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GENERAL WEAK TURBULENCE THEORY  
OF RESONANT FOUR-WAVE PROCESSES\*

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ABSTRACT

The time evolution of wave correlations (cumulants) in a uniformly turbulent ensemble of weakly nonlinear, dispersive systems is discussed. With closure of the hierarchy for wave correlations appropriate to the inclusion of resonant four-wave processes, a kinetic equation for the spectral energy density of the waves is derived in situations where the resonant three-wave decay condition cannot be satisfied. The resulting kinetic equation is a nonlinear integro-differential equation with driving terms trilinear in the energy density. Some general properties of this equation are discussed including associated conservation relations and the law of increase of entropy.

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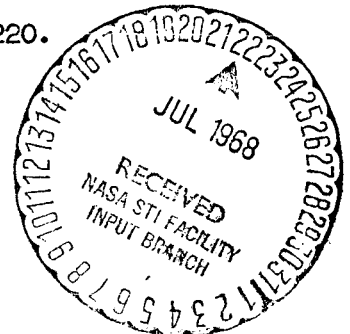
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## I: INTRODUCTION

A large number of nonlinear dispersive systems are described by dynamical equations of the form

$$\frac{\partial A_\alpha}{\partial t}(k_1, t) = \sum_{\beta, \gamma} \iint dk_2 dk_3 \delta(k_1 - k_2 - k_3) A_\beta(k_2, t) A_\gamma(k_3, t) \times K^{\alpha\beta\gamma}(k_1, k_2, k_3) \exp(i(\omega_\alpha(k_1) - \omega_\beta(k_2) - \omega_\gamma(k_3))t), \quad (1.1)$$

where  $A_\alpha(k_1, t)$  is the amplitude of the  $\alpha$ 'th mode of oscillation in the appropriate representation,  $\omega_\alpha(k_1)$  is the corresponding oscillation frequency as a function of wave vector  $k_1$ , and  $K^{\alpha\beta\gamma}(k_1, k_2, k_3)$  is the interaction kernel for the triplet of modes  $(\alpha, \beta, \gamma)$ . The functional dependence of  $\omega_\alpha(k_1)$  and  $K^{\alpha\beta\gamma}(k_1, k_2, k_3)$  on wave vector depends on the details of the problem under consideration. The dynamical equation (1.1) arises in a variety of circumstances where a fluid-like description is applicable, ranging from plasma models in which a magnetohydrodynamic description is used,<sup>1</sup> to the nonlinear interaction of gravity waves in a fluid of constant depth.<sup>2,3</sup> The formal derivation of (1.1) for a broad class of nonlinear dispersive systems, together with references to explicit examples has been given elsewhere and need not be included in our considerations here.<sup>4</sup>

There are essentially two types of nonlinear wave-wave interaction problems one can consider in relation to (1.1). First, there is the coherent initial-value problem in which we are given the initial amplitudes  $\{A_\alpha(k_1, 0)\}$  and determine from (1.1) the subsequent time evolution of  $\{A_\alpha(k_1, t)\}$ . The nonlinear nature of the dynamical equation

makes a mathematical analysis untractable except in small-amplitude situations where  $|A_{\alpha}(\underline{k}_1, 0)| \sim \lambda \ll 1$ , say. In this case one can formally generate the perturbation solution to (1.1), i.e.,

$$A_{\alpha}(\underline{k}_1, t) \approx \lambda A_{\alpha}^{(1)}(\underline{k}_1, t) + \lambda^2 A_{\alpha}^{(2)}(\underline{k}_1, t) + \dots, \quad (1.2)$$

to some desired order,  $\lambda^n$ . In lowest approximation, the wave amplitude  $A_{\alpha}$  does not change with time. In higher order, however, the nonlinear terms act as perturbations which cause the amplitude to change during the course of time through the interaction between waves of differing wave number. The second type of problem, and the one that shall concern us here, corresponds to weakly turbulent situations where the time evolution of a statistical ensemble of systems, each evolving according to (1.1), is considered. For the case of uniform turbulence (i.e. a spatially homogeneous ensemble) it is of paramount interest to ascertain the time behavior of the spectral energy density,  $2G_{\alpha\alpha}$ , associated with the oscillation amplitude where

$$2G_{\alpha\alpha}(\underline{k}_1, t) \delta(\underline{k}_1 + \underline{k}_2) = \langle A_{\alpha}(\underline{k}_1, t) A_{\alpha}(\underline{k}_2, t) \rangle \quad (1.3)$$

and  $\langle \rangle$  denotes ensemble average. As will be discussed below, there are two procedures which may be used to obtain a kinetic equation for  $2G_{\alpha\alpha}$ . In the conventional approach, the coherent solution (1.2) is obtained to some order and appropriate statistical averages are then taken over a spatially uniform ensemble in order to give a kinetic equation for the spectral energy density.<sup>5,6</sup> The other approach, which leads simply and directly to the desired kinetic equation, consists of considering at the

outset dynamical equations for wave correlations in the ensemble.<sup>4</sup>

Before describing the details of the formalism to be used, we remind the reader of some salient features of existing kinetic theories for resonant three-wave processes pertinent to the general dynamical equation (1.1). The discussion will be brief in this regard since the weak turbulence theory of resonant three-wave interactions has been discussed in detail elsewhere,<sup>4-7</sup> and, as suggested by the title and abstract, the principal concern here will be the construction of a kinetic theory for resonant four-wave processes within the framework of (1.1). At this point we record the sign conventions and symmetries to be used throughout the remainder of this article, i.e.,

$$\omega_{\alpha}(-\tilde{k}_1) = -\omega_{\alpha}(\tilde{k}_1) , \quad (1.4)$$

$$A_{\alpha}(\tilde{k}_1, t) = A_{\alpha}^{*}(-\tilde{k}_1, t) , \quad (1.5)$$

$$K^{\alpha\beta\gamma}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = K^{*\alpha\beta\gamma}(-\tilde{k}_1, -\tilde{k}_2, -\tilde{k}_3) . \quad (1.6)$$

The oscillation frequency  $\omega_{\alpha}(\tilde{k})$  is assumed real. Moreover, the response  $\mu^{\alpha\beta\gamma}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3)$  defined in terms of the interaction kernel by

$$\mu^{\alpha\beta\gamma}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = \omega_{\beta}(\tilde{k}_2)\omega_{\gamma}(\tilde{k}_3)K^{\alpha\beta\gamma}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) \quad (1.7)$$

will be assumed to enjoy the symmetries

$$\mu^{\alpha\beta\gamma}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = \mu^{\alpha\gamma\beta}(\tilde{k}_1, \tilde{k}_3, \tilde{k}_2) = \mu^{\beta\alpha\gamma}(-\tilde{k}_2, -\tilde{k}_1, \tilde{k}_3) = \mu^{\gamma\beta\alpha}(-\tilde{k}_3, \tilde{k}_2, -\tilde{k}_1) , \quad (1.8)$$

for  $\tilde{k}_1 = \tilde{k}_2 + \tilde{k}_3$ , which is the case for a large number of physical processes of interest. In terms of the action density  $n_{\alpha}(\tilde{k}_1, t)$  defined by

$$n_{\alpha}(\tilde{k}_1, t) = 2G_{\alpha\alpha}(\tilde{k}_1, t)/\omega_{\alpha}(\tilde{k}_1) , \quad (1.9)$$

the leading-order kinetic equation for  $n_\alpha$  is given by<sup>4</sup>

$$\begin{aligned} \frac{\partial}{\partial t} n_\alpha(k_{\sim 1}, t) = & 4\pi \sum_{\beta, \gamma} \iint dk_{\sim 2} dk_{\sim 3} \frac{|\mu^{\alpha\beta\gamma}(k_{\sim 1}, k_{\sim 2}, k_{\sim 3})|^2}{\omega_\alpha(k_{\sim 1})\omega_\beta(k_{\sim 2})\omega_\gamma(k_{\sim 3})} \\ & \delta(k_{\sim 1} - k_{\sim 2} - k_{\sim 3}) \delta(\omega_\alpha(k_{\sim 1}) - \omega_\beta(k_{\sim 2}) - \omega_\gamma(k_{\sim 3})) \\ & \times \left( n_\beta(k_{\sim 2}, t) n_\gamma(k_{\sim 3}, t) - n_\alpha(k_{\sim 1}, t) n_\beta(k_{\sim 2}, t) - n_\alpha(k_{\sim 1}, t) n_\gamma(k_{\sim 3}, t) \right), \end{aligned} \quad (1.10)$$

where (1.10) holds for a spatially uniform ensemble in weakly turbulent situations. Equation (1.10) is correct to  $O(n^2)$  and the basic process causing the action density to change with time is that of resonant three-wave interactions where

$$\begin{aligned} \omega_\alpha(k_{\sim 1}) &= \omega_\beta(k_{\sim 2}) + \omega_\gamma(k_{\sim 3}), \\ k_{\sim 1} &= k_{\sim 2} + k_{\sim 3}. \end{aligned} \quad (1.11)$$

This may be viewed as the "decay" of  $(\omega_\alpha(k_{\sim 1}), k_{\sim 1})$  into two further waves  $(\omega_\beta(k_{\sim 2}), k_{\sim 2})$  and  $(\omega_\gamma(k_{\sim 3}), k_{\sim 3})$ , or as the inverse process of the merging of two waves into one. Although Eq. (1.10) is non-trivial in a large number of situations of interest, there are physical problems where the linear dispersive properties, as manifest in  $\{\omega_\alpha(k_{\sim 1})\}$ , are such that the resonant three-wave decay condition (1.11) cannot be satisfied. In this case  $(\partial/\partial t)n_\alpha(k_{\sim 1}, t) = 0$  trivially from (1.10), and clearly a higher-order kinetic theory must be constructed to describe the time evolution of the action density. Whether or not (1.11) can be satisfied depends on the explicit functional forms of  $\{\omega_\alpha(k_{\sim 1})\}$ . As particular examples, we remind the reader that the dispersive properties of gravity waves in a fluid of constant depth, as well as the dispersive properties of (long wavelength) electrostatic electron plasma oscillations, are such that

resonant three-wave interactions are forbidden. In any case it will be assumed in the present analysis that (1.11) cannot be satisfied. Since it is not necessary to specialize to a particular problem in deriving the kinetic equation appropriate to these circumstances, the above-mentioned examples may be kept in mind as possible applications.

In this case the leading-order process which causes the action density to change will be that of resonant four-wave interactions. We will see in Sec. III that for a statistical ensemble of systems evolving according to the dynamical equation (1.1), the kinetic behavior of  $n_\alpha(\mathbf{k}, t)$  is governed by stoss terms trilinear in the action density when the resonance condition (1.11) cannot be satisfied. In particular, the relevant kinetic equation (4.15) may be written in the form (dropping the  $\epsilon^2 t$  notation)

$$\begin{aligned} \frac{\partial n_\alpha}{\partial t}(\mathbf{k}_1, t) = & \sum_{\beta, \gamma, \delta} \iiint d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \frac{|D^{\alpha\beta\gamma\delta}(-\mathbf{k}_1, -\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)|^2}{\omega_\alpha(\mathbf{k}_1)\omega_\beta(\mathbf{k}_2)\omega_\gamma(\mathbf{k}_3)\omega_\delta(\mathbf{k}_4)} \\ & \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(\omega_\alpha(\mathbf{k}_1) + \omega_\beta(\mathbf{k}_2) - \omega_\gamma(\mathbf{k}_3) - \omega_\delta(\mathbf{k}_4)) \\ & \times \left( n_\beta(\mathbf{k}_2, t) n_\gamma(\mathbf{k}_3, t) n_\delta(\mathbf{k}_4, t) + n_\alpha(\mathbf{k}_1, t) n_\gamma(\mathbf{k}_3, t) n_\delta(\mathbf{k}_4, t) \right. \\ & \left. - n_\alpha(\mathbf{k}_1, t) n_\beta(\mathbf{k}_2, t) n_\delta(\mathbf{k}_4, t) - n_\alpha(\mathbf{k}_1, t) n_\beta(\mathbf{k}_2, t) n_\gamma(\mathbf{k}_3, t) \right), \quad (1.12) \end{aligned}$$

where  $D^{\alpha\beta\gamma\delta}(-\mathbf{k}_1, -\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$  is related to the three-wave response through Eq. (4.16). It is clear from (1.12) that  $n_\alpha(\mathbf{k}_1, t)$  evolves due to resonant behavior manifest by the condition

$$\begin{aligned} \omega_\alpha(\mathbf{k}_1) + \omega_\beta(\mathbf{k}_2) &= \omega_\gamma(\mathbf{k}_3) + \omega_\delta(\mathbf{k}_4), \\ \mathbf{k}_1 + \mathbf{k}_2 &= \mathbf{k}_3 + \mathbf{k}_4. \end{aligned} \quad (1.13)$$

The resonant four-wave condition (1.13) has a simple interpretation. It should be kept in mind that three-wave interactions are still basic to the dynamical equation (1.1). As shown schematically in Fig. 1, the resonant four-wave behavior associated with (1.12) and (1.13) consists of two successive three-wave processes. Namely, the two states  $(\omega_{\alpha}(k_1), k_1)$  and  $(\omega_{\beta}(k_2), k_2)$  merge into a virtual state  $(\omega_{\eta}(k_1+k_2), k_1+k_2)$  which instantaneously decays into two further states  $(\omega_{\gamma}(k_3), k_3)$  and  $(\omega_{\delta}(k_4), k_4)$ . Other possible combinations are of course possible. It should be noted that although  $\omega_{\alpha}(k_1) + \omega_{\beta}(k_2) \neq \omega_{\eta}(k_1+k_2)$  by hypothesis, the over-all process microscopically conserves energy and momentum according to (1.13).

The purpose of the present article is to put the derivation of the kinetic equation for resonant four-wave processes on a systematic and rigorous basis, within the framework of the general dynamical equation (1.1). The details of the formalism to be used have been developed in Ref. 4 where a hierarchy was obtained describing the time evolution of wave correlations (cumulants) in a spatially homogeneous ensemble of systems evolving according to (1.1). The kinetic equation (1.10) for resonant three-wave interactions was shown to be very simply accessible within this hierarchy for wave correlations. The analysis here is an extension of this formalism to one higher order, in situations where (1.11) cannot be satisfied and hence the basic process for causing the action density to change is that of resonant four-wave interactions. We reiterate the basic philosophy for using the hierarchy approach. As mentioned earlier, the conventional method for treating the weak turbulence problem is to obtain the perturbation solution to the coherent initial-value problem to some desired order. Then suitable statistical averages (usually referred to as the random phase approximation) of the

coherent solution are performed over a spatially uniform ensemble. In particular, the kinetic equation for the spectral energy density is obtained in the conventional approach by considering the average change in the transition probability per unit time. Objections have been raised to this type of approach, not only regarding the basis of validity of the random phase approximation, but also because the formalism inherently involves considerably more information (and hence algebra) than is necessary to describe the turbulent ensemble.<sup>4,8</sup> For example, the solution to the coherent problem entails phase information which is unnecessary in describing properties of the ensemble. The hierarchy formalism offers considerable simplification in this regard since dynamical equations for s-wave correlations  $G_s$ ,  $s = 2, 3, \dots$ , characterizing the ensemble are considered at the outset. A simplification also occurs regarding the relevant expansion parameters. Whereas the natural expansion parameter encountered in obtaining the coherent solution to (1.1) is  $\lambda$ , a measure of the oscillation amplitude, the expansion parameter appropriate to the hierarchy framework is  $\epsilon \sim \lambda^2$ , a measure of the spectral energy density. Consequently, in order to obtain a kinetic equation for resonant four-wave processes it will be necessary to carry out an analysis of the hierarchy for wave correlations only to order  $\epsilon^3$ . However, if the conventional approach were used it would be necessary to first obtain the coherent solution to (1.1) correct to  $O(\lambda^5)$ , as was done by Hasselmann in relation to the particular problem of the nonlinear interaction of gravity waves in a fluid of constant depth.<sup>2,3</sup>

In Sec. II the hierarchy formalism is briefly reviewed. The ordering appropriate to weakly turbulent situations is discussed and the equations expanded in a multiple-time perturbation analysis. It should be noted that no assumption regarding random phases or gaussianity of statistics is made. The



only assumption is that the ensemble is spatially homogeneous corresponding to uniformly turbulent situations. The expanded hierarchy equations (2.13)-(2.18) are solved order by order in Sec. III. For simplicity of presentation the mode labels are omitted in the analysis of Secs. II and III, but reinstated in the final version of the kinetic equation for the action density, Eq. (4.15). Various general properties of this kinetic equation are discussed in Sec. V including energy and momentum conservation relations. In addition, the appropriate non-equilibrium entropy for the kinetic equation (1.12) is shown to be

$$S = \sum_{\alpha} \int d\tilde{k}_{\alpha} \ln |n_{\alpha}(\tilde{k}_{\alpha}, t)| .$$

## II. HIERARCHY FORMALISM AND WEAK TURBULENCE ORDERING

### (a) Review of Hierarchy Formalism

The moment hierarchy for a statistical ensemble of systems evolving according to (1.1) forms an interconnected chain in which the average of the product of  $s$  amplitudes,  $\langle A_{\alpha_1}(\tilde{k}_1, t) \dots A_{\alpha_s}(\tilde{k}_s, t) \rangle$ , is advanced in terms of the average of the product of  $s+1$  amplitudes,  $\langle A_{\alpha_1}(\tilde{k}_1, t) \dots A_{\alpha_{s+1}}(\tilde{k}_{s+1}, t) \rangle$ , where  $s = 1, 2, \dots$ . As elaborated in Ref. 4, from this moment hierarchy an alternate hierarchy may be constructed describing the time evolution of irreducible  $s$ -wave correlations (cumulants),  ${}_s G$ ,  $s = 2, 3, \dots$ , defined by

$${}_2 G(\tilde{k}_1, t) \delta(\tilde{k}_1 + \tilde{k}_2) \equiv \langle A(\tilde{k}_1, t) A(\tilde{k}_2, t) \rangle , \quad (2.1)$$

$${}_3 G(\tilde{k}_1, \tilde{k}_2, t) \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3) \equiv \langle A(\tilde{k}_1, t) A(\tilde{k}_2, t) A(\tilde{k}_3, t) \rangle , \quad (2.2)$$

$${}_4 G(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, t) \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 + \tilde{k}_4) \equiv \langle A(\tilde{k}_1, t) \dots A(\tilde{k}_4, t) \rangle$$

$$- {}_2 G(\tilde{k}_1, t) {}_2 G(\tilde{k}_3, t) \delta(\tilde{k}_1 + \tilde{k}_2) \delta(\tilde{k}_3 + \tilde{k}_4)$$

$$\begin{aligned}
& - {}_2G(\underline{k}_1, t) {}_2G(\underline{k}_2, t) \delta(\underline{k}_1 + \underline{k}_3) \delta(\underline{k}_2 + \underline{k}_4) \\
& - {}_2G(\underline{k}_1, t) {}_2G(\underline{k}_2, t) \delta(\underline{k}_1 + \underline{k}_4) \delta(\underline{k}_2 + \underline{k}_3) \quad , \quad (2.3) \\
& \quad \quad \quad \vdots
\end{aligned}$$

for a spatially uniform ensemble. Similarly,  ${}_5G$  is defined by subtracting from the average of the product of five amplitudes all distinct permutations of  ${}_2G {}_3G$  products, and so on. In Eqs. (2.1)-(2.3) and throughout the analysis of Secs. II and III mode labels have been dropped for simplicity of presentation. These serve principally to complicate the notation and not to modify the basic formalism, and will be appropriately reinstated in the final version of the kinetic equation. Moreover, in writing the definitions (2.1)-(2.3) it has been assumed that  $\langle A(\underline{k}_1, t) \rangle = 0$ , i.e. that  $A(\underline{k}_1, t)$  corresponds to a fluctuation with zero mean. This remains true for all times if so initially provided  $K(0, -\underline{k}_2, \underline{k}_2) = 0$ . It should also be noted that the delta-function dependence in the above definitions is just a manifestation of translational invariance of ensemble averages in  $\underline{x}$ -space, i.e. that the ensemble is spatially uniform.<sup>9</sup> The resulting hierarchy for  ${}_sG$ ,  $s = 2, 3, \dots$ , forms an interconnected chain in which  ${}_2G$  is advanced in terms of  ${}_3G$ ,  ${}_3G$  is advanced in terms of  ${}_4G$  and  ${}_2G {}_2G$  driving terms, etc. The first few members are given by<sup>4</sup>

$$\begin{aligned}
& \delta(\underline{k}_1 + \underline{k}_2) \frac{\partial}{\partial t} {}_2G(\underline{k}_1, t) \\
& = \delta(\underline{k}_1 + \underline{k}_2) \left\{ \left( \int d\underline{k}' K(\underline{k}_1, \underline{k}', \underline{k}_1 - \underline{k}') {}_3G(\underline{k}', \underline{k}_1 - \underline{k}', t) \right. \right. \\
& \left. \left. \times \exp\{i(\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}'))t\} \right) + (1 \leftrightarrow 2) \right\} \quad , \quad (2.4)
\end{aligned}$$

$$\begin{aligned}
& \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3) \frac{\partial}{\partial t} {}_3G(\tilde{k}_1, \tilde{k}_2, t) \\
&= \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3) \left\{ \left( [K(\tilde{k}_1, -\tilde{k}_2, \tilde{k}_1 + \tilde{k}_2)]_2 G(\tilde{k}_2, t) {}_2G(\tilde{k}_3, t) \right. \right. \\
&\quad \times \exp\{i(\omega(\tilde{k}_1) + \omega(\tilde{k}_2) - \omega(\tilde{k}_1 + \tilde{k}_2))t\} + [2 \leftrightarrow 3] \\
&\quad \left. \left. + (2 \leftrightarrow 1) + (3 \leftrightarrow 1) \right\} \right. \\
&\quad \left. + \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3) \left\{ \left( \int d\tilde{k}' K(\tilde{k}_1, \tilde{k}', \tilde{k}_1 - \tilde{k}') \right)_4 G(\tilde{k}', \tilde{k}_1 - \tilde{k}', \tilde{k}_2, t) \right. \right. \\
&\quad \left. \left. \times \exp\{i(\omega(\tilde{k}_1) - \omega(\tilde{k}') - \omega(\tilde{k}_1 - \tilde{k}'))t\} + (2 \leftrightarrow 1) + (3 \leftrightarrow 1) \right\} , \tag{2.5}
\end{aligned}$$

$$\begin{aligned}
& \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 + \tilde{k}_4) \frac{\partial}{\partial t} {}_4G(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, t) \\
&= \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 + \tilde{k}_4) {}_2 \sum_{s'=2}^4 \left\{ \left( [K(\tilde{k}_1, \tilde{k}_1 + \tilde{k}_2, -\tilde{k}_2)]_2 G(\tilde{k}_2, t) {}_3G(\tilde{k}_3, \tilde{k}_4, t) \right. \right. \\
&\quad \times \exp\{i(\omega(\tilde{k}_1) + \omega(\tilde{k}_2) - \omega(\tilde{k}_1 + \tilde{k}_2))t\} + [2 \leftrightarrow 3] + [2 \leftrightarrow 4] \\
&\quad \left. \left. + (1 \leftrightarrow s') \right\} \right. \\
&\quad \left. + \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 + \tilde{k}_4) \sum_{s'=2}^4 \left\{ \left( \int d\tilde{k}' K(\tilde{k}_1, \tilde{k}', \tilde{k}_1 - \tilde{k}') \right)_5 G(\tilde{k}', \tilde{k}_1 - \tilde{k}', \tilde{k}_2, \tilde{k}_3, t) \right. \right. \\
&\quad \left. \left. \times \exp\{i(\omega(\tilde{k}_1) - \omega(\tilde{k}') - \omega(\tilde{k}_1 - \tilde{k}'))t\} + (1 \leftrightarrow s') \right\} . \\
&\quad \vdots
\end{aligned} \tag{2.6}$$

Similarly  ${}_5G$  is driven by  ${}_3G$ ,  ${}_4G$  and  ${}_6G$  terms, and in general  ${}_sG$  is driven by  ${}_{s+1}G$  and  ${}_pG$   ${}_{s+1-p}G$  terms,  $p = 2, \dots, s-1$ . The resulting hierarchy of equations is structurally quite analogous to the B.B.G.K.Y. hierarchy for irreducible correlations. Equations (2.4), (2.5)... represent

the dynamical system of equations to be used in describing the time evolution of wave correlations in a spatially homogeneous ensemble of systems, each evolving according to (1.1). Such a description is clearly practical only if a meaningful closure of this hierarchy can be obtained at some level of description, i.e.  ${}_s G$  becomes small for increasing  $s$ . That this is the case in weakly turbulent situations will now be demonstrated.

(b) Weak Turbulence Ordering and the Multiple-Time Perturbation Expansion

In the weak turbulence case corresponding to a small-amplitude analysis of (1.1) we assume

$${}_2 G \sim \epsilon \quad \epsilon \ll 1 \quad (2.7)$$

in leading order. From (2.5), since  ${}_3 G$  is driven by  ${}_2 G {}_2 G$  terms we assume  ${}_3 G \sim \epsilon^2$  to leading order. Similarly  ${}_4 G$  is driven by  ${}_2 G {}_3 G$  terms and we take  ${}_4 G \sim \epsilon^3$ , and in general

$${}_s G \sim \epsilon^{s-1} . \quad (2.8)$$

Whether or not an initial ordering of the form (2.8) is violated during the course of time may of course be checked. Of paramount interest in present considerations is the evolution of the spectral energy density,  ${}_2 G$ . It is clear from (2.4) that the degree of accuracy with which we describe  ${}_2 G$  depends in detail on the level of sophistication with which we describe  ${}_3 G$ , etc. For purposes of obtaining the kinetic equation for resonant four-wave processes it will be necessary to obtain  ${}_2 G$  to order  $\epsilon^3$  and describe the leading-order  ${}_2 G$  for times  $t \sim 1/\epsilon^2$ . Consequently, as may be seen from (2.4)-(2.6),  ${}_3 G$  and  ${}_4 G$  are needed correct to order  $\epsilon^3$ ;  ${}_5 G$  and higher correlations, however, may be omitted from the analysis. In general, to calculate  ${}_2 G$  to order  $\epsilon^n$  and describe

the leading-order  ${}_2G$  for times  $t \sim 1/\epsilon^{n-1}$ , closure may be obtained by neglecting  ${}_{n+2}G$  and higher correlations.

Within the context of the above estimates, Eqs. (2.4)-(2.6) may be expanded using the multiple-time perturbation techniques of Frieman and Sandri.<sup>10,11,12</sup> In particular,  ${}_2G, {}_3G, \dots$  are written

$${}_2G \cong \epsilon {}_2G^{(1)}(\underline{k}_1, t, \epsilon t, \dots) + \epsilon^2 {}_2G^{(2)}(\underline{k}_1, t, \epsilon t, \dots) + \epsilon^3 {}_2G^{(3)}(\underline{k}_1, t, \epsilon t, \dots) + \dots, \quad (2.9)$$

$${}_3G \cong \epsilon^2 {}_3G^{(2)}(\underline{k}_1, \underline{k}_2, t, \epsilon t, \dots) + \epsilon^3 {}_3G^{(3)}(\underline{k}_1, \underline{k}_2, t, \epsilon t, \dots) + \dots, \quad (2.10)$$

$${}_4G \cong \epsilon^3 {}_4G^{(3)}(\underline{k}_1, \underline{k}_2, \underline{k}_3, t, \epsilon t, \dots) + \dots, \quad (2.11)$$

$${}_5G \cong \epsilon^4 {}_5G^{(4)}(\underline{k}_1, \underline{k}_2, \underline{k}_3, \underline{k}_4, t, \epsilon t, \dots) + \dots, \quad (2.12)$$

⋮

Operationally, the time scales  $t, \epsilon t, \epsilon^2 t, \dots$  are treated as independent variables, and the freedom inherent in the multiple-time formalism will be used to obtain a uniformly valid solution. To the order appropriate to obtaining a kinetic equation for resonant four-wave interactions only the time scales  $t, \epsilon t, \epsilon^2 t$  are relevant, and  $\epsilon^n t$   $n \geq 3$  may be suppressed in the notation below. Evidently

$$\frac{\partial}{\partial t} {}_2G^{(1)}(\underline{k}_1, t, \epsilon t, \epsilon^2 t) = 0, \quad (2.13)$$

$$\begin{aligned} & \delta(\underline{k}_1 + \underline{k}_2) \left( \frac{\partial}{\partial t} {}_2G^{(2)}(\underline{k}_1, t, \epsilon t, \epsilon^2 t) + \frac{\partial}{\partial \epsilon t} {}_2G^{(1)}(\underline{k}_1, t, \epsilon t, \epsilon^2 t) \right) \\ &= \delta(\underline{k}_1 + \underline{k}_2) \left\{ \left( \int d\underline{k}' K(\underline{k}_1, \underline{k}', \underline{k}_1 - \underline{k}') {}_3G^{(2)}(\underline{k}', \underline{k}_1 - \underline{k}', t, \epsilon t, \dots) \right. \right. \\ & \quad \left. \left. \times \exp\{i(\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}'))t\} \right) + (1 \leftrightarrow 2) \right\}, \quad (2.14) \end{aligned}$$

$$\begin{aligned}
& \delta(\underline{k}_{\sim 1} + \underline{k}_{\sim 2}) \left( \frac{\partial}{\partial t} {}_2G^{(3)}(\underline{k}_{\sim 1}, t, \epsilon t, \epsilon^2 t) + \frac{\partial}{\partial \epsilon t} {}_2G^{(2)}(\underline{k}_{\sim 1}, t, \epsilon t, \epsilon^2 t) \right. \\
& \quad \left. + \frac{\partial}{\partial \epsilon^2 t} {}_2G^{(1)}(\underline{k}_{\sim 1}, t, \epsilon t, \epsilon^2 t) \right) \\
& = \delta(\underline{k}_{\sim 1} + \underline{k}_{\sim 2}) \left\{ \left( \int d\underline{k}' K(\underline{k}_{\sim 1}, \underline{k}', \underline{k}_{\sim 1} - \underline{k}') {}_3G^{(3)}(\underline{k}', \underline{k}_{\sim 1} - \underline{k}', t, \epsilon t, \dots) \right. \right. \\
& \quad \left. \left. \times \exp[i(\omega(\underline{k}_{\sim 1}) - \omega(\underline{k}') - \omega(\underline{k}_{\sim 1} - \underline{k}'))t] \right) + (1 \leftrightarrow 2) \right\}, \quad (2.15) \\
& \quad \vdots
\end{aligned}$$

$$\begin{aligned}
& \delta(\underline{k}_{\sim 1} + \underline{k}_{\sim 2} + \underline{k}_{\sim 3}) \frac{\partial}{\partial t} {}_3G^{(2)}(\underline{k}_{\sim 1}, \underline{k}_{\sim 2}, t, \epsilon t, \dots) \\
& = \delta(\underline{k}_{\sim 1} + \underline{k}_{\sim 2} + \underline{k}_{\sim 3}) \left\{ \left( [K(\underline{k}_{\sim 1}, -\underline{k}_{\sim 2}, \underline{k}_{\sim 1} + \underline{k}_{\sim 2})] {}_2G^{(1)}(\underline{k}_{\sim 2}, t, \epsilon t, \dots) {}_2G^{(1)}(\underline{k}_{\sim 3}, t, \epsilon t, \dots) \right. \right. \\
& \quad \left. \left. \times \exp[i(\omega(\underline{k}_{\sim 1}) + \omega(\underline{k}_{\sim 2}) - \omega(\underline{k}_{\sim 1} + \underline{k}_{\sim 2}))t] \right] + [2 \leftrightarrow 3] \right) \\
& \quad \left. + (2 \leftrightarrow 1) + (3 \leftrightarrow 1) \right\}, \quad (2.16)
\end{aligned}$$

$$\begin{aligned}
& \delta(\underline{k}_{\sim 1} + \underline{k}_{\sim 2} + \underline{k}_{\sim 3}) \left\{ \frac{\partial}{\partial t} {}_3G^{(3)}(\underline{k}_{\sim 1}, \underline{k}_{\sim 2}, t, \epsilon t, \dots) + \frac{\partial}{\partial \epsilon t} {}_3G^{(2)}(\underline{k}_{\sim 1}, \underline{k}_{\sim 2}, t, \epsilon t, \dots) \right\} \\
& = \delta(\underline{k}_{\sim 1} + \underline{k}_{\sim 2} + \underline{k}_{\sim 3}) \left\{ \left( [K(\underline{k}_{\sim 1}, -\underline{k}_{\sim 2}, \underline{k}_{\sim 1} + \underline{k}_{\sim 2})] [{}_2G^{(1)}(\underline{k}_{\sim 2}, t, \epsilon t, \dots) {}_2G^{(2)}(\underline{k}_{\sim 3}, t, \epsilon t, \dots) \right. \right. \\
& \quad \left. \left. + {}_2G^{(2)}(\underline{k}_{\sim 2}, t, \epsilon t, \dots) {}_2G^{(1)}(\underline{k}_{\sim 3}, t, \epsilon t, \dots) \right] \exp[i(\omega(\underline{k}_{\sim 1}) + \omega(\underline{k}_{\sim 2}) - \omega(\underline{k}_{\sim 1} + \underline{k}_{\sim 2}))t] \right] \\
& \quad \left. + [2 \leftrightarrow 3] \right) + (2 \leftrightarrow 1) + (3 \leftrightarrow 1) \left\{ \right. \\
& \quad \left. + \delta(\underline{k}_{\sim 1} + \underline{k}_{\sim 2} + \underline{k}_{\sim 3}) \left\{ \left( \int d\underline{k}' K(\underline{k}_{\sim 1}, \underline{k}', \underline{k}_{\sim 1} - \underline{k}') {}_4G^{(3)}(\underline{k}', \underline{k}_{\sim 1} - \underline{k}', \underline{k}_{\sim 2}, t, \epsilon t, \dots) \right. \right. \right. \\
& \quad \left. \left. \times \exp[i(\omega(\underline{k}_{\sim 1}) - \omega(\underline{k}') - \omega(\underline{k}_{\sim 1} - \underline{k}'))t] \right) + (2 \leftrightarrow 1) + (3 \leftrightarrow 1) \right\}, \quad (2.17) \\
& \quad \vdots
\end{aligned}$$

$$\begin{aligned}
& \delta(k_{\sim 1} + k_{\sim 2} + k_{\sim 3} + k_{\sim 4}) \frac{\partial}{\partial t} {}_4G^{(3)}(k_{\sim 1}, k_{\sim 2}, k_{\sim 3}, t, \epsilon t, \dots) \\
&= \delta(k_{\sim 1} + k_{\sim 2} + k_{\sim 3} + k_{\sim 4}) {}_2G^{(1)}(k_{\sim 2}, t, \epsilon t, \dots) \\
&\quad \times {}_3G^{(2)}(k_{\sim 3}, k_{\sim 4}, t, \epsilon t, \dots) \exp\{i(\omega(k_{\sim 1}) + \omega(k_{\sim 2}) - \omega(k_{\sim 1} + k_{\sim 2}))t\} \\
&\quad + [2 \leftrightarrow 3] + [2 \leftrightarrow 4] + (1 \leftrightarrow s') \} \quad . \quad (2.18) \\
&\quad \vdots
\end{aligned}$$

Although Eqs. (2.13)-(2.18) may appear somewhat formidable, the analysis is quite straightforward. One solves (2.13)-(2.18) order by order, commencing with the lowest order first. The freedom inherent in the multiple-time formalism allows for the removal of secular behavior on short time-scales, which in turn determines the kinetic behavior on longer time-scales. We note from (2.13) that the leading-order two-wave correlation,  ${}_2G^{(1)}$ , does not vary on the short (oscillation) time-scale  $t$ , i.e.,

$${}_2G^{(1)}(k_{\sim 1}, t, \epsilon t, \epsilon^2 t) = {}_2G^{(1)}(k_{\sim 1}, 0, \epsilon t, \epsilon^2 t) . \quad (2.19)$$

Consequently  ${}_2G^{(1)}$  may be treated as constant as far as any  $t$  integrations are concerned in the analysis that follows.

### III. FORMAL SOLUTION

We now systematically seek the solutions to Eqs. (2.13)-(2.18) for the correlation perturbations. The symmetries and definitions (1.4), (1.6), (1.7) and (1.8), will be freely used throughout this section.

#### (a) Lowest Order

Since  ${}_2G^{(1)}$  does not vary on the short time-scale  $t$  according to (2.13), Eq. (2.16) may be integrated directly to give the leading-order three-wave

correlations on the  $t$  time-scale. This gives

$$\begin{aligned}
 {}_3G^{(2)}(\underline{k}_1, \underline{k}_2, t, \epsilon t, \dots) &= {}_3G^{(2)}(\underline{k}_1, \underline{k}_2, 0, \epsilon t, \dots) \\
 &+ \left\{ \frac{\exp(i(\omega(\underline{k}_1) + \omega(\underline{k}_2) - \omega(\underline{k}_1 + \underline{k}_2))t) - 1}{i(\omega(\underline{k}_1) + \omega(\underline{k}_2) - \omega(\underline{k}_1 + \underline{k}_2))} \times {}_3\hat{G}^{(2)}(\underline{k}_1, \underline{k}_2, \epsilon t, \epsilon^2 t) \right\}, \quad (3.1)
 \end{aligned}$$

where we have introduced the convenient notation

$$\begin{aligned}
 {}_3\hat{G}^{(2)}(\underline{k}_1, \underline{k}_2, \epsilon t, \epsilon^2 t) &\equiv 2 \left\{ K(\underline{k}_1, \underline{k}_1 + \underline{k}_2, -\underline{k}_2) {}_2G^{(1)}(\underline{k}_2, \epsilon t, \epsilon^2 t) {}_2G^{(1)}(\underline{k}_1 + \underline{k}_2, \epsilon t, \epsilon^2 t) \right. \\
 &+ K(-\underline{k}_1 - \underline{k}_2, -\underline{k}_2, -\underline{k}_1) {}_2G^{(1)}(\underline{k}_1, \epsilon t, \epsilon^2 t) {}_2G^{(1)}(\underline{k}_2, \epsilon t, \epsilon^2 t) \\
 &\left. + K(\underline{k}_2, -\underline{k}_1, \underline{k}_1 + \underline{k}_2) {}_2G^{(1)}(\underline{k}_1, \epsilon t, \epsilon^2 t) {}_2G^{(1)}(\underline{k}_1 + \underline{k}_2, \epsilon t, \epsilon^2 t) \right\}. \quad (3.2)
 \end{aligned}$$

In writing (3.1) and (3.2) use has been made of appropriate symmetries discussed in the introduction; moreover, we remind the reader that in light of the definition (2.1),  ${}_2G$  is real and an even function of its  $\underline{k}$ -argument. It should be noted that  ${}_3\hat{G}^{(2)}$  as defined in (3.2) in general varies on the  $\epsilon t, \epsilon^2 t, \dots$  time-scales through the slow time variation of  ${}_2G^{(1)}$ . It will also be useful to have an expression for  ${}_3\hat{G}^{(2)}$  in terms of the action density  $n(\underline{k}, \epsilon t, \epsilon^2 t)$  associated with the leading-order spectral energy density where

$$n(\underline{k}, \epsilon t, \epsilon^2 t) \equiv {}_2G^{(1)}(\underline{k}, \epsilon t, \epsilon^2 t) / \omega(\underline{k}) \quad (3.3)$$

Making use of (1.7), (1.8), and the oddness of  $n(\underline{k}, t, \epsilon t, \epsilon^2 t)$  as a function of its  $\underline{k}$  argument, Eq. (3.2) may be rewritten as

$$\begin{aligned}
 {}_3\hat{G}^{(2)}(\underline{k}_1, \underline{k}_2, \epsilon t, \epsilon^2 t) &\equiv -2\mu(\underline{k}_1, \underline{k}_1 + \underline{k}_2, -\underline{k}_2) \times \left\{ n(\underline{k}_1, \epsilon t, \epsilon^2 t) n(\underline{k}_1 + \underline{k}_2, \epsilon t, \epsilon^2 t) \right. \\
 &\left. + n(\underline{k}_2, \epsilon t, \epsilon^2 t) n(\underline{k}_1 + \underline{k}_2, \epsilon t, \epsilon^2 t) - n(\underline{k}_1, \epsilon t, \epsilon^2 t) n(\underline{k}_2, \epsilon t, \epsilon^2 t) \right\}. \quad (3.4)
 \end{aligned}$$



Expression (3.1) for the leading-order three-wave correlations may be substituted directly into (2.18) to determine the leading-order four-wave correlations on the short time-scale  $t$ . In addition (3.1) may be substituted into (2.14) to determine the fast time variation of  ${}_2G^{(2)}$  as well as the evolution of  ${}_2G^{(1)}$  on the  $\epsilon t$  scale. The latter calculation has been discussed in Ref. 4, and leads directly to the kinetic equation for resonant three-wave interactions. We remind the reader of the relevant results. Provided the initial-value term in Expression (3.1) for  ${}_3G^{(2)}(\underline{k}', \underline{k}_1 - \underline{k}', t, \epsilon t, \epsilon^2 t)$  is a sufficiently smooth function of its Fourier arguments, the associated effect in the  $\underline{k}'$ -integration on the right-hand side of (2.14) phase mixes to zero on the  $t$  time-scale. In particular, this mixing process yields long time behavior in the  $\underline{k}'$  integration of the form<sup>4,13</sup>

$$\frac{(\text{oscillation})}{t^{3/2}}, \quad (3.5)$$

in three-dimensional situations. However, the last term in (3.1) when substituted (2.14), gives a steady contribution to the  $\underline{k}'$ -integration, as well as terms which phase mix to zero. In particular, terms of the form

$$\frac{\exp\{i(\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}'))t\} - 1}{i(\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}'))}, \quad (3.6)$$

behave effectively as

$$\frac{i}{\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}') + i\Delta} \Big|_{\Delta \rightarrow 0_+}, \quad (3.7)$$

as  $t \rightarrow \infty$ , insofar as integrations over relatively smooth functions of  $\underline{k}'$  are concerned. Consequently, it is clear that when (2.14) is integrated on the  $t$  time-scale,  $(\partial/\partial \epsilon t) {}_2G^{(1)}$  (which is independent of  $t$ ) as well as the steady

contribution from the  $\tilde{k}'$ -integration will lead to secular behavior in  ${}_2G^{(2)}$  proportional to  $t$ . Removal of this secular behavior gives the evolution of  ${}_2G^{(1)}$  on the  $\epsilon t$  time scale, namely

$$\begin{aligned} \frac{\partial}{\partial \epsilon t} {}_2G^{(1)}(\tilde{k}_1, \epsilon t, \epsilon^2 t) &= \left( i \int d\tilde{k}' \frac{\mu(\tilde{k}_1, \tilde{k}', \tilde{k}_1 - \tilde{k}')}{\omega(\tilde{k}') \omega(\tilde{k}_1 - \tilde{k}')} \frac{\hat{G}^{(2)}(\tilde{k}_1, \tilde{k}_1 - \tilde{k}', \epsilon t, \epsilon^2 t)}{\omega(\tilde{k}_1) - \omega(\tilde{k}') - \omega(\tilde{k}_1 - \tilde{k}') + i\Delta} \right) \\ &+ (\tilde{k}_1 \leftrightarrow -\tilde{k}_1), \quad \Delta \rightarrow 0_+ . \end{aligned} \quad (3.3)$$

With the secular behavior removed in this manner,

$$\frac{\partial}{\partial t} {}_2G^{(2)}(\tilde{k}_1, t, \epsilon t, \epsilon^2 t) = 0, \quad (3.9)$$

as  $t \rightarrow \infty$ . Equation (3.9) may be written in a more familiar form in terms of the action density as

$$\begin{aligned} \frac{\partial}{\partial \epsilon t} n(\tilde{k}_1, \epsilon t, \epsilon^2 t) &= 4\pi \int d\tilde{k}' \frac{|\mu(\tilde{k}_1, \tilde{k}', \tilde{k}_1 - \tilde{k}')|^2}{\omega(\tilde{k}_1) \omega(\tilde{k}') \omega(\tilde{k}_1 - \tilde{k}')} \delta(\omega(\tilde{k}_1) - \omega(\tilde{k}') - \omega(\tilde{k}_1 - \tilde{k}')) \\ &\times \left( n(\tilde{k}', \epsilon t, \epsilon^2 t) n(\tilde{k}_1 - \tilde{k}', \epsilon t, \epsilon^2 t) - n(\tilde{k}_1, \epsilon t, \epsilon^2 t) n(\tilde{k}', \epsilon t, \epsilon^2 t) \right. \\ &\left. - n(\tilde{k}_1, \epsilon t, \epsilon^2 t) n(\tilde{k}_1 - \tilde{k}', \epsilon t, \epsilon^2 t) \right). \end{aligned} \quad (3.10)$$

where use has been made (3.4) and the symmetries (1.6) and (1.8). Equation (3.10) will be recognized as the conventional kinetic equation for resonant three-wave processes. If the appropriate mode labels are reinstated in the preceding analysis, then this kinetic equation may be written in the form of Eq. (1.10) with time variations on the  $\epsilon t$  time scale. It should be emphasized that once the hierarchy formalism is established, the derivation of (3.10) is a very simple two-step process. Equation (2.16) is solved for the leading-order three-wave correlations and the resulting expression substituted into (2.14), which directly yields the kinetic equation.

The concern here, however, is with situations where the resonant three-wave decay condition (1.11) cannot be satisfied, and hence it is necessary to carry out an analysis of the hierarchy equations for the correlation perturbations to higher order. In this case the action density does not vary on the  $\epsilon t$  time scale, i.e.  $(\partial/\partial \epsilon t)n(\underline{k}_1, \epsilon t, \epsilon^2 t) = 0$  from (3.10). Or in terms of the leading-order spectral energy density we have that

$$\frac{\partial}{\partial \epsilon t} {}_2G^{(1)}(\underline{k}_1, \epsilon t, \epsilon^2 t) = 0. \quad (3.11)$$

It should be noted from the definition (3.2) that  ${}_3\hat{G}^{(2)}$  is also independent of  $\epsilon t$  by virtue of (3.11).

(b) Next Order

In carrying out the analysis to next order, we now successively solve (2.13), (2.17) and (2.15), using the information regarding the two- and three-wave correlations given in Eqs. (3.1), (3.9) and (3.11). The kinetic equation for four-wave interactions will be manifest as a nonlinear integro-differential equation for  ${}_2G^{(1)}$  on the  $\epsilon^2 t$  time-scale.

First, the leading-order four-wave correlations,  ${}_4G^{(3)}$ , may be simply obtained from (2.13) as a functional of the leading-order spectral energy density and initial conditions using expression (3.1) for the three-wave correlations,  ${}_3G^{(2)}$ . This gives

$$\begin{aligned} \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3 + \underline{k}_4) {}_4G^{(3)}(\underline{k}_1, \underline{k}_2, \underline{k}_3, t, \epsilon t, \epsilon^2 t) &= \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3 + \underline{k}_4) {}_4G^{(3)}(\underline{k}_1, \underline{k}_2, \underline{k}_3, 0, \epsilon t, \epsilon^2 t) \\ &+ \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3 + \underline{k}_4) 2 \sum_{s'=2} \left\{ \left( \left[ K(\underline{k}_1, \underline{k}_1 + \underline{k}_2, -\underline{k}_2) {}_2G^{(1)}(\underline{k}_2, \epsilon^2 t) \right. \right. \right. \\ &\times \int_0^t dt'' \exp\{i(\omega(\underline{k}_1) + \omega(\underline{k}_2) - \omega(\underline{k}_1 + \underline{k}_2))t''\} [{}_3G^{(2)}(\underline{k}_3, \underline{k}_4, 0, \epsilon t, \epsilon^2 t) \end{aligned}$$

$$\begin{aligned}
& + {}_3\hat{G}^{(2)}(\underline{k}_3, \underline{k}_4, \epsilon^2 t) \int_0^{t''} dt' \exp\{i(\omega(\underline{k}_3) + \omega(\underline{k}_4) - \omega(\underline{k}_3 + \underline{k}_4))t'\} \Big] \\
& + \left[ 2 \leftrightarrow 3 \right] + \left[ 2 \leftrightarrow 4 \right] + (1 \leftrightarrow s') \Big\} . \tag{3.12}
\end{aligned}$$

Expression (3.12) for the four-wave correlations when used in Eq. (2.17) determines the  $t$  and  $\epsilon t$  behavior of  ${}_3G^{(3)}$  and  ${}_3G^{(2)}$  respectively. The resulting expression for  ${}_3G^{(3)}$  when substituted into (2.15) then gives further information regarding the time evolution of the two-wave correlations, in particular the kinetic behavior of  ${}_2G^{(1)}$  on the  $\epsilon^2 t$  time-scale.

In order to shorten algebraic presentation in the subsequent analysis, we point out at this time some simplifications which occur when the above mentioned series of substitutions is carried out. These may be verified by the reader. It will be noted that the initial-value terms  ${}_3G^{(2)}(\underline{k}_1, \underline{k}_2, 0, \epsilon t, \dots)$  and  ${}_4G^{(3)}(\underline{k}_1, \underline{k}_2, \underline{k}_3, 0, \epsilon t, \dots)$  have been retained in writing (3.1) and (3.12) as should always be done in any rigorous multiple-time perturbation formalism. However, when the contribution of the initial-value term  ${}_4G^{(3)}(\underline{k}_1, \underline{k}_2, \underline{k}_3, 0, \epsilon t, \dots)$  to the three-wave correlations,  ${}_3G^{(3)}(\underline{k}_1, \underline{k}_2, t, \epsilon t)$ , is calculated from (2.17) and the result substituted into (2.15), it is found that the corresponding contribution to the  $\underline{k}'$ -integration in (2.15) phase mixes to zero on the short time-scale, on the basis of previous arguments.<sup>4,13</sup> Consequently, for all intensive purposes of obtaining a kinetic equation for the spectral energy density, the initial-value term associated with the leading order four-wave correlations may be omitted from the analysis, and, in particular, set equal to zero. Similar conclusions also pertain in regard to the initial-value term,  ${}_3G^{(3)}(\underline{k}_1, \underline{k}_2, 0, \epsilon t, \dots)$  which occurs in integrating (2.17) on the short time-scale. Moreover,  ${}_3G^{(2)}(\underline{k}_1, \underline{k}_2, 0, \epsilon t, \dots)$  may be omitted from the analysis.

Keeping in mind that  $(\partial/\partial\epsilon t)_3 \hat{G}^{(2)} = 0$  in light of (3.11), it may be shown from (2.17) that the condition for removing secular behavior in the solution  ${}_3G^{(3)}(k_{\sim 1}, k_{\sim 2}, t, \epsilon t, \dots)$  is simply

$$\frac{\partial}{\partial\epsilon t} {}_3G^{(2)}(k_{\sim 1}, k_{\sim 2}, 0, \epsilon t, \dots) = 0. \quad (3.13)$$

That is, if  ${}_3G^{(2)}$  is initially zero, it remains so at least on the  $t$  and  $\epsilon t$  scales. We thus omit the initial-value terms associated with  ${}_4G^{(3)}$ ,  ${}_3G^{(3)}$  and  ${}_3G^{(2)}$  from the subsequent analysis. The phase mixing of these initial-value terms as regards the time evolution of the spectral energy density is a most encouraging feature, and in fact a necessary one for deriving a closed kinetic equation for the two-wave correlations.<sup>14</sup>

With the omission of initial-value terms, the four-wave correlations (3.12) become simply

$$\begin{aligned} & \delta(k_{\sim 1} + k_{\sim 2} + k_{\sim 3} + k_{\sim 4}) {}_4G^{(3)}(k_{\sim 1}, k_{\sim 2}, k_{\sim 3}, t, \epsilon t, \dots) \\ &= \delta(k_{\sim 1} + k_{\sim 2} + k_{\sim 3} + k_{\sim 4})^2 \sum_{s'=2} \left\{ \left( [K(k_{\sim 1}, k_{\sim 1} + k_{\sim 2}, -k_{\sim 2})] {}_2G^{(1)}(k_{\sim 2}, \epsilon^2 t) {}_3\hat{G}^{(2)}(k_{\sim 3}, k_{\sim 4}, \epsilon^2 t) \right. \right. \\ & \times \int_0^t dt'' \exp\{i(\omega(k_{\sim 1}) + \omega(k_{\sim 2}) - \omega(k_{\sim 1} + k_{\sim 2}))t''\} \int_0^{t''} dt' \exp\{i(\omega(k_{\sim 3}) + \omega(k_{\sim 4}) - \omega(k_{\sim 3} + k_{\sim 4}))t'\} \left. \right. \\ & \left. \left. + [2 \leftrightarrow 3] + [2 \leftrightarrow 4] \right) + (1 \leftrightarrow s') \right\}, \quad (3.14) \end{aligned}$$

which is a trilinear functional of the leading-order spectral energy density,  ${}_2G^{(1)}(k, \epsilon^2 t)$ , in view of definition (3.2) of  ${}_3\hat{G}^{(2)}$ . We now integrate (2.17) on the short time-scale to determine the three-wave correlations  ${}_3G^{(3)}$  (keep in mind  $(\partial/\partial\epsilon t) {}_3G^{(2)} = 0$  from previous discussion). This gives

$$\begin{aligned}
& \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) {}_3G^{(3)}(\underline{k}_1, \underline{k}_2, t, \epsilon t, \dots) \\
&= \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) \left\{ \left( \left[ K(\underline{k}_1, -\underline{k}_2, \underline{k}_1 + \underline{k}_2) [ {}_2G^{(1)}(\underline{k}_2, \epsilon^2 t) {}_2G^{(2)}(\underline{k}_3, \epsilon t, \dots) \right. \right. \right. \\
&+ \left. \left. \left. {}_2G^{(2)}(\underline{k}_2, \epsilon t, \epsilon^2 t) {}_2G^{(1)}(\underline{k}_3, \epsilon^2 t) \right] \frac{\exp\{i(\omega(\underline{k}_1) + \omega(\underline{k}_2) - \omega(\underline{k}_1 + \underline{k}_2))t\} - 1}{i(\omega(\underline{k}_1) + \omega(\underline{k}_2) - \omega(\underline{k}_1 + \underline{k}_2))} \right) \right. \\
&\quad \left. + \left[ 2 \leftrightarrow 3 \right] + (2 \leftrightarrow 1) + (3 \leftrightarrow 1) \right\} \\
&+ \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) \left\{ \left( \int_{\underline{k}''} d\underline{k}'' K(\underline{k}_1, \underline{k}'', \underline{k}_1 - \underline{k}'') \int_0^t dt {}_4G^{(3)}(\underline{k}'', \underline{k}_1 - \underline{k}'', \underline{k}_2, t, \epsilon t, \dots) \right. \right. \\
&\times \left. \left. \exp\{i(\omega(\underline{k}_1) - \omega(\underline{k}'') - \omega(\underline{k}_1 - \underline{k}''))t\} + (2 \leftrightarrow 1) + (3 \leftrightarrow 1) \right\} \tag{3.15}
\end{aligned}$$

where use has been made of (3.9), and  ${}_4G^{(3)}$  appearing in the last term of (3.15) is given by (3.14) with appropriate Fourier arguments. Expression (3.15) may then be substituted into (2.15) to give further information on the time evolution of the spectral energy density. It will be recognized that the  ${}_2G^{(1)} {}_2G^{(2)}$  driving terms on the right-hand side of Expression (3.15) for  ${}_3G^{(3)}$  represent a higher-order version of the ternary correlation effects calculated in part (a) of Sec. III. Consequently, for large  $t$  these terms give zero contribution to the  $\underline{k}'$ -integration in (2.15), in view of our assumption that the resonant three-wave decay condition cannot be satisfied. Only the contribution of the four-wave correlations in (3.15) remain, and Eq. (2.15) becomes

$$\begin{aligned}
& \frac{\partial}{\partial t} {}_2G^{(3)}(\underline{k}_1, t, \epsilon t, \epsilon^2 t) + \frac{\partial}{\partial \epsilon t} {}_2G^{(2)}(\underline{k}_1, \epsilon t, \epsilon^2 t) + \frac{\partial}{\partial \epsilon^2 t} {}_2G^{(1)}(\underline{k}_1, \epsilon^2 t) \\
&= \left( \iint d\underline{k}' d\underline{k}'' K(\underline{k}_1, \underline{k}', \underline{k}_1 - \underline{k}'') \left\{ \left[ K(\underline{k}', \underline{k}'', \underline{k}' - \underline{k}'') \exp\{i(\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}''))t\} \right. \right. \right. \\
&\times \left. \left. \left. \int_0^t dt' {}_4G^{(3)}(\underline{k}'', \underline{k}_1 - \underline{k}'', -\underline{k}_1, t', \epsilon^2 t) \exp\{i(\omega(\underline{k}') - \omega(\underline{k}'') - \omega(\underline{k}' - \underline{k}''))t'\} \right] \right) \right)
\end{aligned}$$

$$\begin{aligned}
& + \left[ \tilde{k}' \leftrightarrow \tilde{k}_1 - \tilde{k}' \right] + \left[ \tilde{k}_1 \leftrightarrow -\tilde{k}' \right] \Big\} + (\tilde{k}_1 \leftrightarrow -\tilde{k}_1) \\
& \equiv \text{St}\{t, {}_2G^{(1)}(\epsilon^2 t), {}_2G^{(1)}(\epsilon^2 t), {}_2G^{(1)}(\epsilon^2 t)\} , \tag{3.16}
\end{aligned}$$

which is the final equation of the perturbation analysis to be considered here. Since  ${}_4G^{(3)}$  is a trilinear functional of  ${}_2G^{(1)}(\tilde{k}_1, \epsilon^2 t)$ , it is clear that the right-hand side of (3.16) varies on the  $t$  and  $\epsilon^2 t$  time-scales, but not on the  $\epsilon t$  scale. We use the abbreviated notation  $\text{St}\{t, {}_2G^{(1)}(\epsilon^2 t), {}_2G^{(1)}(\epsilon^2 t), {}_2G^{(1)}(\epsilon^2 t)\}$  for the right-hand side of (3.16) to emphasize these features.

Evidently, when Eq. (3.16) is integrated on the  $t$  time-scale, both  $(\partial/\partial \epsilon t)$   ${}_2G^{(2)}$  and  $(\partial/\partial \epsilon^2 t)$   ${}_2G^{(1)}$  lead to secularities in  ${}_2G^{(3)}$  proportional to  $t$ . Moreover, that portion of the stoss term in (3.16) which tends to a steady asymptotic value as  $t \rightarrow \infty$ ,<sup>15</sup> will also lead to secular behavior (proportional to  $t$ ) in  ${}_2G^{(3)}$ . Using the freedom inherent in the multiple-time formalism to yield a uniformly valid solution, we thus have the condition that

$$\begin{aligned}
& \frac{\partial}{\partial \epsilon t} {}_2G^{(2)}(\tilde{k}_1, \epsilon t, \epsilon^2 t) + \frac{\partial}{\partial \epsilon^2 t} {}_2G^{(1)}(\tilde{k}_1, \epsilon^2 t) \\
& = \lim_{t \rightarrow \infty} \text{St}\{t, {}_2G^{(1)}(\epsilon^2 t), {}_2G^{(1)}(\epsilon^2 t), {}_2G^{(1)}(\epsilon^2 t)\} \tag{3.17}
\end{aligned}$$

where the limit in (3.17) is with respect to the short time-scale  $t$ . Moreover,

$$\frac{\partial}{\partial t} {}_2G^{(3)}(\tilde{k}_1, t, \epsilon t, \epsilon^2 t) = 0 , \tag{3.18}$$

as  $t \rightarrow \infty$ . In addition, since  ${}_2G^{(1)}$  and the right-hand side of (3.17) are independent of  $\epsilon t$ , these terms will give secular contributions (proportional to  $\epsilon t$ ) to  ${}_2G^{(2)}$  when Eq. (3.17) is integrated on the  $\epsilon t$  time-scale. The condition for removing this secular behavior is simply

$$\frac{\partial}{\partial \epsilon^2 t} {}_2G^{(1)}(\underline{k}_1, \epsilon^2 t) = \lim_{t \rightarrow \infty} \text{St}\{t, {}_2G^{(1)}(\epsilon^2 t), {}_2G^{(1)}(\epsilon^2 t), {}_2G^{(1)}(\epsilon^2 t)\} \quad , \quad (3.19)$$

in which case  $(\partial/\partial \epsilon t) {}_2G^{(2)}(\underline{k}_1, \epsilon t, \epsilon^2 t) = 0$ . Equation (3.19), which forms a closed nonlinear integro-differential equation for  ${}_2G^{(1)}$  on the  $\epsilon^2 t$  time-scale, thus describes the kinetic behavior of the leading-order spectral energy density. It should be noted that the analysis of this section has indicated that the  $\epsilon t$  time-scale is irrelevant in describing the evolution of ensemble correlations when the resonant three-wave decay condition cannot be satisfied. This completes the formal analysis of the hierarchy (2.13)-(2.18) for the correlation perturbations. In the next section, the  $t \rightarrow \infty$  limit of the stoss term given by the right-hand side of Eq. (3.16) will be carried out, and the kinetic equation (3.19) reduced to a useful form.

#### IV. THE KINETIC EQUATION FOR FOUR-WAVE INTERACTIONS

##### (a) Reduction of Eq. (3.19)

It is convenient to rewrite Eq. (3.19) in terms of the action density, which in this case varies on the  $\epsilon^2 t$  scale through

$$n(\underline{k}_1, \epsilon^2 t) = {}_2G^{(1)}(\underline{k}_1, \epsilon^2 t) / \omega(\underline{k}_1) \quad . \quad (4.1)$$

Writing out explicitly the stoss term from (3.16), Eq. (3.19) then becomes

$$\begin{aligned} \frac{\partial}{\partial \epsilon^2 t} n(\underline{k}_1, \epsilon^2 t) = \lim_{t \rightarrow \infty} \frac{1}{\omega(\underline{k}_1)} & \left[ \left( \iint d\underline{k}' d\underline{k}'' \frac{\mu(\underline{k}_1, \underline{k}', \underline{k}_1 - \underline{k}')}{\omega(\underline{k}') \omega(\underline{k}_1 - \underline{k}')} \left\{ \frac{\mu(\underline{k}', \underline{k}'', \underline{k}' - \underline{k}'')}{\omega(\underline{k}'') \omega(\underline{k}' - \underline{k}'')} \right. \right. \right. \\ & \times \exp\{i(\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}'))t\} \int_0^t dt'' \exp\{i(\omega(\underline{k}') - \omega(\underline{k}'') - \omega(\underline{k}' - \underline{k}''))t''\} \\ & \left. \left. \left. \times {}_4G^{(3)}(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, t'', \epsilon^2 t) \right\} + [\underline{k}' \leftrightarrow \underline{k}_1 - \underline{k}'] + [\underline{k}_1 \leftrightarrow -\underline{k}'] \right) + (\underline{k}_1 \leftrightarrow -\underline{k}_1) \right] \quad , \end{aligned} \quad (4.2)$$



where the response  $\mu$  defined by (1.7) has been introduced in (4.2). The leading-order four-wave correlation  ${}_4G^{(3)}$  appearing in (4.2) is given by Expression (3.14) with appropriate Fourier arguments. Making use of the definition (3.4),  ${}_4G^{(3)}$  may be written explicitly in terms of the action density as

$$\begin{aligned}
& \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3 + \underline{k}_4) {}_4G^{(3)}(\underline{k}_1, \underline{k}_2, \underline{k}_3, t'', \epsilon^2 t) \\
&= {}_4 \sum_{s'=2}^4 \left[ \left( \int_0^{t''} dt'' \exp\{i(\omega(\underline{k}_1) + \omega(\underline{k}_2) - \omega(\underline{k}_1 + \underline{k}_2))t''\} \right) \right. \\
&\times \int_0^{t''} dt' \exp\{i(\omega(\underline{k}_3) + \omega(\underline{k}_4) - \omega(\underline{k}_3 + \underline{k}_4))t'\} \\
&\times \frac{1}{\omega(\underline{k}_3 + \underline{k}_4)} \mu(\underline{k}_1, \underline{k}_1 + \underline{k}_2, -\underline{k}_2) \mu(\underline{k}_3, \underline{k}_3 + \underline{k}_4, -\underline{k}_4) n(\underline{k}_2, \epsilon^2 t) \cdot \{n(\underline{k}_3, \epsilon^2 t) n(\underline{k}_3 + \underline{k}_4, \epsilon^2 t) \\
&+ n(\underline{k}_4, \epsilon^2 t) n(\underline{k}_3 + \underline{k}_4, \epsilon^2 t) - n(\underline{k}_3, \epsilon^2 t) n(\underline{k}_4, \epsilon^2 t)\} + [2 \leftrightarrow 3] + [2 \leftrightarrow 4] \\
&\left. + (1 \leftrightarrow s') \right] \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3 + \underline{k}_4). \tag{4.3}
\end{aligned}$$

It is clear from (4.2) and (4.3) that there are three successive time integrals over oscillations to consider on the short time-scale before taking the  $t \rightarrow \infty$  limit. As indicated in Appendix A, several of the associated terms phase mix to zero in the  $\underline{k}'$  and  $\underline{k}''$  integrations. However, a steady portion also remains as  $t \rightarrow \infty$ . In particular, it is shown that insofar as integrations over  $\underline{k}'$  and  $\underline{k}''$  are concerned,

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \exp\{i(\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}'))t\} \int_0^t dt'' \exp\{i(\omega(\underline{k}') - \omega(\underline{k}'') - \omega(\underline{k}' - \underline{k}''))t''\} \\
&\times {}_4G^{(3)}(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}', t'', \epsilon^2 t) \equiv \frac{2}{\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}')}
\end{aligned}$$

$$\times \frac{i}{\omega(\underline{k}_1) - \omega(\underline{k}'') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'') + i\Delta} {}_4\hat{G}^{(3)}(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, \underline{k}' - \underline{k}'', \epsilon^2 t),$$

$$\Delta \rightarrow 0_+, \quad (4.4)$$

where  ${}_4\hat{G}^{(3)}(\underline{k}_1, \underline{k}_2, \underline{k}_3, \underline{k}_4, \epsilon^2 t)$  is the trilinear functional of action density defined by

$${}_4\hat{G}^{(3)}(\underline{k}_1, \underline{k}_2, \underline{k}_3, \underline{k}_4, \epsilon^2 t)$$

$$= \left( \left[ \frac{2/\omega(\underline{k}_3 + \underline{k}_4)}{\omega(\underline{k}_3) + \omega(\underline{k}_4) - \omega(\underline{k}_3 + \underline{k}_4)} \mu(\underline{k}_1, \underline{k}_1 + \underline{k}_2, -\underline{k}_2) \mu(\underline{k}_3, \underline{k}_3 + \underline{k}_4, -\underline{k}_4) n(\underline{k}_2, \epsilon^2 t) \right. \right.$$

$$\times \{ n(\underline{k}_3, \epsilon^2 t) n(\underline{k}_3 + \underline{k}_4, \epsilon^2 t) + n(\underline{k}_4, \epsilon^2 t) n(\underline{k}_3 + \underline{k}_4, \epsilon^2 t) - n(\underline{k}_3, \epsilon^2 t) n(\underline{k}_4, \epsilon^2 t) \} \left. \right]$$

$$+ [2 \leftrightarrow 3] + [2 \leftrightarrow 4] \Big) + (1 \leftrightarrow 2) + (1 \leftrightarrow 3) + (1 \leftrightarrow 4), \quad (4.5)$$

with  $\underline{k}_1 + \underline{k}_2 + \underline{k}_3 + \underline{k}_4 = 0$ . In light of (4.4), the kinetic equation (4.2) is given by

$$\frac{\partial}{\partial \epsilon^2 t} n(\underline{k}_1, \epsilon^2 t) = 2\text{Re} \left( \iint d\underline{k}' d\underline{k}'' \left\{ \left[ \frac{2/\omega(\underline{k}')}{\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}')} \right. \right. \right.$$

$$\times \frac{\mu(\underline{k}_1, \underline{k}', \underline{k}_1 - \underline{k}') \mu(\underline{k}', \underline{k}'', \underline{k}' - \underline{k}'')}{\omega(\underline{k}_1) \omega(\underline{k}_1 - \underline{k}') \omega(\underline{k}'') \omega(\underline{k}' - \underline{k}'')} \left. \right]$$

$$\times \frac{i {}_4\hat{G}^{(3)}(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, \underline{k}' - \underline{k}'', \epsilon^2 t)}{\omega(\underline{k}_1) - \omega(\underline{k}'') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'') + i\Delta} \Big] + [\underline{k}' \leftrightarrow \underline{k}_1 - \underline{k}'] + [\underline{k}' \leftrightarrow -\underline{k}_1] \Big\} \Big)$$

$$\Delta \rightarrow 0_+, \quad (4.6)$$

where the factor  $[\omega(\underline{k}_1) \omega(\underline{k}') \omega(\underline{k}_1 - \underline{k}')]^{-1} \mu(\underline{k}_1, \underline{k}', \underline{k}_1 - \underline{k}')$  has been absorbed into the square bracket of (4.6).<sup>16</sup> Twice the real part has been taken in Eq. (4.6) since the  $(\underline{k}_1 \leftrightarrow -\underline{k}_1)$  interchange in (4.2) just adds the complex conjugate. It should also be noted that the various factors of the form

$[\omega(\underline{k}_1) + \omega(\underline{k}_2) - \omega(\underline{k}_1 + \underline{k}_2)]^{-1}$  in (4.5) and (4.6), are non-divergent since it has been assumed that the three-wave resonance condition is not satisfied.

Equation (4.6) appended by the definition (4.5) is thus the closed kinetic equation for the action density. We see that four-wave interactions are manifest through the appearance of

$$\frac{1}{\omega(\underline{k}_1) - \omega(\underline{k}''') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'') + i\Delta} \Big|_{\Delta \rightarrow 0_+}$$

$$\equiv \frac{P}{\omega(\underline{k}_1) - \omega(\underline{k}''') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'')} - i\pi\delta(\omega(\underline{k}_1) - \omega(\underline{k}''') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'')) , \quad (4.7)$$

in Eq. (4.6). The  $\delta$ -function contribution from (4.7) corresponds to interactions of the resonant four-wave type discussed in the introduction. The principal-value term, however, corresponds to non-resonant or adiabatic four-wave processes. Both effects will in general be present. However, in a large number of situations of interest, the effects of non-resonant four-wave interactions, i.e. the principal-value contributions in (4.6), are small. This occurs, for example, in circumstances where the response  $\mu$  is pure real corresponding to the case of a lossless medium.<sup>17</sup> In this situation we see from (4.5) that  ${}_4\hat{G}^{(3)}$  is real. Moreover, the remaining integrand in (4.6) is also real except for the  $iP[\omega(\underline{k}_1) - \omega(\underline{k}''') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'')]^{-1}$  term, which gives zero contribution upon taking the real part in (4.6). Even in the general case where  $\mu$  is complex, it is clear that if  $\text{Im}\mu$  is sufficiently small, the principal value contribution to the kinetic equation (4.6) will be negligible in comparison to the resonant four-wave contribution from  $\delta(\omega(\underline{k}_1) - \omega(\underline{k}''') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}''))$ .

In many case, we shall retain only the resonant four-wave contribution for present considerations. After a modest amount of algebra which is

summarized in Appendix B the kinetic equation (4.6) may then be written in the convenient form

$$\begin{aligned} \frac{\partial}{\partial \epsilon^2 t} n(\tilde{k}_1, \epsilon^2 t) &= \iiint d\tilde{k}_2 d\tilde{k}_3 d\tilde{k}_4 \frac{|D(-\tilde{k}_1, -\tilde{k}_2, \tilde{k}_3, \tilde{k}_4)|^2}{\omega(\tilde{k}_1)\omega(\tilde{k}_2)\omega(\tilde{k}_3)\omega(\tilde{k}_4)} \\ &\times \delta(\omega(\tilde{k}_1) + \omega(\tilde{k}_2) - \omega(\tilde{k}_3) - \omega(\tilde{k}_4)) \delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k}_3 - \tilde{k}_4) \\ &\times \left( n(\tilde{k}_2, \epsilon^2 t) n(\tilde{k}_3, \epsilon^2 t) n(\tilde{k}_4, \epsilon^2 t) + n(\tilde{k}_1, \epsilon^2 t) n(\tilde{k}_3, \epsilon^2 t) n(\tilde{k}_4, \epsilon^2 t) \right. \\ &\left. - n(\tilde{k}_1, \epsilon^2 t) n(\tilde{k}_2, \epsilon^2 t) n(\tilde{k}_4, \epsilon^2 t) - n(\tilde{k}_1, \epsilon^2 t) n(\tilde{k}_2, \epsilon^2 t) n(\tilde{k}_3, \epsilon^2 t) \right), \end{aligned} \quad (4.8)$$

where D is defined in terms of the response  $\mu$  by

$$\begin{aligned} D(-\tilde{k}_1, -\tilde{k}_2, \tilde{k}_3, \tilde{k}_4) &= \sqrt{\frac{i\pi}{3}} \left\{ \left( \frac{2/\omega(\tilde{k}_1 + \tilde{k}_2)}{\omega(\tilde{k}_1) + \omega(\tilde{k}_2) - \omega(\tilde{k}_1 + \tilde{k}_2)} \mu(-\tilde{k}_1, -\tilde{k}_1 - \tilde{k}_2, \tilde{k}_2) \right. \right. \\ &\left. \left. \mu(\tilde{k}_3, \tilde{k}_3 + \tilde{k}_4, -\tilde{k}_4) \right) + (\tilde{k}_2 \leftrightarrow -\tilde{k}_3) + (\tilde{k}_2 \leftrightarrow -\tilde{k}_4) \right\} \end{aligned} \quad (4.9)$$

Equation (4.8) is analogous to (1.12) in the absence of mode labels, and with  $t$  replaced by  $\epsilon^2 t$ . It should be noted that  $D(-\tilde{k}_1, -\tilde{k}_2, \tilde{k}_3, \tilde{k}_4)$  is symmetric under interchange of any two of its Fourier arguments when the resonance condition

$$\tilde{k}_1 + \tilde{k}_2 = \tilde{k}_3 + \tilde{k}_4$$

$$\omega(\tilde{k}_1) + \omega(\tilde{k}_2) = \omega(\tilde{k}_3) + \omega(\tilde{k}_4)$$

is satisfied. Moreover,

$$D(-\tilde{k}_1, -\tilde{k}_2, \tilde{k}_3, \tilde{k}_4) = D^*(\tilde{k}_1, \tilde{k}_2, -\tilde{k}_3, -\tilde{k}_4). \quad (4.10)$$

(b) Many Mode Extension

The considerations of Secs. II and III can be extended in a straight forward manner to include mode labels. We now have that

$${}_2G_{\alpha\beta}(\tilde{k}_1, t) \delta(\tilde{k}_1 + \tilde{k}_2) \equiv \langle A_{\alpha}(\tilde{k}_1, t) A_{\beta}(\tilde{k}_2, t) \rangle, \quad (4.11)$$

$${}_3G_{\alpha\beta\gamma}(\tilde{k}_1, \tilde{k}_2, t) \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3) \equiv \langle A_{\alpha}(\tilde{k}_1, t) A_{\beta}(\tilde{k}_2, t) A_{\gamma}(\tilde{k}_3, t) \rangle, \quad (4.12)$$

⋮

etc. The resulting hierarchy for  ${}_2G_{\alpha\beta}$ ,  ${}_3G_{\alpha\beta\gamma}$ ,  ${}_4G_{\alpha\beta\gamma\delta}$ , ... may then be expanded in a multiple-time perturbation scheme, and the analysis carried out in a completely analogous fashion. As was the case in the derivation of the kinetic equation for resonant three-wave interactions using the hierarchy formalism,<sup>4</sup> it is found that if the two-wave correlations between unlike modes are initially zero, they remain so in the time-scales of interest. That is to say, if

$${}_2G_{\alpha\beta} = 0, \quad \alpha \neq \beta \quad (4.13)$$

initially, then (4.13) remains true at subsequent times, at least within the range of validity of the third-order perturbation analysis. We assume (4.13) to be the case. As before, in situations where the resonance three-wave decay condition (1.11) cannot be satisfied, it is found that the leading-order spectral energy density,  ${}_2G_{\alpha\alpha}^{(1)}$ , does not vary on the  $\epsilon t$  time-scale, but does exhibit kinetic behavior on the  $\epsilon^2 t$  scale through the effects of four-wave correlations. As stated earlier, the analysis proceeds in an analogous manner, and we leave it to the reader to reinstate the appropriate mode labels and mode summations in the previous sections. Introducing the action density associated

with the  $\alpha$ 'th mode, which varies on the  $\epsilon^2 t$  scale, i.e.,

$$n_{\alpha}(k_{\alpha}, \epsilon^2 t) = 2^G G_{\alpha\alpha}(k_{\alpha}, \epsilon^2 t) / \omega_{\alpha}(k_{\alpha}), \quad (4.14)$$

the kinetic equation is given by

$$\begin{aligned} \frac{\partial}{\partial \epsilon^2 t} n_{\alpha}(k_{\alpha}, \epsilon^2 t) = & \sum_{\beta, \gamma, \delta} \iiint_{\tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4} dk_2 dk_3 dk_4 \frac{|D^{\alpha\beta\gamma\delta}(-k_1, -k_2, k_3, k_4)|^2}{\omega_{\alpha}(k_1) \omega_{\beta}(k_2) \omega_{\gamma}(k_3) \omega_{\delta}(k_4)} \\ & \times \delta(\omega_{\alpha}(k_1) + \omega_{\beta}(k_2) - \omega_{\gamma}(k_3) - \omega_{\delta}(k_4)) \delta(k_1 + k_2 - k_3 - k_4) \\ & \times \left( n_{\beta}(k_2, \epsilon^2 t) n_{\gamma}(k_3, \epsilon^2 t) n_{\delta}(k_4, \epsilon^2 t) + n_{\alpha}(k_1, \epsilon^2 t) n_{\gamma}(k_3, \epsilon^2 t) n_{\delta}(k_4, \epsilon^2 t) \right. \\ & \left. - n_{\alpha}(k_1, \epsilon^2 t) n_{\beta}(k_2, \epsilon^2 t) n_{\delta}(k_4, \epsilon^2 t) - n_{\alpha}(k_1, \epsilon^2 t) n_{\beta}(k_2, \epsilon^2 t) n_{\gamma}(k_3, \epsilon^2 t) \right), \end{aligned} \quad (4.15)$$

where only the resonant four-wave contribution has been retained in (4.15)

(i.e. the principal-value, non-resonant four-wave contribution has been omitted as before). Equation (4.15) is the obvious generalization of (4.8). The coupling coefficients  $D^{\alpha\beta\gamma\delta}$ , however, involve a summation over virtual states and are given by

$$\begin{aligned} D^{\alpha\beta\gamma\delta}(-k_1, -k_2, k_3, k_4) \equiv & \sqrt{\frac{4\pi}{3}} \sum_{\eta} \left\{ \left( \frac{2/\omega_{\eta}(k_1+k_2)}{\omega_{\alpha}(k_1) + \omega_{\beta}(k_2) - \omega_{\eta}(k_1+k_2)} \right) \right. \\ & \times \mu^{\gamma\eta\delta}(k_3, k_3+k_4, -k_4) \mu^{\alpha\eta\beta}(-k_1, -k_1-k_2, +k_2) \left. + \left( \begin{matrix} k_2 \leftrightarrow -k_3 \\ \beta \leftrightarrow \gamma \end{matrix} \right) + \left( \begin{matrix} k_2 \leftrightarrow -k_4 \\ \beta \leftrightarrow \delta \end{matrix} \right) \right\}. \end{aligned} \quad (4.16)$$

The states  $\eta$  are just the intermediate virtual states referred to in relation to Fig. 1. In view of the symmetries (1.4) and (1.6) we see from (4.16) that

$$D^{\alpha\beta\gamma\delta}(k_1, k_2, -k_3, -k_4) = D^{*\alpha\beta\gamma\delta}(-k_1, -k_2, k_3, k_4). \quad (4.17)$$

Moreover, from (4.16) and (1.8),  $D^{\alpha\beta\gamma\delta}(-\tilde{k}_1, -\tilde{k}_2, \tilde{k}_3, \tilde{k}_4)$  is symmetric under the interchange of any two of  $(\alpha, -\tilde{k}_1)$ ,  $(\beta, -\tilde{k}_2)$ ,  $(\gamma, \tilde{k}_3)$ , and  $(\delta, \tilde{k}_4)$ , when the four-wave resonance condition (1.13) is satisfied, i.e.

$$\begin{aligned} D^{\alpha\beta\gamma\delta}(-\tilde{k}_1, -\tilde{k}_2, \tilde{k}_3, \tilde{k}_4) &= D^{\beta\alpha\gamma\delta}(-\tilde{k}_2, -\tilde{k}_1, \tilde{k}_3, \tilde{k}_4) \\ &= D^{\gamma\beta\alpha\delta}(\tilde{k}_3, -\tilde{k}_2, -\tilde{k}_1, \tilde{k}_4) = D^{\delta\beta\gamma\alpha}(\tilde{k}_4, -\tilde{k}_2, \tilde{k}_3, -\tilde{k}_1) = \dots, \end{aligned} \quad (4.18)$$

when

$$\begin{aligned} \omega_\alpha(\tilde{k}_1) + \omega_\beta(\tilde{k}_2) &= \omega_\gamma(\tilde{k}_3) + \omega_\delta(\tilde{k}_4), \\ \tilde{k}_1 + \tilde{k}_2 &= \tilde{k}_3 + \tilde{k}_4. \end{aligned}$$

## V. GENERAL PROPERTIES OF THE KINETIC EQUATION

We now discuss various general properties of the kinetic equation (4.15) including the preservation of the non-negative nature of the spectral energy density, energy and momentum conservation relations, and the appropriate law of increase of entropy.

### (a) Preservation of the Sign of the Spectral Energy Density

The spectral energy density  ${}_2G_{\alpha\alpha}$  associated with the  $\alpha$ 'th mode, in addition to being real and an even function of its Fourier argument, is manifestly non-negative as may be demonstrated from the definition (4.11) with  $\alpha = \beta$ . Consequently, one of the minimum demands we can make of the kinetic equation (4.15), in order that the kinetic theory be acceptable, is that this property be preserved during the course of time. That is to say, if

$$n_\alpha(\tilde{k})\omega_\alpha(\tilde{k}) = {}_2G_{\alpha\alpha}^{(1)}(\tilde{k}) \geq 0, \text{ each } \alpha, \quad (5.1)$$

initially, then  $n_{\alpha} \omega_{\alpha}$  does not turn negative at subsequent times. We check that this is the case by the following reductio ad absurdum argument.

Take relation (5.1) to be true initially. Assume that the first correlation to turn negative is associated with the mode  $\alpha$ , and that this occurs for  $k_1 = k_0$ . It follows that at the instant  $2G_{\alpha\alpha}^{(1)}$  is passing through zero

$$\begin{aligned} 2G_{\alpha\alpha}^{(1)}(k_0) &= 0, \\ 2G_{\alpha\alpha}^{(1)}(k_1 \neq k_0) &\geq 0, \end{aligned} \quad (5.2)$$

and  $2G_{\beta\beta}^{(1)}(k_1) \geq 0$   $\beta$  distinct from  $\alpha$ .

Also, at the instant  $2G_{\alpha\alpha}^{(1)}$  is passing through zero, (4.15) gives

$$\begin{aligned} \frac{\partial}{\partial \epsilon^2 t} 2G_{\alpha\alpha}^{(1)}(k_0, \epsilon^2 t) &= \omega_{\alpha}(k_0) \frac{\partial}{\partial \epsilon^2 t} n_{\alpha}(k_0, \epsilon^2 t) \\ &= \sum_{\beta, \gamma, \delta} \iiint_{\tilde{k}_2, \tilde{k}_3, \tilde{k}_4} dk_2 dk_3 dk_4 \frac{|D^{\alpha\beta\gamma\delta}(-k_0, -k_2, k_3, k_4)|^2}{(\omega_{\beta}(k_2) \omega_{\gamma}(k_3) \omega_{\delta}(k_4))^2} \\ &\times \delta(\omega_{\alpha}(k_0) + \omega_{\beta}(k_2) - \omega_{\gamma}(k_3) - \omega_{\delta}(k_4)) \delta(k_0 + k_2 - k_3 - k_4) \\ &\times 2G_{\beta\beta}^{(1)}(k_2, \epsilon^2 t) 2G_{\gamma\gamma}^{(1)}(k_3, \epsilon^2 t) 2G_{\delta\delta}^{(1)}(k_4, \epsilon^2 t) \\ &\geq 0, \end{aligned} \quad (5.3)$$

where use has been made of Relations (5.2). This contradicts our hypothesis of  $2G_{\alpha\alpha}^{(1)}$  turning negative. This argument may of course be extended to show that none of the correlations,  $2G_{\alpha'\alpha'}^{(1)}$ , turn negative.

#### (b) Conservation Laws

There are simple conservation laws associated with the kinetic equation (4.15). We note that resonant four-wave processes conserve energy and momentum



on the microscopic scale according to (1.13). Furthermore, the total energy density,  $\tilde{C}_0$ , and momentum density,  $\tilde{P}_0$ , are conserved by Eq. (4.15), where

$$\tilde{C}_0 \equiv \sum_{\alpha} \int d\tilde{k}_1 n_{\alpha}(\tilde{k}_1, \epsilon^2 t) \omega_{\alpha}(\tilde{k}_1) , \quad (5.4)$$

and

$$\tilde{P}_0 \equiv \sum_{\alpha} \int d\tilde{k}_1 n_{\alpha}(\tilde{k}_1, \epsilon^2 t) \tilde{k}_1 . \quad (5.5)$$

In particular,

$$\begin{aligned} \frac{\partial}{\partial \epsilon^2 t} \tilde{C}_0 &= \sum_{\alpha} \int d\tilde{k}_1 \frac{\partial}{\partial \epsilon^2 t} n_{\alpha}(\tilde{k}_1, \epsilon^2 t) \omega_{\alpha}(\tilde{k}_1) \\ &= \frac{1}{4} \sum_{\substack{\alpha, \beta \\ \gamma, \delta}} \iiint d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 d\tilde{k}_4 (\omega_{\alpha}(\tilde{k}_1) + \omega_{\beta}(\tilde{k}_2) - \omega_{\gamma}(\tilde{k}_3) - \omega_{\delta}(\tilde{k}_4)) \\ &\times \frac{|D^{\alpha\beta\gamma\delta}(-\tilde{k}_1, -\tilde{k}_2, \tilde{k}_3, \tilde{k}_4)|^2}{\omega_{\alpha}(\tilde{k}_1) \omega_{\beta}(\tilde{k}_2) \omega_{\gamma}(\tilde{k}_3) \omega_{\delta}(\tilde{k}_4)} \delta(\omega_{\alpha}(\tilde{k}_1) + \omega_{\beta}(\tilde{k}_2) - \omega_{\gamma}(\tilde{k}_3) - \omega_{\delta}(\tilde{k}_4)) \delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k}_3 - \tilde{k}_4) \\ &\times \left( n_{\beta}(\tilde{k}_2, \epsilon^2 t) n_{\gamma}(\tilde{k}_3, \epsilon^2 t) n_{\delta}(\tilde{k}_4, \epsilon^2 t) + n_{\alpha}(\tilde{k}_1, \epsilon^2 t) n_{\gamma}(\tilde{k}_3, \epsilon^2 t) n_{\delta}(\tilde{k}_4, \epsilon^2 t) \right. \\ &\left. - n_{\alpha}(\tilde{k}_1, \epsilon^2 t) n_{\beta}(\tilde{k}_2, \epsilon^2 t) n_{\delta}(\tilde{k}_4, \epsilon^2 t) - n_{\alpha}(\tilde{k}_1, \epsilon^2 t) n_{\beta}(\tilde{k}_2, \epsilon^2 t) n_{\gamma}(\tilde{k}_3, \epsilon^2 t) \right) \\ &\equiv 0 , \end{aligned} \quad (5.6)$$

where use has been made of the symmetries (4.18), (1.4), and the oddness of the action density as a function of its Fourier argument,  $n_{\alpha}(-\tilde{k}, \epsilon^2 t) = -n_{\alpha}(\tilde{k}, \epsilon^2 t)$ . In a similar manner it may be shown that  $\tilde{P}_0$  is conserved, i.e.,

$$\frac{\partial}{\partial \epsilon^2 t} \tilde{P}_0 = \sum_{\alpha} \int d\tilde{k}_1 \frac{\partial}{\partial \epsilon^2 t} n_{\alpha}(\tilde{k}_1, \epsilon^2 t) \tilde{k}_1 \equiv 0 .$$

(c) Non-Equilibrium Entropy

As is the case for the kinetic equation for resonant three-wave processes, the appropriate non-equilibrium entropy density in relation to (4.15) is given by

$$S \equiv \sum_{\alpha} \int d\tilde{k}_1 \ln |n_{\alpha}(\tilde{k}_1, \epsilon^2 t)|. \quad (5.7)$$

Making use of the oddness of  $\omega_{\alpha}$  and  $n_{\alpha}$  as functions of their Fourier arguments, together with the symmetries (4.18), it follows that

$$\begin{aligned} \frac{\partial}{\partial \epsilon^2 t} S &= \sum_{\alpha} \int d\tilde{k}_1 \frac{1}{n_{\alpha}(\tilde{k}_1, \epsilon^2 t)} \frac{\partial}{\partial \epsilon^2 t} n_{\alpha}(\tilde{k}_1, \epsilon^2 t) \\ &= \frac{1}{4} \sum_{\substack{\alpha, \beta \\ \gamma, \delta}} \iiint d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 d\tilde{k}_4 \frac{|D^{\alpha\beta\gamma\delta}(-\tilde{k}_1, -\tilde{k}_2, \tilde{k}_3, \tilde{k}_4)|^2}{(\omega_{\alpha}(\tilde{k}_1)\omega_{\beta}(\tilde{k}_2)\omega_{\gamma}(\tilde{k}_3)\omega_{\delta}(\tilde{k}_4))^2} \\ &\quad \times \delta(\omega_{\alpha}(\tilde{k}_1) + \omega_{\beta}(\tilde{k}_2) - \omega_{\gamma}(\tilde{k}_3) - \omega_{\delta}(\tilde{k}_4)) \delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k}_3 - \tilde{k}_4) \\ &\quad \times (n_{\alpha}(\tilde{k}_1, \epsilon^2 t)\omega_{\alpha}(\tilde{k}_1)) (n_{\beta}(\tilde{k}_2, \epsilon^2 t)\omega_{\beta}(\tilde{k}_2)) (n_{\gamma}(\tilde{k}_3, \epsilon^2 t)\omega_{\gamma}(\tilde{k}_3)) (n_{\delta}(\tilde{k}_4, \epsilon^2 t)\omega_{\delta}(\tilde{k}_4)) \\ &\quad \times \left( \frac{1}{n_{\alpha}(\tilde{k}_1, \epsilon^2 t)} + \frac{1}{n_{\beta}(\tilde{k}_2, \epsilon^2 t)} - \frac{1}{n_{\gamma}(\tilde{k}_3, \epsilon^2 t)} - \frac{1}{n_{\delta}(\tilde{k}_4, \epsilon^2 t)} \right)^2. \end{aligned} \quad (5.8)$$

Since  $n_{\alpha}\omega_{\alpha} \geq 0$  for each  $\alpha$ , the right-hand side of (5.8) is clearly non-negative, i.e.,

$$\frac{\partial}{\partial \epsilon^2 t} S \geq 0. \quad (5.9)$$

Consequently,  $S$  is a monotonic increasing function of time, and reaches a steady asymptotic value only when

$$\frac{1}{n_{\alpha}(\tilde{k}_1)} + \frac{1}{n_{\beta}(\tilde{k}_2)} = \frac{1}{n_{\gamma}(\tilde{k}_3)} + \frac{1}{n_{\delta}(\tilde{k}_4)}, \quad (5.10)$$

for  $k_1, k_2, k_3, k_4$  satisfying the four-wave resonance condition

$$\begin{aligned} \omega_{\alpha}(k_1) + \omega_{\beta}(k_2) &= \omega_{\gamma}(k_3) + \omega_{\delta}(k_4) , \\ k_1 + k_2 &= k_3 + k_4 . \end{aligned} \quad (5.11)$$

Relations (5.10) and (5.11) will also be recognized as the conditions for a stationary (time-independent) solution to the kinetic equation (4.15). Evidently the "modified" Rayleigh-Jeans distribution<sup>18</sup>

$$\frac{1}{n_{\alpha}(k_1)} = \frac{\omega_{\alpha}(k) + b \cdot k}{\oplus} , \quad \text{each } \alpha , \quad (5.12)$$

where  $\oplus$  and  $b$  are constants, solves (5.10) subject to the resonance condition (5.11). However, Eq. (5.12) is not an acceptable stationary or time-asymptotic solution for non-zero  $\oplus$ , since it corresponds to infinite energy density  $\sum_{\alpha} \int n_{\alpha}(k) \omega_{\alpha}(k) dk$ . We saw earlier that  $\mathcal{E}_0 = \sum_{\alpha} \int n_{\alpha}(k) \omega_{\alpha}(k) dk$  is a constant of the motion; consequently, if the initial preparation corresponds to finite  $\mathcal{E}_0$ , the system cannot pass to a state of the form (5.12).

(d) Generalizations and Limitations of the Theory

Throughout the analysis of the preceding sections it has been assumed that  $\mu^{\alpha\beta\gamma}$ , and hence  $D^{\alpha\beta\gamma\delta}$ , do not vary with time. However, it is clear that the results may be trivially extended to situations where  $\mu^{\alpha\beta\gamma}$  has a slow time variation on the  $\epsilon^2 t$  scale. The analysis of Secs. II and III remains exactly the same, and the final kinetic equation is still given by (4.15), with

$$D^{\alpha\beta\gamma\delta} \rightarrow D^{\alpha\beta\gamma\delta}(\epsilon^2 t) . \quad (5.13)$$

Moreover, the conservation relations and law of increase of entropy which were derived above, remain valid.

A further simple extension of the analysis occurs with the inclusion of a sufficiently weak linear instability. The modification of the hierarchy formalism of Sec. II has previously been considered<sup>4</sup> in situations where

$$\omega_{\alpha}(\underline{k}) = \omega_{\alpha}^R(\underline{k}) + i\gamma_{\alpha}(\underline{k}) \quad (5.14)$$

with  $\omega_{\alpha}^R$  and  $\gamma_{\alpha}$  real, and  $\omega_{\alpha}(-\underline{k}) = -\omega_{\alpha}^*(\underline{k})$ . A meaningful perturbation analysis of the hierarchy for wave correlations is possible provided  $|\gamma_{\alpha}/\omega_{\alpha}^R| \ll 1$ .

In particular, let us assume

$$\gamma_{\alpha}(\underline{k}_1) \cong \epsilon^2 \gamma_{\alpha}^{(2)}(\underline{k}_1) + \dots \quad (5.15)$$

Since the generalization of the kinetic equation (4.15) is the obvious one, we only quote the results here, namely

$$\begin{aligned} \frac{\partial}{\partial \epsilon^2 t} n_{\alpha}(\underline{k}_1, \epsilon^2 t) &= 2\gamma_{\alpha}^{(2)}(\underline{k}_1) n_{\alpha}(\underline{k}_1, \epsilon^2 t) \\ &+ \{\text{right-hand side of Eq. (4.15) with } \omega_{\alpha} \rightarrow \omega_{\alpha}^R\}. \end{aligned} \quad (5.16)$$

In addition, the growth rate  $\gamma_{\alpha}^{(2)}(\underline{k}_1)$  may be allowed to vary on the slow time scale  $\epsilon^2 t$  and not alter the result (5.16).

Finally, we remind the reader of a restriction for the validity of Eq. (4.15). Certainly, in order for the kinetic equation to be meaningful, the right-hand side of (4.15) must not be divergent. Let us examine Eq. (4.15) for a fixed  $\underline{k}_1$ , and imagine carrying out the  $\underline{k}_4$ ,  $\underline{k}_3$  and  $\underline{k}_2$  integrations, respectively. The  $\underline{k}_4$ -integration just replaces  $\underline{k}_4$  by  $\underline{k}_1 + \underline{k}_2 - \underline{k}_3$  in the remaining integrand. The  $\underline{k}_3$ -integration over  $\delta(\omega_{\alpha}(\underline{k}_1) + \omega_{\beta}(\underline{k}_2) - \omega_{\gamma}(\underline{k}_3) - \omega_{\delta}(\underline{k}_1 + \underline{k}_2 - \underline{k}_3))$  is then restricted to a surface (or surfaces)  $S(\underline{k}_1, \underline{k}_2)$  determined from

$$\omega_{\alpha}(\underline{k}_1) + \omega_{\beta}(\underline{k}_2) = \omega_{\gamma}(\underline{k}_3) + \omega_{\delta}(\underline{k}_1 + \underline{k}_2 - \underline{k}_3) \quad (5.17)$$

This gives a contribution in the integrand proportional to

$$\left| \frac{d}{dk_3} \omega_\gamma(k_3) + \frac{d}{dk_3} \omega_\delta(k_1 + k_2 - k_3) \right|^{-1} \quad (5.18)$$

evaluated on  $S(k_1, k_2)$ . Clearly the integrand will be singular in regions where

$$\frac{d}{dk_3} \omega_\gamma(k_3) + \frac{d}{dk_3} \omega_\delta(k_1 + k_2 - k_3) = 0 \quad (5.19)$$

for  $k_3$  satisfying the resonance condition (5.17). In order for the right-hand side of (4.15) to be finite, we thus require

$$\frac{d}{dk_3} \omega_\gamma(k_3) + \frac{d}{dk_3} \omega_\delta(k_1 + k_2 - k_3) \neq 0 \quad (5.20)$$

for  $k_3$  satisfying (5.17), except possibly for regions of zero measure insofar as the integrations over  $S$  and  $k_2$  are concerned. Condition (5.20) thus poses a restriction on the class of problems to which the theory is applicable.

However, this does not appear to be a serious limitation.

Physically, the restriction (5.20) corresponds to situations where the group velocities are different for the final states on the right in Fig. 1; consequently, the corresponding wave disturbances move away from one another and do not further interact effectively. If condition (5.19) were possible, the wave disturbances would move off together with the same group velocity and be capable of additional multiple interactions.<sup>4,5</sup> In this case the wave-coupling is strong, and the weak turbulence analysis of preceding sections is no longer valid.

## VI. CONCLUDING REMARKS

In conclusion we reiterate some features of the analysis. First, the hierarchy formalism for weak turbulence is inherently simple in comparison with the conventional approach of solving Eq. (1.1) to some order, and then carrying out appropriate statistical averages over a spatially uniform ensemble. As pointed out earlier, in order to derive the kinetic equation (4.15), the latter approach would involve a perturbation analysis of (1.1) to order  $\lambda^5$  in the oscillation amplitude. However, the natural expansion parameter in the hierarchy for wave correlations is  $\epsilon \sim \lambda^2$ , a measure of the spectral energy density; moreover, it is necessary to carry out the analysis only to order  $\epsilon^3$ . The hierarchy formalism also has the desirable feature that no additional statistical averaging need be carried out. Second, we remind the reader that no random phase approximations, assumptions of gaussian statistics, or golden rule of transition probabilities, have been invoked in the present analysis. The kinetic equation (4.15) has been systematically derived solely within the assumptions of weak nonlinearity, and spatial homogeneity of the ensemble.

Application of the present kinetic theory to the nonlinear interaction of long wavelength electron plasma oscillations is the subject of an additional article.<sup>19</sup>

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## APPENDIX A

The factor

$$\begin{aligned} & \exp\{i(\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}'))t\} \\ & \times \int_0^t dt'' \exp\{i(\omega(\underline{k}') - \omega(\underline{k}'') - \omega(\underline{k}' - \underline{k}''))t''\} {}_4G^{(3)}(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, t'', \epsilon^2 t) \end{aligned} \quad (\text{A.1})$$

in the integrand of (4.2), contains three successive time integrals, in light of Expression (4.3) for the leading-order four-wave correlations. For the time being we consider the contribution to Expression (A.1) made by the first square bracket term in Eq. (4.3). Inserting the appropriate Fourier arguments, the corresponding portion of (A.1) is given by

$$\begin{aligned} I(t) & \cdot \frac{4}{\omega(\underline{k}_1 - \underline{k}' + \underline{k}'')} \mu(\underline{k}'', \underline{k}_1 - \underline{k}' + \underline{k}'', \underline{k}' - \underline{k}_1) \mu(-\underline{k}_1, \underline{k}' - \underline{k}'' - \underline{k}_1, \underline{k}'' - \underline{k}') \\ & \times n(\underline{k}_1 - \underline{k}', \epsilon^2 t) \{ n(-\underline{k}_1, \epsilon^2 t) n(\underline{k}' - \underline{k}'' - \underline{k}_1, \epsilon^2 t) + n(\underline{k}' - \underline{k}'', \epsilon^2 t) n(\underline{k}' - \underline{k}'' - \underline{k}_1, \epsilon^2 t) \\ & \quad - n(-\underline{k}_1, \epsilon^2 t) n(\underline{k}' - \underline{k}'', \epsilon^2 t) \} , \end{aligned} \quad (\text{A.2})$$

where we have factored out the explicit dependence on the short time-scale  $t$  through

$$\begin{aligned} I(t) & \equiv \exp\{i(\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}'))t\} \\ & \times \int_0^t dt'' \exp\{i(\omega(\underline{k}') - \omega(\underline{k}'') - \omega(\underline{k}' - \underline{k}''))t''\} \int_0^{t''} dt''' \\ & \quad \times \exp\{i(\omega(\underline{k}'') + \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}_1 - \underline{k}' + \underline{k}''))t'''\} \\ & \quad \times \int_0^{t'''} dt'''' \exp\{i(\omega(\underline{k}' - \underline{k}''') - \omega(\underline{k}_1) + \omega(\underline{k}_1 - \underline{k}' + \underline{k}'''))t''''\} . \end{aligned} \quad (\text{A.3})$$

Using the abbreviated notation

$$\begin{aligned}
 f_1 &= \omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}') , \\
 f_2 &= \omega(\underline{k}') - \omega(\underline{k}'') - \omega(\underline{k}' - \underline{k}'') , \\
 f_3 &= \omega(\underline{k}'') + \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}_1 - \underline{k}' + \underline{k}'') , \\
 f_4 &= \omega(\underline{k}' - \underline{k}'') - \omega(\underline{k}_1) + \omega(\underline{k}_1 - \underline{k}' + \underline{k}'') ,
 \end{aligned} \tag{A.4}$$

it is convenient to take the Laplace transform of (A.3) with respect to  $t$ .

This yields

$$\begin{aligned}
 I(s) &= \int_0^\infty e^{-st} I(t) dt , \quad \text{Res} > 0 , \\
 &= \frac{1}{s - if_1} \cdot \frac{1}{s - if_1 - if_2} \cdot \frac{1}{s - if_1 - if_2 - if_3} \cdot \frac{1}{s - if_1 - if_2 - if_3 - if_4} .
 \end{aligned} \tag{A.5}$$

Since  $f_1 + f_2 + f_3 + f_4 = 0$  from (A.4),  $I(s)$  may be rewritten as

$$I(s) = \frac{1}{s} \cdot \frac{1}{s - if_1} \cdot \frac{1}{s + if_4} \cdot \frac{1}{s - if_1 - if_2} . \tag{A.6}$$

When Expression (A.6) is inverted, various of the oscillatory terms may be shown (by standard stationary-phase techniques) to phase-mix to zero as  $t \rightarrow \infty$ , when the  $\underline{k}'$  and  $\underline{k}''$  integrations are carried out in the kinetic equation (4.2). In particular, these oscillatory terms give contributions which decay for large  $t$  as

$$\frac{(\text{oscillation})}{t^{3/2}} \quad \text{or} \quad \frac{(\text{oscillation})}{t^3} , \tag{A.7}$$

in three-dimensional situations. However, there is also a portion of  $I(t)$  which gives a steady contribution as  $t \rightarrow \infty$  insofar as the  $\underline{k}'$  and  $\underline{k}''$  integrations are concerned. This contribution may be simply obtained from (A.6) as



$$\begin{aligned} \lim_{t \rightarrow \infty} I(t) &\equiv \lim_{s \rightarrow 0_+} sI(s) \\ &= \frac{1}{f_1 f_4} \cdot \frac{1}{\Delta - if_1 - if_2}, \quad \Delta \rightarrow 0_+, \end{aligned} \quad (\text{A.8})$$

where  $f_i \neq 0$  by hypothesis. Reinstating the definitions of  $f_i$  we have that

$$\begin{aligned} \lim_{t \rightarrow \infty} I(t) &\equiv \left\{ \frac{1}{\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}')} \cdot \frac{i}{\omega(\underline{k}_1) - \omega(\underline{k}'') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'') + i\Delta} \right\} \\ &\cdot \frac{1}{\omega(\underline{k}' - \underline{k}'') - \omega(\underline{k}_1) + \omega(\underline{k}_1 - \underline{k}' + \underline{k}'')} \quad \Delta \rightarrow 0_+ \end{aligned} \quad (\text{A.9})$$

This determines the steady contribution of (A.2) as  $t \rightarrow \infty$ , to Expression (A.1).

The above arguments may also be carried out for the successive permutations of terms in Expression (4.3) for the four-wave correlations. When all terms are grouped together, it follows that (insofar as integrations over  $\underline{k}'$  and  $\underline{k}''$  are concerned)

$$\begin{aligned} \lim_{t \rightarrow \infty} \exp\{i(\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}'))t\} \int_0^t dt'' \exp\{i(\omega(\underline{k}') - \omega(\underline{k}'') - \omega(\underline{k}' - \underline{k}''))t''\} \\ \cdot {}_4G^{(3)}(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, t'', \epsilon^2 t) \\ \equiv \left\{ \frac{2}{\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}')} \cdot \frac{i}{\omega(\underline{k}_1) - \omega(\underline{k}'') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'') + i\Delta} \right\} \\ {}_4\hat{G}^{(3)}(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, \underline{k}' - \underline{k}'', \epsilon^2 t), \\ \Delta \rightarrow 0_+, \end{aligned} \quad (\text{A.10})$$

where  ${}_4\hat{G}^{(3)}$  is defined by (4.5) with appropriate Fourier arguments.

## APPENDIX B

We consider Expression (4.5) for  ${}_4\hat{G}^{(3)}(\underline{k}'' , \underline{k}_1 - \underline{k}' , -\underline{k}_1 , \underline{k}' - \underline{k}'' , \epsilon^2 t)$  when the resonance condition

$$\omega(\underline{k}_1) - \omega(\underline{k}'') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'') = 0 \quad (\text{B.1})$$

is satisfied. After some straightforward algebra making use of the symmetries (1.4), (1.8), and  $n(-\underline{k}, \epsilon^2 t) = -n(\underline{k}, \epsilon^2 t)$ , it may be shown that the various permutations group to give

$$\begin{aligned} & {}_4\hat{G}^{(3)}(\underline{k}'' , \underline{k}_1 - \underline{k}' , -\underline{k}_1 , \underline{k}' - \underline{k}' , \epsilon^2 t) \\ &= \sqrt{\frac{3}{4\pi}} D(\underline{k}'' , \underline{k}_1 - \underline{k}' , -\underline{k}_1 , \underline{k}' - \underline{k}'') \{ n(\underline{k}'', \epsilon^2 t) n(\underline{k}_1 - \underline{k}' , \epsilon^2 t) n(\underline{k}' - \underline{k}' , \epsilon^2 t) \\ & - n(\underline{k}_1 , \epsilon^2 t) n(\underline{k}' , \epsilon^2 t) n(\underline{k}_1 - \underline{k}' , \epsilon^2 t) - n(\underline{k}_1 , \epsilon^2 t) n(\underline{k}_1 - \underline{k}' , \epsilon^2 t) n(\underline{k}' - \underline{k}' , \epsilon^2 t) \\ & - n(\underline{k}_1 , \epsilon^2 t) n(\underline{k}' , \epsilon^2 t) n(\underline{k}' - \underline{k}' , \epsilon^2 t) \} , \quad (\text{B.2}) \end{aligned}$$

where

$$\begin{aligned} & D(\underline{k}'', \underline{k}_1 - \underline{k}' , -\underline{k}_1 , \underline{k}' - \underline{k}'') \\ & \equiv \sqrt{\frac{4\pi}{3}} \left\{ \left( \frac{2/\omega(\underline{k}_1 - \underline{k}' + \underline{k}'')}{\omega(\underline{k}'') + \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}_1 - \underline{k}' + \underline{k}'')} \cdot \mu(-\underline{k}_1 , \underline{k}' - \underline{k}'' - \underline{k}_1 , \underline{k}'' - \underline{k}') \right) \right. \\ & \left. \times \mu(\underline{k}'', \underline{k}_1 - \underline{k}' + \underline{k}' , \underline{k}' - \underline{k}_1) \right) + \left( \underline{k}_1 - \underline{k}' \leftrightarrow -\underline{k}_1 \right) + \left( \underline{k}_1 - \underline{k}' \leftrightarrow \underline{k}' - \underline{k}'' \right) \left. \right\} , \quad (\text{B.3}) \end{aligned}$$

and the resonance condition (B.1) is satisfied. It should be noted that

$$D(\underline{k}'', \underline{k}_1 - \underline{k}' , -\underline{k}_1 , \underline{k}' - \underline{k}'') = D^*(-\underline{k}'', \underline{k}' - \underline{k}_1 , \underline{k}_1 , \underline{k}'' - \underline{k}') , \quad (\text{B.4})$$

in view of (1.4) and the relation

$$\mu(\underline{k}_1', \underline{k}_1' + \underline{k}_2', -\underline{k}_2') = \mu^*(-\underline{k}_1', -\underline{k}_1' - \underline{k}_2', \underline{k}_2') , \quad (\text{B.5})$$

with appropriate Fourier arguments. Moreover, it may be shown from (B.3) and (B.1) that D is symmetric under interchange of any two of its Fourier arguments, i.e.,

$$\begin{aligned} D(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, \underline{k}' - \underline{k}'') &= D(\underline{k}_1 - \underline{k}'', \underline{k}'', -\underline{k}_1, \underline{k}' - \underline{k}'') \\ &\equiv D(-\underline{k}_1, \underline{k}_1 - \underline{k}'', \underline{k}'', \underline{k}' - \underline{k}'') = D(\underline{k}' - \underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, \underline{k}'') = \dots \end{aligned} \quad (\text{B.6})$$

Use is also made of the symmetries (1.8) in verifying (B.6). In view of (B.2), the resonant four-wave contribution in the kinetic equation (4.6) becomes

$$\begin{aligned} \frac{\partial}{\partial \epsilon^2 t} n(\underline{k}_1, \epsilon^2 t) &= 2\pi \text{Re} \left( \iint d\underline{k}' d\underline{k}'' \left\{ \left[ \frac{2/\omega(\underline{k}')}{\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}')} \right. \right. \right. \\ &\quad \times \frac{\mu(\underline{k}_1, \underline{k}', \underline{k}_1 - \underline{k}') \mu(\underline{k}'', \underline{k}'', \underline{k}' - \underline{k}'')}{\omega(\underline{k}_1) \omega(\underline{k}_1 - \underline{k}') \omega(\underline{k}'') \omega(\underline{k}' - \underline{k}'')} \\ &\quad \times \sqrt{\frac{3}{4\pi}} D(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, \underline{k}' - \underline{k}'') \delta(\omega(\underline{k}_1) - \omega(\underline{k}'') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'')) \\ &\quad \times \{ n(\underline{k}'', \epsilon^2 t) n(\underline{k}_1 - \underline{k}', \epsilon^2 t) n(\underline{k}' - \underline{k}'', \epsilon^2 t) \\ &\quad - n(\underline{k}_1, \epsilon^2 t) n(\underline{k}'', \epsilon^2 t) n(\underline{k}_1 - \underline{k}', \epsilon^2 t) - n(\underline{k}_1, \epsilon^2 t) n(\underline{k}_1 - \underline{k}', \epsilon^2 t) n(\underline{k}' - \underline{k}'', \epsilon^2 t) \\ &\quad \left. \left. \left. - n(\underline{k}_1, \epsilon^2 t) n(\underline{k}'', \epsilon^2 t) n(\underline{k}' - \underline{k}'', \epsilon^2 t) \right\} \right] \right. \\ &\quad \left. + [\underline{k}' \leftrightarrow \underline{k}_1 - \underline{k}'] + [\underline{k}' \leftrightarrow -\underline{k}_1] \right\} ) . \end{aligned} \quad (\text{B.7})$$

Equation (B.7) may be further reduced to the convenient form (4.8). We outline here some of the intermediate algebraic steps, and write (B.7) schematically as

$$\frac{\partial}{\partial \epsilon^2 t} n(\underline{k}_1, \epsilon^2 t) = T_1 + T_2 + T_3 \quad (\text{B.8})$$

where  $T_1$ ,  $T_2$  and  $T_3$  denote the contributions from the successive square bracket interchanges in the order they appear in (B.7).

We first demonstrate that  $T_3 \equiv 0$ . From the third square bracket interchange in (B.7),

$$\begin{aligned}
 T_3 = & 2\pi \text{Re} \iint \underline{\underline{dk'}} \underline{\underline{dk''}} \frac{2/\omega(\underline{\underline{k}}_1)}{\omega(\underline{\underline{k}}_1) - \omega(\underline{\underline{k}}') - \omega(\underline{\underline{k}}_1 - \underline{\underline{k}}')} \frac{\mu(-\underline{\underline{k}}', -\underline{\underline{k}}_1, \underline{\underline{k}}_1 - \underline{\underline{k}}') \mu(-\underline{\underline{k}}_1, \underline{\underline{k}}'', -\underline{\underline{k}}_1 - \underline{\underline{k}}'')}{\omega(\underline{\underline{k}}') \omega(\underline{\underline{k}}_1 - \underline{\underline{k}}') \omega(\underline{\underline{k}}'') \omega(\underline{\underline{k}}_1 + \underline{\underline{k}}'')} \\
 & \times \sqrt{\frac{3}{4\pi}} D(\underline{\underline{k}}'', \underline{\underline{k}}_1 - \underline{\underline{k}}', \underline{\underline{k}}', -\underline{\underline{k}}_1 - \underline{\underline{k}}'') \delta(\omega(\underline{\underline{k}}') + \omega(\underline{\underline{k}}'') + \omega(\underline{\underline{k}}_1 - \underline{\underline{k}}') - \omega(\underline{\underline{k}}_1 + \underline{\underline{k}}'')) \\
 & \times \left\{ n(\underline{\underline{k}}'', \epsilon^2 t) n(\underline{\underline{k}}_1 - \underline{\underline{k}}', \epsilon^2 t) n(\underline{\underline{k}}_1 + \underline{\underline{k}}'', \epsilon^2 t) \right. \\
 & - n(\underline{\underline{k}}', \epsilon^2 t) n(\underline{\underline{k}}'', \epsilon^2 t) n(\underline{\underline{k}}_1 - \underline{\underline{k}}', \epsilon^2 t) + n(\underline{\underline{k}}', \epsilon^2 t) n(\underline{\underline{k}}_1 - \underline{\underline{k}}', \epsilon^2 t) n(\underline{\underline{k}}_1 + \underline{\underline{k}}'', \epsilon^2 t) \\
 & \left. + n(\underline{\underline{k}}', \epsilon^2 t) n(\underline{\underline{k}}'', \epsilon^2 t) n(\underline{\underline{k}}_1 + \underline{\underline{k}}'', \epsilon^2 t) \right\}, \tag{B.9}
 \end{aligned}$$

where the symmetries  $\omega(-\underline{\underline{k}}) = -\omega(\underline{\underline{k}})$ ,  $n(-\underline{\underline{k}}, \epsilon^2 t) = -n(\underline{\underline{k}}, \epsilon^2 t)$  have been used.

Equation (B.9) may also be written in an alternate form by interchanging the variables  $\underline{\underline{k}}' \leftrightarrow -\underline{\underline{k}}''$ . Making use of the symmetries (1.8), (B.6) and conditions (B.4) and (B.5), this gives

$$\begin{aligned}
 T_3 = & 2\pi \text{Re} \iint \underline{\underline{dk'}} \underline{\underline{dk''}} \frac{2/\omega(\underline{\underline{k}}_1)}{\omega(\underline{\underline{k}}_1) + \omega(\underline{\underline{k}}'') - \omega(\underline{\underline{k}}_1 + \underline{\underline{k}}'')} (-) \mu(-\underline{\underline{k}}_1, \underline{\underline{k}}'', -\underline{\underline{k}}_1 - \underline{\underline{k}}'') \mu(-\underline{\underline{k}}', -\underline{\underline{k}}_1, \underline{\underline{k}}_1 - \underline{\underline{k}}') \\
 & \times \sqrt{\frac{3}{4\pi}} D(\underline{\underline{k}}'', \underline{\underline{k}}_1 - \underline{\underline{k}}', \underline{\underline{k}}', -\underline{\underline{k}}_1 - \underline{\underline{k}}'') \delta(\omega(\underline{\underline{k}}') + \omega(\underline{\underline{k}}'') + \omega(\underline{\underline{k}}_1 - \underline{\underline{k}}') - \omega(\underline{\underline{k}}_1 + \underline{\underline{k}}'')) \\
 & \times \left\{ \right\} \tag{B.10}
 \end{aligned}$$

where  $\{ \}$  is the curly bracket in (B.9). Adding (B.9) and (B.10) yields

$T_3 = 0$ , since the resulting integrand contains the factor

$$(\omega(\underline{\underline{k}}') + \omega(\underline{\underline{k}}'') + \omega(\underline{\underline{k}}_1 - \underline{\underline{k}}') - \omega(\underline{\underline{k}}_1 + \underline{\underline{k}}'')) \delta(\omega(\underline{\underline{k}}') + \omega(\underline{\underline{k}}'') + \omega(\underline{\underline{k}}_1 - \underline{\underline{k}}') - \omega(\underline{\underline{k}}_1 + \underline{\underline{k}}'')) \tag{B.11}$$

which vanishes in the  $\underline{k}'$  or  $\underline{k}''$  integration.

Consequently, the kinetic equation becomes  $(\partial/\partial\epsilon^2 t) n(\underline{k}_1, \epsilon^2 t) = T_1 + T_2$ . The  $T_2$  contribution, which corresponds to the interchange  $\underline{k}' \leftrightarrow \underline{k}_1 - \underline{k}'$ , just represents a simple change of variables in the  $\underline{k}'$  integration, and hence reproduces  $T_1$ . Thus, the kinetic equation may be written

$$\frac{\partial}{\partial\epsilon^2 t} n(\underline{k}_1, \epsilon^2 t) = 2T_1, \quad (\text{B.12})$$

where  $T_1$  corresponds to the contribution from the first square bracket term in (B.7). After appropriate changes of integration variables in the expression for  $T_1$ , it may be demonstrated that the factor,

$$\frac{2/\omega(\underline{k}')}{\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}')} \mu(\underline{k}_1, \underline{k}', \underline{k}_1 - \underline{k}') \mu(\underline{k}', \underline{k}'', \underline{k}' - \underline{k}''),$$

in the integrand of (B.7) may be replaced by

$$\begin{aligned} & \frac{1}{3} \left\{ \left( \frac{2/\omega(\underline{k}')}{\omega(\underline{k}_1) - \omega(\underline{k}') - \omega(\underline{k}_1 - \underline{k}')} \mu(\underline{k}_1, \underline{k}', \underline{k}_1 - \underline{k}') \mu(\underline{k}', \underline{k}'', \underline{k}' - \underline{k}'') \right) \right. \\ & \quad \left. + (\underline{k}' \leftrightarrow \underline{k}_1 - \underline{k}' - \underline{k}'') + (\underline{k}' \leftrightarrow \underline{k} - \underline{k}'') \right\} \\ & \equiv \frac{1}{3} \sqrt{\frac{3}{4\pi}} D^*(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, \underline{k}' - \underline{k}''). \end{aligned} \quad (\text{B.10})$$

Use has been made of all the usual symmetries in obtaining this result. Consequently the kinetic equation  $(\partial/\partial\epsilon^2 t) n(\underline{k}_1, \epsilon^2 t) = 2T_1$  becomes

$$\begin{aligned} \frac{\partial}{\partial\epsilon^2 t} n(\underline{k}_1, \epsilon^2 t) &= \iint \underline{dk}' \underline{dk}'' \frac{|D(\underline{k}'', \underline{k}_1 - \underline{k}', -\underline{k}_1, \underline{k}' - \underline{k}'')|^2}{\omega(\underline{k}_1) \omega(\underline{k}_1 - \underline{k}') \omega(\underline{k}'') \omega(\underline{k}' - \underline{k}'')} \\ & \quad \times \delta(\omega(\underline{k}_1) - \omega(\underline{k}'') - \omega(\underline{k}_1 - \underline{k}') - \omega(\underline{k}' - \underline{k}'')) \\ & \quad \times \left\{ n(\underline{k}'', \epsilon^2 t) n(\underline{k}_1 - \underline{k}', \epsilon^2 t) n(\underline{k}' - \underline{k}'', \epsilon^2 t) \right. \end{aligned}$$

$$\begin{aligned}
& - n(\underline{k}_1, \epsilon^2 t) n(\underline{k}'', \epsilon^2 t) n(\underline{k}_1 - \underline{k}', \epsilon^2 t) - n(\underline{k}_1, \epsilon^2 t) n(\underline{k}_1 - \underline{k}'', \epsilon^2 t) n(\underline{k}' - \underline{k}'', \epsilon^2 t) \\
& - n(\underline{k}_1, \epsilon^2 t) n(\underline{k}'', \epsilon^2 t) n(\underline{k}' - \underline{k}'', \epsilon^2 t) \} . \tag{B.11}
\end{aligned}$$

Changing the  $\underline{k}'$  and  $\underline{k}''$  integration variables to  $\underline{k}_2$  and  $\underline{k}_3$  integration variables where  $\underline{k}' = \underline{k}_1 + \underline{k}_2$  and  $\underline{k}'' = \underline{k}_3$ , Eq. (B.11) may be written in the convenient form (4.8), since  $D$  is symmetric under interchange of any two of its Fourier arguments.

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14. In the discussion of this paragraph it has been assumed that these initial-value terms are sufficiently smooth functions of their Fourier arguments. The special initial preparation corresponding to singular initial conditions can of course lead to non-mixing contributions.
15. Those terms which do not tend to a steady value as  $t \rightarrow \infty$  phase mix to zero in the  $\underline{k}'$  and  $\underline{k}''$  integrations.
16. This factor is symmetric under the interchanges indicated in successive square brackets of (4.6) because of the symmetries (1.4) and (1.8).

17. A case in point where  $\mu$  is pure real, and hence the principal-value terms vanish in the kinetic equation (3.25), occurs in the description of the nonlinear interaction of gravity waves in a fluid of constant depth.
18. The Rayleigh-Jeans distribution is modified through the appearance of the term  $\tilde{b} \cdot \tilde{k} / \oplus$  in (5.12).
19. R. C. Davidson (to be published).



## FIGURE CAPTIONS

Fig. 1. Basic resonant four-wave process consisting of the merging of two waves into an intermediate virtual state, followed by the (instantaneous) decay of this virtual state into two further states.

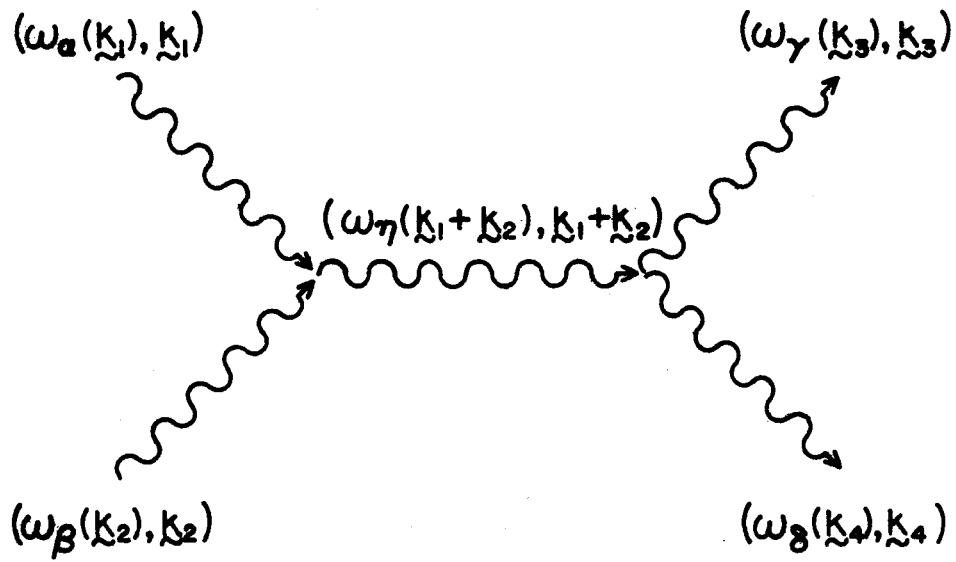


Figure 1