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METHOD OF PARTICULAR SOLUTIONS FOR LINEAR, TWO-POINT BOUNDARY-VALUE PROBLEMS PART 2 - GENERAL THEORY

by

ANGELO MIELE

RICE UNIVERSITY

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Method of Particular Solutions for Linear, Two-Point Boundary-Value Problems

Part 2 - General Theory[1]

ANGELO MIELE[2]

Abstract. The methods commonly employed for solving linear, two-point boundary-value problems require the use of two sets of differential equations: the original set and the derived set. This derived set is the adjoint set if the method of adjoint equations is used, the Green's functions set if the method of Green's functions is used, and the homogeneous set if the method of complementary functions is used.

With particular regard to high-speed digital computing operations, this report explores an alternate method, the method of particular solutions, in which only the original, nonhomogeneous set is used. A general theory is presented for a linear differential system of nth order. The boundary-value problem is solved by combining linearly several particular solutions of the original, nonhomogeneous set. Both the case of an uncontrolled system and the case of a controlled system are considered.

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[2] Professor of Astronautics and Director of the Aero-Astronautics Group, Department of Mechanical and Aerospace Engineering and Materials Science, Rice University, Houston, Texas.

1. Introduction

In a previous paper (Ref. 1), the two-point boundary-value problem was formulated for a linear differential system of the second order and solved using the method of particular solutions. Here, the theory is extended to a system of *n*th order. Both the case of an uncontrolled system and the case of a controlled system are analyzed.

2. Uncontrolled System

In this section, we consider the following linear, nonhomogeneous system of order n:[3]

$$
\begin{aligned}
\dot{x}^1 &= A_{11}x^1 + A_{12}x^2 + \ldots\ldots + A_{1n}x^n + \alpha^1 \\
\dot{x}^2 &= A_{21}x^1 + A_{22}x^2 + \ldots\ldots + A_{2n}x^n + \alpha^2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\dot{x}^n &= A_{n1}x^1 + A_{n2}x^2 + \ldots\ldots + A_{nn}x^n + \alpha^n
\end{aligned}
\tag{1}
$$

in which t is the independent variable, x^j are the dependent variables, and the dot sign denotes a derivative with respect to t. We assume that the coefficients A_{jk} and α^j are time-dependent and continuous. We also assume that the following p conditions are prescribed at $t = 0$:

$$
\begin{aligned}
B_{11}x^1(0) + B_{12}x^2(0) + \ldots\ldots + B_{1n}x^n(0) &= \beta^1 \\
B_{21}x^1(0) + B_{22}x^2(0) + \ldots\ldots + B_{2n}x^n(0) &= \beta^2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
B_{p1}x^1(0) + B_{p2}x^2(0) + \ldots\ldots + B_{pn}x^n(0) &= \beta^p
\end{aligned}
\tag{2}
$$

[3] The system (1) can be called uncontrolled in that its trajectory in the tx^j-space is completely determined once the initial conditions are given.

where the coefficients B_{jk} and β^j are constant. We also assume that the following q conditions are to be met at $t = \tau$:

$$
\begin{aligned}
C_{11}x^1(\tau) + C_{12}x^2(\tau) + \ldots\ldots + C_{1n}x^n(\tau) &= \gamma^1 \\
C_{21}x^1(\tau) + C_{22}x^2(\tau) + \ldots\ldots + C_{2n}x^n(\tau) &= \gamma^2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
C_{q1}x^1(\tau) + C_{q2}x^2(\tau) + \ldots\ldots + C_{qn}x^n(\tau) &= \gamma^q
\end{aligned}
\tag{3}
$$

where τ is given and where the coefficients C_{jk} and γ^j are constant. Finally, we suppose that

$$p + q = n \tag{4}$$

and that[4]

$$p \geq q \tag{5}$$

With this understanding, we formulate the following problem: Find the functions

$$x^1 = x^1(t), \quad x^2 = x^2(t), \quad \ldots\ldots, \quad x^n = x^n(t) \tag{6}$$

which satisfy the differential system (1), the initial conditions (2), and the final conditions (3).

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[4] If $p < q$, it is convenient to integrate Eqs. (1) backward in order to reduce the computational effort.

2.1. <u>Matrix Formulation.</u> In order to simplify the formal treatment, we define the following matrices:

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \vdots \\ \dot{x}^n \end{bmatrix} \tag{7}$$

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^n \end{bmatrix} \tag{8}$$

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{p1} & B_{p2} & \cdots & B_{pn} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^p \end{bmatrix} \tag{9}$$

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{q1} & C_{q2} & \cdots & C_{qn} \end{bmatrix}, \quad \gamma = \begin{bmatrix} \gamma^1 \\ \gamma^2 \\ \vdots \\ \gamma^q \end{bmatrix} \tag{10}$$

Clearly, the matrices $x, \dot{x}$ are $n \times 1$; the matrices A, B, C are $n \times n$, $p \times n$, $q \times n$, respectively; and the matrices α, β, γ are $n \times 1$, $p \times 1$, $q \times 1$, respectively. With this understanding, Eqs. (1)-(3) become[5]

$$\dot{x} = Ax + \alpha \tag{11}$$

$$Bx(0) = \beta \tag{12}$$

$$Cx(\tau) = \gamma \tag{13}$$

where A, α are time-dependent, while B, β and C, γ are constant. Thus, the previous problem consists of finding the function $x(t)$ which satisfies the differential equation (11) subject to the initial condition (12) and the final condition (13).

2.2. Solution Process. In order to solve the proposed problem, we integrate Eq. (11) forward $q + 1$ times from $t = 0$ using $q + 1$ different sets of initial conditions and the stopping condition $t = \tau$. From these integrations, we obtain the functions[6]

$$x_i = x_i(t), \qquad i = 1, \ldots\ldots, q+1 \tag{14}$$

each of which is a particular integral of (11), that is, it satisfies the relation

$$\dot{x}_i = Ax_i + \alpha, \qquad i = 1, \ldots\ldots, q+1 \tag{15}$$

In each integration, the prescribed initial condition (12) is employed. That is, $x_i(0)$ is such that

$$Bx_i(0) = \beta, \qquad i = 1, \ldots\ldots, q+1 \tag{16}$$

---

[5] According to the accepted terminology, column matrices are called vectors. Therefore, $x, \dot{x}, \alpha$ are n-vectors, β is a p-vector, and γ is a q-vector.

[6] The subscript i denotes the generic integration.

We note that the matrices B and β are p x n and p x 1; hence, Eq. (16) is equivalent to p scalar conditions. Since n initial conditions are needed, Eq. (16) must be completed by the relation

$$\tilde{B}x_i(0) = \tilde{\beta}_i, \qquad i = 1, \ldots\ldots, q+1 \tag{17}$$

where the q x n constant matrix $\tilde{B}$ and the q x 1 constant matrix $\tilde{\beta}_i$ are arbitrarily prescribed. Hence, Eq. (17) is equivalent to q scalar conditions. As an example, the matrices $\tilde{B}$ and $\tilde{\beta}_i$ can be chosen to be

$$\tilde{B} = \left[\begin{array}{c|c} 0 \ldots\ldots 0 & 1\; 0 \ldots\ldots 0 \\ 0 \ldots\ldots 0 & 0\; 1 \ldots\ldots 0 \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ 0 \ldots\ldots 0 & 0\; 0 \ldots\ldots 1 \end{array}\right], \qquad \tilde{\beta}_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \vdots \\ \delta_{iq} \end{bmatrix} \tag{18}$$

In the q x n matrix (18-1), the vertical partition generates two submatrices, a q x p submatrix to the left and a q x q submatrix to the right. In the q x 1 matrix (18-2), the symbol δ_{ik} denotes the Kroenecker delta. As a consequence, for each i, Eq. (18) leads to the q scalar conditions

$$x_i^{p+k}(0) = \delta_{ik}, \qquad i = 1, \ldots\ldots, q+1, \qquad k = 1, \ldots\ldots, q \tag{19}$$

Next, we introduce the q + 1 undetermined, scalar constants k_i and form the linear combination

$$x = \sum_{i=1}^{q+1} k_i x_i \tag{20}$$

Then, we inquire whether, by an appropriate choice of the constants, this linear combination can satisfy the differential equation (11), the initial condition (12), and the final condition (13).

By substituting (20) into (11) and rearranging terms, we obtain the relation

$$\sum_{i=1}^{q+1} k_i[\dot{x}_i - Ax_i] = \alpha \tag{21}$$

Since the functions (14) satisfy Eqs. (15), Eq. (21) becomes

$$\sum_{i=1}^{q+1} k_i \alpha = \alpha \tag{22}$$

and is satisfied providing the constants are such that

$$\sum_{i=1}^{q+1} k_i = 1 \tag{23}$$

By substituting (20) into the initial condition (12) and rearranging terms, we obtain the relation

$$\sum_{i=1}^{q+1} k_i[Bx_i(0)] = \beta \tag{24}$$

which, in the light of (16), can be rewritten as

$$\sum_{i=1}^{q+1} k_i \beta = \beta \tag{25}$$

and is satisfied providing the constants are consistent with (23).

Finally, by substituting (20) into the final condition (13) and rearranging terms, we obtain the relation

$$\sum_{i=1}^{q+1} k_i [C x_i(\tau)] = \gamma \tag{26}$$

which is equivalent to q scalar equations. Hence, (23) and (26) constitute a system of q + 1 scalar equations in the q + 1 constants k_i. In this way, the proposed problem is solved in principle.

2.3. Remarks. The following comments are pertinent to the previous discussion:

(a) The particular solutions (14) must be linearly independent. This can be achieved by a proper choice of the matrices $\tilde{B}$ and $\tilde{\beta}_i$ appearing in Eqs. (17).

(b) Because of the arbitrariness of the initial conditions for the particular solutions, it is conceivable that the matrix of the coefficients in Eqs. (23) and (26) may be ill-conditioned. Should this situation arise, corrective steps can be taken by changing some of the matrices in Eqs. (17).

(c) Thus far, the continuity of the coefficients A_{jk} and α^j has been assumed. If this restriction is removed, that is, if the coefficients exhibit a finite number of discontinuities, the previous results are still valid. The only difference is that, in the continuous case, the derivative $\dot{x}$ is a continuous function of time; while, in the discontinuous case, $\dot{x}$ exhibits discontinuities even though x is continuous.

2.4. Relation to the Method of Complementary Functions. Here, we establish a connection between the method of particular solutions and the method of complementary functions. First, we solve Eq. (23) in terms of the constant k_{q+1} as follows:

$$k_{q+1} = 1 - \sum_{i=1}^{q} k_i \tag{27}$$

Next, we rewrite Eq. (20) in the form

$$x = \sum_{i=1}^{q} k_i y_i + x_{q+1} \tag{28}$$

where, by definition,

$$y_i = x_i - x_{q+1}, \qquad i = 1,\ldots\ldots,q \tag{29}$$

We note that the complementary functions

$$y_i = y_i(t), \qquad i = 1,\ldots\ldots,q \tag{30}$$

are solutions of the following homogeneous system derived from (11):

$$\dot{y} = Ay \tag{31}$$

and that they are subject to the initial conditions[7]

$$By_i(0) = 0 \tag{32}$$

$$\tilde{B}y_i(0) = \tilde{\beta}_i - \tilde{\beta}_{q+1} \tag{33}$$

[7] Since the initial condition (17) is arbitrary, the initial condition (33) is arbitrary and can be changed, if necessary.

We also note that the function

$$x_{q+1} = x_{q+1}(t) \tag{34}$$

is a solution of the complete system (11) subject to the initial conditions

$$Bx_{q+1}(0) = \beta \tag{35}$$

$$\tilde{B}x_{q+1}(0) = \tilde{\beta}_{q+1} \tag{36}$$

The q constants k_i must be determined from the final condition

$$\sum_{i=1}^{q} k_i[Cy_i(\tau)] + Cx_{q+1}(\tau) = \gamma \tag{37}$$

which is equivalent to q scalar equations. Therefore, in the method of complementary functions, the solution of (11) can be obtained by combining linearly the solutions (30) of the homogeneous system (31) and the solution (34) of the complete system (11). However, different initial conditions must be used: specifically, conditions (32)-(33) apply to the homogeneous system and conditions (35)-(36) to the complete system.

2.5. Final Time Unspecified. It is now assumed that the final time τ is unspecified and that the differential equation (11) is subject to the initial condition (12), the final condition (13), and the stopping condition

$$D_1x^1(\tau) + D_2x^2(\tau) + \ldots\ldots + D_nx^n(\tau) = \delta \tag{38}$$

in which the constant coefficients D_k and δ are prescribed and τ is to be determined.

In matrix form, Eq. (38) can be rewritten as

$$Dx(\tau) = \delta \tag{39}$$

where D denotes the 1 x n constant matrix

$$D = [D_1, D_2, \ldots\ldots, D_n] \tag{40}$$

and δ is a scalar.

Once more, we integrate Eq. (11) forward $q + 1$ times from $t = 0$ using the initial conditions (16) and (17). After obtaining the solutions (14), we form the linear combination (20). Then, we note that this linear combination satisfies the differential equation (11) and the initial condition (12) providing the constants k_i are consistent with (23).

Next, we turn our attention to the final conditions. By substituting (20) into (13) and (39) and rearranging terms, we obtain the relations

$$\sum_{i=1}^{q+1} k_i[Cx_i(\tau)] = \gamma \tag{41}$$

$$\sum_{i=1}^{q+1} k_i[Dx_i(\tau)] = \delta \tag{42}$$

Equations (23), (41), and (42) are equivalent to $q + 2$ scalar equations in which the unknowns are the $q + 1$ constants k_i and the time τ. Elimination of the constants k_i from (23), (41), and (42) yields a determinantal equation which can be used in place of (39) as the stopping condition for the integration process and determines the final time τ. Once τ is known, the $q + 1$ constants k_i can be determined by solving (41) and (42).

3. Controlled System

Here, we consider the following modification of the previous system:[8]

$$
\begin{aligned}
\dot{x}^1 &= A_{11}x^1 + A_{12}x^2 + \ldots\ldots + A_{1n}x^n + \alpha^1 + E_{11}u^1 + E_{12}u^2 + \ldots\ldots + E_{1m}u^m \\
\dot{x}^2 &= A_{21}x^1 + A_{22}x^2 + \ldots\ldots + A_{2n}x^n + \alpha^2 + E_{21}u^1 + E_{22}u^2 + \ldots\ldots + E_{2m}u^m \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\dot{x}^n &= A_{n1}x^1 + A_{n2}x^2 + \ldots\ldots + A_{nn}x^n + \alpha^n + E_{n1}u^1 + E_{n2}u^2 + \ldots\ldots + E_{nm}u^m
\end{aligned}
\tag{43}
$$

in which t is the independent variable, x^j are the dependent variables, u^j are controls, and the dot sign denotes a derivative with respect to t. We assume that the coefficients A_{jk}, α^j, E_{jk} are time-dependent. We also assume that the following p conditions are prescribed at $t = 0$:

$$
\begin{aligned}
B_{11}x^1(0) + B_{12}x^2(0) + \ldots\ldots + B_{1n}x^n(0) &= \beta^1 \\
B_{21}x^1(0) + B_{22}x^2(0) + \ldots\ldots + B_{2n}x^n(0) &= \beta^2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
B_{p1}x^1(0) + B_{p2}x^2(0) + \ldots\ldots + B_{pn}x^n(0) &= \beta^p
\end{aligned}
\tag{44}
$$

where the coefficients B_{jk} and β^j are constant. We also assume that the following q conditions are to be met at $t = \tau$:

---

[8] The system (43) can be called controlled in that its trajectory in the tx^j-space depends not only on the initial conditions but also on the time-history of the controls $u^j(t)$.

$$
\begin{aligned}
C_{11}x^1(\tau) + C_{12}x^2(\tau) + \ldots\ldots + C_{1n}x^n(\tau) &= \gamma^1 \\
C_{21}x^1(\tau) + C_{22}x^2(\tau) + \ldots\ldots + C_{2n}x^n(\tau) &= \gamma^2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots& \\
C_{q1}x^1(\tau) + C_{q2}x^2(\tau) + \ldots\ldots + C_{qn}x^n(\tau) &= \gamma^q
\end{aligned}
\tag{45}
$$

where τ is given and where the coefficients C_{jk} and γ^j are constant. Finally, we suppose that

$$p \le n, \qquad q \le n \tag{46}$$

With this understanding, we formulate the following problem: Find a set of functions

$$u^1 = u^1(t), \quad u^2 = u^2(t), \quad \ldots\ldots, \quad u^m = u^m(t) \tag{47}$$

$$x^1 = x^1(t), \quad x^2 = x^2(t), \quad \ldots\ldots, \quad x^n = x^n(t) \tag{48}$$

which satisfy the differential system (43), the initial conditions (44), and the final conditions (45). We emphasize that (43) subject to (44)-(45) admits an infinite number of solutions. Nevertheless, we are concerned here with finding <u>only one</u> among these infinite solutions.

3.1. <u>Matrix Formulation</u>. In order to simplify the formal treatment, we define the following matrices:

$$
x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \qquad \dot{x} = \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \vdots \\ \dot{x}^n \end{bmatrix}
\tag{49}
$$

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots\cdots & A_{1n} \\ A_{21} & A_{22} & \cdots\cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots\cdots & A_{nn} \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^n \end{bmatrix} \tag{50}$$

$$E = \begin{bmatrix} E_{11} & E_{12} & \cdots\cdots & E_{1m} \\ E_{21} & E_{22} & \cdots\cdots & E_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ E_{n1} & E_{n2} & \cdots\cdots & E_{nm} \end{bmatrix}, \quad u = \begin{bmatrix} u^1 \\ u^2 \\ \vdots \\ u^m \end{bmatrix} \tag{51}$$

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots\cdots & B_{1n} \\ B_{21} & B_{22} & \cdots\cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{p1} & B_{p2} & \cdots\cdots & B_{pn} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^p \end{bmatrix} \tag{52}$$

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots\cdots & C_{1n} \\ C_{21} & C_{22} & \cdots\cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{q1} & C_{q2} & \cdots\cdots & C_{qn} \end{bmatrix}, \quad \gamma = \begin{bmatrix} \gamma^1 \\ \gamma^2 \\ \vdots \\ \gamma^q \end{bmatrix} \tag{53}$$

Clearly, the matrices x, $\dot{x}$ are $n \times 1$; the matrices A, E, B, C are $n \times n$, $n \times m$, $p \times n$, and $q \times n$, respectively; and the matrices α, u, β, γ are $n \times 1$, $m \times 1$, $p \times 1$, and $q \times 1$, respectively. With this understanding, Eqs. (43)-(45) become[9]

$$\dot{x} = Ax + \alpha + Eu \tag{54}$$

$$Bx(0) = \beta \tag{55}$$

$$Cx(\tau) = \gamma \tag{56}$$

where A, α, E are time-dependent, while B, β and C, γ are constant. Thus, the previous problem consists of finding a pair of functions $u(t)$, $x(t)$ which satisfy the differential equation (54) subject to the initial condition (55) and the final condition (56).

3.2. Solution Process. In order to solve this problem, we integrate Eq. (54) forward $q + 1$ times from $t = 0$ using the initial condition (55), the stopping condition $t = \tau$, and $q + 1$ different time-histories of the control. In each integration, the control employed is[10]

$$u_i = u_i(t), \qquad i = 1, \ldots\ldots, q+1 \tag{57}$$

and the corresponding solution of Eq. (54) is denoted by

$$x_i = x_i(t), \qquad i = 1, \ldots\ldots, q+1 \tag{58}$$

[9] According to the accepted terminology, column matrices are called vectors. Therefore, $x, \dot{x}, \alpha$ are n-vectors, u is an m-vector, β is a p-vector, and γ is a q-vector.

[10] The subscript i denotes the generic integration.

Since each function (58) is a particular integral of (54), we have

$$\dot{x}_i = Ax_i + \alpha + Eu_i, \qquad i = 1, \ldots\ldots, q+1 \tag{59}$$

In each integration, the prescribed initial condition (55) is employed. That is, $x_i(0)$ is such that

$$Bx_i(0) = \beta, \qquad i = 1, \ldots\ldots, q+1 \tag{60}$$

We note that the matrices B and β are p x n and p x 1; hence, Eq. (60) is equivalent to p scalar conditions. Since n initial conditions are needed, Eq. (60) must be completed by the relation

$$\tilde{B}x_i(0) = \tilde{\beta}, \qquad i = 1, \ldots\ldots, q+1 \tag{61}$$

where the matrix $\tilde{B}$ is (n - p) x n and the matrix $\tilde{\beta}$ is (n - p) x 1. Both matrices are constant and arbitrarily prescribed. As an example, the matrices $\tilde{B}$ and $\tilde{\beta}$ can be chosen to be

$$\tilde{B} = \left[\begin{array}{c|c} 0 \ldots. 0 & 1\ 0 \ldots. 0 \\ 0 \ldots. 0 & 0\ 1 \ldots. 0 \\ \ldots\ldots. & \ldots\ldots.. \\ 0 \ldots. 0 & 0\ 0 \ldots. 1 \end{array}\right], \qquad \tilde{\beta} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{62}$$

where the submatrices in (62-1) are q x p and q x q, respectively. Therefore, for each i, Eq. (62) leads to the q scalar conditions

$$x_i^{p+k}(0) = 1, \qquad i = 1, \ldots\ldots, q+1, \qquad k = 1, \ldots, q \tag{63}$$

Next, we introduce the q + 1 undetermined, scalar constants k_i and form the linear combinations

$$u = \sum_{i=1}^{q+1} k_i u_i, \qquad x = \sum_{i=1}^{q+1} k_i x_i \tag{64}$$

Then, we inquire whether, by an appropriate choice of the constants, these linear combinations can satisfy the differential equation (54), the initial condition (55), and the final condition (56).

By substituting (64) into (54) and rearranging terms, we obtain the relation

$$\sum_{i=1}^{q+1} k_i[\dot{x}_i - Ax_i - Eu_i] = \alpha \tag{65}$$

Since the functions (57)-(58) satisfy Eqs. (59), Eq. (65) becomes

$$\sum_{i=1}^{q+1} k_i \alpha = \alpha \tag{66}$$

and is satisfied providing the constants are such that

$$\sum_{i=1}^{q+1} k_i = 1 \tag{67}$$

By substituting (64-2) into the initial condition (55) and rearranging terms, we obtain the relation

$$\sum_{i=1}^{q+1} k_i[Bx_i(0)] = \beta \tag{68}$$

which, in the light of (60), can be rewritten as

$$\sum_{i=1}^{q+1} k_i\beta = \beta \tag{69}$$

and is satisfied providing the constants are consistent with (67).

Finally, by substituting (64-2) into the final condition (56) and rearranging terms, we obtain the relation

$$\sum_{i=1}^{q+1} k_i[Cx_i(\tau)] = \gamma \tag{70}$$

which is equivalent to q scalar equations. Hence, (67) and (70) constitute a system of $q + 1$ scalar equations in the $q + 1$ constants k_i. In this way, the proposed problem is solved in principle.

3.3. Final Time Unspecified. It is now assumed that the final time τ is unspecified and that the differential equation (54) is subject to the initial condition (55), the final condition (56), and the stopping condition

$$D_1x^1(\tau) + D_2x^2(\tau) + \ldots\ldots + D_nx^n(\tau) = \delta \tag{71}$$

in which the constant coefficients D_k and δ are prescribed and τ is to be determined. In matrix form, Eq. (71) can be rewritten as

$$Dx(\tau) = \delta \tag{72}$$

where D denotes the 1 x n constant matrix

$$D = [D_1, D_2, \ldots\ldots, D_n] \tag{73}$$

and δ is a scalar.

Once more, we integrate Eq. (54) forward q + 1 times from t = 0 using the initial conditions (60)-(61) and q + 1 different time-histories of the control. In each integration, the control employed is (57), and (58) is the corresponding solution. We note that the linear combinations (64) satisfy the differential equation (54) and the initial condition (55) providing the constants k_i are consistent with (67).

Next, we turn our attention to the final conditions. By substituting (64-2) into (56) and (72) and rearranging terms, we obtain the relations

$$\sum_{i=1}^{q+1} k_i [Cx_i(\tau)] = \gamma \tag{74}$$

$$\sum_{i=1}^{q+1} k_i [Dx_i(\tau)] = \delta \tag{75}$$

Equations (67), (74), and (75) are equivalent to q + 2 scalar equations in which the unknowns are the q + 1 constants k_i and the time τ. Elimination of the constants k_i from (67), (74), and (75) yields a determinantal equation which can be used in place of (72) as the stopping condition for the integration process and determines the final time τ. Once τ is known, the $q + 1$ constants k_i can be determined by solving (74) and (75).

APPENDIX A

General Solution for an Uncontrolled System

The technique derived in Section 2 can also be employed to find the general solution of (11) in the closed interval $[0, \tau]$. To do so, we integrate the differential equation (11) $n + 1$ times from $t = 0$ using $n + 1$ different sets of initial conditions, for instance,

$$\hat{B}x_i(0) = \hat{\beta}_i(0), \qquad i = 1, \ldots\ldots, n+1 \tag{76}$$

where the matrix $\hat{B}$ is $n \times n$ and the matrix $\hat{\beta}$ is $n \times 1$. Both matrices are constant and arbitrarily prescribed. From the integrations, we obtain the particular solutions[11]

$$x_i = x_i(t), \qquad i = 1, \ldots\ldots, n+1 \tag{77}$$

Next, we introduce the $n + 1$ undetermined, scalar constants k_i and form the linear combination

$$x = \sum_{i=1}^{n+1} k_i x_i \tag{78}$$

Then, we inquire whether, by an appropriate choice of the constants, this linear combination can satisfy the differential equation (11). Simple manipulations, omitted for the sake of brevity, show that this is precisely the case providing the constants are such that

$$\sum_{i=1}^{n+1} k_i = 1 \tag{79}$$

[11] The initial conditions (76) are assumed to be such that the particular solutions (77) are linearly independent.

A.1. Relation to the Method of Complementary Functions. Here, we establish a connection between the method of particular solutions and the method of complementary functions. First, we combine Eqs. (78) and (79) to obtain

$$x = \sum_{i=1}^{n} k_i y_i + x_{n+1} \tag{80}$$

where, by definition

$$y_i = x_i - x_{n+1}, \qquad i = 1, \ldots\ldots, n \tag{81}$$

We note that each complementary function

$$y_i = y_i(t), \qquad i = 1, \ldots\ldots, n \tag{82}$$

is a solution of the homogeneous equation (31) derived from (11). Therefore, Eq. (80) expresses a well-known theorem: The general solution of a linear, nonhomogeneous equation is the sum of the general solution of the corresponding homogeneous equation and a particular solution of the complete equation.

A.2. Remark. The general solution (78) of Eq. (11) contains n + 1 independent solutions. On the other hand, in the boundary-value problem represented by Eqs. (11)-(13), q + 1 independent solutions were employed. This apparent anomaly is now explained. If Eq. (78) is combined with the initial condition (12) and the final condition (13), the following relations are obtained:

$$\sum_{i=1}^{n+1} k_i[Bx_i(0)] = \beta \tag{83}$$

$$\sum_{i=1}^{n+1} k_i[Cx_i(\tau)] = \gamma \tag{84}$$

and, together with (79), determine the constants k_i.

Assume now that the functions (77) satisfy the initial condition (12). This is equivalent to stating that the matrix $\hat{B}$ can be partitioned into the matrices B and $\tilde{B}$ and that the matrix $\hat{\beta}_i$ can be partitioned into the matrices β and $\tilde{\beta}_i$ (see Section 2). Therefore, the initial condition (76) splits into the separate conditions

$$Bx_i(0) = \beta, \qquad i = 1, \ldots\ldots, n+1 \tag{85}$$

$$\tilde{B}x_i(0) = \tilde{\beta}_i, \qquad i = 1, \ldots\ldots, n+1 \tag{86}$$

Equation (83) becomes

$$\sum_{i=1}^{n+1} k_i\beta = \beta \tag{87}$$

and, therefore, is identical with (79). Since the system composed of Eqs. (79), (83), and (84) admits an infinite number of solutions, it is entirely permissible to set

$$k_i = 0, \qquad i = q+2, \ldots\ldots, n+1 \tag{88}$$

that is, integrate the system (11) q + 1 times[12]. This was precisely done in Section 2.

[12] Clearly, only q + 1 independent solutions satisying the initial condition (12) exist.

References

1. MIELE, A., Method of Particular Solutions for Linear, Two-Point Boundary-Value Problems, Part 1: Preliminary Examples, Rice University, Aero-Astronautics Report No. 48, 1968.