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THE LAPLACE TRANSFORM OF THE DERIVATIVE OF A FUNCTION WITH FINITE JUMPS

by

Leon Y. Bahar

Prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Lewis Research Center

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SUMMARY

Generalized functions as developed by Laurent Schwartz are used to include finite jump contributions to transforms of derivatives. The expression of the Bilateral Laplace Transform of the derivative of a piecewise continuous function is first derived and then specialized to the unilateral transform. It is shown that the use of the Laplace Transform of a generalized derivative resolves some well-known apparent contradictions. In this new context the distinction between "prescribed" and "actual" initial values disappears. Although the present investigation is restricted to ordinary differential equations, it can be extended to partial differential equations.

I. INTRODUCTION

In the analytical study of meteoroid impact on space vehicles, including fuel tanks, the integral transform technique is one of the available methods. The impact loadings, whether on the bumper or on the main structural wall, are always abruptly applied. These abrupt loadings may be mathematically represented by discontinuous functions. In the application of transform methods, the treatment of discontinuous functions exhibiting finite jumps as given in most textbooks and papers is less than satisfactory. In this paper, a more adequate treatment utilizing the notion of generalized functions is given.

The derivative of a function with finite discontinuities cannot be transformed in accordance with the well known expression:

$$L\left\{f'(t)\right\} = s\,\bar{f}(s) - f(o^{+}) \tag{1-1}$$

which only applies to continuous functions. The contribution of finite jumps to the righthand side of this equation is usually given as an exercise of no particular significance in most textbooks (1,2). In physical applications, such as hypervelocity impact problems, piecewise continuous functions are of great importance, and their behavior will usually contain at least one jump occurring at zero. The expression (1-1) does not include the contribution due to this term, since the transform is defined as the limit of the Laplace Integral as the lower limit tends to zero from the right. To include such effects the range of integration can be extended to minus infinity, thus introducing the Bilateral Laplace Transform (3). This procedure has limited usefulness, since it can only be used to transform a known function. In the solution of differential equations one has, in general, no advance information about the location and magnitude of discontinuities that will appear in the response as they must be part of the solution. Therefore, an extension of the notion of function becomes necessary in order to transform such abrupt changes of unknown location and magnitude in the response.

Generalized functions as introduced by Laurent Schwarz (4) are ideally suited to handle the particular problem under consideration. The expression (1-1) becomes upon such a generalization:

$$L\left\{g'(t)\right\} = s\,L\left\{g(t)\right\} - g(o^-) \qquad (1\text{-}2)$$

where the jumps are implicitly contained in the definition of the generalized function.

The expression given in (1-2) can be easily extended to the derivative of order n:

$$L\left\{g^{(n)}(t)\right\} = s^n\,L\left\{g(t)\right\} - s^{n-1}\,g(o^-) - \ldots\ldots - g^{(n-1)}(o^-) \qquad (1\text{-}3)$$

The equation (1-3) is both simpler and more general than its corresponding expression for an ordinary function, as will be established in later sections.

A careful distinction between ordinary and generalized functions can also resolve some apparent contradictions in partial differential equations such as those recently studied by Boley (5) and Reid (6). It is expected that the investigation of this problem will be the subject of another study.

In order to emphasize the underlying ideas, well known sufficient conditions governing equation (1-1) will not be given. These conditions are stated with precision and correct perspective in LePage (7).

II BACKGROUND AND PRELIMINARIES

A function is said to have a finite discontinuity at t_o if $f(t_o^+)$ and $f(t_o^-)$ exist, and $f(t_o^+) \neq f(t_o^-)$. The quantity $J\{f(t_o)\} = f(t_o^+) - f(t_o^-)$ is known as the "jump" or"saltus" of the function at the point t_o, and may be represented by $J\{f(t_o)\}$, $J(t_o)$ or J_o. It can also be considered as a function J(t) of the continuous variable t, which is zero except at a point of discontinuity. This is in essence the "Saltus-function" whose properties have been studied by Hobson (5). The jump can be considered as a directed quantity and may be represented by an upward arrow if $J\{f(t_o)\} > o$ and a downward arrow if $J\{f(t_o)\} < o$. This representation is particularly useful in the formal graphical differentiation of the jump multiplied by the unit step function.

The only paper discussing the inclusion of jumps in the Laplace transform of the derivative of a function seems to be that of Rasof (9).

Rasof (9) has incorporated the effect of jumps in f(t) into the expression for $L\{f'(t)\}$, by considering the definitions

$$L\{f(t)\} = L\{f_1(t)\} + \int_{t_1}^{\infty} [f_2(t) - f_1(t)]e^{-st}dt \qquad (2\text{-}1)$$

$$L\{f(t)\} = L\{f_2(t)\} + \int_{o}^{t_1} [f_1(t) - f_2(t)]e^{-st}dt \qquad (2\text{-}2)$$

where $f_1(t)$ and $f_2(t)$ have been defined in the open intervals (o, t_1) and (t_1, ∞) as shown in Figure 1.

The paper makes the incorrect assumption that the notation $L\{f(t)\}$ is ambiguous, because it does not state which of the two forms (2-1) or (2-2) must be ascribed to f(t) in interpreting $L\{f(t)\}$. Actually, no such difficulty exists, and the forms (2-1) and (2-2) introduce the additional complication of making it impossible to determine the Laplace transforms from tables.

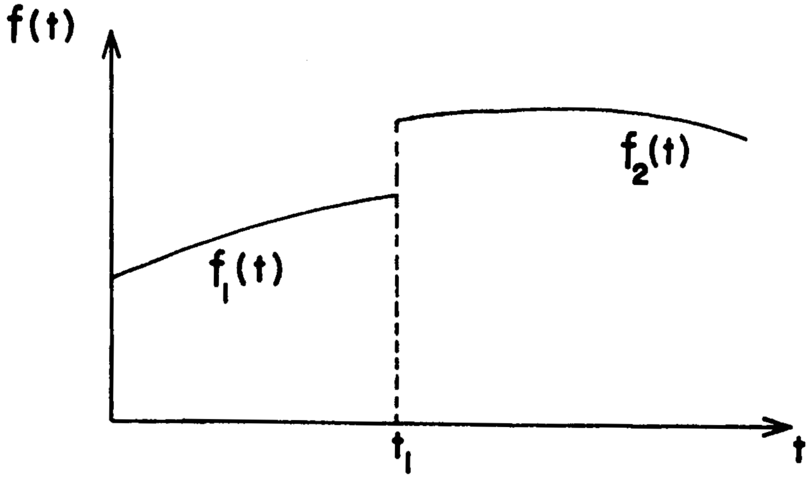

Figure 1

$$f(t) = \begin{cases} f_1(t) & o < t < t_1 \\ f_2(t) & t > t_1 \end{cases}$$

It is not difficult to see how complicated the expressions given in (9) would become as the set of functions $\{f_1, f_2, \ldots, f_n\}$ defining the piecewise continuous function f increases. This would lead to n different representations for L $\{f(t)\}$, with the ensuing complicated expressions for L $\{f'(t)\}$ to be found in the same paper (9).

The complication in the proposed problem does not arise from the source indicated by Rasof (9). It is usually due to an incorrect identification of the initial values in the transform of the derivative of a generalized function.

It should again be pointed out that in a given physical problem, if the function f(t) is the forcing function it is not difficult to obtain the transform of its derivatives, even if f(t) is not continuous. This, to some extent is the problem treated in (9). It is far more important and difficult to identify the location and magnitude of the jumps that appear in the response. This paper will be mainly devoted to a formulation that will automatically accomplish this aim.

III ELEMENTARY FORMULATION AND SOLUTION OF THE PROBLEM

Consider the function represented in Figure 2. If this function is considered an ordinary function it has the following representation in the open intervals I_i:

$$f(t) = f_i(t) \qquad t \in I_i \qquad (i = 1, 2, 3, 4)$$

where the following definitions are used:

$$-\infty < t < t_1 \qquad t \in I_1$$

$$t_1 < t < o \qquad t \in I_2$$

$$o < t < t_2 \qquad t \in I_3$$

$$t_2 < t < \infty \qquad t \in I_4$$

The same function can also be represented in the form of a regular generalized function, by introducing the unit step function, so as to produce the functions $f_i(t)$, $t \in I_i$ in the various intervals of definition. This function may be denoted by g(t) and written explicitly as:

$$g(t) = [1-u(t-t_1)]\ f_1(t) + [u(t-t_1) - u(t)]\ f_2(t)$$
$$+ [u(t) - u(t-t_2)]\ f_3(t) + u(t-t_2)\ f_4(t) \qquad (3\text{-}2)$$

where the use of the gate function defined as the difference of two step functions has been made in order to annihilate the function outside the gate.

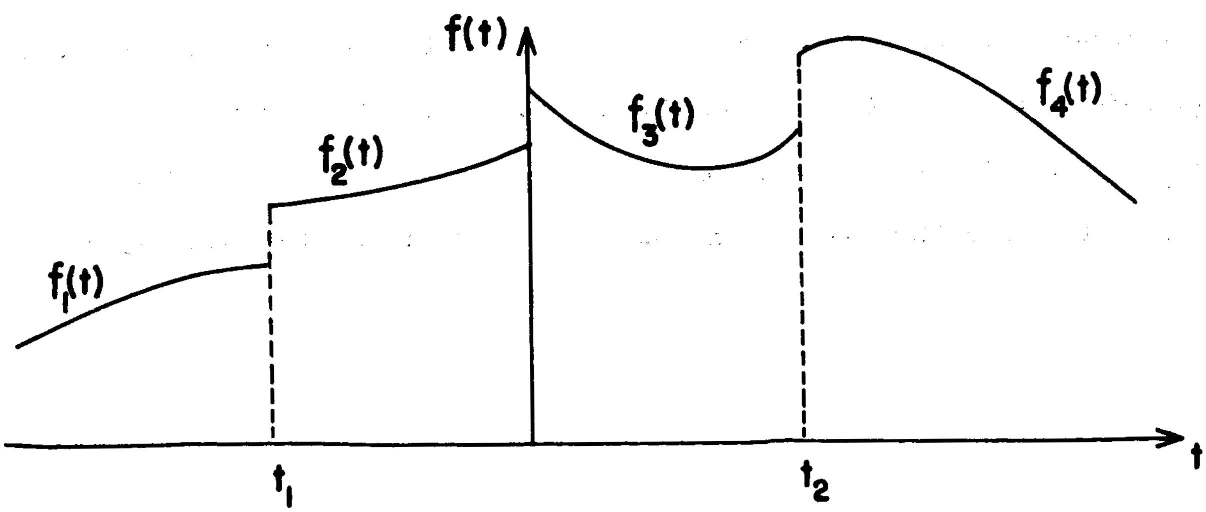

Figure 2

It is well known that while there is no reason to distinguish between the two different representations of the same function, namely the ordinary function f(t) and the generalized function g(t); these two functions exhibit marked differences under differentiation, due to the introduction of the "Dirac delta" as the derivative of the unit step function. The ordinary derivative of f(t) at a point of finite discontinuity such as t_1, o, and t_2 is undefined; the generalized derivatives at those points are given by $J\left\{f(t_i)\right\}\delta(t-t_i)$.

In order to derive the expression of the generalized derivative g'(t) in terms of the ordinary derivative f'(t) and the contribution of the finite jumps, one can formally differentiate the expression for g(t) as given by (3-2).

In order to achieve the desired result, it is more convenient to rewrite the expression for g(t) in a form that will exhibit the jumps, namely

$$g(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-t_1) + [f_3(t) - f_2(t)]\,u(t) + [f_4(t) - f_3(t)]\,u(t-t_2)$$

The interpretation of this equation is of interest in itself. It simply indicates that the function g(t) can be constructed by a stepwise continuation procedure starting with the function $f_1(t)$. To $f_1(t)$ is added $[f_2(t) - f_1(t)]\,u(t-t_1)$ which brings the function up $f_2(t)$ from t_1 to infinity, the next term builds the function up to $f_3(t)$ from zero to infinity, and so on.

The formal application of the differentiation of a product to (3-3) yields:

$$g'(t) = f_1'(t) + [f_2'(t) - f_1'(t)]\ u(t-t_1) + [f_3'(t) - f_2'(t)]\ u(t) + [f_4'(t) - f_3'(t)]\ u(t-t_2) + [f_2(t) - f_1(t)]\ \delta(t-t_1) + [f_3(t) - f_2(t)]\ \delta(t) + [f_4(t) - f_3(t)]\ \delta(t-t_2) \qquad (3-4)$$

It is clear that the terms multiplied by the unit step function in the expression (3-4) represent the ordinary derivative $f'(t)$ and the terms involving the Dirac delta can be rewritten in a form that will produce the same integral according to the sifting property.

$$g'(t) = f'(t) + [f_2(t_1) - f_1(t_1)]\ \delta(t-t_1) + [f_3(o) - f_2(o)]\ \delta(t) + [f_4(t_2) - f_3(t_2)]\ \delta(t-t_2) \qquad (3-5)$$

It is clear that the coefficients of the delta functions represent the jumps of the function at the points of discontinuity. This yields:

$$g'(t) = f'(t) + J\ \{f(t_1)\}\ \delta(t-t_1) + J\ \{f(o)\}\ \delta(t) + J\ \{f(t_2)\}\ \delta(t-t_2) \qquad (3-6)$$

There is no loss of generality in this derivation and it was presented in detail in conformity with Figure 2. Summation signs at this stage would have served no useful purpose in further clarification of the derivation.

Generalization now gives:

$$g'(t) = f'(t) + \sum_{i=1}^{n} J\ \{f(t_i)\}\ \delta(t-t_i) \qquad (3-7)$$

Although the above derivation was formal, it can be shown that it is in agreement with that obtained via the Theory of Distributions (10) and finds its rigorous justification only in such an interpretation.

It would be of interest to determine the bilateral transform of the generalized derivative of $g'(t)$ after determining it for the ordinary derivative $f'(t)$.

Following Figure 2, this expression can be written as:

$$L_2 \{f'(t)\} = \int_{-\infty}^{t_1^-} f'_1(t)e^{-st}dt + \int_{t_1^+}^{o^-} f'_2(t)e^{-st}dt + \int_{o^+}^{t_2^-} f'_3(t)e^{-st}dt$$

$$+ \int_{t_2^+}^{\infty} f'_4(t)e^{-st}dt \qquad (3\text{-}8)$$

Integration by parts, gives under the usual assumptions found in (7)

$$L_2 \{f'(t)\} = s\, L_2 \{f(t)\} - [f_2(t_1^+) - f_1(t_1^-)]\, e^{-st_1}$$

$$- [f_3(o^+) - f_2(o^-)] - [f_4(t_2^+) - f_3(t_2^-)]\, e^{-st_2} \qquad (3\text{-}9)$$

The quantities within the brackets are the terms indicating jump contributions, and the result can be easily generalized to:

$$L_2 \{f'(t)\} = s\, L_2 \{f(t)\} - \sum_{i=1}^{n} J \{f(t_i)\}e^{-st_i} \qquad (3\text{-}10)$$

The bilateral transform of the generalized derivative (3-7) gives on the other hand:

$$L_2 \{g'(t)\} = L_2 \{f'(t)\} + \sum_{i=1}^{n} J \{f(t_i)\}e^{-st_i} \qquad (3\text{-}11)$$

Combining the two expressions (3-10) and (3-11) yields

$$L_2 \{g'(t)\} = s\, L_2 \{f(t)\} \qquad (3\text{-}12)$$

Clearly $L_2 \{f(t)\} = L_2 \{g(t)\}$ since the function $g(t)$ is a regular generalized function containing only unit step functions in its description.

Therefore

$$L_2 \{g'(t)\} = s\, L_2 \{g(t)\} \tag{3-13}$$

The simplicity of this result is remarkable. It does not explicitly contain any jump contributions, as these are implicit in the representation of the generalized function and its derivatives in terms of the ordinary function and its derivatives, as given by the previous expressions (3-4) to (3-7).

It is of great practical interest to determine the simpler expressions these reduce to in the case of a unilateral transform.

The transform of the derivative of the ordinary function becomes using (3-10)

$$L \{f'(t)\} = s\, L \{f(t)\} - \sum_{j=1}^{n} J \{f(t_j)\} - f(o^+) \tag{3-14}$$

where the Laplace integral is taken with a righthand lower limit, and $t_j > 0$.

The unilateral transform of $g'(t)$ can be written as:

$$L \{g'(t)\} = L \{f'(t)\} + \sum_{j=1}^{n} J \{f(t_j)\} e^{-st_j} + f(o^+) - f(o^-) \tag{3-15}$$

following (3-7), with $t_j > o$.

Elimination of $L \{f'(t)\}$ between the two expressions (3-14), (3-15); and insertion of $L \{f(t)\} = L \{g(t)\}$ and $f(o^-) = g(o^-)$ leads to the important result:

$$L \{g'(t)\} = s\, L \{g(t)\} - g\,(o^-) \tag{3-16}$$

It is interesting to note that as was already pointed out before, the jumps do not explicitly appear in this formula, but are implicitly contained in g(t), and that the initial value appearing in (3-16), is the initial value from the left. Transform expressions for higher order derivatives of ordinary functions become progressively more unwieldy due to the presence of jumps in the various derivatives, but no such problem exists with generalized derivatives. Successive substitutions lead to:

$$L\ \{g''(t)\} = s^2\ L\ \{g(t)\} - s\ g(o^-) - g'(o^-) \qquad (3\text{-}17)$$

and finally for the derivative of order n:

$$L\ \{g^{(n)}(t)\} = s^n\ L\ \{g(t)\} - s^{n-1}\ g(o^-) - \ldots\ldots -g^{(n-1)}(o^-) \qquad (3\text{-}18)$$

This expression seems to be neither well-known, nor widely used. It is in agreement, however, with that given by Zadeh and Desoer (11). In view of the importance of (3-18) in applications, various alternatives to the preceding development leading to the same final result will be explored in the next sections.

IV METHOD BASED ON THE CONSTRUCTION OF THE EQUIVALENT CONTINUOUS FUNCTION

It is possible to construct an equivalent continuous function $f_c(t)$ from f(t) as represented in Figure 2, by sliding the curve $f_2(t)$, $f_3(t)$, and $f_4(t)$ down by the jumps, $J\ \{f(t_1)\}$, $J\ \{f(t_1)\} + J\ \{f(o)\}$ and $J\ \{f(t_1)\} + J\ \{f(o)\} + J\ \{f(t_2)\}$ respectively. This gives the explicit expression:

$$\begin{aligned} f_c(t) = f(t) - J\ \{f(t_1)\}\ u(t-t_1) - [J\ \{f(t_1)\} + J\ \{f(o)\}]\ u(t) \\ - [J\ \{f(t_1)\} + J\ \{f(o)\} + J\ \{f(t_2)\}]u(t-t_2) \end{aligned} \qquad (4\text{-}1)$$

Recalling that for a continuous function $f_c(t)$

$$L_2\ \{f'_c(t)\} = s\ L_2\ \{f_c(t)\} \qquad (4\text{-}2)$$

and taking into account the action of the unit step function on the various terms leads to

$$L_2 \{f_c'(t)\} = s\, L_2\, [f(t) - J\{f(t_1)\}\, u(t-t_1) - J\{f(o)\}\, u(t) - J\{f(t_2)\}\, u(t-t_2)] \tag{4-3}$$

It should be noted that no difference exists between $f'(t)$ and $f_c'(t)$ when they are considered as functions in the ordinary sense, since they are left undefined at the points where $f(t)$ suffers a jump discontinuity. The result (4-3) can now be generalized to:

$$L_2\ \{f'(t)\} = s\ L_2\ \{f(t)\} - \sum_{i=1}^{n} J\ \{f(t_i)\} e^{-st_i} \tag{4-4}$$

which confirms the result (3-10) previously obtained. If the function is now considered as a generalized function it can be written as:

$$g(t) = f_c(t) + J\ \{f(t_1)\}\ u(t-t_1) + J\ \{f(o)\}\ u(t) + J\ \{f(t_2)\}\ u(t-t_2) \tag{4-5}$$

Upon differentiation, generalization and making use of $f'(t) = f_c'(t)$, it follows that:

$$g'(t) = f'(t) + \sum_{i=1}^{n} J\ \{f(t_i)\}\ \delta(t-t_i) \tag{4-6}$$

Taking transforms of both sides of (4.6)

$$L_2\ \{g'(t)\} = L_2\ \{f'(t)\} + \sum_{i=1}^{n} J\ \{f(t_i)\} e^{-st_i} \tag{4-7}.$$

Elimination of $f'(t)$ between the two expressions (4-4) and (4-7) leads, recalling that $L_2\ \{g(t)\} = L_2\ \{f(t)\}$, to the following:

$$L_2 \{g'(t)\} = s\, L_2 \{g(t)\} \tag{4-8}$$

This is precisely the expression obtained in (3-13), and from that point on the derivation is as given in Section III.

V METHOD BASED ON THE STIELTJES INTEGRAL

It is possible to consider the Bilateral Laplace Transform of the derivative of a function exhibiting finite jumps as the Stieltjes integral

$$L_2 \{f'(t)\} = \int_{-\infty}^{\infty} e^{-st}\, d\, \{f(t)\} \tag{5-1}$$

following Widder (12).

Inserting the expression (4-5) into (5-1) results in the following:

$$L_2 \{g'(t)\} = \int_{-\infty}^{\infty} e^{-st}\, d\, \{f_c(t)\} + \sum_{i=1}^{n} J\, \{f(t_i)\} \int_{-\infty}^{\infty} e^{-st} d\, u(t-t_i)$$

or rewriting:

$$L_2 \{g'(t)\} = L_2 \{f'(t)\} + \sum_{i=1}^{n} J\, \{f(t_i)\}\, e^{-st_i} \tag{5-2}$$

where use has been made of a well known theorem in Stieltjes integration in the evaluation of the second term (10). Combining (5-2) with (4-4) yields

$$L_2 \{g'(t)\} = s\, L_2 \{g(t)\} \tag{5-3}$$

which is the same as (4-8).

VI METHOD BASED ON THE THEORY OF DISTRIBUTIONS

By definition the generalized derivative of a function f(t) with finite jumps can be defined as follows in the sense distributions (4)

$$< f', \psi > = - < f, \psi' > = - \int_{-\infty}^{\infty} f(t)\, \psi'(t)\, dt \tag{6-1}$$

where $\psi(x)$ is a testing function. Particularizing this to the function represented in Figure 2, the integral can be split over the appropriate domains, yielding

$$- \int_{-\infty}^{\infty} f(t)\, \psi'(t)\, dt = - \int_{-\infty}^{t_1} f(t)\, \psi'(t)\, dt - \int_{t_1}^{o} f(t)\, \psi'(t)\, dt$$

$$- \int_{o}^{t_2} f(t)\, \psi'(t)\, dt - \int_{t_2}^{\infty} f(t)\, \psi'(t)\, dt \tag{6-2}$$

Integrating (6-2) by parts in order to transfer all the differentiations from $\psi(t)$ to $f(t)$ one obtains:

$$< f', \psi > = \int_{-\infty}^{t_1} f'(t)\, \psi(t)\, dt + \int_{t_1}^{o} f'(t)\, \psi(t)\, dt$$

$$+ \int_{o}^{t_2} f'(t)\, \psi(t)\, dt + \int_{t_2}^{\infty} f'(t)\, \psi(t)\, dt \tag{6-3}$$

$$+ \sum_{i=1}^{3} J\{f(t_i)\}\, \psi(t_i)$$

where use has been made of the fact that the testing function vanishes outside a finite interval. Upon generalizing, it follows that:

$$< f', \psi > = \int_{-\infty}^{\infty} f'(t)\, \psi(t)\, dt + \sum_{i=1}^{n} J\{f(t_i)\}\, \psi(t_i) \tag{6-4}$$

or denoting by $g'(t)$ the generalized derivative given by the expression (6-4)

$$g'(t) = f'(t) + \sum_{i=1}^{n} J\{f(t_i)\}\, \delta(t-t_i) \tag{6-5}$$

where use of the sifting property has been made. Here the symbol $f'(t)$

stands for the derivative of f(t) interpreted in the ordinary sense, wherever it exists and it can be arbitrarily assigned any finite value where it does not exist.

Therefore the generalized derivative is equal to the ordinary derivative plus the jump contributions. It is this last term that insures that the integral of g'(t) will yield the discontinuous function f(t) rather than the equivalent continuous function $f_c(t)$ that would have been obtained by considering the integral of f'(t) alone. In other words, the integration of g'(t) automatically introduces the correct integration constant.

It should be noted that equation (6-5) is identical to (4-6) and its transform is:

$$L_2 \{g'(t)\} = L_2 \{f'(t)\} + \sum_{i=1}^{n} J \{f(t_i)\} e^{-st_i} \tag{6-6}$$

On the other hand (4-4) gives:

$$L_2 \{f'(t)\} = s\, L_2 \{f(t)\} - \sum_{i=1}^{n} J\{f(t_i)\} e^{-st_i} \tag{6-7}$$

Eliminating f'(t) between (6-6) and (6-7) and recalling that $L_2 \{f(t)\} = L_2 \{g(t)\}$ results in

$$L_2 \{g'(t)\} = s\, L_2 \{g(t)\}$$

The specialization of this result to the unilateral transform as well as its extension to higher order derivatives have been developed in Section III.

VII SOLUTION AND DISCUSSION OF SOME ELEMENTARY PROBLEMS

1. Transform of the Space Variable

Consider the problem of a simply supported beam subjected to a concentrated load P as shown in figure 3

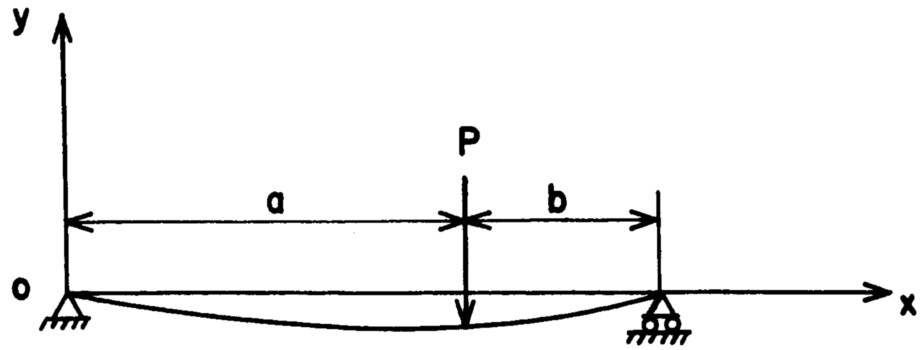

Figure 3

The Euler-Bernoulli equation for the deflection of the beam is

$$EI\ y^{(4)}\ (x) = w(x) \tag{7-1}$$

where EI has been assumed constant. The usual operational solution for this beam is to write $w(x) = P\ \delta(x-a)$

$$EI\ \ y^{(4)}\ (x) = -\ P\ \delta(x-a) \tag{7-2}$$

This equation is usually transformed into:

$$EI\ [s^4\ \bar{y}(s) - s^2 y'(o^+) - y'''(o^+)] = -\ P\ \bar{e}^{as} \tag{7-3}$$

where $y(o^+) = y''(o^+) = 0$ has been incorporated into (7-3). $y(x)$ is now obtained by a simple inversion, and produces the expected result. The boundary conditions $y(L^-) = y''(L^-) = 0$ are taken into account in order to determine $y'(o^+)$ and $y'''\ (o^+)$. Unfortunately, this correct result is due to the compounding of two errors that cancel each other out. First, if $y(x)$ is considered an ordinary function, its third derivative $y'''(x)$ will exhibit a jump equal to the shear $-P$ at the point $x = a$, and its contribution must be included in the expression of the transform (3-14). This, however, would lead to the incorrect solution if the righthand side of (7-2) is retained. It is not difficult to see the reason for which one would not obtain the correct solution. Consider the equation

$$EI\ y'''(x) = \frac{Pb}{L} - P\ u(x-a) \tag{7-4}$$

If y(x) in this equation is considered an ordinary function then the derivative is $EI\ y^{(4)} = 0$. Since u(x-a) is discontinuous at x = a, and therefore not differentiable there, its value can be arbitrarily taken as zero.

Taking the transform of y(x) and inserting the appropriate jump contribution yields:

$$EI\ [s^4\bar{y}(s) - s^3y(o^+) - s^2y'(o^+) - s\ y''(o^+)$$

$$- y'''(o^+) - J\ \{y'''(a)\}e^{-as}] = 0$$

Recalling that

$$J\ \{y'''(a)\} = -\frac{P}{EI} \tag{7-5}$$

and incorporating $y(o^+) = y''(o^+) = 0$ yields

$$EI\ [s^4\bar{y}(s) - s^2y'(o^+) - y'''(o^+)] = -P\ e^{-as} \tag{7-6}$$

This is precisely the equation obtained before as a result of neglecting the jump contribution, and replacing it by the derivative of the unit step function which should have been taken as zero. It is seen that the net effect of these two errors annihilate each other, thereby producing a correct result. Although the inclusion of the jump due to the shear would solve the particular problem under consideration, this procedure cannot admit any generalization. The deflection of the beam can be considered as the response of the beam to the concentrated load input P and therefore, should be uniquely determined, once the system parameters, initial conditions and the input are completely known. In other words, the jump in the shear is part of the response and should be reflected explicitly in the expression of the deflection. To incorporate the jump condition in the shear into the transformed deflection equation, is to assume in an a priori fashion the partial solution of the problem. It is not difficult to see that such anticipatory knowledge would be much more difficult in the case of linear differential equations with variable coefficients, where the location and magnitude of the jumps is not as easy to ascertain. The theory of the Green's Function would be of help here, but this is equivalent to obtaining the complete solution of the problem.

To the knowledge of this writer, Churchill (14) is the only author to have pointed out the inclusion of the jump in the shear as an alternative to the classical treatment of the problem usually given in books on Transform Theory. The literature seems to give no indication as to the effect that the classical treatment is incorrect or that the introduction of the concept of a generalized function is necessary to treat the problem satisfactorily.

Rewriting the equation for the shear in $0^- < x < L^+$ as:

$$EI\, y'''(x) = \frac{Pb}{L}\, u(x) - P\, u(x-a) + \frac{Pa}{L}\, u(x-L) \tag{7-7}$$

If y(x) is now considered a generalized function, its derivative becomes

$$EI\, y^{(4)}(x) = \frac{Pb}{L}\, \delta(x) - P\, \delta(x-a) + \frac{Pa}{L}\, \delta(x-L) \tag{7-8}$$

Clearly, since righthand side is a generalized function, the lefthand side has to be considered as a generalized function and treated as such in its transform. For the sake of clarity, if this generalized function is denoted by $y_g(x)$, one obtains the following transformed equation in accordance with (3-18).

$$EI\,[s^4\bar{y}_g(s) - s^3y_g(o^-) - s^2y_g'(o^-) - sy''_g(o^-) - y'''_g(o^-)]$$

$$= \frac{Pb}{L} - Pe^{-as} + \frac{Pa}{L}e^{-Ls} \qquad (7\text{-}9)$$

It should be noted that the reaction at the left was introduced, since lefthanded limits now make this reaction part of the input. Clearly, if one considers the beam as extending beyond the left support $y''(o^-) = y'''(o^-)$ since the bending moment and shear vanish along the extension of the beam to the left, which is a straight line. In addition, if the origin is taken to coincide with the left support $y(o^-) = o$. Therefore the equation reduces to:

$$\bar{y}_g(s) = \frac{y'(o^-)}{s^2} + \frac{1}{s^4}\frac{P\,b}{EIL} - \frac{P\,e^{-as}}{s^4EI} + \frac{Pa}{s^4EIL}e^{-Ls} \qquad (7\text{-}10)$$

Inverting

$$y_g(x) = y'(o^-)\,x + \frac{P\;\;b\;x^3}{6EIL} - \frac{P\langle x-a\rangle^3}{6EI} + \frac{Pa\langle x-L\rangle^3}{6EIL} \qquad (7\text{-}11)$$

where $\langle x-a\rangle^3$ is the Macaulay Bracket notation for $(x-a)^3\,u(x-a)$, and the last term has no contribution to the deflection unless the beam extends beyond the right support.

The boundary condition $y(L) = 0$ serves to determine the only unknown $y'(o^-)$ remaining in the equation. Inserting this value which is:

$$y'(o^-) = \frac{Pb}{6EIL} (b^2-L^2) \qquad (7\text{-}12)$$

into the deflection equation yields:

$$y(x) = \frac{Pb}{6EIL} [(b^2-L^2)\, x + x^3 - \frac{L}{b} <x-a>^3] \qquad (7\text{-}13)$$

which produces the jump in the shear as part of the response, without any such a priori assumption.

2. Transform of the Time Variable

Consider the equation of motion of a harmonic oscillator

$$\ddot{x}(t) + p^2 x(t) = o \qquad (7\text{-}14)$$

where $p^2 = k/m$ subject to initial conditions $x(o^-) = \dot{x}(o^-) = 0$ and struck by a blow of impulse I_o. The problem can be handled in two different ways via the Laplace Transform.

Let x(t) be an ordinary function. Then

$$\ddot{x}(t) + p^2 x(t) = 0 \qquad (7\text{-}15)$$

Subject to $x(o^+) = 0 \quad \dot{x}(0^+) = \frac{I_o}{m}$ where the second condition has been obtained by a consideration of linear momentum. The transformed equation is

$$\bar{x}(s) = \frac{I_o/m}{s^2 + p^2} \qquad (7\text{-}16)$$

after insertion of initial conditions, it follows that

$$x(t) = \frac{I_o}{m\,p} \sin pt \qquad t > 0 \qquad (7\text{-}17)$$

Let $x(t)$ now be a generalized function. The equation of motion can now be written as:

$$\ddot{x}(t) + p^2 x(t) = \frac{I_o}{m} \delta(t) \tag{7-18}$$

subject to $x(o^-) = \dot{x}(o^-) = 0$.

The transformed equation now reads

$$\bar{x}(s) = \frac{I_o/m}{s^2 + p^2} \tag{7-19}$$

and its inverse is

$$x(t) = \frac{I_o}{m\,p} \sin pt\, u(t) \tag{7-20}$$

It is easy to show that this solution satisfies the given equation and the initial conditions either by formal substitution or by a simple application of distribution theory.

A number of textbooks such as (13) use the equation

$$\ddot{x}(t) + p^2 x(t) = \frac{I_o}{m} \delta(t) \tag{7-21}$$

with rightsided initial conditions $x(o^+) = \dot{x}(o^+) = 0$, and explain the discrepancy between the assumed zero initial velocity and the actual initial velocity I_o/m by appealing to momentum considerations. It is clear that no contradiction actually exists, and the apparent contradiction so frequently encountered in the literature can be easily resolved by considering generalized functions which automatically exhibit lefthanded limits in the transforms of their derivatives.

VIII CONCLUSIONS

It has been shown that the failure to recognize the difference between an ordinary function and a generalized function, particularly of their derivatives, and transforms of derivatives; has led to well-known discrepancies between the assigned initial values and the actual initial values. This has been discussed at length in the literature by several authors such as (16). This difficulty can be avoided by the use of the concept of generalized functions or distributions developed by Laurent Schwartz (4).

An elementary derivation of the transform of the generalized derivative is developed in this paper and applied to typical problems in which it is shown that some of the classical contradictions disappear. It is hoped that a more widespread understanding of these relatively new concepts by those interested in applications, will give them a more powerful method of approach to boundary value problems, particularly those leading to partial differential equations.

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13. ABSTRACT

Generalized functions as developed by Laurent Schwartz are used to include finite jump contributions to transforms of derivatives. The expression of Bilateral Laplace Transform of the derivative of a piecewise continuous function is first derived and then specialized to the unilateral transform. It is shown that the use of the Laplace Transform of a generalized derivative resolves some well-known apparent contradictions. In this new context the distinction between "prescribed" and "actual" initial values disappears. Although the present investigation is restricted to ordinary differential equations, it can be extended to partial differential equations.

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