Resonant Four-Wave Interaction of Electron Plasma Oscillations*

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The time evolution of a uniformly turbulent ensemble of electron fluids is studied in the electrostatic approximation. A kinetic equation for the action density is obtained in the long wavelength limit, in which resonant four-wave processes cause the nonlinear transfer of energy in the oscillation spectrum. It is shown that the entire region of $\underset{\sim}{k}$-space (within the limits of the simple fluid model) is accessible to resonant four-wave interactions. Such four-wave mode-coupling serves as a mechanism for the transfer of wave energy into shorter wavelengths.

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I. INTRODUCTION

The derivation of the kinetic equations describing the time evolution of wave correlations due to resonant three-wave and resonant four-wave interactions in a uniformly turbulent ensemble of weakly nonlinear, dispersive systems, has previously been put on a rigorous and systematic basis.[1,2] The results are applicable to a broad class of problems in which a fluid-like description may be used. Let us denote by $\{\omega_\alpha(\underset{\sim}{k})\}$ the set of possible eigenfrequencies as a function of wave vector $\underset{\sim}{k}$ describing the linear response in a given problem for which the analyses in Refs. 1 and 2 may be applied. In situations where the resonant three-wave decay conditions

$$\begin{aligned} \omega_\alpha(\underset{\sim}{k}_1) + \omega_\beta(\underset{\sim}{k}_2) &= \omega_\gamma(\underset{\sim}{k}_3) \ , \\ \underset{\sim}{k}_1 + \underset{\sim}{k}_2 &= \underset{\sim}{k}_3 \ , \end{aligned} \tag{1.1}$$

cannot be satisfied for a triplet of modes (α, β, γ), then the principal mechanism for the nonlinear transfer of energy between modes is resonant four-wave coupling. This is manifest in the resulting kinetic equation[2] through resonant behavior for $\omega(\underset{\sim}{k})$ and $\underset{\sim}{k}$ satisfying

$$\begin{aligned} \omega_\alpha(\underset{\sim}{k}_1) + \omega_\beta(\underset{\sim}{k}_2) &= \omega_\gamma(\underset{\sim}{k}_3) + \omega_\delta(\underset{\sim}{k}_4) \ , \\ \underset{\sim}{k}_1 + \underset{\sim}{k}_2 &= \underset{\sim}{k}_3 + \underset{\sim}{k}_4 \ . \end{aligned} \tag{1.2}$$

The fundamental process in relation to (1.2) is the merging of two waves of frequencies $\omega_\alpha(\underset{\sim}{k}_1)$ and $\omega_\beta(\underset{\sim}{k}_3)$, say, into a virtual state

$\omega_\eta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2)$, followed by the (instantaneous) decay of $\omega_\eta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2)$ into two oscillations $\omega_\gamma(\underset{\sim}{k}_3)$ and $\omega_\delta(\underset{\sim}{k}_4)$. This is shown schematically in Fig. 1. An excellent case in point arises in the study of a random sea of gravity waves in a channel of constant depth h.[3,4] In this example the dispersion relation is of the form $\omega^2(\underset{\sim}{k}) = g|\underset{\sim}{k}| \tanh |\underset{\sim}{k}|h$, where $g = |g|$ is the acceleration due to gravity. The resonance condition (1.1) cannot be satisfied for such a dispersion relation, whereas (1.2) can be satisfied. Consequently, in weakly turbulent situations, the principal mechanism for the nonlinear transfer of energy in the oscillation spectrum is that of resonant four-wave scattering. The situation in a plasma is, of course, considerably more complicated since there are many modes of oscillation possible in general. In certain simple models, however, situations occur where the dispersive properties do not permit a solution to (1.1). This is the case for long wavelength electrostatic electron plasma oscillations with

$$\omega^2(\underset{\sim}{k}) \cong \omega_o^2(1 + 3\underset{\sim}{k}^2\lambda_D^2) \ , \qquad (1.3)$$

where the electron plasma frequency, ω_o, and the electron Debye length, λ_D, are given by

$$\omega_o^2 = \frac{4\pi n_o e^2}{m_e} \ , \qquad \lambda_D^2 = \frac{\theta_e}{4\pi n_o e^2} \ . \qquad (1.4)$$

In Eq. (1.4), n_o is the uniform density of the background ions (assumed fixed and singly ionized), and θ_e is the electron temperature in units of ergs; m_e and $-e$ are the mass and charge of the electron. Equation

(1.3) is valid in the absence of external magnetic field for sufficiently long wavelength disturbances, i.e., $k^2\lambda_D^2$ small compared to unity. Although resonant three-wave interactions are forbidden, the four-wave condition (1.2) may be satisfied since $|\omega(k)| \cong \omega_o$.

The nonlinear interaction of coherent electron plasma oscillations has been extensively studied in a simple fluid approximation.[5-9] In this article we consider the time behavior of a uniformly turbulent ensemble of such electron fluids, which evolve according to (2.1)-(2.4) in the absence of magnetic fields. The model lacks sufficient sophistication to recover the effects of linear and nonlinear Landau damping which appears in a Vlasov analysis.[10,11] However, it does serve to illustrate the essential features of resonant four-wave coupling which is a fundamental process in relation to the nonlinear interaction of electron plasma oscillations. Moreover, it is a useful example to demonstrate the techniques which may be used in reducing a particular problem to a form in which the weak turbulence formalisms of Refs. 1 and 2 may be applied. We remind the reader that the effects of resonant four-wave scatterings of electron plasma oscillations were first estimated within a one-dimensional Vlasov framework.[11] The analysis, however, was incomplete in this regard. Upon close examination of the problem, it is apparent that a fifth-order perturbation analysis of the Vlasov equation (in powers of the electric field amplitude), instead of third-order, would have been necessary to properly include the totality of resonant four-wave interactions which cause the wave energy to change with time. The necessity of a fifth-order perturbation analysis in the oscillation amplitude has been shown to be the case in relation to the shallow water wave problem,[3] and is discussed in some generality elsewhere.[2]

In Section II, the simple electron fluid model used in the present analysis is discussed. There results a coupled system of nonlinear equations (2.19) and (2.20) with bilinear nonlinearities describing the time evolution of the density and velocity fluctuations in the spatially uniform ensemble. Analogous to the techniques employed in time-dependent perturbation theory in quantum mechanics, the problem is reformulated in Section III in a representation where the basic vectors are solutions to the linear versions of Eqs. (2.19) and (2.20). The resulting nonlinear equation for the fluctuation amplitudes is given by (3.10) and is of the general form used in Ref. 2; consequently, the appropriate kinetic equation (3.16) describing the time evolution of the spectral energy density of the fluctuations may be taken over directly from this latter reference. Although the general solution to the kinetic equation is not tractable, some simple observations are made in Section IV. It is shown that the entire region of $\underset{\sim}{k}$-space (within the limits of applicability of the model) is accessible to resonant four-wave scatterings of electron plasma oscillations. In particular, if the wave vectors composing an initial preparation are $|\underset{\sim}{k}| < |\underset{\sim}{k}_o|$, say, then it is found that energy may be transferred into the region of $\underset{\sim}{k}$-space for $|\underset{\sim}{k}| > |\underset{\sim}{k}_o|$, i.e., to shorter wavelengths.

II. MODEL OF ELECTRON FLUID

The model we use to describe the nonlinear interaction of electron plasma oscillations is a very simple one. Namely, the positive ions are assumed to form a fixed uniform background of density n_o. Moreover, the electrons are described by truncated moment equations. In the electrostatic approximation, the electron fluid evolves according to

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \underset{\sim}{x}} \cdot (n\underset{\sim}{v}) = 0 , \tag{2.1}$$

$$\frac{\partial}{\partial t}\underset{\sim}{v} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\underset{\sim}{v} = - \frac{e}{m_e}\underset{\sim}{E} - \frac{1}{nm_e}\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{P} , \tag{2.2}$$

$$\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{E} = -4\pi e(n - n_o) , \tag{2.3}$$

$$\frac{\partial}{\partial \underset{\sim}{x}} \times \underset{\sim}{E} = 0 , \tag{2.4}$$

where n, $\underset{\sim}{v}$ and $\underset{\approx}{P}$ are the electron density, mean velocity, and pressure tensor (defined relative to the mean electron velocity), respectively. The electric field $\underset{\sim}{E}$ is self-consistent through Poisson's equation (2.3). It should be kept in mind in relation to (2.2) that the forces associated with the thermal stresses are smaller by $O(k^2\lambda_D^2)$ in comparison with the electrostatic restoring forces, where k is the wave number typical of the disturbance under consideration. To complete the system (2.1)-(2.4), information must be specified regarding the electron pressure tensor $\underset{\approx}{P}$. This may be done by using the equation of evolution for the corresponding moment of the Vlasov equation, and achieving closure by neglecting the effects of heat flow.[12] However, for present purposes, it is adequate to use a model in which the electron pressure is scalar, and behaves adiabatically with the density. In particular, we take $P = (\theta_e/n_o^2)n^3$ in order to recover the linear dispersion relation (1.3) correct to $O(k^2\lambda_D^2)$. Although this procedure is technically incorrect except in one dimension, it has the obvious advantage of simplifying the problem as well as recovering the correct long wavelength dispersion relation. We will see at a later point that the nonlinear contribution of the force associated with the above scalar pressure model may for all intensive purposes be omitted from the analysis, although we shall make no assumption <u>a priori</u> in this regard. The equation of motion for the electron fluid is simply

$$\frac{\partial}{\partial t}\underset{\sim}{v} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\underset{\sim}{v} = -\frac{e}{m_e}\underset{\sim}{E} - 3\frac{\theta_e}{m_e}\frac{n}{n_o}\frac{\partial}{\partial \underset{\sim}{x}}\frac{n}{n_o} . \qquad (2.5)$$

Equations (2.1) and (2.3)-(2.5) then form a closed set. It should be noted in regard to (2.5) that if the velocity flow is initially irrotational, i.e.,

$$\frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{\times} \underset{\sim}{v} = 0 , \qquad (2.6)$$

it remains so for all times. We assume that this is the case. This is tantamount to omitting zero frequency shear waves from the analysis.

Let us now consider the problem of uniform turbulence in a statistical ensemble evolving according to the system (2.1) and (2.3)-(2.6). Each fluid and field quantity is written as an average plus a fluctuation, i.e.,

$$\begin{aligned} n &= \langle n \rangle + \delta n \ , \\ \underset{\sim}{v} &= \langle \underset{\sim}{v} \rangle + \delta \underset{\sim}{v} \ , \\ \underset{\sim}{E} &= \langle \underset{\sim}{E} \rangle + \delta \underset{\sim}{E} \ . \end{aligned} \tag{2.7}$$

In Eq. (2.7), $\langle n \rangle$, $\langle \underset{\sim}{v} \rangle$ and $\langle \underset{\sim}{E} \rangle$ are independent of position by the assumption of spatial uniformity of the ensemble. Moreover, the correlations between fluctuations are invariant under translation. Averages may be viewed as averages over a probability distribution of systems, or alternatively as the arithmetic mean of the quantity under consideration taken over a large number of systems. Clearly the average of the continuity equation (2.1) gives $(\partial/\partial t)\langle n \rangle = 0$ because of spatial uniformity of the ensemble. That is to say

$$\langle n \rangle = n_o \ , \tag{2.8}$$

for all times if so initially. Moreover, as indicated in Appendix A, it may be demonstrated within the context of the model that no average flow velocity or average electric field is generated in the ensemble if

$$\langle \underset{\sim}{v} \rangle = \langle \underset{\sim}{E} \rangle = 0 \ , \tag{2.9}$$

initially. We assume this to be the case. In light of (2.7)-(2.9), the fluctuations in the ensemble evolve according to

$$\frac{\partial}{\partial t}\frac{\delta n}{n_o} = -\frac{\partial}{\partial \underset{\sim}{x}}\cdot\delta\underset{\sim}{v} - \left\{\frac{\partial}{\partial \underset{\sim}{x}}\cdot\left(\frac{\delta n}{n_o}\,\delta\underset{\sim}{v}\right)\right\}, \tag{2.10}$$

$$\frac{\partial}{\partial t}\delta\underset{\sim}{v} = -\frac{e}{m}\delta\underset{\sim}{E} - \frac{3\theta_e}{m_e}\frac{\partial}{\partial \underset{\sim}{x}}\frac{\delta n}{n_o} - \left\{\delta\underset{\sim}{v}\cdot\frac{\partial}{\partial \underset{\sim}{x}}\delta\underset{\sim}{v} + \frac{3\theta_e}{m_e}\frac{\delta n}{n_o}\frac{\partial}{\partial \underset{\sim}{x}}\frac{\delta n}{n_o}\right\}, \tag{2.11}$$

where

$$\frac{\partial}{\partial \underset{\sim}{x}}\cdot\delta\underset{\sim}{E} = -4\pi e\,\delta n\ , \tag{2.12}$$

$$\frac{\partial}{\partial \underset{\sim}{x}}\underset{\sim}{\times}\delta\underset{\sim}{E} = 0\ , \tag{2.13}$$

$$\frac{\partial}{\partial \underset{\sim}{x}}\underset{\sim}{\times}\delta\underset{\sim}{v} = 0\ . \tag{2.14}$$

The only nonlinearities in Eqs. (2.10) and (2.11) appear within curly brackets and are bilinear in nature.

We now Fourier transform with respect to the variables $\underset{\sim}{x}$ according to the convention

$$\delta A(\underset{\sim}{x},t) = \int\frac{d\underset{\sim}{k}}{(2\pi)^3}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\,\delta A(\underset{\sim}{k},t)\ ,$$

$$\delta A(\underset{\sim}{k},t) = \int d\underset{\sim}{x}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\,\delta A(\underset{\sim}{x},t)\ , \tag{2.15}$$

where $\delta A(\underset{\sim}{x},t)$ may represent any of the fluctuations appearing in Eqs. (2.10)-(2.14). Since the fluctuations are real-valued,

$$\delta A(-\underset{\sim}{k},t) = \delta A^*(\underset{\sim}{k},t) \ . \tag{2.16}$$

In addition, it is convenient to introduce the quantities $\delta V(\underset{\sim}{k},t)$ and $\delta N(\underset{\sim}{k},t)$ related to the Fourier transforms of the velocity and density fluctuations, $\delta\underset{\sim}{v}(\underset{\sim}{k},t)$ and $\delta n(\underset{\sim}{k},t)$, by

$$\frac{i\underset{\sim}{k}}{|\underset{\sim}{k}|}\,\delta V(\underset{\sim}{k},t) = \delta\underset{\sim}{v}(\underset{\sim}{k},t) \ , \tag{2.17}$$

$$\delta N(\underset{\sim}{k},t) = \frac{|\omega(\underset{\sim}{k})|}{|\underset{\sim}{k}|}\,\frac{\delta n(\underset{\sim}{k},t)}{n_o} \ , \tag{2.18}$$

where $\omega^2(\underset{\sim}{k})$ is given by (1.3). Equation (2.17) follows since the velocity field is irrotational; and the function $\delta N(\underset{\sim}{k},t)$ in (2.18) is constructed to have the same dimensions as $\delta V(\underset{\sim}{k},t)$. Moreover, the symmetries $\delta V(-\underset{\sim}{k},t) = \delta V^*(\underset{\sim}{k},t)$ and $\delta N(-\underset{\sim}{k},t) = \delta N^*(\underset{\sim}{k},t)$ hold true. The Fourier transforms of Eqs. (2.10) and (2.11) may then be readily reduced to

$$\frac{\partial}{\partial t}\,\delta N(\underset{\sim}{k},t) = |\omega(\underset{\sim}{k})|\delta V(\underset{\sim}{k},t) + \int \frac{d\underset{\sim}{k}'}{(2\pi)^3}\left(\frac{|\omega(\underset{\sim}{k})|\,|\underset{\sim}{k}'|\,\underset{\sim}{k}\cdot(\underset{\sim}{k}-\underset{\sim}{k}')}{|\underset{\sim}{k}|\,|\omega(\underset{\sim}{k}')|\,|\underset{\sim}{k}-\underset{\sim}{k}'|}\right)$$
$$\times\ \delta N(\underset{\sim}{k}',t)\,\delta V(\underset{\sim}{k}-\underset{\sim}{k}',t) \ , \tag{2.19}$$

and

$$\frac{\partial}{\partial t}\,\delta V(\underset{\sim}{k},t) = -|\omega(\underset{\sim}{k})|\,\delta N(\underset{\sim}{k},t) + \int \frac{d\underset{\sim}{k}'}{(2\pi)^3}\left\{\left(\frac{|\underset{\sim}{k}|\underset{\sim}{k}'\cdot(\underset{\sim}{k}-\underset{\sim}{k}')}{2|\underset{\sim}{k}'|\,|\underset{\sim}{k}-\underset{\sim}{k}'|}\right)\right.$$
$$\times\ \delta V(\underset{\sim}{k}',t)\,\delta V(\underset{\sim}{k}-\underset{\sim}{k}',t) - \frac{3\theta_e}{2m_e}\left(\frac{|\underset{\sim}{k}|\,|\underset{\sim}{k}'|\,|\underset{\sim}{k}-\underset{\sim}{k}'|}{|\omega(\underset{\sim}{k}')|\,|\omega(\underset{\sim}{k}-\underset{\sim}{k}')|}\right)$$
$$\left.\times\ \delta N(\underset{\sim}{k}',t)\,\delta N(\underset{\sim}{k}-\underset{\sim}{k}',t)\right\}. \tag{2.20}$$

Equations (2.19) and (2.20) are exact within the context of (2.10) and (2.11), and the corresponding nonlinear terms are easily identified.

As previously discussed, in a conventional weak turbulence analysis, it would be necessary to obtain the perturbation solution to (2.19) and (2.20) to fifth order in the fluctuation amplitude,[13] followed by the appropriate statistical averaging, in order to obtain the kinetic equation describing the time evolution of the spectral energy density in the fluctuations due to resonant four-wave processes. The principal thesis in Refs. 1 and 2, however, is to eliminate the vast amount of information and algebra associated with the aforementioned method by studying at the outset the time behavior of correlations in the ensemble. We now rewrite Eqs. (2.19) and (2.20) in canonical form similar to the general dynamical equation previously studied.[1,2]

III. THE KINETIC EQUATION

Analogous to the techniques used in time-dependent perturbation theory, it is convenient to formulate the problem in a representation where the basis vectors are solutions of the linear versions of Eqs. (2.19) and (2.20). Introducing the column vector, $\underset{\sim}{\psi}(\underset{\sim}{k},t)$, where

$$\underset{\sim}{\psi}(\underset{\sim}{k},t) \equiv \begin{pmatrix} \delta V(\underset{\sim}{k},t) \\ \delta N(\underset{\sim}{k},t) \end{pmatrix}, \qquad (3.1)$$

Eqs. (2.19) and (2.20) may be written in the form[1]

$$\frac{\partial}{\partial t} \underset{\sim}{\psi}(\underset{\sim}{k},t) = \underset{\approx}{H}_o(\underset{\sim}{k}) \cdot \underset{\sim}{\psi}(\underset{\sim}{k},t) + \iint d\underset{\sim}{k}_1 \, d\underset{\sim}{k}_2 \, \delta(\underset{\sim}{k} - \underset{\sim}{k}_1 - \underset{\sim}{k}_2)$$

$$\times \underset{\sim}{H}_1[\underset{\sim}{k}_1,\underset{\sim}{k}_2;\underset{\sim}{\psi}(\underset{\sim}{k}_1,t),\underset{\sim}{\psi}(\underset{\sim}{k}_2,t)] \ . \qquad (3.2)$$

The 2×2 matrix $\underset{\approx}{H}_o$ and column vector $\underset{\sim}{H}_1$ in (3.2) are given by

$$\underset{\approx}{H}_o(\underset{\sim}{k}) = \begin{pmatrix} 0 & -i|\omega(\underset{\sim}{k})| \\ i|\omega(\underset{\sim}{k})| & 0 \end{pmatrix}, \qquad (3.3)$$

and

$$2(2\pi)^3(-i)\underset{\sim}{H}_1[\underset{\sim}{k}_1,\underset{\sim}{k}_2;\underset{\sim}{\psi}(\underset{\sim}{k}_1,t),\underset{\sim}{\psi}(\underset{\sim}{k}_2,t] =$$

$$\left\{ (1,0) \cdot \underset{\sim}{\psi}(\underset{\sim}{k}_1,t) \begin{pmatrix} \frac{1}{2} \frac{|\underset{\sim}{k}|}{|\underset{\sim}{k}_1|} \frac{\underset{\sim}{k}\cdot\underset{\sim}{k}_2}{|\underset{\sim}{k}_2|} & 0 \\ 0 & \frac{|\omega(\underset{\sim}{k})|}{|\omega(\underset{\sim}{k}_2)|} \frac{|\underset{\sim}{k}_2| \underset{\sim}{k}\cdot\underset{\sim}{k}_1}{|\underset{\sim}{k}|\,|\underset{\sim}{k}_1|} \end{pmatrix} \cdot \underset{\sim}{\psi}(\underset{\sim}{k}_2,t) \right.$$

$$\left. + (0,1) \cdot \underset{\sim}{\psi}(\underset{\sim}{k}_1,t) \begin{pmatrix} 0 & -\frac{3\theta_e}{2m_e} \frac{|\underset{\sim}{k}|\,|\underset{\sim}{k}_1|\,|\underset{\sim}{k}_2|}{|\omega(\underset{\sim}{k}_1)|\,|\omega(\underset{\sim}{k}_2)|} \\ 0 & 0 \end{pmatrix} \cdot \underset{\sim}{\psi}(\underset{\sim}{k}_2,t) \right\}$$

$$+ \left\{ \underset{\sim}{k}_1 \longleftrightarrow \underset{\sim}{k}_2 \right\} . \qquad (3.4)$$

The eigenfrequencies associated with the linear ($\underset{\sim}{H}_1 = 0$) version of Eq. (3.2) are solutions to $\omega^2 = \omega^2(\underset{\sim}{k}) = \omega_o^{\,2}(1 + 3k^2\lambda_D^{\,2})$. We denote the two possible modes by $\omega_+(\underset{\sim}{k})$ and $\omega_-(\underset{\sim}{k})$ $(= -\omega_+(\underset{\sim}{k}))$ and use the sign convention

$$\omega_\alpha(-\underset{\sim}{k}) = -\omega_\alpha(\underset{\sim}{k}) \ ; \qquad \alpha = +,- \qquad (3.5)$$

throughout the remainder of this article. The eigenvector of the linear version of Eq. (3.2) corresponding to mode α is simply

$$\underset{\sim}{U}_{\alpha}(\underset{\sim}{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \dfrac{i\omega_{\alpha}(\underset{\sim}{k})}{|\omega(\underset{\sim}{k})|} \end{pmatrix} , \tag{3.6}$$

It should be noted in relation to (3.6) that $\underset{\sim}{U}_{\alpha}(-\underset{\sim}{k}) = U^{*}_{\alpha}(\underset{\sim}{k})$ and that the normalization is of the form

$$\underset{\sim}{U}^{\beta}(\underset{\sim}{k}) \cdot \underset{\sim}{U}_{\alpha}(\underset{\sim}{k}) = \delta_{\alpha\beta} , \tag{3.7}$$

where $\underset{\sim}{U}^{\beta}(\underset{\sim}{k})$ is the Hermitian conjugate of $\underset{\sim}{U}_{\beta}(\underset{\sim}{k})$ and $\delta_{\alpha\beta}$ is the Kronecker delta. Introducing the amplitude $A_{\alpha}(\underset{\sim}{k},t)$ associated with the α'th mode, then, with

$$\underset{\sim}{\psi}(\underset{\sim}{k},t) = \sum_{\alpha} A_{\alpha}(\underset{\sim}{k}_1,t)\underset{\sim}{U}_{\alpha}(\underset{\sim}{k})e^{-i\omega_{\alpha}(\underset{\sim}{k})t} , \tag{3.8}$$

Eq. (3.2) can be rewritten

$$\sum_{\alpha} i\left(\frac{\partial A_{\alpha}}{\partial t}(\underset{\sim}{k},t)\right)\underset{\sim}{U}_{\alpha}(\underset{\sim}{k})e^{-i\omega_{\alpha}(\underset{\sim}{k})t} =$$

$$\sum_{\beta,\gamma} \iint d\underset{\sim}{k}_1\, d\underset{\sim}{k}_2\, \delta(\underset{\sim}{k} - \underset{\sim}{k}_1 - \underset{\sim}{k}_2)\underset{\sim}{H}_1[\underset{\sim}{k}_1,\underset{\sim}{k}_2;\underset{\sim}{U}_{\beta}(\underset{\sim}{k}_1),\underset{\sim}{U}_{\gamma}(\underset{\sim}{k}_2)]$$
$$\times A_{\beta}(\underset{\sim}{k}_1,t)A_{\gamma}(\underset{\sim}{k}_2,t)e^{-i(\omega_{\beta}(\underset{\sim}{k}_1) + \omega_{\gamma}(\underset{\sim}{k}_2))t} , \tag{3.9}$$

where the summations in (3.8) and (3.9) are over +,- polarizations. Multiplying Eq. (3.9) by the Hermitian conjugate of $\underset{\sim}{U}_{\alpha}(\underset{\sim}{k})$, we have

$$\frac{\partial}{\partial t} A_\alpha(\underset{\sim}{k},t) = \sum_{\beta,\gamma} \iint d\underset{\sim}{k}_1\, d\underset{\sim}{k}_2\, \delta(\underset{\sim}{k} - \underset{\sim}{k}_1 - \underset{\sim}{k}_2)\, \overset{\alpha\beta\gamma}{K}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2)$$

$$\times A_\beta(\underset{\sim}{k}_1,t)A_\gamma(\underset{\sim}{k}_2,t) \exp\{i(\omega_\alpha(\underset{\sim}{k}) - \omega_\beta(\underset{\sim}{k}_1) - \omega_\gamma(\underset{\sim}{k}_2))t\}, \quad (3.10)$$

where the interaction kernel $\overset{\alpha\beta\gamma}{K}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2) \equiv -i\underset{\sim}{U}^\alpha(\underset{\sim}{k}) \cdot \underset{\sim}{H}_1[\underset{\sim}{k}_1,\underset{\sim}{k}_2;$ $U_\beta(\underset{\sim}{k}_1),U_\gamma(\underset{\sim}{k}_2)]$, i.e.,

$$\overset{\alpha\beta\gamma}{K}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2) \equiv \frac{\omega_\alpha(\underset{\sim}{k})}{(2\pi)^3 4\sqrt{2}|\underset{\sim}{k}||\underset{\sim}{k}_1||\underset{\sim}{k}_2|} \left\{ \frac{|\underset{\sim}{k}|^2 \underset{\sim}{k}_1\cdot\underset{\sim}{k}_2}{\omega_\alpha(\underset{\sim}{k})} + \frac{|\underset{\sim}{k}_2|^2 \underset{\sim}{k}\cdot\underset{\sim}{k}_1}{\omega_\gamma(\underset{\sim}{k}_2)} + \right.$$

$$\left. + \frac{|\underset{\sim}{k}_1|^2 \underset{\sim}{k}\cdot\underset{\sim}{k}_2}{\omega_\beta(\underset{\sim}{k}_1)} + 3\frac{\theta_e}{m_e} \frac{|\underset{\sim}{k}|^2|\underset{\sim}{k}_1|^2|\underset{\sim}{k}_2|^2}{\omega_\alpha(\underset{\sim}{k})\omega_\beta(\underset{\sim}{k}_1)\omega_\gamma(\underset{\sim}{k}_2)} \right\}, \quad (3.11)$$

which follows from Eq. (3.4). It should be noted that the term proportional to θ_e in expression (3.11) arises from the nonlinear contribution of the electron pressure in Eq. (2.11). In an order-of-magnitude estimate, this is smaller by $O(k^2\lambda_D^2)$ than the remaining terms in (3.11), and may be omitted from the long wavelength analysis for all practical purposes.

Equation (3.10) , advancing the amplitudes $A_\alpha(\underset{\sim}{k},t)$ in time, is exact within the context of the original model; the form, however, is more amenable to a direct analysis than that of Eqs. (2.19) and (2.20). The general procedure outlined above can, of course, be applied in situations where there are any finite number of coupled nonlinear equations describing the problem under consideration. It should be noted that in a small-amplitude theory of (3.10), $A_\alpha(\underset{\sim}{k},t)$ does not change with time in the lowest approximation, thus giving a $\underset{\sim}{\psi}(\underset{\sim}{k},t)$ in which waves of different wavenumber propagate independently. In higher order, however, the nonlinear terms act as perturbations causing $A_\alpha(\underset{\sim}{k},t)$ to change in the course of time through the interaction between waves of differing wavenumber.

Since the time evolution of the fluctuations has been reduced to the canonical form (3.10), the general results from previous analyses may be used directly to give the kinetic behavior of the spectral energy density associated with the fluctuations in the ensemble. The spectral energy density associated with the α'th mode, ${}_2G_{\alpha\alpha}(\underset{\sim}{k}_1,t)$, is given by[2]

$$\langle A_\alpha(\underset{\sim}{k}_1,t)A_\alpha(\underset{\sim}{k}_2,t)\rangle = {}_2G_{\alpha\alpha}(\underset{\sim}{k}_1,t)\delta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2) \qquad (3.12)$$

for a spatially uniform ensemble. Moreover, we introduce the action density, $n_\alpha(\underset{\sim}{k}_1,t)$, associated with the α'th mode where

$$n_\alpha \equiv {}_2G_{\alpha\alpha}(\underset{\sim}{k}_1,t)/\omega_\alpha(\underset{\sim}{k}_1) \;, \qquad (3.13)$$

and the response, $\overset{\alpha\beta\gamma}{\mu}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2)$, defined in terms of the interaction kernel by $\overset{\alpha\beta\gamma}{\mu}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2) = \omega_\beta(\underset{\sim}{k}_1)\omega_\gamma(\underset{\sim}{k}_2)\overset{\alpha\beta\gamma}{K}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2)$, i.e.,

$$\overset{\alpha\beta\gamma}{\mu}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2) = \frac{\omega_\alpha(\underset{\sim}{k})\omega_\beta(\underset{\sim}{k}_1)\omega_\gamma(\underset{\sim}{k}_2)}{(2\pi)^3 4\sqrt{2}\,|\underset{\sim}{k}|\,|\underset{\sim}{k}_1|\,|\underset{\sim}{k}_2|}\left\{\frac{|\underset{\sim}{k}|^2\underset{\sim}{k}_1\cdot\underset{\sim}{k}_2}{\omega_\alpha(\underset{\sim}{k})} + \frac{|\underset{\sim}{k}_2|^2\underset{\sim}{k}\cdot\underset{\sim}{k}_1}{\omega_\gamma(\underset{\sim}{k}_2)} + \right.$$

$$\left. + \frac{|\underset{\sim}{k}_1|^2\underset{\sim}{k}\cdot\underset{\sim}{k}_2}{\omega_\beta(\underset{\sim}{k}_1)} + 3\,\frac{\theta_e}{m_e}\,\frac{|\underset{\sim}{k}|^2|\underset{\sim}{k}_1|^2|\underset{\sim}{k}_2|^2}{\omega_\alpha(\underset{\sim}{k})\omega_\beta(\underset{\sim}{k}_1)\omega_\gamma(\underset{\sim}{k}_2)}\right\} . \qquad (3.14)$$

It should be noted that $n_\alpha(-\underset{\sim}{k},t) = -n_\alpha(\underset{\sim}{k},t) = -n^*_\alpha(\underset{\sim}{k},t)$, and that the response, in addition to being real, enjoys the symmetries

$$\mu^{\alpha\beta\gamma}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2) = \mu^{\alpha\beta\gamma}(-\underset{\sim}{k},-\underset{\sim}{k}_1,-\underset{\sim}{k}_2) = \mu^{\beta\alpha\gamma}(-\underset{\sim}{k}_1,-\underset{\sim}{k},\underset{\sim}{k}_2) = \mu^{\gamma\beta\alpha}(-\underset{\sim}{k}_2,\underset{\sim}{k}_1,-\underset{\sim}{k}). \quad (3.15)$$

In addition, we remind the reader that ${}_2G_{++}$ and ${}_2G_{--}$ may be expressed directly in terms of the density and velocity correlations, $\langle \delta N_{k_1} \delta N_{k_2} \rangle$, $\langle \delta V_{k_1} \delta V_{k_2} \rangle$ and $\langle \delta N_{k_1} \delta V_{k_2} \rangle$ although this is not necessary for present considerations.

As stated in the introduction, long wavelength electron plasma oscillations cannot satisfy the resonant three-wave decay condition. Hence, in the simple model used here, the leading-order process causing $n_\alpha(\underset{\sim}{k},t)$ to change with time is that of resonant four-wave interactions. In light of Eq. (3.10) and the symmetries (3.15), we may use the kinetic equation (4.15) of Ref. 2 to describe the evolution of $n_\alpha(\underset{\sim}{k},t)$. In particular, the kinetic equation for the action density is given to $O(n^3)$ by

$$\frac{\partial}{\partial t} n_\alpha(\underset{\sim}{k}_1,t) = \sum_{\beta,\gamma,\delta} \iiint d\underset{\sim}{k}_2\, d\underset{\sim}{k}_3\, d\underset{\sim}{k}_4\, \delta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2 - \underset{\sim}{k}_3 - \underset{\sim}{k}_4)\delta(\omega_\alpha(\underset{\sim}{k}_1)$$

$$+ \omega_\beta(\underset{\sim}{k}_2) - \omega_\gamma(\underset{\sim}{k}_3) - \omega_\delta(\underset{\sim}{k}_4)) \frac{|D^{\alpha\beta\gamma\delta}(-\underset{\sim}{k}_1,-\underset{\sim}{k}_2,\underset{\sim}{k}_3,\underset{\sim}{k}_4)|^2}{\omega_\alpha(\underset{\sim}{k}_1)\omega_\beta(\underset{\sim}{k}_2)\omega_\gamma(\underset{\sim}{k}_3)\omega_\delta(\underset{\sim}{k}_4)}$$

$$\times\Big(n_\beta(\underset{\sim}{k}_2,t)n_\gamma(\underset{\sim}{k}_3,t)n_\delta(\underset{\sim}{k}_4,t) + n_\alpha(\underset{\sim}{k}_1,t)n_\gamma(\underset{\sim}{k}_3,t)n_\delta(\underset{\sim}{k}_4,t)$$

$$- n_\alpha(\underset{\sim}{k}_1,t)n_\beta(\underset{\sim}{k}_2,t)n_\gamma(\underset{\sim}{k}_3,t) - n_\alpha(\underset{\sim}{k}_1,t)n_\beta(\underset{\sim}{k}_2,t)n_\delta(\underset{\sim}{k}_4,t)\Big), \qquad (3.16)$$

where

$$\overset{\alpha\beta\gamma\delta}{D}(-\underset{\sim}{k}_1,-\underset{\sim}{k}_2,\underset{\sim}{k}_3,\underset{\sim}{k}_4) = \left(\frac{4\pi}{3}\right)^{1/2} \sum_{\eta} \left\{\left(\frac{2/\omega_\eta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2)}{\omega_\alpha(\underset{\sim}{k}_1) + \omega_\beta(\underset{\sim}{k}_2) - \omega_\eta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2)} \right.\right.$$

$$\left.\left. \times\, \overset{\gamma\ \eta\ \delta}{\mu}(\underset{\sim}{k}_3,\underset{\sim}{k}_3 + \underset{\sim}{k}_4,-\underset{\sim}{k}_4)\,\overset{\alpha\ \eta\ \beta}{\mu}(-\underset{\sim}{k}_1,-\underset{\sim}{k}_1-\underset{\sim}{k}_2,\underset{\sim}{k}_2)\right) + \begin{pmatrix} \underset{\sim}{k}_2 \leftrightarrow -\underset{\sim}{k}_3 \\ \beta \leftrightarrow \gamma \end{pmatrix} + \begin{pmatrix} \underset{\sim}{k}_2 \leftrightarrow -\underset{\sim}{k}_4 \\ \beta \leftrightarrow \delta \end{pmatrix} \right\} . \tag{3.17}$$

The response D defined in (3.17) and appearing in the kinetic equation (3.16) may be written explicitly in terms of its $\underset{\sim}{k}$-arguments and $\{\omega_\alpha(\underset{\sim}{k})\}$ through the definition of μ in Eq. (3.14). It should be noted that the summation in (3.17) is over virtual states η. Moreover, $\overset{\alpha\beta\gamma\delta}{D}(-\underset{\sim}{k}_1,-\underset{\sim}{k}_2,\underset{\sim}{k}_3,\underset{\sim}{k}_4)$ is symmetric under interchange of any two of the quantities $(\alpha,-\underset{\sim}{k}_1)$, $(\beta,-\underset{\sim}{k}_2)$, $(\gamma,\underset{\sim}{k}_3)$ and $(\delta,\underset{\sim}{k}_4)$ when the resonance condition (1.2) is satisfied. We also remind the reader that the stoss term in the kinetic equation (3.16) is <u>trilinear</u> in the action density, in contrast to the kinetic equation for resonant three-wave processes[1] where the driving term is <u>bilinear</u> in the action density.

IV. DISCUSSION OF RESULTS

The kinetic equation (3.16) thus describes the time evolution of the action density $n_\alpha(\underset{\sim}{k}_1,t)$ associated with the α'th mode. The general conservation relations and law of entropy increase discussed in Ref. 2 apply in relation to Eq. (3.16). Although the resulting kinetic equation is a nonlinear integro-differential equation whose solution, in general, is not tractable, certain observations may be made. It is of particular interest to determine the region of $\underset{\sim}{k}$-space accessible to resonant four-wave scatterings.

We examine Eq. (3.16) for a fixed $\underset{\sim}{k}_1$ and carry out the $\underset{\sim}{k}_4$ integration over $\delta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2 - \underset{\sim}{k}_3 - \underset{\sim}{k}_4)$, which replaces $\underset{\sim}{k}_4$ by $\underset{\sim}{k}_1 + \underset{\sim}{k}_2 - \underset{\sim}{k}_3$ in the remainder of the integrand. For example, $\delta(\omega_\alpha(\underset{\sim}{k}_1) + \omega_\beta(\underset{\sim}{k}_2) - \omega_\gamma(\underset{\sim}{k}_3) - \omega_\delta(\underset{\sim}{k}_4)) \rightarrow \delta(\omega_\alpha(\underset{\sim}{k}_1) + \omega_\beta(\underset{\sim}{k}_2) - \omega_\gamma(\underset{\sim}{k}_3) - \omega_\delta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2 - \underset{\sim}{k}_3))$. We now imagine doing the $\underset{\sim}{k}_3$ and $\underset{\sim}{k}_2$ integrations successively. For each $\underset{\sim}{k}_2$ (keep in mind $\underset{\sim}{k}_1$ is fixed), the resonant region in the $\underset{\sim}{k}_3$-integration is determined from

$$\omega_\alpha(\underset{\sim}{k}_1) + \omega_\beta(\underset{\sim}{k}_2) = \omega_\gamma(\underset{\sim}{k}_3) + \omega_\delta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2 - \underset{\sim}{k}_3) \ . \qquad (4.1)$$

Keeping in mind that $|\omega(\underset{\sim}{k})| \cong \omega_o$ in the long wavelength circumstances considered here, then, depending on the various polarizations, there are three distinct ways in which the resonance condition (4.1) can be satisfied. Namely

$$|\omega(\underset{\sim}{k}_1)| + |\omega(\underset{\sim}{k}_2)| = |\omega(\underset{\sim}{k}_3)| + |\omega(\underset{\sim}{k}_1 + \underset{\sim}{k}_2 - \underset{\sim}{k}_3)| \ , \qquad (4.2a)$$

$$|\omega(\underset{\sim}{k}_1)| - |\omega(\underset{\sim}{k}_2)| = |\omega(\underset{\sim}{k}_3)| - |\omega(\underset{\sim}{k}_1 + \underset{\sim}{k}_2 - \underset{\sim}{k}_3)| \ , \qquad (4.2b)$$

or

$$|\omega(\underset{\sim}{k}_1)| - |\omega(\underset{\sim}{k}_2)| = -|\omega(\underset{\sim}{k}_3)| + |\omega(\underset{\sim}{k}_1 + \underset{\sim}{k}_2 - \underset{\sim}{k}_3)| \ . \qquad (4.2c)$$

For the purpose of illustration, it is sufficient to determine the resonant $\underset{\sim}{k}_3$-region in case (4.2a). With $|\omega(\underset{\sim}{k})| \cong \omega_o(1 + \frac{3}{2}\underset{\sim}{k}^2\lambda_D^{\ 2})$ in the long wavelength limit, it follows from (4.2a) that

$$\underset{\sim}{k}_1^{\ 2} + \underset{\sim}{k}_2^{\ 2} = \underset{\sim}{k}_3^{\ 2} + (\underset{\sim}{k}_1 + \underset{\sim}{k}_2 - \underset{\sim}{k}_3)^2 \ . \qquad (4.3)$$

Equation (4.3) may be rewritten

$$\left(\underset{\sim}{k}_3 - \frac{\underset{\sim}{k}_1 + \underset{\sim}{k}_2}{2}\right)^2 = \frac{1}{4}\,(\underset{\sim}{k}_1 - \underset{\sim}{k}_2)^2 .$$

That is to say, for each $\underset{\sim}{k}_2$ the resonant $\underset{\sim}{k}_3$-region is the surface of a sphere of radius $\frac{1}{2}\,|\underset{\sim}{k}_1 - \underset{\sim}{k}_2|$ centered at $(\underset{\sim}{k}_1 + \underset{\sim}{k}_2)/2$ as depicted in Fig. 2. In Eq. (3.16), the $\underset{\sim}{k}_3$-integration over the surface of this sphere may, in principle, be carried out leaving only the $\underset{\sim}{k}_2$-integration. When integrating over $\underset{\sim}{k}_2$, the location as well as the radius of this "resonant sphere" varies, covering the entire region of available phase space. Of course, the model is limited in the range of wave numbers to which it is applicable. In particular, $0 < |\underset{\sim}{k}| < |\underset{\sim}{k}|_{max}$, where $|\underset{\sim}{k}|_{max}\lambda_D < 1$. For wavelengths shorter than $2\pi/|\underset{\sim}{k}|_{max}$, collisionless dissipation through Landau damping becomes important.

Since the resonant region covers all of phase space (within the limits of the model), resonant four-wave scattering serves as a mechanism for the transfer of energy into shorter wavelengths. Without presenting any of the algebra here, one can show from Eq. (3.16) and the preceding arguments that, if the wavenumbers composing an initial preparation are $|\underset{\sim}{k}| < |\underset{\sim}{k}_o|$ say, then the region of $\underset{\sim}{k}$-space $|\underset{\sim}{k}| > |\underset{\sim}{k}_o|$ becomes populated for times greater than zero. This corresponds to a transfer of energy to shorter wavelengths. In a more sophisticated model, the ultimate fate of the wave energy would be dissipation through Landau damping at sufficiently high wave numbers.

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APPENDIX A

Since the velocity field is irrotational according to Eq. (2.6), the convective term $\underset{\sim}{v} \cdot (\partial/\partial \underset{\sim}{x})\underset{\sim}{v}$ may be rewritten as $(\partial/\partial \underset{\sim}{x})(\underset{\sim}{v}^2/2)$. Then the average of Eq. (2.5) over a spatially uniform ensemble just gives

$$\frac{\partial}{\partial t} \langle \underset{\sim}{v} \rangle = - \frac{e}{m_e} \langle \underset{\sim}{E} \rangle \ . \tag{A.1}$$

Moreover, combining the equations of continuity (2.1) and motion (2.5) readily gives

$$\frac{\partial}{\partial t} \langle n\underset{\sim}{v} \rangle = - \frac{e}{m_e} \langle n\underset{\sim}{E} \rangle \ . \tag{A.2}$$

If we supplement the model with the curl $\underset{\sim}{B}$ Maxwell equation in the absence of magnetic field, i.e.,

$$0 = -4\pi e n\underset{\sim}{v} + \frac{\partial}{\partial t} \underset{\sim}{E} \ , \tag{A.3}$$

then

$$\frac{\partial}{\partial t} \langle \underset{\sim}{E} \rangle = 4\pi e \langle n\underset{\sim}{v} \rangle \ , \tag{A.4}$$

trivially. Combining (A.2) and (A.4), it follows that

$$\frac{\partial^2}{\partial t^2} \langle \underset{\sim}{E} \rangle = - \frac{4\pi e^2}{m_e} \langle n\underset{\sim}{E} \rangle \ . \tag{A.5}$$

However, the average $\langle n\underset{\sim}{E} \rangle$ may be rewritten from Poisson's equation (2.3) as

$$\langle n\underset{\sim}{E}\rangle = n_o\langle\underset{\sim}{E}\rangle - \frac{1}{4\pi e}\langle\underset{\sim}{E}\frac{\partial}{\partial\underset{\sim}{x}}\cdot\underset{\sim}{E}\rangle \ . \qquad (A.6)$$

Using the identity $\underset{\sim}{E}(\partial/\partial\underset{\sim}{x})\cdot\underset{\sim}{E} = (\partial/\partial\underset{\sim}{x})\cdot(\underset{\sim}{E}\underset{\sim}{E}) - (\partial/\partial\underset{\sim}{x})(\underset{\sim}{E}^2/2)$ when $(\partial/\partial\underset{\sim}{x})\times\underset{\sim}{E} = 0$, it is clear that the last average in (A.6) vanishes for a spatially uniform ensemble. Consequently, Eq. (A.5) may be written as

$$\frac{\partial^2}{\partial t^2}\langle\underset{\sim}{E}\rangle + \omega_o^2\langle\underset{\sim}{E}\rangle = 0 \ . \qquad (A.7)$$

Thus, the uniform ensemble supports average electric fields oscillating exactly at the plasma frequency.[14] In the event that the average fields and currents are absent initially, i.e.,

$$\langle\underset{\sim}{E}\rangle = 0 = \langle n\underset{\sim}{v}\rangle \ , \qquad (A.8)$$

they remain so for all times. Moreover, from (A.1)

$$\langle\underset{\sim}{v}\rangle = 0 \ , \qquad (A.9)$$

for all times if Eqs. (A.8) and (A.9) are satisfied initially.

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13. In terms of the fluctuations in (2.10) and (2.11), the relevant expansion parameter is $\lambda \sim |\delta n/n_o| \sim |\delta \underset{\sim}{v}/v_p| \ll 1$, where v_p is the phase velocity characteristic of the waves under consideration. That is, $v_p \sim \omega_o/k$, where k is a typical wave number making up the disturbance.

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FIGURE CAPTIONS

Figure 1: Basic resonant four-wave process consisting of the merging of two waves into an intermediate virtual state, followed by the (instantaneous) decay of this virtual state into two further states.

Figure 2: "Resonant sphere" over which the $\underset{\sim}{k}_3$ integration in Eq. (3.16) is to be carried out.

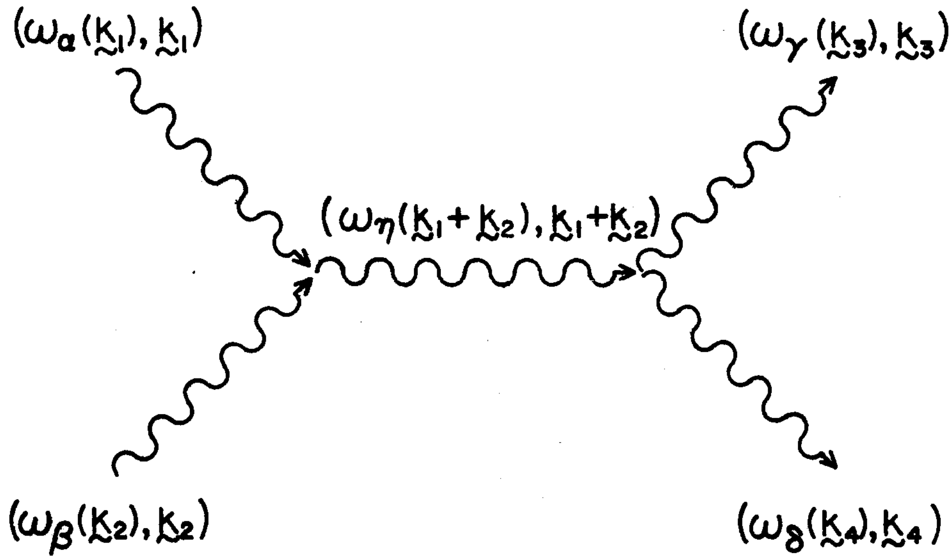

Figure 1

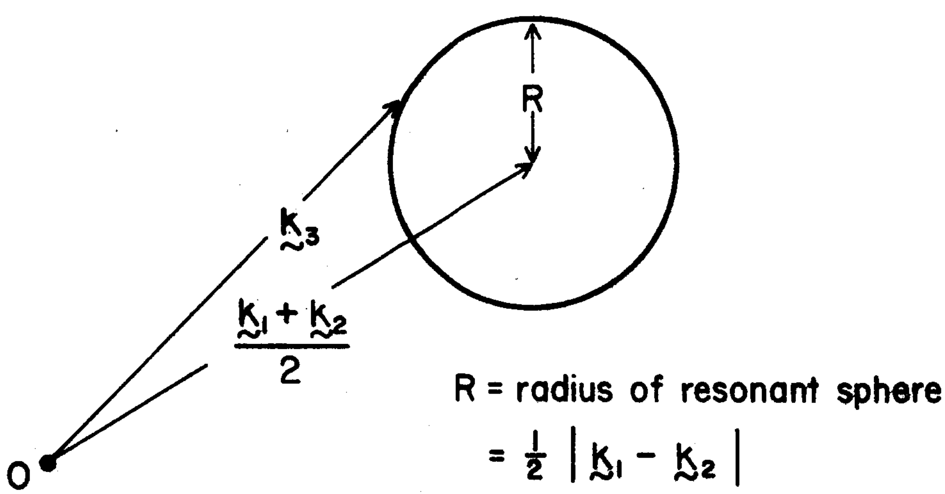

Figure 2