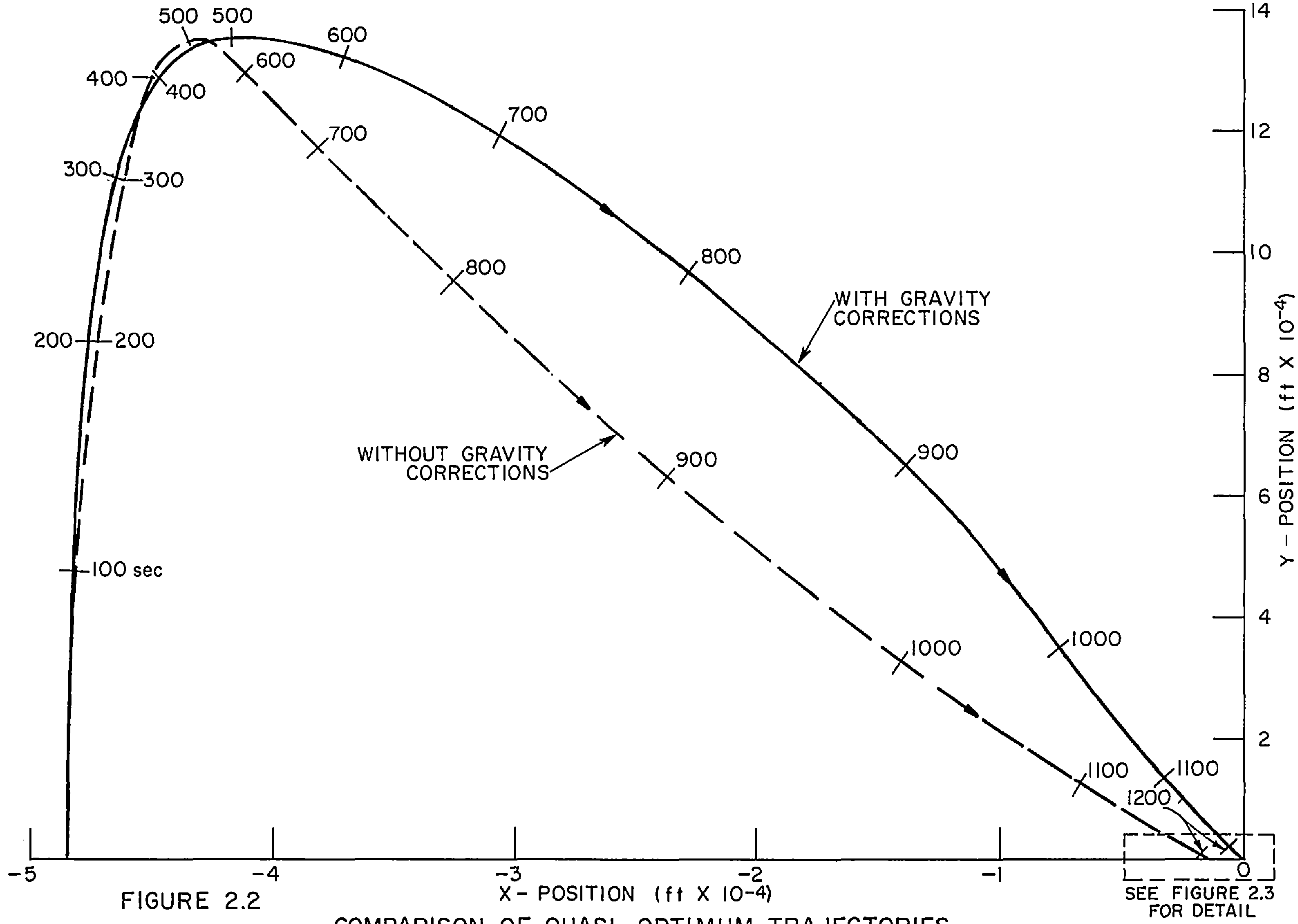


FIGURE 2.2

COMPARISON OF QUASI-OPTIMUM TRAJECTORIES WITH AND WITHOUT GRAVITY CORRECTIONS. RENDEZVOUS; $f/m = 10 \text{ ft/sec}^2$

where

$$p_1 = P_1 + X_5[\{-UP_1\}\{\mathcal{I}_2(T,t_s) - UP_2\mathcal{I}_3(T,t)\} + \mathcal{I}_2(t_s,t) + \frac{P_2^3}{2} X_1\mathcal{I}_4(T,t)] \quad (2\text{-}66)$$

$$p_3 = \frac{X_3P_1U}{X_2}\left[X_1 - \frac{\sqrt{2}\,U}{P_2}\right] + X_5\left[\frac{UP_2\mathcal{I}_5(T,t) - h_3P_1(P_2X_1 - \sqrt{2}\,U)}{\sqrt{2}}\right] \quad (2\text{-}67)$$

Performance With Quasi-Optimum Controller - The performance of the rendezvous control system using the quasi-optimum control law (2-8), (2-66) and (2-67) was simulated with the aid of a digital computer. For purposes of comparison, the performance without the gravity correction ($X_5 = 0$) was also simulated.

The first example chosen illustrates a rendezvous between a target in an 80 nautical mile circular orbit about the moon and a vehicle starting on the lunar surface. The thrust to mass ratio (a) used is 10 ft/sec^2. Lift-off occurs as the target passes directly over head. The trajectories for the quasi-optimum and simplified control laws are illustrated in Figures 2-2 and 2-3. Using the quasi-optimum control law, the vehicle is steered very close to the target and the radial and tangential velocities are simultaneously reduced very nearly to zero.

The control law without gravity corrections caused the vehicle to miss the target on the first pass and then recovered to complete the rendezvous (see Figure 2-2). The engine burn times were:

With gravity corrections:	1,267.8 seconds
Without gravity corrections:	1,347.0 seconds

The trajectories in a second example of rendezvous between a vehicle in a 160 nautical mile circular orbit and a target point on the lunar surface is illustrated in Figure 2-4. Powered descent is initiated when the vehicle passes over the target. The touchdown parameters for the illustrated trajectories are listed below:

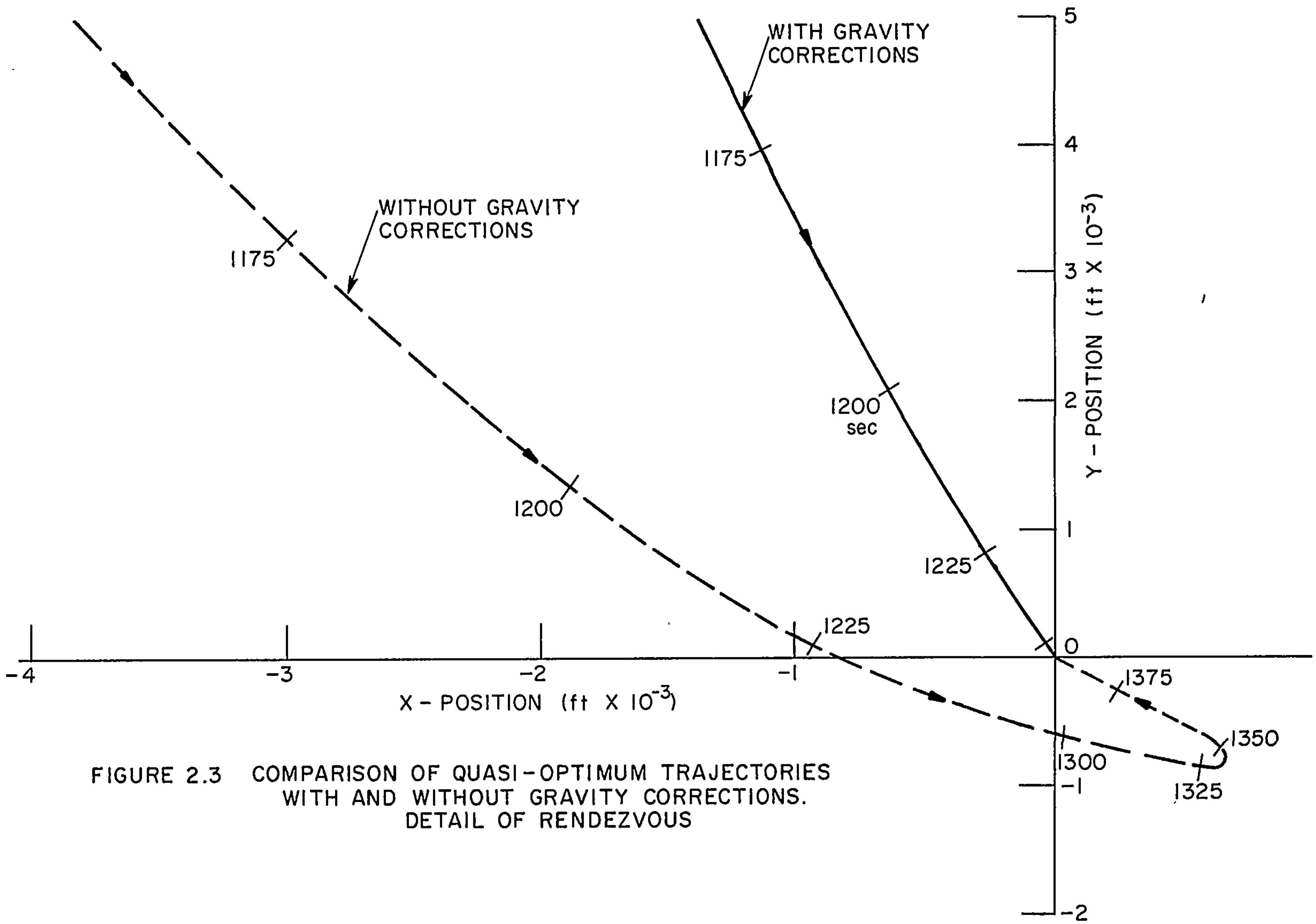


FIGURE 2.3 COMPARISON OF QUASI-OPTIMUM TRAJECTORIES WITH AND WITHOUT GRAVITY CORRECTIONS. DETAIL OF RENDEZVOUS

Guidance Law	Time (Sec.)	Miss Distance (Feet)	Impact Velocity (FPS)	Attitude (Deg)	Attitude Rate (Deg/Sec)
With gravity corrections	1152.4	8	7	+ 50	+ 3
Without gravity corrections	1153.5	300	750	- 60	- 58

The unsatisfactory performance of the controller without gravity correction is illustrated by the high impact velocity of 750 fps and attitude rate of -58 degrees per second. The performance using the quasi-optimum control law which accounts for gravitation, however, is quite satisfactory.

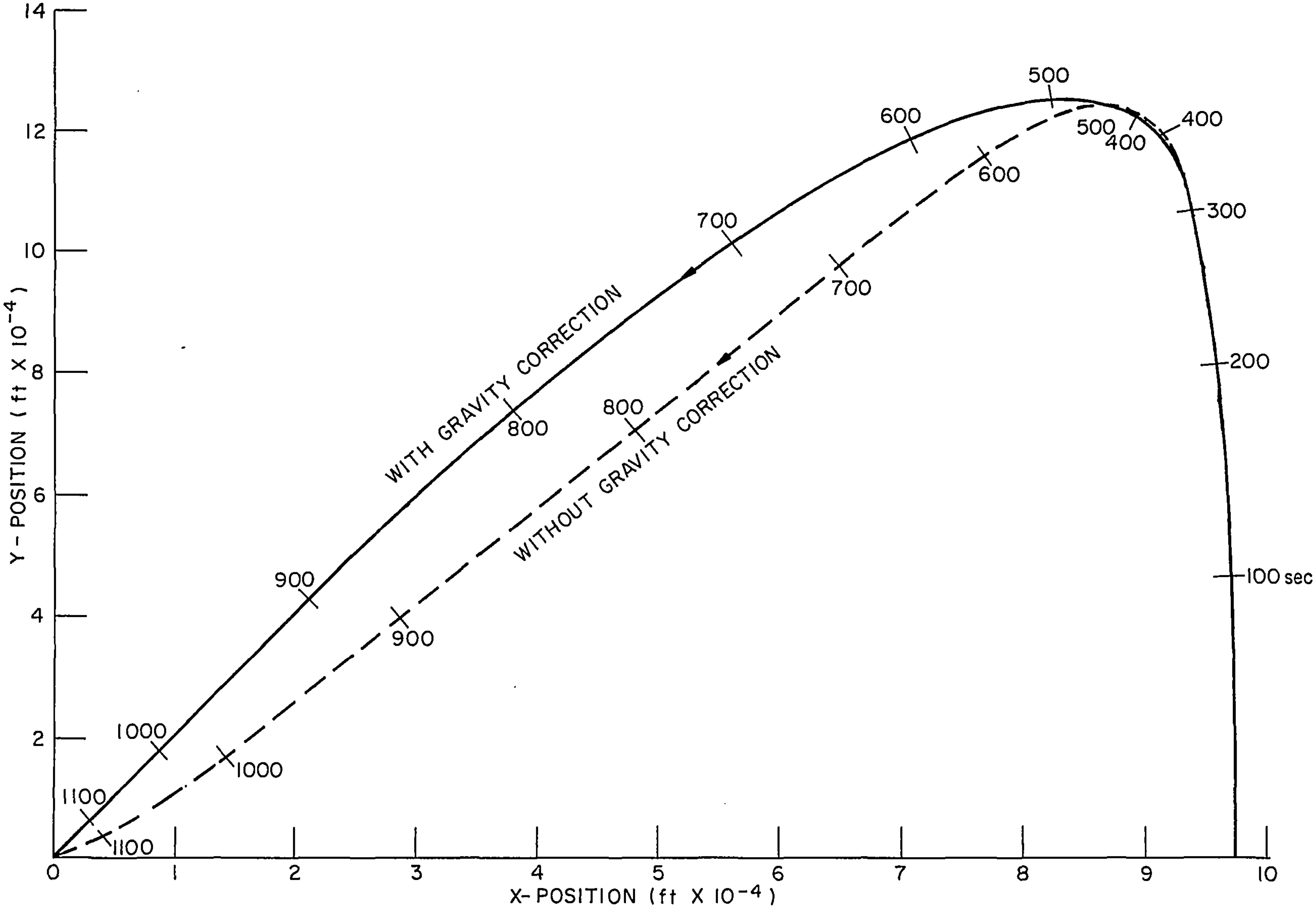


FIGURE 2.4 COMPARISON OF QUASI-OPTIMUM TRAJECTORIES WITH AND WITHOUT GRAVITY CORRECTIONS. LUNAR DESCENT, f/m = 10 ft/sec

2.3 AIRCRAFT LANDING PROBLEM

Problem Statement - Another application of the quasi-optimum control technique which was considered is the control of the longitudinal motion of an aircraft during the final approach phase. It is assumed that the "ideal" descent path is specified, along which the glide path angle is small (Figure 3.1a). It is also assumed that the forward aircraft speed is maintained essentially constant by utilizing throttle control. Thus, the longitudinal motion of the aircraft is governed entirely by the elevator deflection, $\delta(t)$, which is the only control variable. These assumptions lead to the so-called "short period" equations of motion of the aircraft. The control problem is formulated to obtain an optimum control of the elevator to guide the aircraft back to ideal descent path if any deviation therefrom is detected.

To obtain the short-period dynamic equations consider motion in the vertical plane. The (rigid body) torque acting at the center of pressure (cp) is

$$M_{cp} = I_{cp}\ddot{\psi} - md(\dot{V}_{GX}\sin\psi - \dot{V}_{GY}\cos\psi) \tag{3-1}$$

where I_{cp} is the moment of inertia about the cp and the other quantities are defined in Figure 3.1b. Now

$$V_{GX} = v(t)\cos(\psi + \alpha) \tag{3-2}$$

$$V_{GY} = v(t)\sin(\psi + \alpha)$$

where α is the angle-of attack. Hence

$$M_{cp} = I_{cp}\ddot{\psi} + mdv\,\dot{\psi}\cos\alpha + mdv\,\dot{\alpha}\cos\alpha + md\dot{v}\,\sin\alpha \tag{3-3}$$

The torque M_{cp} is induced by the nonlinear aerodynamic resistance forces,the elevator deflection δ, and gravity force. The respective torques are M_R, M_δ and M_g;

$$M_{cp} = M_R + M_\delta + M_g$$

It is a general practice to represent the aerodynamic resistance in terms of functions of the angular velocity of aircraft. Assuming that the angular velocity $\dot{\psi}$ is of such

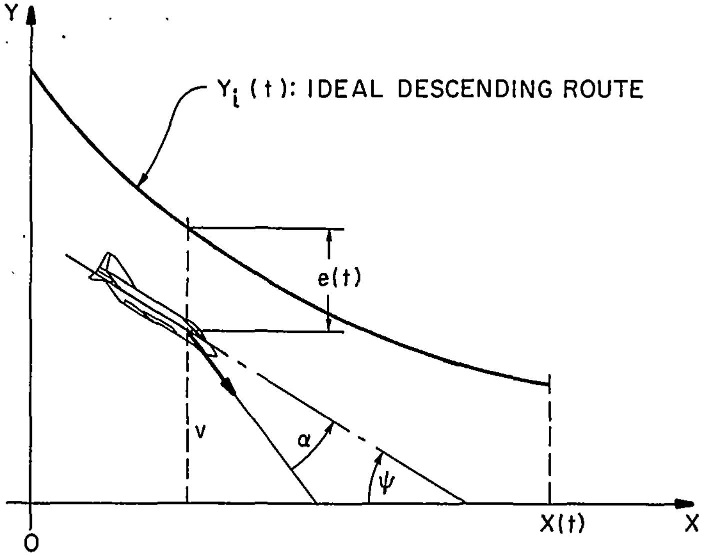


FIGURE 3.1(a) AIRCRAFT LANDING SCHEME

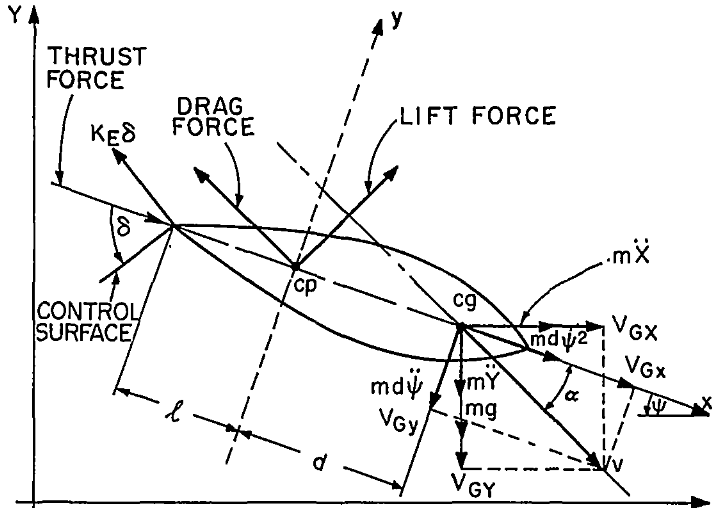


FIGURE 3.1(b) FORCE AND VELOCITY DIAGRAM

a magnitude that the higher order terms $\dot{\psi}^3$, $\dot{\psi}^4$, ... can be neglected, then

$$M_R = -2K_{D_1}\dot{\psi} - K_{D_2}\dot{\psi}|\dot{\psi}|$$

where K_{D_1} and K_{D_2} are constants.

The moment M_δ induced by elevator deflection is usually expressed as a linear function of the deflection δ , i.e.

$$M_\delta = K_E \ell \delta$$

where K_E is a constant coefficient and ℓ is as defined in Figure 3.1(b). Thus

$$M_{cp} = -2K_{D_1}\dot{\psi} - K_{D_2}\dot{\psi}|\dot{\psi}| + K_E \ell \delta + mgd \cos \psi \tag{3-4}$$

The vertical displacement from the ideal path is denoted by $e(t)$; it is seen that

$$\dot{e}(t) = v(t) \sin (\alpha + \psi) \tag{3-5}$$

To consider the dynamics of the angle-of-attack α, sum the forces in y-direction:

$$\begin{aligned} mg \cos \psi - K_D \sin \alpha - K_L \cos \alpha - K_E \delta &= m\frac{d}{dt}(v \sin \alpha) \\ &= m\dot{v} \sin \alpha + mv\dot{\alpha} \cos \alpha \end{aligned} \tag{3-6}$$

where

K_L is the lift force per unit angle of attack

K_D is the drag force per unit angle of attack

Thus, (3-3), (3-4), (3-5) and (3-6) completely specify the system dynamics.

Assuming small α and ψ , i.e. $\sin \alpha \approx \alpha$, $\sin \psi \approx \psi$, $\cos \alpha \approx 1$, $\cos \psi \approx 1$, (3-3), (3-4) and (3-6), after a bit of manipulation, become

$$\ddot{\psi} + \frac{K_{D_2}}{I_{cp}}\dot{\psi}|\dot{\psi}| + \frac{2K_{D_1} + mdv}{I_{cp}}\dot{\psi} = \frac{K_D d}{I_{cp}}\alpha + \frac{K_L d}{I_{cp}} + \frac{(\ell + d)K_E}{I_{cp}}\delta \tag{3-7}$$

$$\dot{\alpha} = - \frac{m\dot{v} + K_D}{mv} \alpha + \frac{mg - K_L}{mv} - \frac{K_E}{mv} \delta \tag{3-8}$$

For this investigation, we assume that the variation of angle-of-attack is negligible, $\dot{\alpha} \simeq 0$, and that the forward aircraft speed v is constant. Then (3-8) gives

$$\alpha = \frac{mg - K_L - K_E \delta}{K_D} \tag{3-9}$$

and (3-7) becomes

$$\ddot{\psi} + \frac{K_{D_2}}{I_{cp}} \dot{\psi} |\dot{\psi}| + \frac{2K_{D_1} + mdv}{I_{cp}} \dot{\psi} = \frac{1}{I_{cp}} (mgd + K_E \ell \delta) \tag{3-10}$$

Observe that the right hand side of (3-10) represents a normalized total torque acting on the aircraft. It is reasonable to keep the total torque at reasonably low value, by proper choice of control torque, and hence to include total torque penalty into the performance criterion.

Since the aircraft altitude and time duration for maneuvering are both limited during last phase of landing (typically altitude is about 100 ft at 20 sec. before touchdown), it is important to maintain the deviation $e(t)$ as small as possible during the control interval and to force the deviation to zero.

With these considerations, possible choice of performance indices are

$$V_1 = \int_0^T [e^2(t) + k(\text{total torque})^2]\, dt$$

or

$$V_2 = \int_0^T e^2(t) dt \qquad \text{with} \qquad |\text{total torque}| \leq \text{constant}$$

or

$$V_3 = \int_0^T [k + (\text{total torque})^2]\, dt$$

or

$$V_4 = \int_0^T [k_1 + k_2 e^2(t) + (\text{total torque})^2]\, dt$$

or

$$V_5 = \int_0^T [k + e^2(t)]\, dt \quad \text{with} \quad |\text{total torque}| \leq \text{constant}$$

For this investigation, the first case will be considered.

From the viewpoint of safety and passenger comfort, the pitch angle $\psi(t)$ should be kept within a small range during the entire interval of last phase landing. Thus, if we let $R(t)$ represent the ideal descending route the glide-path angle $\beta(t)$ is

$$\beta(t) = \tan^{-1} \dot{R}(t)$$

hence, if we let $\psi(T) \simeq 0$, the ideal path $R(t)$ should be so chosen to satisfy

$$\psi(T) = \tan^{-1} \dot{R}(T) + \alpha \simeq 0$$

but T can be any instant along the ideal route, therefore, in general

$$\tan^{-1} \dot{R}(t) + \alpha \simeq 0$$

and $R(t)$ is approximately a straight line for α = constant.

<u>Exact Problem</u> - In the application of the quasi-optimum control technique we have selected V_1 as the performance criterion, and have defined the state variables as follows:

$$\begin{aligned} x_1(t) &= e & x_4 &= v \\ x_2(t) &= \psi + \alpha & x_5 &= a = K_{D_2}/I_{cp} \\ x_3(t) &= \dot{\psi} & x_6 &= b = (2K_{D_1} + mdv)/I_{cp} \end{aligned}$$

$$u = (mgd + K_E \ell \delta)/I_{cp}$$

Note the use of extraneous state variables x_4, x_5, and x_6 to represent the constant parameters of the process. This device is used frequently in the application of the quasi-optimum control technique.

The assumption of constant angle-of-attack and constant forward speed thus lead to the following dynamic equations:

$$
\begin{aligned}
\dot{x}_0 &= \frac{1}{2}(x_1^2 + k^2u^2) \qquad \text{(i.e., } x_0(T) = V_1\text{)} \\
\dot{x}_1 &= x_4 \sin x_2 \\
\dot{x}_2 &= x_3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3\text{-}11) \\
\dot{x}_3 &= -x_5x_3|x_3| - x_6x_3 - u \\
\dot{x}_4 &= \dot{x}_5 = \dot{x}_6 = 0
\end{aligned}
$$

It is desired to minimize $x_0(T)$ with $x_1(T) = x_2(T) = x_3(T) = 0$.

The Hamiltonian for system (3-11) is

$$h = \frac{p_0}{2}(x_1^2 + k^2u^2) + p_1x_4 \sin x_2 + p_2x_3 - p_3(x_5x_3|x_3| + x_6x_3 + u) \tag{3-12}$$

and the maximum principle gives

$$u = -\frac{p_3}{k^2} \tag{3-13}$$

The corresponding adjoint equations are

$$\begin{aligned}
\dot{p}_0 &= 0 \\
\dot{p}_1 &= x_1 \\
\dot{p}_2 &= -p_1 x_4 \cos x_2 \\
\dot{p}_3 &= -p_2 + 2x_5 |x_3| p_3 + x_6 p_3 \\
\dot{p}_4 &= -p_1 \sin x_2 \\
\dot{p}_5 &= x_3 |x_3| p_3 \\
\dot{p}_6 &= x_3 p_3
\end{aligned} \tag{3-14}$$

Simplified Problem - Assuming that the constant coefficients v, a, b are small, and letting these constants equal zero, (i.e. $x_4 = x_5 = x_6 = 0$) in (3-11) and (3-14), we obtain a simplified system whose exact solution is readily available. Let the state and adjoint variables of the simplified problem be denoted by

$$X = [X_0, X_1, X_2, X_3, 0, 0, 0]$$

and (3-15)

$$P = [P_0, P_1, P_2, P_3, 0, 0, 0]$$

Then we have the simplified system

$$\begin{aligned}
\dot{X}_0 &= \frac{1}{2}(X_1^2 + k^2U^2) \\
\dot{X}_1 &= 0 \\
\dot{X}_2 &= X_3 \\
\dot{X}_3 &= -U
\end{aligned} \tag{3-16}$$

with Hamiltonian

$$H = \frac{P_0}{2}(X_1^2 + k^2U^2) + P_2X_3 - P_3U \tag{3-17}$$

and the adjoint equations

$$\begin{aligned} \dot{P}_0 &= 0 \\ \dot{P}_1 &= X_1 \\ \dot{P}_2 &= 0 \\ \dot{P}_3 &= -P_2 \end{aligned} \tag{3-18}$$

with the mixed initial and boundary conditions

$$X_0(0) = 0 , \quad X_1(0) = X_{10} , \quad X_2(0) = X_{20} , X_3(0) = X_{30} \tag{3-19}$$

$$\begin{aligned} &X_0(T) \text{ free} \\ &X_1(T) \text{ free} \\ &X_2(T) = 0 \\ &X_3(T) = 0 \\ &P_0(T) = -1 \\ &P_1(T) = 0 \end{aligned} \tag{3-20}$$

The exact solution for the simplified system can be summarized as follows:

$$U^* = -\frac{P_{30}}{k^2} \tag{3-21}$$

$$P_{10} = -X_{10}T \tag{3-22}$$

$$P_{20} = -\frac{6k^2}{T^3}(TX_{30} + 2X_{20}) \tag{3-23}$$

$$P_{30} = -\frac{2k^2}{T^2}(2TX_{30} + 3X_{20}) \tag{3-24}$$

$$S = \frac{1}{2}X_{10}T + \frac{2k^2}{T^3}(X_{30}^2T^2 + 3X_{20}X_{30}T + 3X_{20}^2) \tag{3-25}$$

where the non-unique expressions for T are given as follows:

(i) If $X_3 \geq 0$ and $X_2 \geq 0$, or, $X_3 < 0$ and $X_2 \geq \frac{kX_3^2}{6X_1}$ then

$$T = \frac{kX_3}{X_1} + \sqrt{\frac{k^2X_3^2}{X_1^2} + \frac{6kX_2}{X_1}} \triangleq T_1 \tag{3-26}$$

(ii) If $X_3 \leq 0$ and $X_2 \leq 0$, or, $X_3 > 0$ and $X_2 \leq -\frac{kX_3^2}{6X_1}$ then

$$T = -\frac{kX_3}{X_1} + \sqrt{\frac{k^2X_3^2}{X_1^2} + \frac{6kX_2}{X_1}} \triangleq T_3 \tag{3-27}$$

(iii) If $X_3 > 0$ and $-\frac{kX_3^2}{6X_1} < X_2 < 0$, or, $X_3 < 0$ and $0 < X_2 < \frac{kX_3^2}{6X_1}$

then

$$T = T_1 \qquad \text{if} \quad S(T_1) < S(T_3)$$

$$T = T_3 \qquad \text{if} \quad S(T_1) > S(T_3)$$

$$T = T_1 \text{ or } T = T_3 \qquad \text{if} \quad S(T_1) = S(T_3)$$

Quasi-Optimum Control - In accordance with the general theory, the quasi-optimum feedback control law is given by

$$u = -\frac{1}{k^2}(P_3 + m_{34}x_4 + m_{35}x_5 + m_{36}x_6) \tag{3-28}$$

where m_{34}, m_{35}, m_{36}, are the correction coefficients to be obtained by solving the auxiliary equations (20) with the coefficient matrixes given by

$$
H_{XX} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_1 \cos X_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\,|X_3|\,P_3 & -P_3 \\ 0 & 0 & P_1 \cos X_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\,|X_3|\,P_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_3 & 0 & 0 & 0 \end{bmatrix} \tag{3-29}
$$

$$
H_{XP} = H'_{PX} = \begin{bmatrix} 0 & X_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin X_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -X_3\,|X_3| & -X_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3-30}
$$

$$
H_{PP} = \begin{bmatrix} \dfrac{P_3^2}{P_0^3 k^2} & 0 & 0 & \dfrac{P_3}{P_0 k^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{P_3}{P_0^2 k^2} & 0 & 0 & -\dfrac{1}{P_0 k^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3-31}
$$

Or, in component form,

$$\left.\begin{aligned}
\dot{\xi}_0 &= X_1\xi_1 - \frac{P_3^2}{k^2}\psi_0 + \frac{P_3}{k^2}\psi_3 && \text{(a)}\\
\dot{\xi}_1 &= \sin X_2\,\xi_4 && \text{(b)}\\
\dot{\xi}_2 &= \xi_3 && \text{(c)}\\
\dot{\xi}_3 &= -X_3|X_3|\xi_5 - X_3\xi_6 + \frac{P_3}{k^2}\psi_0 + \frac{1}{k^2}\psi_3 && \text{(d)}\\
\dot{\xi}_4 &= \dot{\xi}_5 = \dot{\xi}_6 = 0 && \text{(e)}
\end{aligned}\right\} \quad (3\text{-}32)$$

and

$$\left.\begin{aligned}
\dot{\psi}_0 &= 0 && \text{(a)}\\
\dot{\psi}_1 &= \xi_1 - X_1\psi_0 && \text{(b)}\\
\dot{\psi}_2 &= -P\cos X_2\,\xi_4 && \text{(c)}\\
\dot{\psi}_3 &= 2\,|X_3|P_3\xi_5 + P_3\xi_6 - \psi_2 && \text{(d)}\\
\dot{\psi}_4 &= -P_1\cos X_2\,\xi_2 - \sin X_2\,\psi_1 && \text{(e)}\\
\dot{\psi}_5 &= 2\,|X_3|P_3\xi_3 + X_3|X_3|\psi_3 && \text{(f)}\\
\dot{\psi}_6 &= P_3\xi_3 + X_3\psi_3 && \text{(g)}
\end{aligned}\right\} \quad (3\text{-}33)$$

with the boundary conditions,

$$\begin{aligned}
\xi_1(T) &= -\dot{X}_1(T)\,dT = 0\\
\xi_2(T) &= -\dot{X}_2(T)\,dT = 0\\
\xi_3(T) &= -X_3(T)\,dT = U(T)\,dT = -\frac{P_3(T)}{k^2}\,dT\\
\psi_0(T) &= -P_0(T)\,dT = 0\\
\psi_4(T) &= -P_4(T)\,dT = 0\\
\psi_5(T) &= -P_5(T)\,dT = 0\\
\psi_6(T) &= -P_6(T)\,dT = 0
\end{aligned} \qquad (3\text{-}34)$$

and

$$\dot{P}_3(T)\xi_3(T) = \dot{X}_3(T)\psi_3(T)$$

or

$$\xi_3(T) = -\frac{P_{20}(P_{30} - P_{20}T)}{k^2}\psi_3(T)$$

Since, for our purpose, we are only interested in obtaining $\psi_3(0)$ in terms of $\xi_i(0)$'s, an observation of equations (3-32), (3-33) and (3-34) reveals that only the set of (3-32c), (3-32d), (3-32e), (3-33a), (3-33c) and (3-33d) has to be solved.

However, the integration of these equations is complicated by the presence of nonlinear terms involving $|X_3|$. To illustrate the procedure we will consider the integration of $P_3|X_3|$. From (3-16) and (3-18) we have

$$\begin{aligned}\int_0^t P_3|X_3|d\tau &= \int_0^t P_3\left|X_{30} + \frac{P_{30}}{k^2}\tau - \frac{P_{20}}{2k^2}\tau^2\right|d\tau \\ &= \frac{|P_{20}|}{2k^2}\int_0^t P_3\left|\frac{2k^2X_{30}}{P_{20}} + \frac{2P_{30}}{P_{20}}\tau - \tau^2\right|d\tau\end{aligned} \tag{3-35}$$

But

$$\frac{2k^2X_{30}}{P_{20}} + \frac{2P_{30}}{P_{20}}\tau - \tau^2 = \left(\frac{P_{30}}{P_{20}}\right)^2 + \frac{2k^2X_{30}}{P_{20}} - \left(t - \frac{P_{30}}{P_{20}}\right)^2 = 0 \tag{3-36}$$

gives

$$\tau = \begin{cases}\tau_2 \\ \tau_1\end{cases} = \frac{P_{30}}{P_{20}} \pm \sqrt{\left(\frac{P_{30}}{P_{20}}\right)^2 + \frac{2k^2X_{30}}{P_{20}}} \tag{3-37}$$

Since $\tau \geq 0$ and is real, we have

$$\left(\frac{P_{30}}{P_{20}}\right)^2 + \frac{2k^2X_{30}}{P_{20}} \geq 0 \tag{3-38}$$

Thus, in general, we would have a curve of the appearance of Fig. 3.2(a) for (3-36).

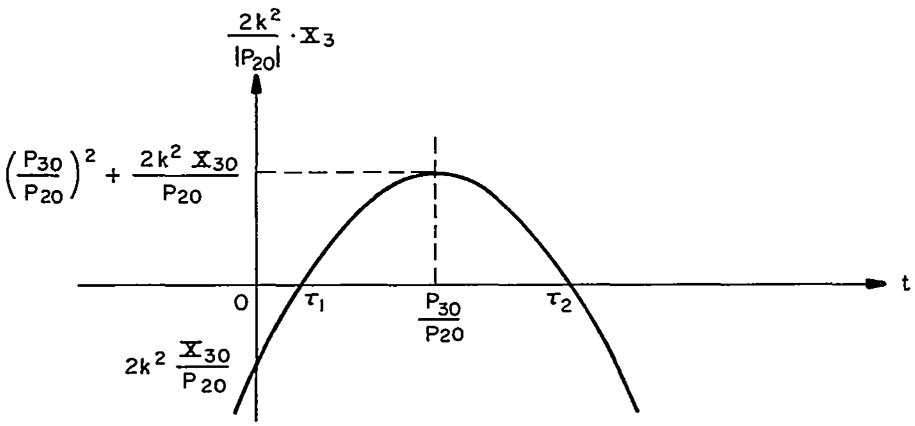


FIGURE 3.2a
PARABOLIC CONFIGURATION OF X_3

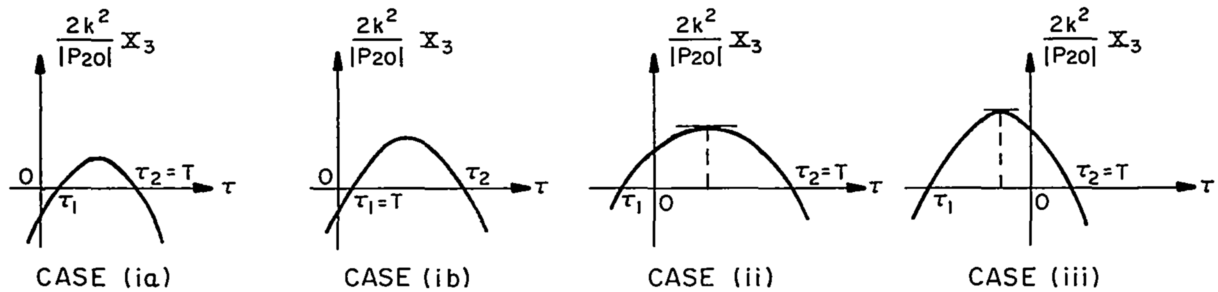


FIGURE 3.2b
DIFFERENT CASES FOR X_3

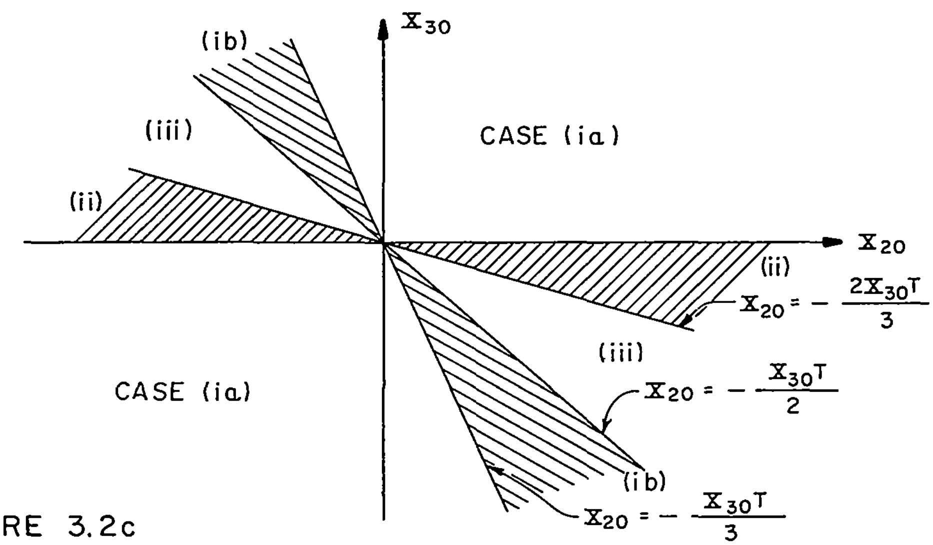


FIGURE 3.2c
REGIONS ON $X_{20} - X_{30}$ PLANE FOR POSSIBLE CASES

Notice that the shape and the location of the curve is governed by P_{30}/P_{20} and $(2k^2X_{30})/(P_{30})$. Hence, we may have the following four cases.

$$\text{(i)} \quad \frac{P_{30}}{P_{20}} \geq 0 \ , \qquad \frac{2k^2X_{30}}{P_{20}} \leq 0$$

$$\text{(ii)} \quad \frac{P_{30}}{P_{20}} \geq 0 \ , \qquad \frac{2k^2X_{30}}{P_{20}} \geq 0$$

$$\text{(iii)} \quad \frac{P_{30}}{P_{20}} \leq 0 \ , \qquad \frac{2k^2X_{30}}{P_{20}} \geq 0$$

$$\text{(iv)} \quad \frac{P_{30}}{P_{20}} \leq 0 \ , \qquad \frac{2k^2X_{30}}{P_{20}} \leq 0$$

Using expressions (3-23) and (3-24) we have

$$\frac{P_{30}}{P_{20}} = \frac{T(2X_{30}T + 3X_{20})}{3(X_{30}T + 2X_{20})} \tag{3-39}$$

$$\frac{2k^2X_{30}}{P_{20}} = -\frac{X_{30}T^3}{3(X_{30}T + 2X_{20})} \tag{3-40}$$

and from the fact that $T > 0$, we can show that case (iv) is impossible.

Furthermore, from (3-38) we have

$$\left\{ \begin{array}{l} \tau_2 \\ \tau_1 \end{array} \right. = \left\{ \begin{array}{l} T \\ \\ \dfrac{X_{30}T^2}{3(X_{30}T + 2X_{20})} \end{array} \right. \tag{3-41}$$

where $\tau_2 > 0$ and $\tau_1 \gtreqless 0$, and it can be observed that $\tau_1 \leq 0$ corresponds to case (ii) or (iii), while $\tau_1 > 0$ corresponds to case (i). Thus, in case (i), it is possible to have $\tau_1 \gtrless \tau_2$. Denote the case when $\tau_1 < \tau_2$ by (ia) and that of $\tau_1 \geq \tau_2$ by (ib) then we can summarize that

Case (i) $\quad \dfrac{P_{30}}{P_{20}} \geq 0 \quad , \quad \dfrac{X_{30}}{P_{20}} \leq 0$

(a) If $X_{30} \leq 0$ and $X_{20} < -\dfrac{X_{30}T}{3}$

or $X_{30} \geq 0$ and $X_{20} > -\dfrac{X_{30}T}{3}$

then $\displaystyle\int_0^T P_3 |X_3| dt = -\int_0^{\tau_1} P_3 X_3 dt + \int_{\tau_1}^T P_3 X_3 dt$

(b) If $X_{30} \leq 0$ and $-\dfrac{X_{30}T}{3} \leq X_{20} < -\dfrac{X_{30}T}{2}$

or $X_{30} \geq 0$ and $-\dfrac{X_{30}T}{3} \geq X_{20} > -\dfrac{X_{30}T}{2}$

then $\displaystyle\int_0^T P_3 |X_3| dt = -\int_0^T P_3 X_3 dt$

Case (ii) $\quad \dfrac{P_{30}}{P_{20}} \geq 0 , \quad \dfrac{X_{30}}{P_{20}} \geq 0$

If $X_{30} \leq 0$ and $X_{20} \geq -\dfrac{2X_{30}T}{3}$

or $X_{30} \geq 0$ and $X_{20} \leq -\dfrac{2X_{30}T}{3}$

then $\displaystyle\int_0^T P_3 |X_3| dt = \int_0^T P_3 X_3 dt$

Case (iii) $\quad \frac{P_{30}}{P_{20}} \le 0 , \qquad \frac{X_{30}}{P_{20}} \ge 0$

If $X_{30} \le 0$ and $- \frac{X_{30}T}{2} < X_{20} \le - \frac{2X_{30}T}{3}$

or $X_{30} \le 0$ and $- \frac{X_{30}T}{2} > X_{20} \ge - \frac{2X_{30}T}{3}$

then $\int_0^T P_3 |X_3| dt = \int_0^T P_3 X_3 dt$

These cases can be easily seen from the shapes of X_3 as shown in Fig. 3.2b. The regions for each case on the X_{20} , X_{30} plane are illustrated in Fig. 3.2c.

Thus, integration in the order, (3-32e), (3-33c), (3-33d), (3-33a), (3-32d) and (3-32c) gives

$$\xi_4(t) = \xi_4(0)$$

$$\xi_5(t) = \xi_5(0)$$

$$\xi_6(t) = \xi_6(0)$$

$$\psi_2(t) = \psi_2(0) + A_1(t)\xi_4(0)$$

$$\psi_3(t) = \psi_3(0) + B_1(t)\psi_2(0) + B_2(t)\xi_4(0) + B_3(t)\xi_5(0) + B_4(t)\xi_6(0) \tag{3-42}$$

$$\psi_0(t) = \psi_0(0) = 0$$

$$\xi_3(t) = \xi_3(0) + C_1(t)\psi_3(0) + C_2(t)\psi_2(0) + C_3(t)\xi_4(0) + C_4(t)\xi_5(0) + C_5(t)\xi_6(0) \tag{3-43}$$

$$\xi_2(t) = \xi_2(0) + D_1(t)\xi_3(0) + D_2(t)\psi_3(0) + D_3(t)\psi_2(0) + D_4(t)\xi_4(0) + D_5(t)\xi_5(0) + D_6(t)\xi_6(0) \tag{3-44}$$

where A_i, B_i, C_i, D_i are time-dependent coefficients resulting from the required integration.

Applying appropriate boundary conditions in (3-34) to equations (3-42), (3-43), (3-44) and solving for $\psi_3(0)$ in terms of $\xi_i(0)$'s yields

$$\psi_3(0) = m_{32}\xi_2(0) + m_{33}\xi_3(0) + m_{34}\xi_4(0) + m_{35}\xi_5(0) + m_{36}\xi_6(0) \tag{3-45}$$

In particular, the required coefficients are given by:

$$\begin{aligned} m_{34} = T^2 \{ & \left(\frac{1}{12} + \frac{K}{6}\right) \ell_1 + \left(\frac{1}{30} + \frac{K}{10}\right) \ell_2 T - \left(\frac{1}{60} + \frac{K}{15}\right) \ell_3 T^2 - \left(\frac{1}{105} + \frac{K}{21}\right) \ell_4 T^3 \\ & - \left(\frac{1}{168} + \frac{K}{28}\right) \ell_5 T^4 + \left(\frac{1}{252} + \frac{K}{36}\right) \ell_6 T^5 + \left(\frac{1}{360} + \frac{K}{45}\right) \ell_7 T^6 \\ & - \left(\frac{1}{495} + \frac{K}{55}\right) \ell_8 T^7 \} \end{aligned} \tag{3-46}$$

where

$$K = \frac{P_2(P_3 - P_2T)}{T + 4P_2(P_3 - P_2T)}$$

$$\ell_1 = P_1\left(1 - \frac{X_2^2}{2}\right)$$

$$\ell_2 = X_1\left(1 - \frac{X_2^2}{2}\right) - P_1X_2X_3$$

$$\ell_3 = X_1X_2X_3 + \frac{P_1}{2}\left(X_3^2 + \frac{P_3X_2}{k^2}\right)$$

$$\ell_4 = \frac{1}{2}\left[\frac{P_1}{k^2}\left(P_3X_3 - \frac{P_2X_2}{3}\right) + X_1\left(X_3^2 + \frac{P_3X_2}{k^2}\right)\right]$$

$$\ell_5 = \frac{1}{2}\left[P_1\left(\frac{P_3^2}{4k^4} - \frac{P_2X_3}{3k^2}\right) + \frac{X_1}{k^2}\left(P_3X_3 - \frac{P_2X_2}{3}\right)\right]$$

$$\ell_6 = \frac{1}{2k^4}\left[\frac{P_1P_2P_3}{6} - X_1\left(\frac{P_3^2}{4} - \frac{k^2P_2X_3}{3}\right)\right]$$

$$\ell_7 = \frac{1}{12k^4}\left(X_1P_2P_3 - \frac{P_1P_2^2}{6}\right)$$

$$\ell_8 = \frac{X_1P_2^2}{72k^4}$$

$$m_{36} = k^2X_3 \tag{3-47}$$

<u>For case (ia)</u>

$$\begin{aligned} m_{35} = C_1\{&T(A_1B_1 + A_2B_2 - A_3B_3 + A_4B_4 + A_5B_5 - A_6B_6 + A_7B_7 - A_8B_8) \\ &+ C_2(A_1B_9 - A_2B_{10} + A_3B_{11} - A_4B_{12} + A_5B_{13} + A_6B_{14} - A_7B_{15} + A_8B_{16})\} \end{aligned} \tag{3-48}$$

where

$$A_1 = k^4 \frac{P_2}{|P_2|} X_3^2$$

$$A_2 = k^2 \frac{P_2}{|P_2|} |X_3P_3|$$

$$A_3 = \frac{1}{3}\left(k^2P_2|X_3| + \frac{P_2}{|P_2|}P_3^2\right)$$

$$A_4 = \frac{1}{4} P_2|P_3|$$

$$A_5 = P_2|P_2|$$

$$A_6 = 2k^2P_3|X_3|$$

$$A_7 = P_3|P_3| + k^2P_2|X_3|$$

$$A_8 = \frac{1}{3}(2P_2|P_3| + |P_2|P_3)$$

$$B_1 = T^2 - 8\tau_1T + 6\tau_1^2$$

$$B_2 = 8\tau_1^2(T - \tau_1)$$

$$B_3 = \frac{1}{2}T^4 + 8\tau_1^3T - 9\tau_1^4$$

$$B_4 = 2\left(\frac{T^5}{5} + 2\tau_1^4T - \frac{12}{5}\tau_1^5\right)$$

$$B_5 = \frac{\tau_1^4}{2}(T^2 - 4\tau_1 T + 3\tau_1^2)$$

$$B_6 = 2\tau_1(T - \tau_1)^2$$

$$B_7 = \frac{T^4}{6} + 2\tau_1^2 T^2 - \frac{16}{3}\tau_1^3 T + 3\tau_1^4$$

$$B_8 = \frac{T^5}{5} + 2\tau_1^3 T^2 - 6\tau_1^4 T + \frac{18}{5}\tau_1^5$$

$$B_9 = 3(T^2 - 4\tau_1 T + 2\tau_1^2)$$

$$B_{10} = 2(T^3 - 6\tau_1^2 T + 4\tau_1^3)$$

$$B_{11} = 3(\frac{T^4}{2} - 4\tau_1^3 T + 3\tau_1^4)$$

$$B_{12} = 6(\frac{T^5}{5} - 2\tau_1^4 T + \frac{8}{5}\tau_1^5)$$

$$B_{13} = \frac{T^6}{4} - 3\tau_1^5 T + \tau_1^4 T^2 + \frac{3}{2}\tau_1^6$$

$$B_{14} = -2(2\tau_1 T^2 - 3\tau_1^2 T + \tau_1^3)$$

$$B_{15} = -\frac{1}{2}T^4 - 4\tau_1^2 T^2 + 8\tau_1^3 T - 3\tau_1^4$$

$$B_{16} = -\frac{7}{10}T^5 - 4\tau_1^3 T^2 + 9\tau_1^4 T - \frac{18}{5}\tau_1^5$$

$$C_1 = \frac{1}{k^2 T^2 [T + 4P_2(P_3 - P_2 T)]}$$

$$C_2 = 2P_2(P_3 - P_2 T)$$

<u>For case (ib)</u>

$$m_{35} = C_1\{T[-A_1 T^2 + \frac{T^4}{2}(A_3 - \frac{A_7}{3}) - \frac{T^5}{5}(4A_4 - A_8)] + C_2[-3A_1 T^2 + 2A_2 T^3 - \frac{T^4}{2}(3A_3 + A_7) + \frac{T^5}{10}(12A_4 + 7A_8) - 4A_5 T^6]\} \qquad (3\text{-}49)$$

<u>For case (ii) and (iii)</u>

$$m_{35} = C_1[A_1 T^3 + C_2 T^2(3A_1 + A_9 T + A_{10} T^2 - A_{11} T^3 + 4A_5 T^4)] \qquad (3\text{-}50)$$

where

$$A_9 = 2k^2 P_3 |X_3|$$

$$A_{10} = \frac{P_2}{|P_2|} P_3^2 - k^2 P_2 |X_3|$$

$$A_{11} = P_3 |P_2|$$

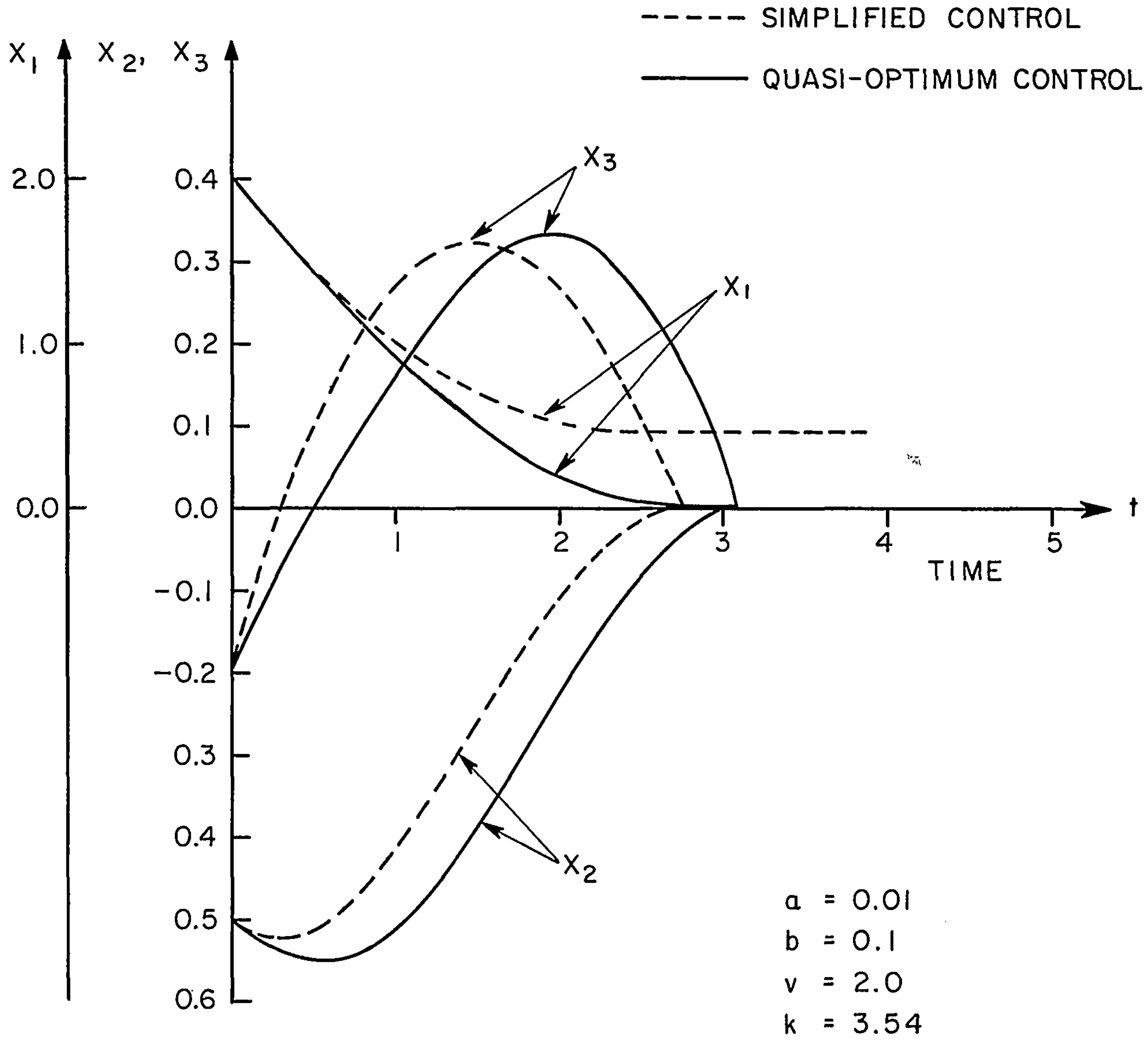


FIGURE 3.3
TRANSIENT RESPONSE OF AIRCRAFT LANDING SYSTEM

Results - Observe that while the simplified control takes the form

$$u_s = f_1(k, x_2, x_3)$$

the quasi-optimum control u_q takes the form

$$u_q = f_2(k, x_1, x_2, x_3)$$

where

$$f_1(k, 0, 0) = f_2(k, x_1, 0, 0) = 0$$

Thus, it can be seen from the system dynamic equations that if both x_2 and x_3 vanish, or become small, simultaneously. The x_1 component of the system may stay at certain steady state value x_{1ss}. This is so because the feedback controls u_s and u_q are mostly dominated by x_2 and x_3 rather than by x_1.

However, for some given initial conditions of x_1, it is possible to adjust the value of k to eliminate the steady-state error x_{1ss}.

A computer simulation study of the control system with $a = 0.01$, $b = 0.1$, $v = 2.0$ and initial conditions $x_{10} = 2.0$, $x_{20} = -0.5$, $x_{30} = -0.2$ the value of k for which $x_1(T) = 0$ is approximately equal to 3.54 for quasi-optimal control and 5.18 for simplified control. Fig 3.3 shows the transient response of x_1, x_2, x_3 for $k = 3.54$. Fig 3.4 shows the plots of $x_1(T)$, $S(T)$ and T vs. k and which indicates that for $x_{1ss} = 0$ the quasi-optimum control system have both better transient time and performance.

For the initial conditions $x_{10} = -2.0$, $x_{20} = 0.5$, $x_{30} = -0.2$ however, the situation is reversed. In this case the quasi-optimal control system has both worse transient time and performance.

For the initial conditions $x_{10} = -2.0$, $x_{20} = -0.5$, $x_{30} = -0.2$ and $x_{10} = 2.0$, $x_{20} = 0.5$, $x_{30} = -0.2$, it is found that no value of k could be found to make $x_{1ss} = 0$.

From these results it can be concluded that the quasi-optimal control system derived gives better performance than the simplified control law only in certain cases. Further study of this problem, utilizing alternative simplified systems, should be undertaken.

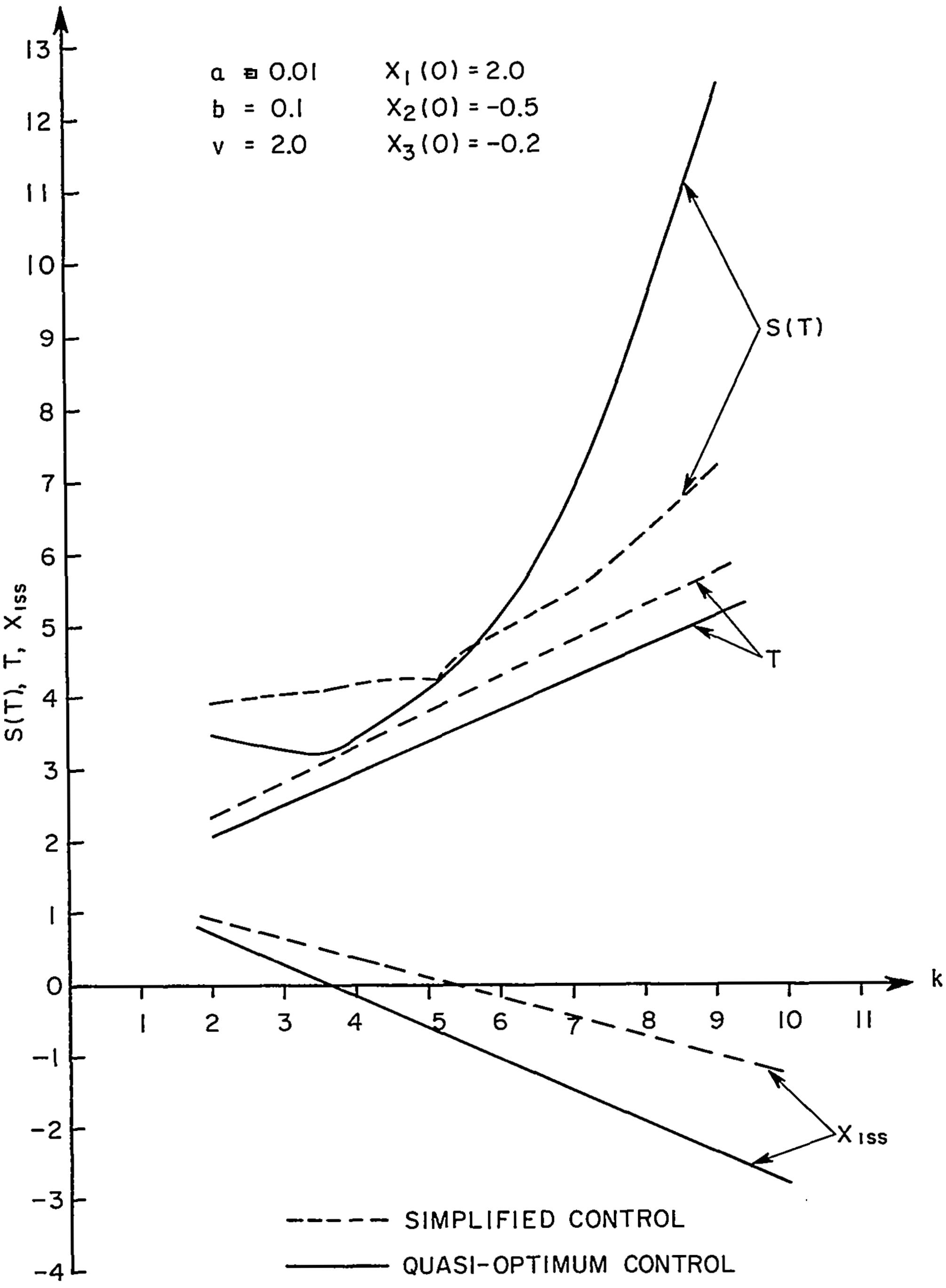


FIGURE 3.4 S(T), T, X_{1SS} vs k CURVES

CONCLUSIONS AND RECOMMENDATIONS

On the basis of the results achieved in the examples considered under Contracts NAS 2-2648 and NAS 2-3636, it is our conclusion that the quasi-optimum control technique described herein is a valuable tool for the design of practical feedback control systems. As indicated in [A1], two conditions must be met in order for our method to be applicable to a particular design problem. First, the actual process must be capable of being approximated by a simpler process, and, second, the exact control law for the simpler process must be found. Experience with the physical problem to be solved is an aid to meeting the first requirement, and familiarity with the solved problems of optimum control is an aid to meeting the second. The successful application of the technique to a particular design problem, however, will ultimately depend on the user's ingenuity. We regard this as an asset, not a shortcoming of the technique.

Although we have shown that for sufficiently small values of the parameter μ in the mildly nonlinear process (34) the performance of the quasi-optimum control law is superior to the simplified control law, a general proof to this effect has as yet not been obtained. It would appear, however, that the approach used to establish the above result can be extended to a wider class of problems in which the exact process reduces to the simplified process when a parameter $\mu \to 0$. It would also appear that the methods used for the mildly-nonlinear process can be used to assess the stability of the quasi-optimum control law. The problems of performance and stability require further investigation.

More attention should also be given to the application of the technique to problems in stochastic optimum control. It would be desirable to calculate the stochastic quasi-optimum control law for a (nontrivial) problem for which the stochastic optimum control law is known, in order that a better comparison between the exact optimum and the quasi-optimum control laws can be made.

Other areas which are worthy of more work are the application of the technique to the treatment of state variable constraints and to the problem of trajectory optimization.

More completed examples will add practical insight into the advantages and limitations of the technique. Consequently we recommend completion of the studies of aircraft landing described in Section 2.3 and of reentry guidance described in [A1]. Studies of the application of the quasi-optimum control technique to other problems in guidance and control should also be undertaken.

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APPENDIX 1

ON THE INVERSE OPTIMUM CONTROL PROBLEM FOR A CLASS OF NONLINEAR AUTONOMOUS SYSTEMS

Fred E. Thau

SUMMARY

Some aspects of the inverse optimum control problem are considered for a class of nonlinear autonomous systems. A closed-loop system with a known control law is given; the problem is to determine performance criteria for which the given control law is optimum. Algebraic conditions that must be satisfied by a class of scalar performance criteria of the form $V = \int_t^\infty [q(x) + h(u)]\,d\tau$ are obtained. It is shown that if the value of the optimum V^o is required to be a quadratic form $V_o = x'Mx/2$ of the current state x, and if certain state variables cannot be measured, then M cannot be positive definite. The inverse optimum control problem corresponding to the problem of Lur'e is considered. Examples are given to illustrate the techniques and to compare the properties of a linear and nonlinear system having the same optimum performance $V^o(x)$.

1. INTRODUCTION

In recent years engineering applications of optimum control theory have to a large degree been confined to linear systems. The principal reason for this situation is that the theory of linear systems with performance criteria of the form

$$V = \frac{1}{2}\int_t^\infty (x'Qx + u'Ru)\, d\tau$$

has been developed to a more advanced point than has the theory of optimum nonlinear systems. Moreover, frequency-domain interpretations of the theoretical results for linear systems have made these results more accessible to engineers who are familiar with the classical frequency-domain techniques of analysis.

A major contribution to the development of linear optimum control theory was the paper of Kalman [1] in which the point of view of the inverse optimum control problem was introduced. The inverse problem of optimum control theory can be stated loosely as follows: "Given a dynamic system and a known control law, find performance criteria (if any) for which this control law is optimum." Kalman considered a precise formulation of this problem for linear autonomous systems and derived many interesting time-domain and frequency-domain properties of linear control systems.

The purpose of this paper is to investigate some aspects of the inverse problem for certain nonlinear control systems. Recent results [3] on the use of higher-order forms as performance criteria for nonlinear systems indicate the usefulness of nonlinear control laws. This study endeavors to contribute to an understanding of the relationship between the specification of a performance criterion of the form

$$V = \int_t^\infty [q(x) + h(u)]\, d\tau$$

(where $q(x)$ and $h(u)$ are scalar functions) and the structure of the resulting optimum control system. The essential assumptions upon which the analysis is based are:

1) The control acts over an infinite time interval.
2) There are no constraints on the control or state variables.
3) The integrand of the performance criterion is a sum of a function of the state $q(x)$ and a function of the control $h(u)$.
4) dh/du is a 1-1 mapping, $h(0) = 0$, and $d^2h/du^2 > 0$.

In Section 2 the inverse optimum control problem is formulated for a general class of systems and performance indices, and in Section 3 general properties implied by optimality are obtained. Two special cases are considered in Section 4: first, the algebraic and frequency-domain characterizations of optimality which were obtained in [1] for single-input linear systems are generalized to multiple-input linear systems, and then it is shown, for linear and nonlinear single-input systems, that if the optimum performance is required to be a positive definite quadratic form in the state variables, then the optimum control must be a function of a linear combination of (at least) those state variables which are directly affected by the control. In Section 5 a class of control laws satisfying the conditions imposed in the problem of Lur'e is considered and a class of performance criteria for which the given control law is optimum are determined.

In Section 6 two examples involving cubic feedback are presented. The first example illustrates the property of single-input optimum systems mentioned above. The second example provides a comparison between a nonlinear system and a linear system having the same optimum performance. It is found that the nonlinear system provides smaller excursions of the state variables than does the linear system. This property may be useful in certain engineering applications.

2. PROBLEM STATEMENT

Consider the nth - order process

$$\dot{x} = f(x) + Gu \qquad (1)$$

where the state of the process x is an n-vector, G is an $n \times m$ constant matrix, and the control function $u(t)$ is a continuous function of time. This paper is concerned with performance criteria of the form

$$V(x(t); u) = \int_t^{\infty} [q(x) + h(u)] \, d\tau \qquad (2)$$

where $q(\cdot)$ and $h(\cdot)$ are smooth functions of their arguments. Additional assumptions that will be required in the subsequent analysis are the following: the vector function $\eta(u) = dh/du$ is a 1-1 mapping, $h(0) = 0$, and $d^2h/du^2 > 0$. The motivation for these conditions will become clear in the next section.

Note that the integral in (2) is a continuous functional of the control function $u(\tau)$. Of particular interest are feedback control laws of the form

$$u(\tau) = \varphi(x(\tau)) \tag{3}$$

which when applied to the process (1) result in an asymptotically stable closed-loop system,

$$\dot{x} = f(x) + G\varphi(x) \tag{4}$$

Thus the origin $x \equiv 0$ is considered the target set and the control law (3) is assumed to be such that

$$\lim_{T\to\infty} x_{\varphi}(T; x(t)) = 0$$

where $x_{\varphi}(T; x(t))$ denotes the trajectory of the asymptotically stable closed-loop system (4).

The inverse optimum control problem can now be formulated as follows: Given a control law (3) with the above properties, find the most general performance functional (if any) of the form (2) which is minimized by (3). Note that an optimum control (3) is assumed to exist; in the next section we will apply the necessary and sufficient conditions for (3) to be optimum which are implied by Hamilton - Jacobi theory [4].

The structure of the given closed-loop system is shown in Fig. 1. Note that two systems may have the same trajectories $x(t)$, $t > 0$, yet, in terms of the above structure, they will be considered as essentially different control systems. (For example, the system

$$\dot{x}_1 = x_2 \tag{5a}$$

$$\dot{x}_2 = -x_1 - x_2 + u$$

where $u = -x_1^3$, and the system

$$\dot{x}_1 = x_2 \tag{5b}$$

$$\dot{x}_2 = -2x_1 - x_2 + u$$

where $u = -(x_1^3 - x_1)$ will have exactly the same trajectories $\{x_1(t), x_2(t)\}$ if they have identical initial states. However, (5a) and (5b) will be treated as essentially different control systems since the vectors $f(x)$ and $G\varphi(x)$ for (5a) and (5b) are clearly not the same.)

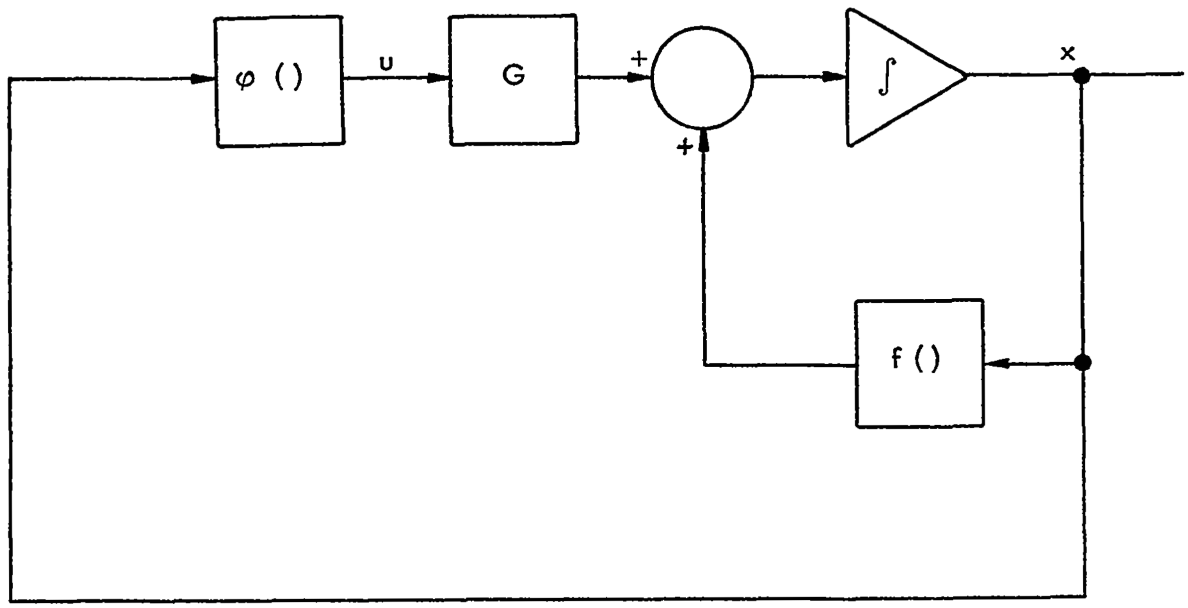

FIGURE 1 (APPENDIX)
STRUCTURE OF GIVEN SYSTEM

3. STRUCTURE OF OPTIMUM SYSTEM

The necessary and sufficient conditions for optimality which were derived by Kalman [4] (restated for the current context) are as follows:

Let

$$H(x, \frac{\partial V}{\partial x}, u) = -[q(x) + h(u)] - \frac{\partial V}{\partial x} \cdot [f(x) + Gu] \tag{6}$$

have an absolute maximum with respect to u at $u = \varphi(x)$ where $\varphi(x)$ is differentiable in x. Then

(1) The twice differentiable function $V^o(x)$ is the optimum performance, and

(2) $\varphi(x)$ is the optimum control law

if, and only if, $V^o(x)$ satisfies the Hamilton-Jacobi equation,

$$\max_u H(x, \frac{\partial V^o}{\partial x}, u) = 0, \quad V^o(0) = 0 \tag{7}$$

It is assumed that there are no constraints on the control u. Then, since $d^2h/du^2 > 0$ and $\eta(u)$ is a 1-1 mapping, H of (6) is maximized by differentiating with respect to u. Thus

$$\varphi(x) = \eta^{-1}(-G' \frac{\partial V}{\partial x}) \tag{8}$$

and (7) yields

$$q(x) = -h(\varphi(x)) - \frac{\partial V^o}{\partial x} \cdot [f(x) + G\varphi(x)] \tag{9}$$

To obtain more explicit results, the value of the optimum performance index V^o, as a function of the current state x, is assumed to be given by

$$V^o(x) = \frac{1}{2} x' Mx \tag{10}$$

where M is a constant symmetric matrix. It is well known that linear optimum systems with performance criteria of the form (14) below yield optimum cost functions of the form (10) with M positive definite. It is shown below that certain optimum nonlinear systems also have optimum cost functions of this form. Thus, from a practical engineering viewpoint, the choice (10) will allow a comparison between the performance of a given nonlinear system and a corresponding linear system to be given in Section 6 below. From a mathematical standpoint the restriction (10) is inessential: other higher-order forms could be assumed and an analysis similar to that presented below would follow with only minor changes in the details.

With V^o given by (10), (8) and (9) become

$$\varphi(x) = \eta^{-1}(-G'Mx) \tag{11}$$

and

$$q(x) = -h(\varphi(x)) - x'M(f(x) + G\varphi(x)) \tag{12}$$

where (11) and (12) must hold for all x. Hence the symmetric matrix M and the scalar function $q(x)$ are determined from the solution to (12) and $h(u)$ is determined as the integral of the function η in (11).

The transformation implied by (11) between state and control-input for the optimum system is shown in Fig. 2.

In general V^o is non-unique, and the problem is to determine consistency conditions regarding the choice of matrix M and functions $q(x)$ and $h(u)$. A technique that might be considered would involve solving the first-order partial differential equation (11) by the method of characteristics. However, since the nonlinear ordinary differential equations that result are, in general, impossible to solve, another approach must be used in the sequel. In Section 5 an explicit form for the control law $\varphi(x)$ will be assumed and (11) and (12) will be used to determine algebraic conditions that are necessary and sufficient for the optimality of the given control law.

4. SPECIAL CASES

Linear Systems - Consider the completely controllable, multiple-input, linear, time-invariant system

$$\dot{x} = Fx + Gu \tag{13}$$

where F is an $n \times n$ matrix and G is an $n \times m$ matrix. The performance index is

$$V = \frac{1}{2}\int_t^\infty (x'H'Hx + u'u)\, d\tau \tag{14}$$

and the optimum performance is required to be $V^o = \frac{1}{2}x'Mx$. The given control law which

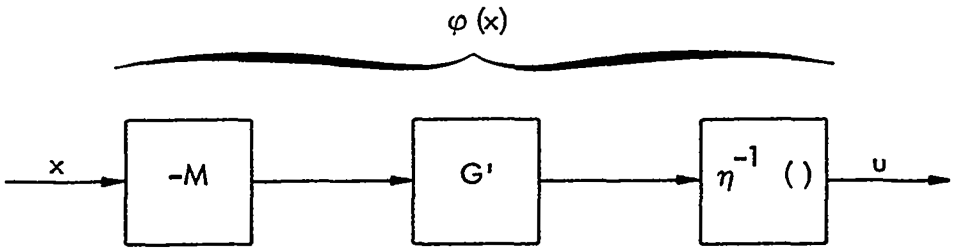

FIGURE 2 (APPENDIX)
CONTROLLER FOR OPTIMUM SYSTEM

drives the system towards the origin is

$$u = -Kx \tag{15}$$

where K is a known constant matrix and M and H are unknown constant matrices. Since (11) and (12) must hold for all x, (11) yields

$$K = G'M \tag{16}$$

and (12) gives

$$H'H + K'K = -MF - F'M + MGK + K'G'M \tag{17}$$

Define

$$F_k = F - GK \tag{18}$$

Then (17) becomes

$$-MF_k - F_k'M = H'H + K'K \tag{19}$$

Thus (16) and (19) are necessary and sufficient algebraic conditions for control law (15) to be optimum for performance index (14).

In [1] Kalman considered single-input linear systems, required M to be positive definite, and (in Theorem 4) presented (16) and (19) along with the positive definite condition on M as necessary and sufficient for (15) to be optimum. Using (17) one can obtain the multiple-input version of the frequency-domain characterization of optimality that was obtained in [1] for single-input linear systems. Using (16), write (17) as

$$-MF - F'M = H'H - MGG'M \tag{20}$$

Add and subtract sM from the left-hand side of (20) to obtain

$$M(s\imath - F) + (-s\imath - F')M = H'H - MGG'M \tag{21}$$

Define

$$\Phi(s) = (s\imath - F)^{-1} \tag{22}$$

and pre-multiply both sides of (21) by $G'\Phi'(-s)$, post-multiply both sides of (21) by $\Phi(s)G$, and add an $m \times m$ identity matrix to both sides of (21), to obtain

$$(\imath + G'\Phi'(-s)K')(\imath + K\Phi(s)G) = \imath + G'\Phi'(-s)H'H\Phi(s)G \tag{23}$$

Let $s = j\omega$ and define

$$T(j\omega) = i + K\Phi(j\omega)G \tag{24}$$

Then (24) becomes

$$T'(j\omega)T(j\omega) = i + G'\Phi'(-j\omega)H'H\Phi(j\omega)G \tag{25}$$

or

$$T'(j\omega)T(j\omega) \leq i \tag{26}$$

which is the frequency-domain characterization of optimality for multiple-input linear systems with performance criteria of the form (14). *

Single-Input Systems – Now consider single-input asymptotically stable systems of the form (1) where $G' = (0 \ldots 0 1)$. The following necessary conditions for M in (10) to be positive definite will be established: *if M is positive definite, then u must be a function of (at least) x_n.* This follows from (11) and Fig. 2 by a simple proof by contradiction: assume u is not a function of x_n. Then, since η is a 1-1 mapping

$$-G'Mx = -(m_{n1}x_1 + \ldots + m_{nn}x_n) \tag{27}$$

where $m_{nn} = 0$ and M is thus not positive definite. (Since one of its main diagonal elements is zero, M could be an indefinite matrix.) This contradiction establishes the above necessary condition for M to be positive definite.

This condition can be generalized to single-input systems in which the control directly affects more than one state variable: if M is positive definite than the optimum control must be a function of a linear combination of (at least) those state variables which are directly affected by the control. For example, if $G' = \underbrace{(0 \ldots 0\underset{k}{1}0 \ldots 0\underset{r}{1}0 \ldots 0)}_{n}$

and u is not a function of x_k and x_r, then

$$-G'Mx = -(m_{k1}x_1 + \ldots + m_{kn}x_n + m_{r1}x_1 + \ldots + m_{rn}x_n) \tag{28}$$

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* This result has been obtained by Anderson in [7] wherein the sensitivity problem for linear systems is also considered.

where $m_{kk} = -m_{rk}$ and $m_{kr} = -m_{rr}$. Therefore, M is not positive definite since one of its principal cofactors is zero.

The above general condition, which holds for both nonlinear and linear single-input systems, is invariant under a nonsingular linear transformation of state variables. For, if $y = Tx$, where T is a constant nonsingular matrix, then (1) and (10) become respectively

$$\dot{y} = Tf(T^{-1}y) + TGu \tag{29}$$

and

$$V = \frac{1}{2}y'\Lambda y \tag{30}$$

where $\Lambda = T'^{-1}MT$ is positive definite if and only if M is positive definite. Equation (11) becomes

$$\varphi(T^{-1}y) = \eta^{-1}(-G'T'\Lambda y) \tag{31}$$

Suppose $G' = (0\ldots0\underset{\uparrow k}{1}0\ldots0\underset{r\uparrow}{1}0\ldots0)$. From the above results it is seen that if M is positive definite, then u must be a function of a linear combination of x_k and x_r. However, if M is positive definite, then from (31) u must be a function of a linear combination of all components of y for which

$$T_{ik} + T_{ir} \neq 0 \tag{32}$$

Since T is nonsingular, there must be at least one value j, $1 \leq j \leq n$, for which $T_{jk} + T_{jr} \neq 0$. Thus, since $y = Tx$, y_j is a linear combination of x_k and x_r, and, as before, u must be a function of a linear combination of x_k and x_r.

This property of single-input systems with performance criteria of the form (2) is significant since it indicates that if the value of the optimum performance is required to be a quadratic form $V^o = \frac{1}{2}x'Mx$ of the current state x, and if each state variable which is directly affected by the control cannot be measured, then M cannot be positive definite. However, one must interpret this property with care. If the optimum performance V^o is required to be of a form other than (10) and if certain state variables cannot be measured, then the problem of finding conditions under which V^o is positive definite in x is currently an open question.

5. PROBLEM OF LUR'E

Since the early 1950's there has been a great deal of interest in determining the asymptotic stability of the origin for a class of systems governed by

$$\dot{x} = Ax + b\theta(\sigma) \tag{33}$$

$$\sigma = g'x \tag{34}$$

where x, b, and g are n-vectors and A is an nxn matrix. The results of the previous sections will now be applied to this class of systems, where $\theta(\sigma)$ is considered to be a known scalar function, defined and continuous for all σ, $\theta(0) = 0$, $\sigma\theta(\sigma) > 0$ for all $\sigma \neq 0$, and

$$\int_0^{\pm\infty} \theta(\sigma)\, d\sigma \text{ diverges}$$

It will be shown that the asymptotically stable system (33) - (34) is the optimum closed-loop system for a class of performance criteria of the form (2).

For this case (11) and (12) become

$$\theta(g'x) = \eta^{-1}(-b'Mx) \tag{35}$$

and

$$q(x) = -\frac{1}{2}x'(MA + A'M)x - x'Mb\theta(g'x) - h(\theta(g'x)) \tag{36}$$

respectively.

Now assume that $\theta(\sigma)$ can be expressed as a power series in odd powers of σ with all positive coefficients, i.e.

$$\theta(\sigma) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} a_i \sigma^i \tag{37}$$

where all $a_i > 0$. It will be shown that if $\eta^{-1}(\sigma)$ is also expressed as a power series

$$\eta^{-1}(\sigma) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} c_i \sigma^i \tag{38}$$

and if M is positive definite, then each coefficient c_i can be determined explicitly in terms of the coefficients a_i and the components of the matrix M. From (35), (37) and (38)

$$\theta(g'x) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} a_i \left(\sum_{j=1}^{n} g_j x_j \right)^i \tag{39}$$

and

$$\eta^{-1}(-b'Mx) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} c_i \left(\sum_{j=1}^{n} h_j x_j \right)^i \tag{40}$$

where $h_j = -(Mb)_j$. Thus (35) becomes

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} a_i \sum_{\text{all } k_j} \frac{i!}{k_1! \ldots k_n!} (g_1 x_1)^{k_1} \ldots (g_n x_n)^{k_n} =$$

$$= \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} c_i \sum_{\text{all } k_j} \frac{i!}{k_1! \ldots k_n!} (h_1 x_1)^{k_1} \ldots (h_n x_n)^{k_n} \tag{41}$$

summed over all $\sum_{j=1}^{n} k_j = i$

Since (41) must hold for all x_i ,

$$c_1 h_i = a_1 g_i \qquad i = 1, \ldots, n \tag{42a}$$

$$\begin{cases} c_2 h_i^2 = a_2 g_i^2 \\ c_2 h_i h_j = a_2 g_i g_j \end{cases} \qquad i, j = 1, \ldots, n \tag{42b}$$

$$\begin{cases} c_3 h_i^3 = a_3 g_i^3 \\ c_3 h_i h_j h_k = a_3 g_i g_j g_k \\ c_3 h_i h_j^2 = a_3 g_i g_j^2 \end{cases} \qquad i, j, k = 1, \ldots, n \tag{42c}$$

$$\vdots$$

Thus from (42a)

$$-c_1 Mb = a_1 g \tag{43}$$

Pre-multiplying both sides by b' yields

$$c_1 = -a_1 b'g/b'Mb \tag{44}$$

Note that, since M is positive definite, $b'Mb$ can be zero only if $b = 0$. However, this is impossible in a meaningful control problem, and thus $b'Mb \neq 0$. Furthermore, since in a meaningful control problem $g \neq 0$ and since $a_1 > 0$ and M is nonsingular, (43) reveals that $c_1 \neq 0$.

From (42b),

$$c_2 h'Kh = a_2 g'Kg \tag{45}$$

where K is any positive definite matrix. Then using $h = (a_1/c_1)g$ (43) yields

$$c_2 = \frac{a_2}{\left(\frac{a_1}{c_1}\right)^2} \tag{46}$$

From (42c),

$$c_3(h'k)(h'Kh) = a_3(g'k)(g'Kg) \tag{47}$$

where K is any positive definite matrix and k is a vector whose components are all unity. Again using $h = (a_1/c_1)g$ gives

$$c_3 = \frac{a_3}{\left(\frac{a_1}{c_1}\right)^3} \tag{48}$$

Similarly, it is easily seen that all the remaining coefficients c_i are given by

$$c_i = \frac{a_i}{\left(\frac{a_1}{c_1}\right)^i} \tag{49}$$

Thus $\eta^{-1}(\sigma)$ is completely determined in terms of the components of the positive definite matrix M and the coefficients of the control law $\theta(g'x)$. The function $\eta(u) = dh/du$ is the inverse series corresponding to $\eta^{-1}(\sigma)$:

$$\eta(\sigma) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} d_i \sigma^i \tag{50}$$

where, as indicated in textbooks on elementary calculus, the coefficients d_i are obtained by substituting (50) into (38). The first few coefficients of the inverse series are

$$\begin{aligned} d_1 &= 1/c_1 \\ d_3 &= -c_3/c_1^4 \\ d_5 &= (3c_1^2c_3^2 - c_1^3c_5)/c_1^9 \\ d_7 &= (8c_1^4c_3c_5 - c_1^5c_7 - 12c_1^3c_3^3)/c_1^{13} \end{aligned} \tag{51}$$

Thus

$$h(\sigma) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} d_i \frac{\sigma^{i+1}}{i+1} \tag{52}$$

and (36) becomes

$$q(x) = -\frac{1}{2}x'(MA + A'M)x - \left[\sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} x'Mba_i(g'x)^i + \frac{d_i}{i+1}(g'x)^{i+1} \right] \tag{53}$$

Hence (52) and (53) with conditions (44) and (49) yield explicit expressions for all performance criteria of the form (2) that are minimized by control laws of the form (37).

Indirect Control – Consider an nth-order system which is again asymptotically stable in the large,

$$\dot{x} = Ax + b\theta(\sigma) \tag{54}$$

$$\dot{\sigma} = g'x - r\theta(\sigma) \tag{55}$$

where r is a scalar and $\theta(\sigma)$ satisfies the same conditions as above. Define $y' = [x' \,\vdots\, \sigma]$. Then (54) and (55) can be written as

$$\dot{y} = \tilde{A}y + \tilde{b}\theta(\sigma) \tag{56}$$

and

$$\sigma = \tilde{g}'y \tag{57}$$

where

$$\tilde{A} = \left[\begin{array}{c|c} A & 0 \\ \hline g' & 0 \end{array}\right], \quad \tilde{b} = \left[\begin{array}{c} b \\ \hline -r \end{array}\right], \quad \tilde{g}' = [0 \ldots 0 \,\vdots\, 1] \tag{58}$$

Note that (56) and (57) are of the same from as (33) and (34). Thus define

$$\tilde{M} = \begin{array}{c} \begin{array}{cc} \overset{n}{\longleftrightarrow} & \overset{1}{\longleftrightarrow} \end{array} \\ \left[\begin{array}{c|c} M_1 & m_2 \\ \hline m_2 & m \end{array}\right] \begin{array}{l} \updownarrow n \\ \updownarrow 1 \end{array} \end{array} \tag{59}$$

so that (42) - (49) remain valid with $\tilde{h} = -\tilde{M}\tilde{b}$.

6. EXAMPLES

The second-order examples of this section are included to illustrate the results derived above. No essential complication would be introduced by considering higher-order systems.

A. Consider the system (33) - (34) where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -a \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \theta(\sigma) = \sigma^3, \quad g' = [-1 \ 0], \quad a > 0 \tag{60}$$

Since the given control $\theta(x) = -x_1^3$ does not depend on x_2 one expects to find that $m_{22} = 0$. Indeed, from (42) $h_2 = 0$; and since $Mb = -h$,

$$\begin{aligned} -m_{12} &= h_1 \\ -m_{22} &= h_2 = 0 \end{aligned} \tag{61}$$

Furthermore,

$$c_1 = c_2 = 0$$

and

$$c_3 = -\frac{1}{h_1^3} = +\frac{1}{m_{12}^3} \tag{62}$$

Thus,

$$\eta^{-1}(\sigma) = \frac{\sigma^3}{m_{12}^3} \quad , \quad \eta(u) = m_{12} u^{1/3} \tag{63}$$

and

$$h(u) = \frac{3}{4} m_{12} u^{4/3} \tag{64}$$

From (36)

$$-q(x) = \frac{1}{2} x' \begin{bmatrix} -2m_{12} & m_{11} - am_{12} \\ m_{11} - am_{12} & 2m_{12} \end{bmatrix} x - m_2 \frac{x_1^4}{4} \tag{65}$$

Thus, the performance index that is optimized by the control $\theta(x) = -x_1^3$ is

$$V = \int_t^\infty \left(\frac{1}{2} x' \begin{bmatrix} +2m_{12} & am_{12} - m_{11} \\ am_{12} - m_{11} & -2m_{12} \end{bmatrix} x + m_2 \frac{x_1^4}{4} + \frac{3}{4} m_{12} u^{4/3} \right) d\tau \tag{66}$$

By starting with the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - x_1 + u \end{aligned} \tag{67}$$

one can easily verify that the control $u = -x_1^3$ does indeed yield

$$V^o(x) = \frac{1}{2} x' \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & 0 \end{bmatrix} x$$

as the solution to the Hamilton-Jacobi equation corresponding to performance index (66). Thus, as indicated in Section 4, $V^o(x)$ is not positive-definite because the control law $\theta(x)$ does not depend on x_2 .

B. Consider the system (33) - (34) where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -a \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \theta(\sigma) = \sigma^3, \quad g' [-1 - \frac{2}{a}], \quad a > 0 \tag{68}$$

Now, since $\theta(x) = (-x_1 - \frac{2}{a}x_2)^3$ depends on x_2 one expects to find $m_{22} \neq 0$.

From (42),

$$\begin{aligned} c_1 &= c_2 = 0 \\ h_1 &= \frac{a}{2}h_2 \\ c_3 &= \frac{1}{h_1^3} \end{aligned} \tag{69}$$

Since $Mb = -h$,

$$m_{12} = \frac{a}{2}m_{22} \tag{70}$$

and

$$c_3 = 8/a^3 m_{22}^3$$

Thus M is a function of two arbitarary constants m_{11} and m_{22},

$$M = \begin{bmatrix} m_{11} & \frac{a}{2}m_{22} \\ \frac{a}{2}m_{22} & m_{22} \end{bmatrix} \tag{71}$$

Clearly m_{11} and m_{22} can be chosen to make M positive definite.

From (40),

$$\eta^{-1}(\sigma) = \frac{8}{a^3 m_{22}^3}\sigma^3 \quad , \quad \eta(u) = \frac{am_{22}}{2}u^{1/3} \tag{72}$$

and

$$h(u) = \frac{3am_{22}}{8}u^{4/3} \tag{73}$$

Thus (36) yields

$$-q(x) = \frac{1}{2}x' \begin{bmatrix} -am_{22} & m_{11} - m_{22} - \frac{a^2}{2}m_{22} \\ m_{11} - m_{22} - \frac{a^2}{2}m_{22} & -am_{22} \end{bmatrix} x + Y \tag{74}$$

where

$$-Y = \frac{a}{2}m_{22}\left(\frac{x_1^4}{4} + \frac{2}{a}x_1^3 x_2 + \frac{6}{a^2}x_1^2 x_2^2 + \frac{8}{a^3}x_1 x_2^3\right) + \frac{2m_{22}}{a^3}x_2^4 \tag{75}$$

Hence the performance index that is optimized by the control $\theta(x) = (-x_1 - \frac{2}{a}x_2)^3$ is

$$V = \int_t^\infty \left(\frac{1}{2}x' \begin{bmatrix} am_{22} & -m_{11} + m_{22} + \frac{a^2}{2}m_{22} \\ -m_{11} + m_{22} + \frac{a^2}{2}m_{22} & + am_{22} \end{bmatrix} x - Y + \frac{3am_{22}}{8}u^{4/3}\right) d\tau \tag{76}$$

Note that the integrand of (76) can be written as

$$\frac{1}{2}x' \begin{bmatrix} am_{22} & -m_{11} + m_{22} + \frac{a^2}{2} m_{22} \\ -m_{11} + m_{22} + \frac{a^2}{2} m_{22} & \end{bmatrix} x + \frac{a}{2} m_{22}(x_1 + \frac{2}{a}x_2)^4 \tag{77}$$

Thus, by proper choice of m_{11} and m_{22} the integrand (77) can be made to be positive semi-definite, and thus V in (76) is a Lyapunov function for the given system.

Again by starting with the system (67) one can easily verify that the control $u = -(x_1 + \frac{2}{a}x_2)^3$ yields

$$V^o(x) = \frac{1}{2}x' \begin{bmatrix} m_{11} & \frac{a}{2}m_{22} \\ \frac{a}{2}m_{22} & m_{22} \end{bmatrix} x \tag{78}$$

as the solution to the Hamilton-Jacobi equation corresponding to performance index (76).

It is of interest to campare the transient response of the above nonlinear system with that of a linear system having the same optimum performance (78). Using the results of [1] one can show that the linear control law,

$$u = -[\frac{a}{2}m_{22}, m_{22}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{79}$$

yields the optimum performance (78) for performance criterion

$$V = \frac{1}{2}\int_t^\infty (x'Qx + u^2)\, d\tau \tag{80}$$

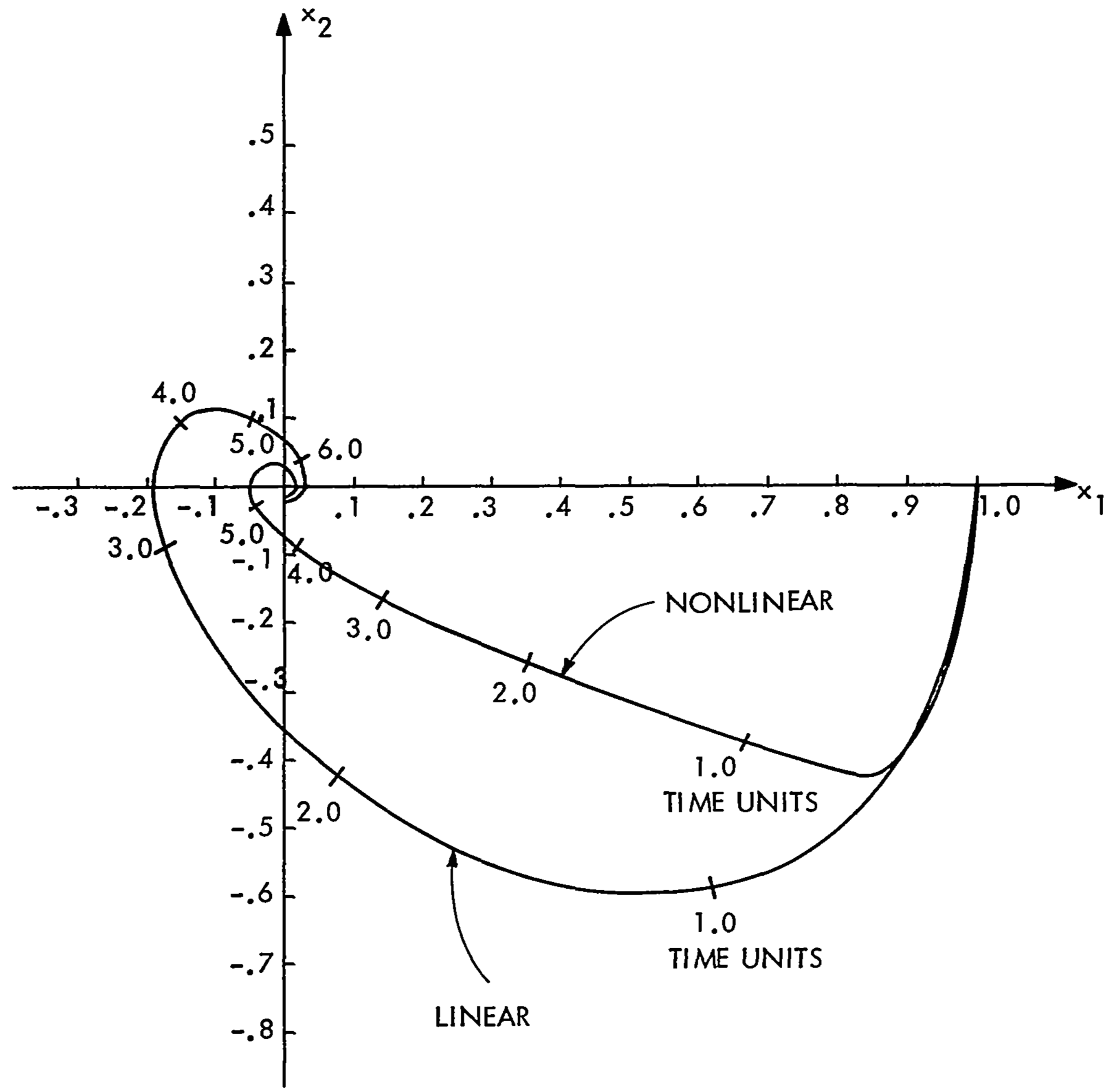

FIGURE 3 (APPENDIX)

TRAJECTORIES FOR NONLINEAR AND LINEAR CONTROL LAWS

where

$$Q = \begin{bmatrix} \frac{a^2}{4} m_{22}^2 + m_{22} & \frac{a}{2} m_{22}^2 + m_{22}(\frac{a^2}{2} + 1) - m_{11} \\ \frac{a}{2} m_{22}^2 + m_{22}(\frac{a^2}{2} + 1) - m_{11} & m_{22}^2 + a m_{22} \end{bmatrix} \tag{81}$$

The values $a = m_{22} = 1/2$ were chosen and a digital computer simulation was used to obtain the phase-plane comparison of the trajectories of the nonlinear and linear systems shown in Fig. 3. From the figure it can be seen that the nonlinear system provides generally smaller excursions of the position coordinate x_1 and the velocity coordinate x_2. Fig. 4 contains a comparison of the control signals required by the two systems. It is clear that the nonlinear system requires a generally greater magnitude of control to provide the smaller excursions of position and velocity noted in Fig. 3. This is due to the fact the performance criterion (80) of the linear system provides a greater penalty on the magnitude of control than does the performance criterion (76) of the nonlinear system. Thus in those engineering applications in which the larger control signals can be tolerated, one might consider the use of nonlinear control laws to prevent large deviations of the state variables.

7. CONCLUSION

Some aspects of the inverse optimum control problem have been examined for a class of nonlinear autonomous systems where the optimum performance criterion

$V = \int_t^\infty [q(x) + h(u)]\, d\tau$ is required to have the form $V = \frac{1}{2}x'Mx$ as function of the current state x. Using assumptions outlined in Section I it was shown that if a given control law $\varphi(x)$ is optimum then two equations, (11) and (12), must be satisfied for all x.

When applied to single-input linear systems the results reduce to conditions already derived by Kalman [1]. These results were extended to multiple-input linear systems.

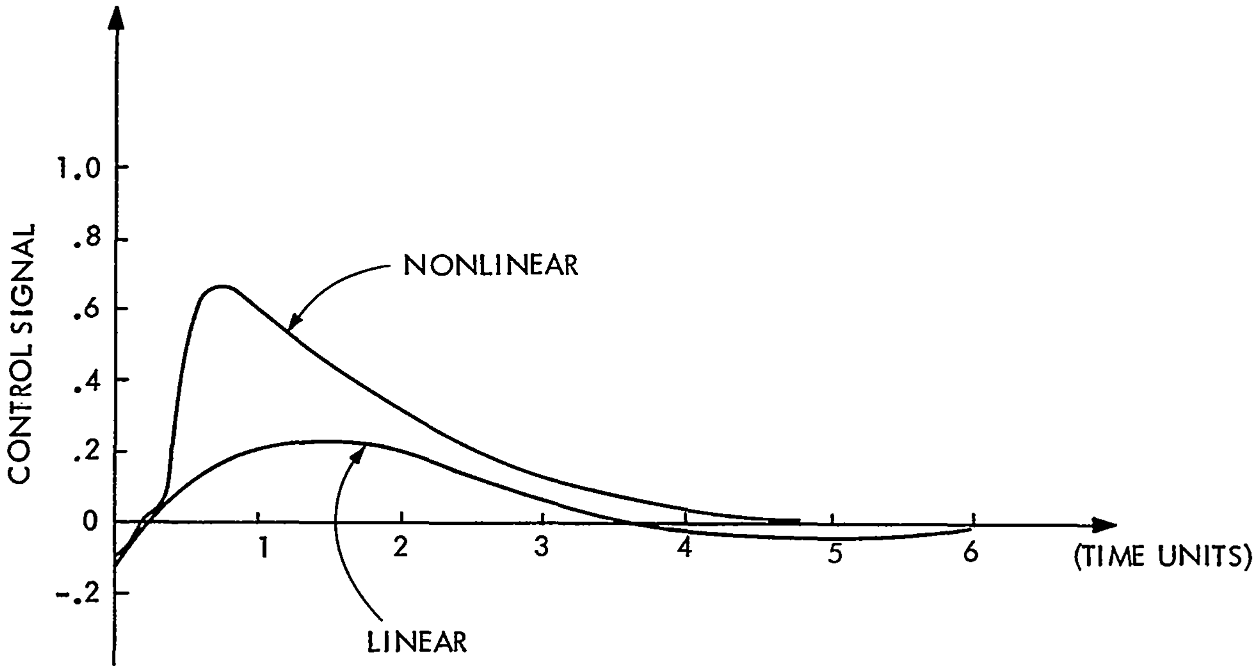

FIGURE 4 (APPENDIX)
CONTROL SIGNAL FOR FIGURE 3

For asymptotically stable nonlinear and linear single-input systems it was shown that if each state variables which is directly affected by the control cannot be measured then M cannot be positive definite. This property is invariant under a nonsingular linear transformation of the state variables.

In examining the problem of Lur'e consistency conditions that must be satisfied by M, $q(x)$, and $h(u)$ were found; two simple examples illustrating the approach were also given.

These results apply to the analysis of sub-optimum control laws which are derived on the basis of certain simplifying assumptions [6]. It is of interest to determine what performance criterion, if any, is optimized by the known suboptimum control law. This is an inverse optimum control problem and is the subject of current research.

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