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ANALOG SIMULATION OF WIND TURBULENCE

By David L. Finn

Prepared for
George C. Marshall Space Flight Center
Huntsville, Alabama

Contract No. NAS8-2473

(Development of New Methods and Applications of Analog Computation)

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ABSTRACT

A mathematical model of random wind velocity is presented for use in the synthesis of an analog computer network to simulate wind turbulence. A synthesis technique, called the covariance-expansion method, is applied to the mechanization of the model. The output of the analog computer network simulates the effect of wind turbulence on a vehicle as it moves on an arbitrary path in space. The inputs to the analog computer network are (a) a Gaussian white-noise random process and (b) appropriate functions of time characterizing the variable position of the moving vehicle.

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I. INTRODUCTION

This technical note summarizes the most important results of the work on nonstationary random processes that has been carried out under Georgia Tech Research Project No. A-588 for the Flight Simulation Branch, Computation Division, of George C. Marshall Space Flight Center. A mathematical model of random wind velocity is presented, and the previously developed "covariance-expansion" synthesis procedure is applied to the mechanization of this mathematical model with an analog computer network. The structure of the mathematical model has been selected to minimize the amount of experimental wind data necessary to determine the parameters of the model and to avoid excessive complexity of the analog computer network that is used to mechanize the model.

The mathematical model characterizes the three components of the vector wind velocity as Gaussian random processes that are dependent on time and on spatial position. A procedure is shown for the synthesis of an analog computer network having as outputs three variables that approximate the components of the vector wind velocity at any specified time and position in space. These outputs simulate the effect of random wind turbulence on a rocket or other aerospace vehicle in flight. The analog computer network has as inputs a Gaussian white-noise random process, a function of time representing the instantaneous altitude of the moving vehicle, and a function of time representing the scalar velocity of the vehicle. A single analog computer network is used in the simulation. It is not necessary to synthesize a different network for each different flight path.

The covariance-expansion synthesis method has been discussed in detail in Project A-588 Technical Notes Nos. 3 and 11 (see Bibliography at end of this report). Details of proof of validity for the method are therefore not included in the present summary. Only those steps are included that are necessary for the implementation of the procedure.

Chapter V of this technical note discusses the mechanization of a simple mathematical model of one horizontal component of wind velocity. This mechanization permits the simulation of the wind disturbance that affects a vehicle moving on an arbitrary path in three-dimensional space. The parameters of this model have been determined by use of experimental wind data. Wind profiles generated by the analog computer network are shown for comparison with actual wind profiles determined by radar observation of an ascending Jimsphere balloon.

II. A MATHEMATICAL MODEL FOR RANDOM WIND VELOCITY

In this chapter a mathematical model of random wind velocity is presented. Wind velocity is characterized as a vector-valued Gaussian random process depending on time and spatial position. The structure of the mathematical model has been selected to minimize the amount of experimental wind data necessary to determine the parameters of the model and to avoid excessive complexity of the analog computer network that is used to mechanize the model.

2-1. Definition of Coordinate System

Wind velocity $\underline{w}$, expressed in meters per second, will be characterized as a vector-valued random process having three components--w_x, w_y, and w_z. The magnitude of the wind velocity will be denoted as w. The random process $\underline{w}$ is assumed to depend on time and on three coordinates describing spatial position. Time t is expressed in seconds. Position is described by a rectangular coordinate system, with the axes chosen in a somewhat unorthodox fashion for reasons of convenience which will be apparent later. The positive y axis, indicating altitude, is directed vertically upward from the surface of the earth. The x axis is chosen to be positive toward the east, while the z axis is taken positive toward the north. The spatial coordinates x, y, and z are expressed in kilometers.

In this technical note, two variations of a mathematical model for characterizing vector wind velocity are presented. In the first of these variations it is advantageous to describe the direction of the vector wind velocity $\underline{w}$ by two angles, θ and Ψ. As is shown in Figure 2.1, θ is the angle between the positive x axis and a radius vector in the horizontal x-z plane. This angle is measured in radians and is taken to be positive in the direction of rotation from the positive x axis to the positive z axis. The radius vector is directed either in the same direction as the projection of $\underline{w}$ in the horizontal x-z plane or in the direction opposite to this projection. As is shown in Figure 2.1, Ψ is the angle between the horizontal radius vector just defined and the vector wind velocity $\underline{w}$. This angle is measured in radians and is taken to be positive in the direction of rotation from the radius vector toward the positive y axis.

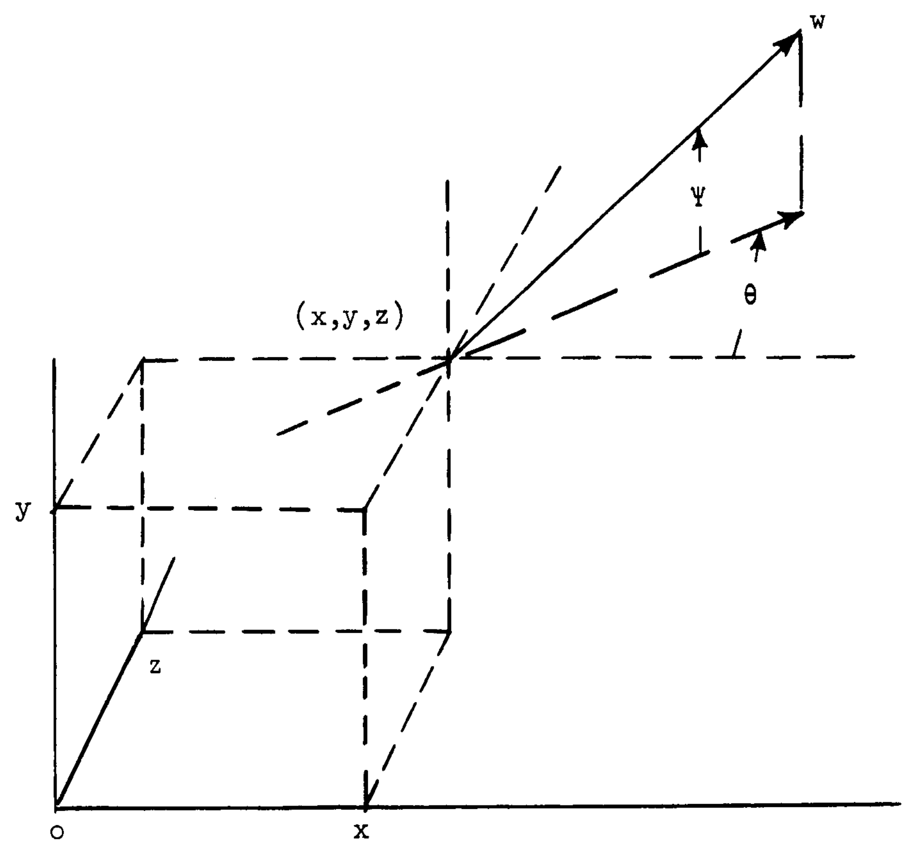


Figure 2.1. Coordinates Describing Position and Direction of Wind Velocity $\underline{w}$.

The angle Ψ describes the vertical orientation of the vector $\underline{w}$, while the angle θ is primarily related to the horizontal orientation of $\underline{w}$. It should be noted that the horizontal orientation of $\underline{w}$ is not uniquely determined unless both Ψ and θ are specified. For implementation of the mathematical model to be presented it is necessary that no constraints be imposed on the range of values that may be assumed by either directional angle.

The wind velocity may be expressed in general as

$$\underline{w} = w_x \, \underline{u}_x + w_y \, \underline{u}_y + w_z \, \underline{u}_z \quad , \tag{2.1}$$

where $\underline{u}_x$, $\underline{u}_y$, and $\underline{u}_z$ are unit vectors in the positive x, y, and z directions. An inspection of Figure 2.1 shows that at any position (x,y,z) the components of the vector wind velocity are given by the relationships

$$w_x = w \cos \Psi \cos \theta; \quad w_y = w \sin \Psi; \quad w_z = w \cos \Psi \sin \theta \quad . \tag{2.2}$$

In order to further define the mathematical model for wind velocity considered as a vector-valued random process in time, assumptions are made concerning various statistical properties of its magnitude and direction characteristics, as noted in the next five sections.

2-2. Assumption 1—Magnitude and Direction

First Variation: The initial simplifying assumption to be made is that the magnitude of the vector wind velocity is statistically independent of its direction, and furthermore that the directional angle θ is statistically independent of the directional angle Ψ. (Stated in another way, it is assumed that how hard the wind is blowing is not affected by the direction in which it is blowing.) It is assumed, in fine, that the three variables w, θ, and Ψ may be characterized as statistically independent, Gaussian random processes.

Second Variation: For this variation, it is assumed that the three rectangular components w_x, w_y, and w_z of the vector wind velocity may be characterized as statistically independent, Gaussian random processes.

Time limitations have not permitted an investigation of possible relationships and differences between the above two variations of Assumption 1. The second variation provides a less complex mechanization system when used with the synthesis procedure presented in this technical note.

2-3. Assumption 2—Variance

According to Assumption 1, the scalar wind-velocity magnitude w is characterized as a Gaussian random process $w(x,y,z,t)$ depending on position and time. Let a random variable w_1 be defined by specification of w for a single position (x_1,y_1,z_1) and time t_1 as

$$w_1 = w(x_1,y_1,z_1,t_1) \quad . \tag{2.3}$$

The variance of w_1 is defined as

$$\sigma_{w_1}^2 = E\left[(w_1 - E[w_1])^2\right] \quad , \tag{2.4}$$

where E is the expected value operator.

It is assumed that the variance of the wind velocity is a function $\sigma_{w_1}{}^2(y_1)$ only of the altitude y_1 and does not vary with x_1, z_1, and t_1.

Stated in another way, it is assumed that the magnitude of the random variations of the wind velocity does not change with time or with alterations of position in a fixed horizontal plane.

An identical assumption is to be made for the five other random processes θ, Ψ, w_x, w_y, and w_z listed in the two variations of Assumption 1.

2-4. Assumption 3--Correlation Coefficient

According to Assumption 1, the scalar wind-velocity magnitude w is characterized as a Gaussian random process $w(x,y,z,t)$ depending on position and time. Let two random variables w_1 and w_2 be defined by specification of w for any two positions and times as

$$\begin{aligned} w_1 &= w(x_1,y_1,z_1,t_1) \quad , \\ w_2 &= w(x_2,y_2,z_2,t_2) \quad . \end{aligned} \tag{2.5}$$

The distance s between the two point locations (x_1,y_1,z_1) and (x_2,y_2,z_2) will in general be given by

$$s = \left[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\right]^{\frac{1}{2}} \quad . \tag{2.6}$$

Similarly, the time interval τ between the two instants t_1 and t_2 may be expressed in general as

$$\tau = |t_2 - t_1| \quad . \tag{2.7}$$

The correlation coefficient ρ of the random variables w_1 and w_2 is defined as

$$\rho = \frac{E\left[(w_1 - E[w_1])(w_2 - E[w_2])\right]}{\sigma_{w_1} \sigma_{w_2}} \quad . \tag{2.8}$$

Here, σ_{w_1} is the standard deviation of the random variable w_1—i.e., the square root of the variance of w_1 as was defined by Equation (2.4). The correlation coefficient is a measure of the correlation or linear dependence between the random variables w_1 and w_2.

It is assumed that the correlation coefficient ρ as defined above is a function $\rho(s,\tau)$ only of the two variables s and τ. Stated in another way, the correlation of the wind velocities at two different positions and times depends only on the distance between the two positions and on the time interval between the two time instants.

An identical assumption is made for the five other random processes θ, Ψ, w_x, w_y, and w_z listed in the two variations of Assumption 1.

2-5. Assumption 4—Mean Value

The mean value m_w of the random process $w(x,y,z,t)$ characterizing the scalar wind-velocity magnitude is defined to be

$$m_w = E\left[w(x,y,z,t)\right] \quad . \tag{2.9}$$

It is assumed that the mean value of the wind velocity is a function $m_w(y)$ only of the altitude y and does not vary with x, z, and t. Stated in another way, it is assumed that the average value of the wind velocity does not change with time or with alterations of position in a fixed horizontal plane.

An identical assumption is made for the five other random processes θ, Ψ, w_x, w_y, and w_z listed in the two variations of Assumption 1.

2-6. Assumption 5—Factorization of Correlation Coefficient

It is assumed that the correlation coefficient $\rho(s,\tau)$ as defined in Assumption 3 above may be expressed as the product of a function depending only on s multiplied by a function depending only on τ. That is,

$$\rho(s,\tau) = \rho_1(s)\ \rho_2(\tau) \quad . \tag{2.10}$$

This assumption is made for the correlation coefficients of all six random processes w, Ψ, θ, w_x, w_y, and w_z listed in the two variations of Assumption 1.

This fifth assumption appears to be considerably more restrictive than the first four. It is postulated because its use permits a simplification both in the experimental determination of the parameters of the mathematical model of the wind velocity and in the analog computer mechanization of the model.

III. EXPERIMENTAL DETERMINATION OF THE PARAMETERS OF THE MATHEMATICAL MODEL FOR RANDOM WIND VELOCITY

3-1. Introduction

In Chapter II, scalar wind-velocity magnitude w has been characterized as a random process $w(x,y,z,t)$ depending on both space and time coordinates. As was mentioned previously, this model has been constructed using assumptions selected to simplify the experimental determination of parameters of the model and also to simplify the physical mechanization of the model by use of an analog computer network. The basic assumption is that the wind velocity is represented by a Gaussian random process. Thus, the process is completely specified by a determination of its first and second order moments. The experimental determination of these moments--as represented by the mean, variance, and correlation coefficient--is discussed in this chapter.

A general investigation of the extent of validity of the mathematical model would be of very substantial magnitude and has not been undertaken as part of the present research. However, a calculation of parameters using experimental data has been carried out using a small number of wind profiles. These results are presented in Chapter V. Also, in that chapter some simulated wind profiles generated by an analog computer network are shown for comparison with experimentally determined wind profiles.

3-2. Determination of Mean Value

According to Assumption 4 of Chapter II, the mean value $m_w(y)$ of the scalar wind velocity is a function only of the altitude y and does not vary with x, z, and t. The mean value at a fixed altitude may be estimated by the use of N measured samples W_k, $k = 1,2,..N$, of the instantaneous wind velocity taken at the specified altitude. The estimate of the mean value $m_w(y)$ may be taken to be the sample mean

$$M(y) = \frac{1}{N} \sum_{k=1}^{N} W_k \quad . \tag{3.1}$$

3-3. Determination of Variance

According to Assumption 2 of Chapter II, the variance $\sigma_w^2(y)$ of the wind velocity is a function only of the altitude y and does not vary with x, z, and t. The variance at a fixed altitude may be estimated by the use of N measured samples W_k, k = 1,2,..N, of the instantaneous wind velocity taken at the specified altitude. The estimate of the variance $\sigma_w^2(y)$ may be taken to be the sample variance

$$V^2(y) = \frac{1}{N} \sum_{k=1}^{N} \left| W_k - M(y) \right|^2 \quad . \tag{3.2}$$

Here M(y) is the sample mean defined by Equation (3.1). This is calculated using the same N samples W_k, k = 1,2,..N, that appear explicitly in Equation (3.2).

3-4. Determination of the Correlation Function

Two random variables w_1 and w_2, representing instantaneous wind velocity at two positions and times, are defined in accordance with Equations (2.5) as

$$\begin{aligned} w_1 &= w(x_1,y_1,z_1,t_1) \\ w_2 &= w(x_2,y_2,z_2,t_2) \quad . \end{aligned} \tag{3.3}$$

The distance between the two positions (x_1,y_1,z_1) and (x_2,y_2,z_2) is given by the variable s defined in Equation (2.6), and the interval between the two time instants t_1 and t_2 is given by the variable τ defined in Equation (2.7).

The correlation coefficient ρ of the random variables w_1 and w_2 is defined by Equation (2.8). According to Assumption 3 in Chapter II, the correlation coefficient is a function $\rho(s,\tau)$ dependent only on the two variables s and τ. The correlation coefficient for the two fixed positions and times may be estimated by the use of two sets of N each experimental samples of the instantaneous wind velocity--viz., $W_k(x_1,y_1,z_1,t_1)$ and $W_k(x_2,y_2,z_2,t_2)$, k = 1,2,..N. The estimate of the correlation coefficient $\rho(s,\tau)$ may be taken to be

$$P(s,\tau) = \frac{- M(y_1)\ M(y_2) + \sum_{k=1}^{N} W_k(x_1,y_1,z_1,t_1)\ W_k(x_2,y_2,z_2,t_2)}{N\ V(y_1)\ V(y_2)} \quad . \tag{3.4}$$

Here again, $M(y_1)$ and $M(y_2)$ are sample means as defined by Equation (3.1), while $V(y_1)$ and $V(y_2)$ are the square roots of variance estimates as defined by Equation (3.2). All of these are to be calculated using the same two sets of N each samples that appear explicitly in Equation (3.4).

The use of Assumption 5 in Chapter II greatly facilitates the estimation of the function $\rho(s,\tau)$. A collection of profiles (sample functions) showing the time variation of the wind velocity experienced by a sensor at a fixed position in space can, by use of Equation (2.10), be used to approximate the function

$$\rho(s=0,\tau) = \rho_1(s=0)\ \rho_2(\tau) \quad . \tag{3.5}$$

Here, in the calculations determining $\tau = |t_2 - t_1|$, the time t_1 may be fixed at any convenient value and t_2 may be allowed to vary. Without loss of generality $\rho_1(s=0)$ may be assumed to equal unity. When calculations are made in the manner just described by using samples taken through use of a stationary sensor, the estimate of the function $\rho_2(\tau)$ may be expressed by the estimator defined in Equation (3.4):

$$\begin{aligned} \rho_2(\tau) &= \rho(s=0,\tau) \\ &\approx P(s=0,\tau) \quad . \end{aligned} \tag{3.6}$$

Next, a collection of profiles (sample functions) showing the time variation of the wind velocity experienced by a sensor moving along any specified path may be used in making an estimate of the function $\rho(s,\tau)$ for the related values of s and τ that characterize the constrained path. For example, the wind sample profiles may be determined by observations on a vertically ascending balloon. Here, the coordinates x and z remain fixed, and the variable s is calculated as the difference of two altitudes y_1 and

y_2. It is necessary to make observations for N independent balloon ascents to accumulate the required number of samples, and it is assumed that the rate at which the balloon rises is the same for each different ascent.

Once the function $\rho(s,\tau)$ is determined for any arbitrary sensor path, the function $\rho_1(s)$ is easily determined by use of Equations (2.10) and (3.6):

$$\begin{aligned} \rho_1(s) &= \rho(s,\tau)/\rho_2(\tau) \\ &= \rho(s,\tau)/\rho(s{=}0,\tau) \quad . \end{aligned} \tag{3.7}$$

The determination of the function $\rho_1(s)$ as expressed in Equation (3.7) and the determination of the function $\rho_2(\tau)$ as expressed in Equation(3.6) permit the calculation of the function $\rho(s,\tau)$ defining the correlation coefficient by use of Equation (2.10):

$$\rho(s,\tau) = \rho_1(s)\ \rho_2(\tau) \quad . \tag{3.8}$$

The assumption that the wind velocity does not vary with time at any fixed position may be incorporated into the mathematical model by assuming the function $\rho_2(\tau)$ in Equation (2.10) to equal the constant unity. In this case, it is not necessary to perform the set of measurements that have been described for the stationary sensor. Here, samples taken in the manner described by use of an ascending balloon are adequate to provide estimates for the mean value, the variance, and the correlation function. For this assumption, the random process characterizing wind velocity depends upon position parameters alone.

In this chapter, the formulas related to the determination of parameters for the wind velocity model have been expressed in terms of the magnitude w of the vector wind velocity. The formulas are equally applicable to the determination of the parameters characterizing the five other random processes θ, Ψ, w_x, w_y, and w_z introduced in the mathematical model.

IV. MECHANIZATION OF THE MATHEMATICAL MODEL

4-1. Introduction

In Chapter II the wind velocity of any position in space and at any time was characterized as a random process $w(x,y,z,t)$. The coordinates of a vehicle or sensing element moving along an arbitrary path through space may be described by three time functions $x(t)$, $y(t)$, and $z(t)$. In this case, the wind velocity experienced by the sensing element is described by the composite, time-parameter random process

$$f(t) = w\Big(x(t), y(t), z(t), t\Big) \quad . \tag{4.1}$$

In this chapter the covariance-expansion method of synthesis that was presented in Project A-588 Technical Notes Nos. 3 and 11 is applied to the approximation of the random process $f(t)$. This method has been discussed in detail in these previous technical notes; hence, no details of proof of validity for the procedure are presented here. Only those steps that are necessary for implementation of the method are included.

The analog computer network synthesized by the procedure discussed in this chapter has as outputs three variables that approximate the components of the vector wind velocity at any specified time and position in space. These outputs may be used to simulate the effect of wind turbulence on a rocket or other aerospace vehicle in flight. The analog computer network has as inputs a Gaussian white-noise waveform, a function of time representing the instantaneous altitude of the moving vehicle, and a function of time representing the scalar velocity of the vehicle.

It should be noted that the mechanization that is presented here allows for a three-dimensional movement of the vehicle, or sensing element, with an analog computer network that is identical to the one that is required for a vertical, one-dimensional motion of the sensing element. This simplified mechanization is accomplished by use of certain approximations for the distance s (see Equations (4.5) and (4.6)).

The synthesis method is presented in this chapter in terms of simulation of the wind-velocity magnitude w. However, the synthesis method is equally applicable to the simulation of the five other random processes θ, Ψ,

w_x, w_y, and w_z listed under Assumption 1 for the mathematical model in Section 2-2 of Chapter II.

4-2. The Covariance-Expansion Synthesis Method

The mean value of the wind velocity, as expressed by Equation (2.9), is assumed to be a function $m_w(y)$ of altitude y only. This is a determinate function--not a random process. Accordingly, the mean value may be generated by a function generator having as input the variable y. This being accomplished, without loss of generality the covariance-expansion method is applied to the generation of a random process f(t) having a mean value equal to zero. The total wind velocity is obtained by adding the random process f(t) to the determinate function $m_w(y)$.

The covariance function $r(t',t)$ of the random process f(t) is defined as

$$r(t',t) = E\left[f(t_1)\ f(t_2)\right] \tag{4.2}$$

where:

t' = larger of (t_1 and t_2)

t = smaller of (t_1 and t_2).

Implementation of the covariance-expansion method requires that the covariance function be expressed as a finite expansion in the form

$$r(t',t) = \sum_{i=1}^{n} \phi_i(t')\ \gamma_i(t) \quad . \tag{4.3}$$

The covariance function for the random wind velocity w, with mean value assumed equal to zero, may be found by rewriting Equation (2.8):

$$E[w_1\ w_2] = \rho(s,\tau)\ \sigma_w(y_1)\ \sigma_w(y_2) \quad . \tag{4.4}$$

As before, $w_1 = w(x_1,y_1,z_1,t_1)$ and $w_2 = w(x_2,y_2,z_2,t_2)$ represent the wind velocity at any two positions and times. An inspection of Equation (4.4) shows that this covariance function depends only on altitude, on the distance s between the two space positions (x_1,y_1,z_1) and (x_2,y_2,z_2), and on the time interval $\tau = |t_2 - t_1| = t' - t$. The covariance function, expressed in Equation (4.2), of the random process f(t) is found from Equation (4.4) by allowing the position coordinates (x,y,z) to become functions of time (x(t), y(t), z(t)) describing the varying position of the sensing element.

In order that the covariance function of f(t) may be expressed in the required form of Equation (4.3) it is expedient to approximate the distance s between any two positions of the moving sensing element in terms of the scalar velocity v(t) of the element. This provides the incidental benefit of allowing a mechanization designed for one-dimensional motion of the sensing element to be used without modification for three-dimensional motion.

Two methods of approximation of the distance s will be presented. First s may be approximated by the expression

$$s \approx \int_0^{t'} v(\lambda)\, d\lambda - \int_0^{t} v(\lambda)\, d\lambda \quad . \tag{4.5}$$

Here, v(t) is the scalar velocity of the sensing element. It is noted that the first term of the expression is a function of t' only, and the second term is a function of t only.

The right-hand side of (4.5) exactly represents total distance traveled along the path of motion of the sensing element during the time interval $\tau = t'-t$. This is obviously a good approximation for the distance between the positions at times t' and t provided motion is almost along a straight line.

The correlation coefficient $\rho(s,\tau)$ tends to zero as distance s and time interval τ increase. For validity, the approximation of (4.5) requires that the movement of the sensing element be approximately along a straight line either for all distances s that are small enough that $\rho(s,\tau)$ is appreciably different from zero or for all time intervals τ small enough that $\rho(s,\tau)$ is appreciably different from zero. For larger values of s or τ an accurate calculation of s is not needed because all calculations of the correlation coefficient provide a value very nearly equal to zero.

A second possible approximation for the distance s is obtained by slightly modifying the relationship that expresses distance as the product of velocity multiplied by time. Thus,

$$s \approx t' \; v(t') - t \; v(t) \quad . \tag{4.6}$$

It is noted that one of the two product terms in (4.6) involves only t', and the other term involves only t.

The approximation of (4.6) is more restrictive than that of (4.5), but in some cases is expected to lead to a less complex mechanization. For validity, the approximation of (4.6) requires that the movement of the sensing element be approximately at constant velocity and along a straight line either for all distances s that are small enough that $\rho(s,\tau)$ is appreciably different from zero or for all time intervals τ small enough that $\rho(s,\tau)$ is appreciably different from zero.

It will be assumed that the covariance function for the composite random process $f(t)$, as found by use of (4.4), can be expressed in the form shown in Equation (4.3) when one of the two approximations discussed above is utilized for the distance variable s. Representation in this form, either exactly or as an approximation, is necessary for the implementation of the covariance-expansion synthesis method.

It is to be emphasized that the functions $\phi_i(t)$ and $\gamma_i(t)$ in Equation (4.3) depend explicitly on the altitude $y(t)$ and on the scalar velocity $v(t)$ of the sensing element because of the use of (4.4) and either (4.5) or (4.6) in the evaluation of $r(t',t)$.

An analog computer network is to be synthesized having three inputs: a Gaussian white-noise waveform, a function of time $y(t)$ representing altitude, and a function of time $v(t)$ representing the instantaneous scalar velocity of the sensing element. The output of the network is to be the composite random process $f(t)$ representing the wind velocity observed by the moving sensing element.

The analog computer network to be synthesized is characterized by the nth-order differential equation

$$\begin{aligned} f^{(n)} + p_{n-1}(t)\, f^{(n-1)} + \ldots + p_1(t)\, f^{(1)} + p_o(t)\, f \\ = q_{n-1}(t)\, g^{(n-1)} + \ldots + q_1(t)\, g^{(1)} + q_o(t)\, g \quad . \end{aligned} \tag{4.7}$$

Here, $f^{(k)}$ denotes the <u>k</u>th derivative of the function f with respect to time. The function g represents the Gaussian white-noise input to the analog computer network.

In order to avoid differentiation of the noise input g(t), the nth-order differential Equation (4.7) may be converted into a set of n first-order differential equations. To make this conversion the following definitions are used:

$$f(t) = f_1(t) \tag{4.8}$$

$$f_1^{(1)} = f_2 - a_{n-1}(t)\, f_1 + b_{n-1}(t)\, g$$

$$f_2^{(1)} = f_3 - a_{n-2}(t)\, f_1 + b_{n-2}(t)\, g$$

$$- - - - -$$

$$f_{n-1}^{(1)} = f_n - a_1(t)\, f_1 + b_1(t)\, g$$

$$f_n^{(1)} = \quad - a_o(t)\, f_1 + b_o(t)\, g$$

This set can be written more concisely in matrix notation as

$$F^{(1)} = A(t)\ F + B(t)\ g \tag{4.9}$$

$$[f] = [f_1] = H\ F$$

where

$$F = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_{n-1} \\ f_n \end{bmatrix} \qquad B(t) = \begin{bmatrix} b_{n-1}(t) \\ b_{n-2}(t) \\ \cdot \\ \cdot \\ b_1(t) \\ b_o(t) \end{bmatrix}$$

$$H = [\,1 \;\; 0 \;\cdot\;\cdot\; 0 \;\; 0\,]$$

$$A(t) = \begin{bmatrix} -a_{n-1}(t) & 1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ -a_{n-2}(t) & 0 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ -a_1(t) & 0 & 0 & 0 & \cdot & \cdot & 0 & 1 \\ -a_o(t) & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \end{bmatrix}$$

The elements a_k and b_k in (4.8) are related to the coefficients p_k and q_k in (4.7) as follows:

$$p_k = \sum_{j=0}^{n-1-k} \frac{(n-1-j)!}{k!\,(n-1-j-k)!}\, a_{n-1-j}^{(n-1-j-k)} \tag{4.10}$$

$$q_k = \sum_{j=0}^{n-1-k} \frac{(n-1-j)!}{k!\,(n-1-j-k)!}\, b_{n-1-j}^{(n-1-j-k)} \; . \tag{4.11}$$

If the p_k and q_k are known, then (4.10) and (4.11) can be solved sequentially for the a_k and b_k.

A determinant L may be defined as

$$L = \begin{vmatrix} f(t) & \phi_1(t) & \cdot & \cdot & \phi_n(t) \\ f^{(1)}(t) & \phi_1^{(1)}(t) & \cdot & \cdot & \phi_n^{(1)}(t) \\ \cdot & & & & \\ \cdot & & & & \\ f^{(n)}(t) & \phi_1^{(n)}(t) & \cdot & \cdot & \phi_n^{(n)}(t) \end{vmatrix} \tag{4.12}$$

If this determinant is equated to zero it can be shown that the coefficient of $f^{(k)}(t)$ in the resulting expression is equal to the corresponding coefficient $p_k(t)$ in Equation (4.7). The elements $a_k(t)$ that appear in the equations of (4.8) and the matrix A(t) of (4.9) can be obtained directly by using (4.10). It is noted that the elements $a_k(t)$ depend explicitly on the altitude y(t) and the velocity v(t) of the sensing element because of the dependence of $\phi_i(t)$ on these functions.

To complete the synthesis procedure, the elements $b_k(t)$ of (4.8) and (4.9) must be determined.

The $\phi_i(t)$ of the covariance function of (4.3) may be used in the construction of a fundamental matrix $\Phi(t)$:

$$\Phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{21}(t) & \cdot & \cdot & \phi_{n1}(t) \\ \phi_{12}(t) & \phi_{22}(t) & \cdot & \cdot & \phi_{n2}(t) \\ \cdot & & & & \\ \cdot & & & & \\ \phi_{1n}(t) & \phi_{2n}(t) & \cdot & \cdot & \phi_{nn}(t) \end{bmatrix} \tag{4.13}$$

where

$$\begin{aligned} \phi_{ij} &= \phi^{(1)}_{i(j-1)} + a_{n-j+1}\,\phi_{i1} \\ \phi_{k1} &= \phi_k(t) \; \{k = 1,2, \,..n\} \quad . \end{aligned} \tag{4.14}$$

Here, the $\phi_k(t)$ of Equation (4.3) are assumed to be linearly independent. However, if one of the $\phi_i(t)$ is not linearly independent, it may be expressed as a combination of the remainder and the index n can be reduced by one.

It must again be emphasized that the $\phi_k(t)$ depend on the functions v(t) and y(t). Thus, the chain rule must be used when performing the differentiation required in (4.14). Both ϕ_k and its derivative depend explicitly on the functions v(t) and y(t).

The elements of the fundamental matrix $\Phi(t)$ can be determined by application of (4.14) once the $a_k(t)$ have been found by the procedure that has just been explained.

A matrix $R(t',t)$ may now be defined as

$$R(t',t) = \Phi(t')\, D(t)\, \Phi^T(t) \qquad (4.15)$$

Here, the elements $d_{ij}(t)$ of the matrix $D(t)$ are defined in terms of the $\phi_i(t)$ and $\gamma_i(t)$ of (4.3) as

$$\begin{aligned} d_{ij}(t) &= \frac{\gamma_i(t)}{\phi_i(t)} \quad \text{for } i = j \\ &= 0 \qquad\quad \text{for } i \neq j \end{aligned} \qquad (4.16)$$

Also, Φ^T denotes the transpose of the matrix Φ.

A matrix $R^*(t',t)$ is now defined as

$$\begin{aligned} R^*(t',t) &= R^T(t,t') \\ &= \Phi(t)\, D(t')\, \Phi^T(t') \quad . \end{aligned} \qquad (4.17)$$

Let $\Delta(t',t)$ denote the difference

$$\Delta(t',t) = R(t',t) - R^*(t',t) \quad . \qquad (4.18)$$

Finally, let $\delta_{ii}(t)$ be the diagonal elements of the matrix

$$\left. \frac{\partial}{\partial t'} \Delta(t',t) \right|_{t' = t} \quad . \qquad (4.19)$$

It can be shown that

$$b_{n-i}(t) = \sqrt{-\delta_{ii}(t)} \quad , \quad i = 1,2,..n \quad . \qquad (4.20)$$

The matrix $B(t)$ is now specified, thus completing the synthesis procedure.

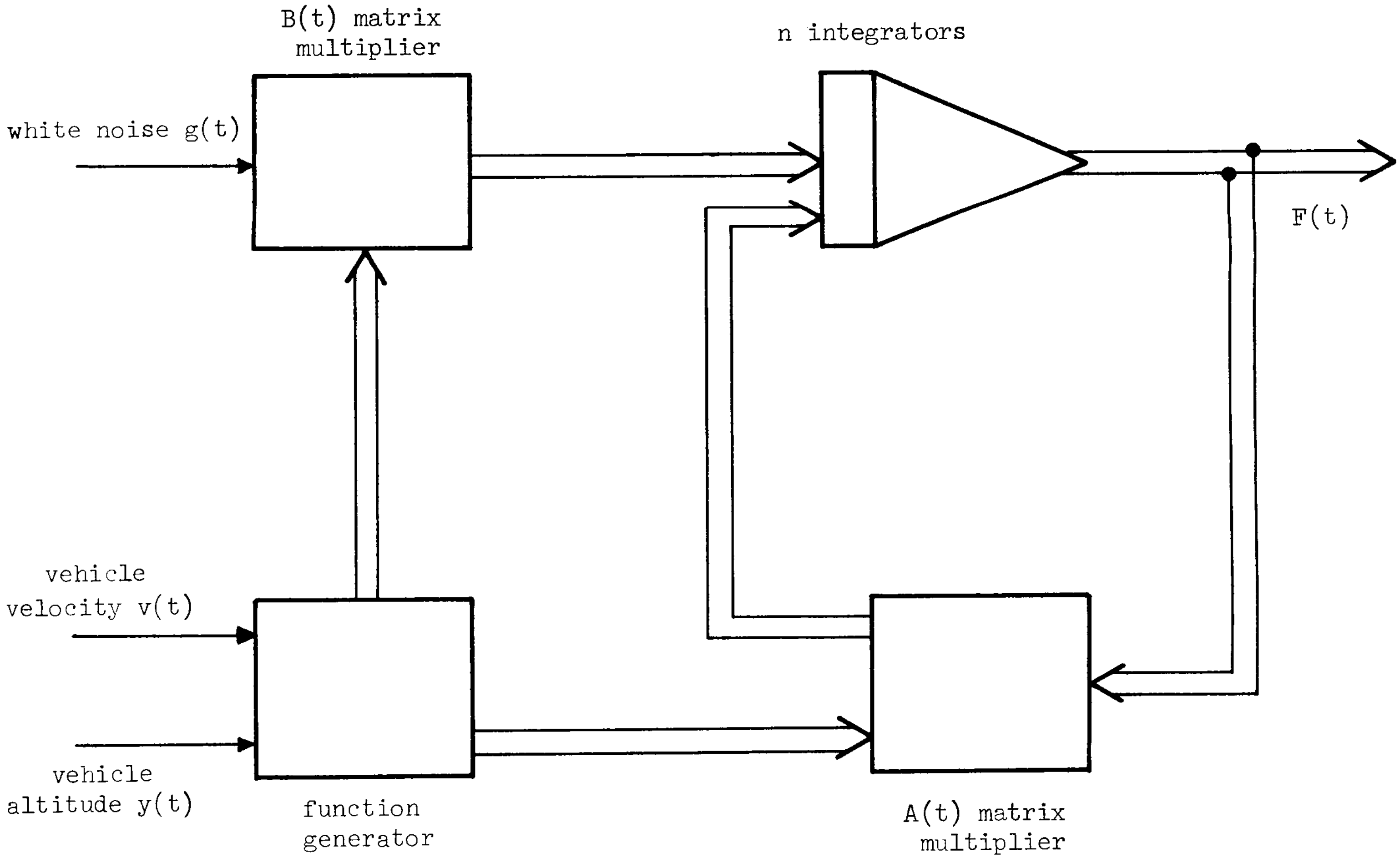
4-3. The Mechanization System

The mechanization system for generating a nonstationary random process that approximates wind turbulence is obtained by constructing an analog computer network that realizes the differential equations of (4.8) or (4.9). A block diagram of the mechanization system is shown in Figure 4.1. It is noted that differentiation of the input white-noise waveform is not required in this system.

It is important to achieve the proper integrator initial conditions at the beginning of a computation cycle (t=0) if the output of the mechanization system is to realize the correct covariance function. Different initial conditions applied to the same mechanization system with the same position-function input may give rise to widely divergent covariance functions. The initial conditions may be altered by manipulation of the white-noise input and the position-function inputs during resetting of the analog computer just prior to $t = 0$. In general, it is required that the steady-state operation obtained during the reset cycle correspond to the desired initial values of the position-function inputs at the beginning of the computation interval.

The mechanization system shown in Figure 4.1 generates a single component of the vector wind velocity. Two additional networks of the same type are required if all three components are to be generated. In case the second variation of Assumption 1 of the mathematical model is used, each of the three velocity components w_x, w_y, and w_z is generated by a separate white-noise generator associated with an analog computer network of the type shown in Figure 4.1.

In case the first variation of Assumption 1 of the mathematical model is used, each of the three variables w, θ, and Ψ is generated by a separate white-noise generator associated with an analog computer network of the type shown in Figure 4.1. In turn, the three velocity components w_x, w_y, and w_z are obtained from the three variables w, θ, and Ψ by a mechanization of the relationships given as Equation (2.2).



Note: Double lines indicate multivariable signal flow.

Figure 4.1. Mechanization System Derived by the Covariance-Expansion Synthesis Procedure.

V. MECHANIZATION OF A SIMPLE MODEL

5-1. Introduction

The covariance-expansion synthesis method is applied in this chapter to the mechanization of a simple mathematical model intended to simulate the behavior of random wind. Wind velocity profiles generated by an analog computer network used to implement the mechanization are presented for comparison with experimentally determined wind profiles.

In the example of this chapter a single component w_x of wind velocity is generated. The output f(t) of the analog computer network approximates the component w_x of random wind velocity that is experienced by a moving vehicle or sensing element. The sensing element may follow an arbitrary path in three-dimensional space. The three required inputs to the simulation network are a Gaussian white-noise waveform, a function y(t) representing the instantaneous altitude of the sensing element, and a function v(t) representing the scalar velocity of the sensing element.

5-2. Application of the Synthesis Procedure

It will be assumed that the standard deviation of the wind velocity w is a constant not varying with time or position:

$$\sigma_w(y) = \sigma \quad . \tag{5.1}$$

It will be assumed that the correlation coefficient of the wind velocity is given by the expression

$$\rho_w(s,\tau) = e^{-\alpha s}\, e^{-\beta\tau} \quad . \tag{5.2}$$

The correlation function of w as expressed by (4.4) becomes

$$\begin{aligned} E[w_1\, w_2] &= \rho_w(s,\tau)\, \sigma_w(y_1)\, \sigma_w(y_2) \\ &= \sigma^2\, e^{-\alpha s}\, e^{-\beta\tau} \quad . \end{aligned} \tag{5.3}$$

By using the approximation for distance s given in (4.5), the definition of τ from (2.7) and (4.2), and the covariance function of (5.3), the covariance function of f(t) as expressed by (4.2) or (4.4) becomes

$$r(t',t) = E[w_1\ w_2] \tag{5.4}$$
$$= \sigma^2 \exp\left[-\beta t' - \alpha \int_0^{t'} v(\lambda)\ d\lambda\right] \exp\left[\beta t + \alpha \int_0^{t} v(\lambda)\ d\lambda\right] .$$

This expression is seen to be in the required form of (4.3) with n = 1. The coefficients $\phi_1(t')$ and $\gamma_1(t)$ of (4.3) are:

$$\phi_1(t') = \sigma \exp\left[-\beta t' - \alpha \int_0^{t'} v(\lambda)\ d\lambda\right] . \tag{5.5}$$

$$\gamma_1(t) = \sigma \exp\left[\beta t - \alpha \int_0^{t} v(\lambda)\ d\lambda\right] . \tag{5.6}$$

The p_k coefficients of (4.7) are determined by use of the determinant L defined in (4.12):

$$L = \begin{vmatrix} f(t) & \phi_1(t) \\ f^{(1)}(t) & \phi_1{}^{(1)}(t) \end{vmatrix} . \tag{5.7}$$

Expanding this determinant by the use of (5.5) and equating the determinant to zero, there results:

$$f^{(1)} + \left(\beta + \alpha\ v(t)\right) f = 0 . \tag{5.8}$$

A comparison of (5.8) with (4.7) shows that

$$p_o(t) = \beta + \alpha\ v(t) . \tag{5.9}$$

Use of (4.10) provides the single coefficient of the A(t) matrix of (4.9):

$$a_o(t) = p_o(t) = \beta + \alpha\ v(t) . \tag{5.10}$$

The single coefficient of the $\Phi(t)$ matrix of (4.13) is found by use of (4.14) and (5.5):

$$\begin{aligned} \phi_{11}(t) &= \phi_1(t) \\ &= \sigma \exp\left[-\beta t - \alpha \int_0^t v(\lambda)\, d\lambda\right] \quad . \end{aligned} \tag{5.11}$$

The single coefficient of the D(t) matrix of (4.15) is found by the use of (4.16), (5.5), and (5.6):

$$\begin{aligned} d_{11}(t) &= \frac{\gamma_1(t)}{\phi_1(t)} \\ &= \exp\left[2\,\beta t + 2\,\alpha \int_0^t v(\lambda)\, d\lambda\right] \quad . \end{aligned} \tag{5.12}$$

The single element matrix R(t′,t) of (4.15) is found by use of (5.11) and (5.12):

$$\begin{aligned} R(t',t) &= \Phi(t')\, D(t)\, \Phi^T(t) \\ &= \sigma^2 \exp\left[-\beta t' - \alpha \int_0^t v(\lambda)\, d\lambda\right] \exp\left[\beta t + \alpha \int_0^t v(\lambda)\, d\lambda\right] \quad . \end{aligned} \tag{5.13}$$

The matrix $R^*(t,'t)$ of (4.17) is

$$\begin{aligned} R^*(t',t) &= R^T(t,t') \\ &= \sigma^2 \exp\left[\beta t' + \alpha \int_0^{t'} v(\lambda)\, d\lambda\right] \exp\left[-\beta t - \alpha \int_0^t v(\lambda)\, d\lambda\right] \quad . \end{aligned} \tag{5.14}$$

The difference matrix of (4.18) is found by use of (5.13) and (5.14).

$$\Delta(t',t) = R(t',t) - R^{*}(t',t)$$

$$= \sigma^2 \exp\left[-\beta t' - \alpha \int_0^{t'} v(\lambda)\, d\lambda\right] \exp\left[\beta t + \alpha \int_0^{t} v(\lambda)\, d\lambda\right] \tag{5.15}$$

$$- \sigma^2 \exp\left[\beta t' + \alpha \int_0^{t'} v(\lambda)\, d\lambda\right] \exp\left[-\beta t - \alpha \int_0^{t} v(\lambda)\, d\lambda\right] .$$

The matrix of (4.19) is found by use of (5.15):

$$\left. \frac{\partial}{\partial t'} \Delta(t',t) \right|_{t' = t} = \left[\delta_{11}(t)\right]$$

$$= \left[-2\, \sigma^2 \left(\beta + \alpha\, v(t)\right)\right] \tag{5.16}$$

Finally, the single coefficient in the B(t) matrix of (4.9) is found by use of (4.20) and (5.16).

$$b_o(t) = \sqrt{-\delta_{11}(t)}$$

$$= \sigma \sqrt{2\left(\beta + \alpha\, v(t)\right)} . \tag{5.17}$$

Specification of this coefficient completes the synthesis procedure. The differential equation characterizing the mechanization system for the realization of the random wind velocity f(t) is obtained by substituting the coefficients of (5.10) and (5.17) in the equation of (5.8):

$$\frac{d}{dt} f(t) = - a_o(t)\, f(t) + b_o(t)\, g(t)$$

$$= - \left[\beta + \alpha\, v(t)\right] f(t) + \sigma\, g(t) \sqrt{2[\beta + \alpha\, v(t)]} . \tag{5.18}$$

The solution of this differential equation realizes the composite random process f(t). As was explained in Section 4-2, the output of a function generator may be added to this solution to produce a random process having nonzero mean value $m_w(y)$. The block diagram of an analog computer network whose output approximates the composite random process f(t) is shown in Figure 5.1.

5-3. Comparison with Experimental Wind Profiles

Experimental wind data taken by radar observation of an ascending Jimsphere balloon* was used to determine parameters for the mechanization presented in Section 5-2. Thirty experimental wind profiles were used to calculate the mean value m(y) and standard deviation σ(y) of the x component w_x of the wind velocity. The estimators of Equations (3.1) and (3.2) were used for these calculations. These experimental values are shown in Figures 5.2 and 5.3. The same profiles were used to calculate the correlation coefficient as a function of the vertical distance interval $s = |y_2 - y_1|$. The estimator of Equation (3.4) was used for this calculation. The experimental values are shown in Figure 5.4.

In order to fit the mathematical model of Section 5-2 to the experimental data just discussed, the following parameter values were selected:

$$\left.\begin{aligned} m_{w_x}(y) &= \frac{40}{13}\,y, \quad 0 \le y \le 13\ , \\ &= \frac{358}{7} - \frac{6}{7}\,y, \quad y > 13\ ; \\ \sigma_{w_x}(y) &= 12.5\ ; \\ \rho_{w_x}(s,\tau) &= e^{-0.1\,s}\ . \end{aligned}\right\} \tag{5.19}$$

These parameter values for the mathematical model are shown as solid lines in Figures 5.2, 5.3, and 5.4.

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* See Reference 7 in the Bibliography.

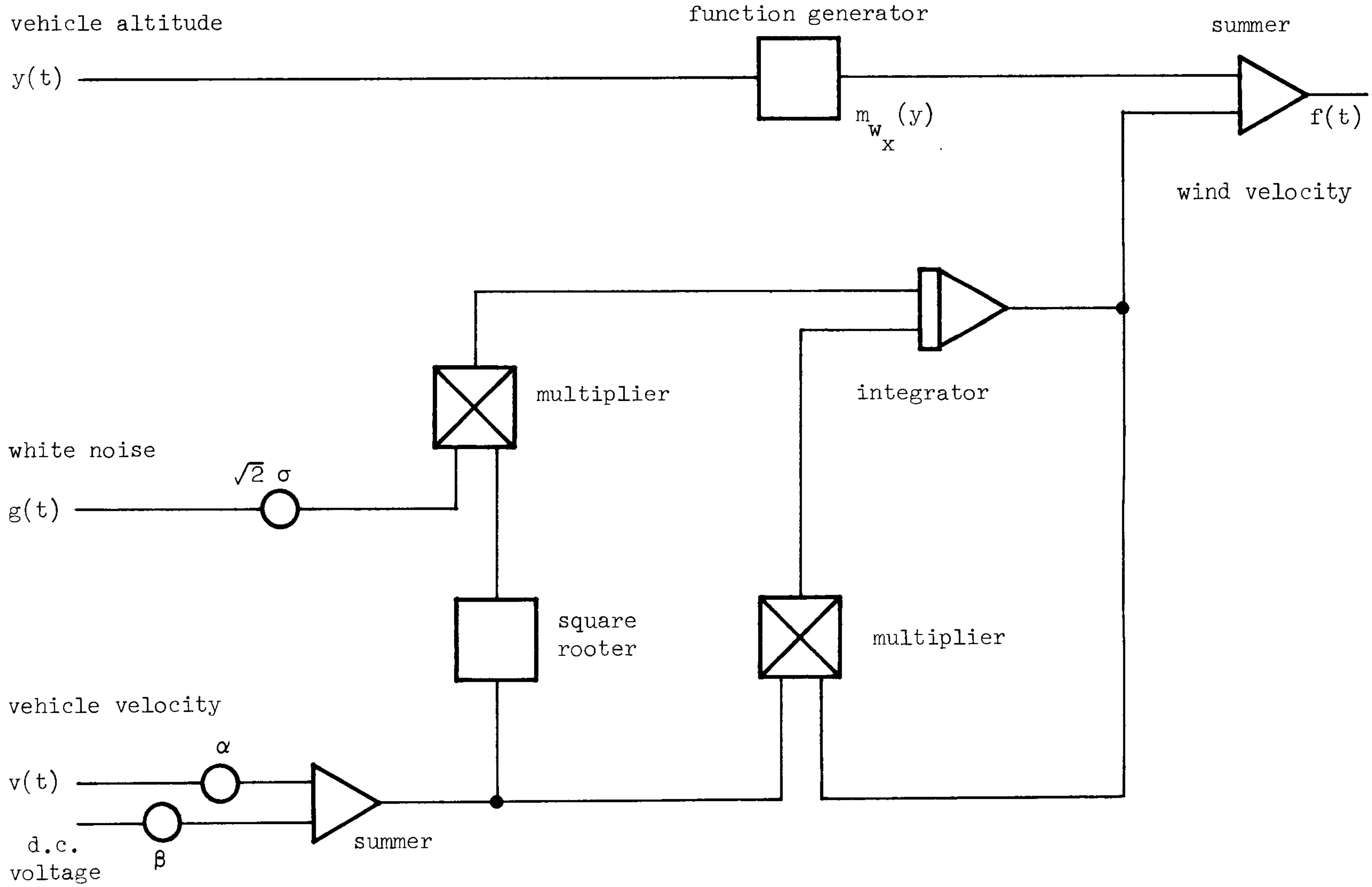

Figure 5.1 Analog Computer Network for the Generation of the Random Wind Velocity Modeled in Chapter V.

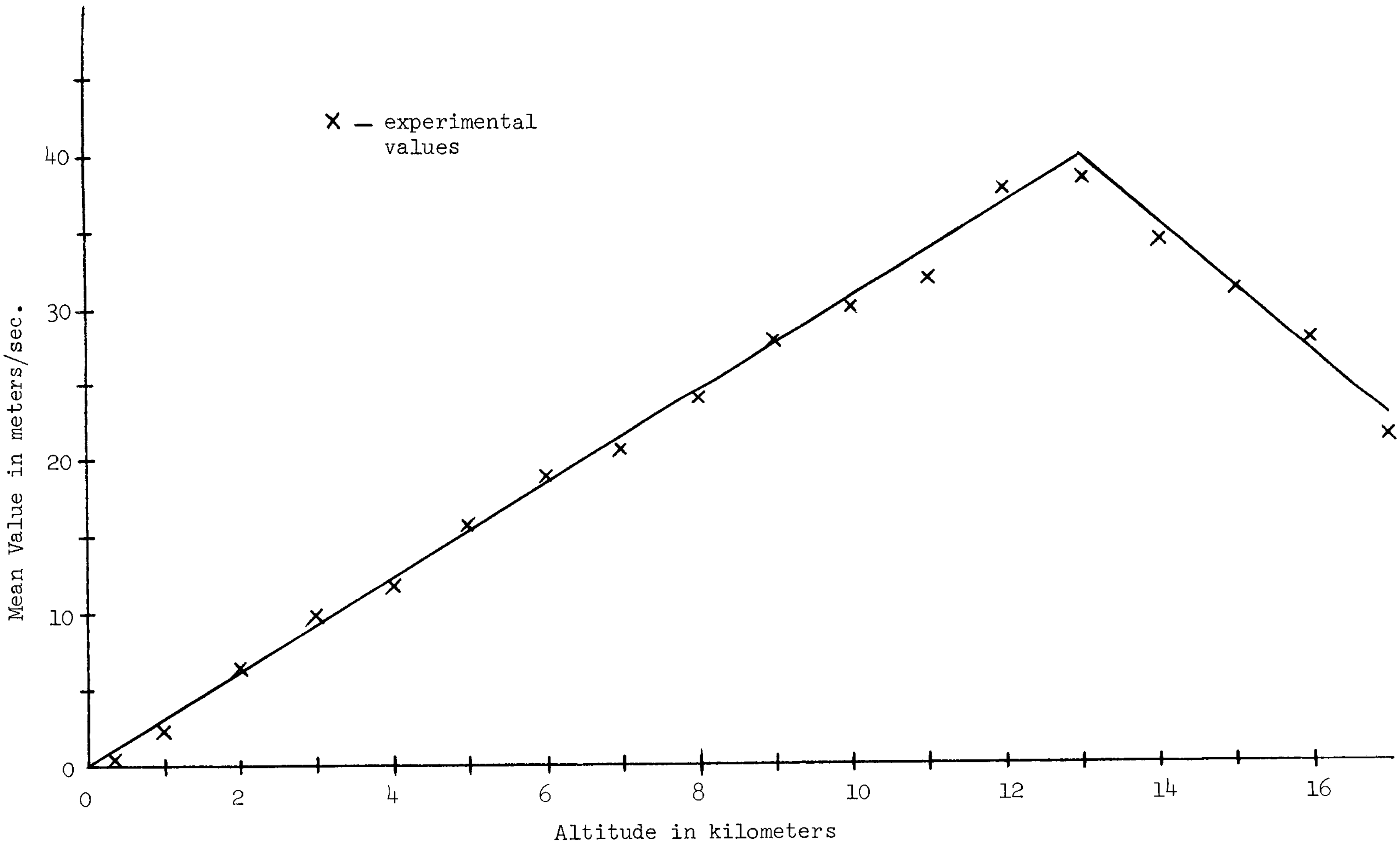

Figure 5.2 Mean Value of the Horizontal Component w_x of Wind Velocity.

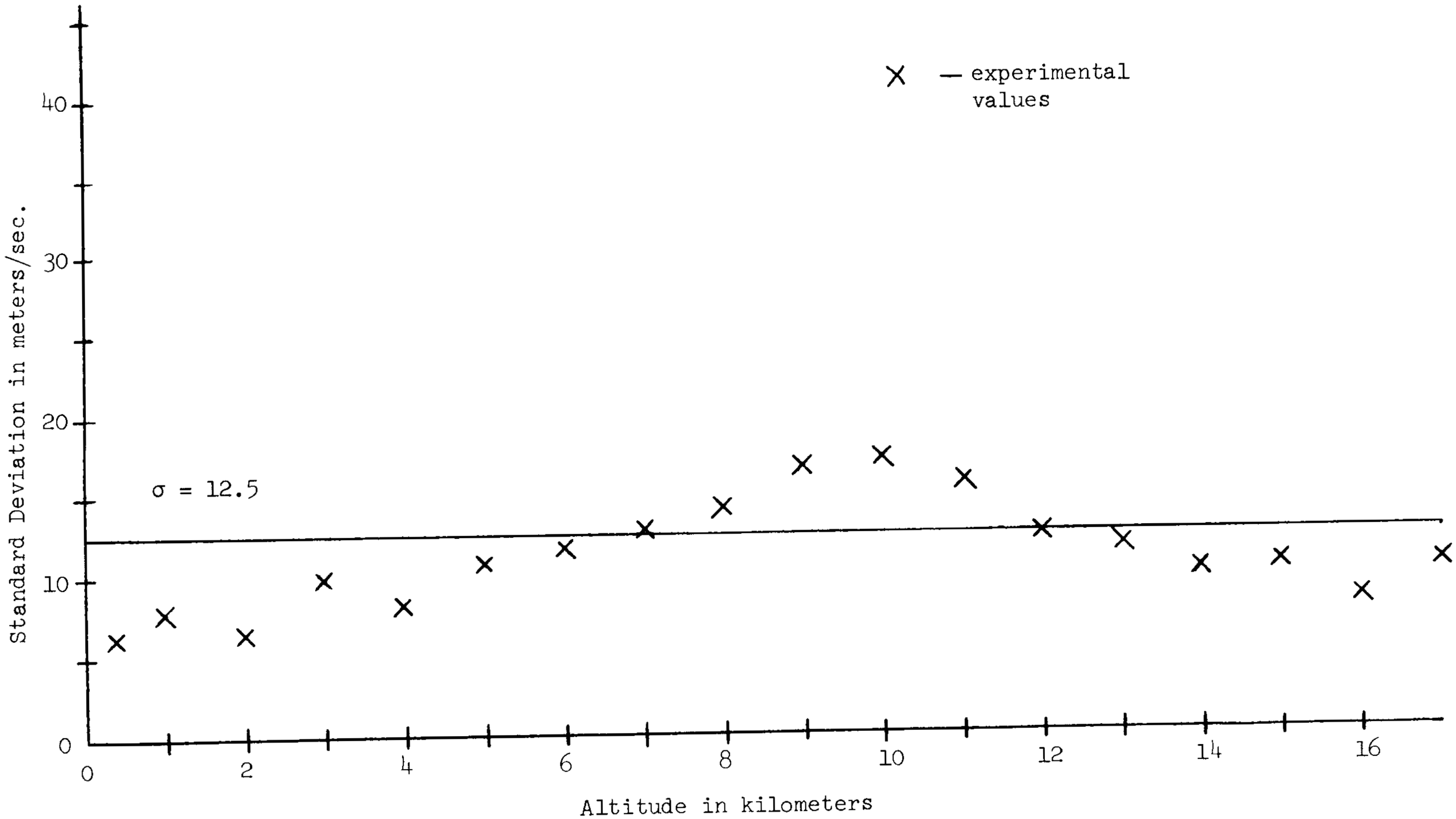

Figure 5.3 Standard Deviation of the Horizontal Component w_x of Wind Velocity.

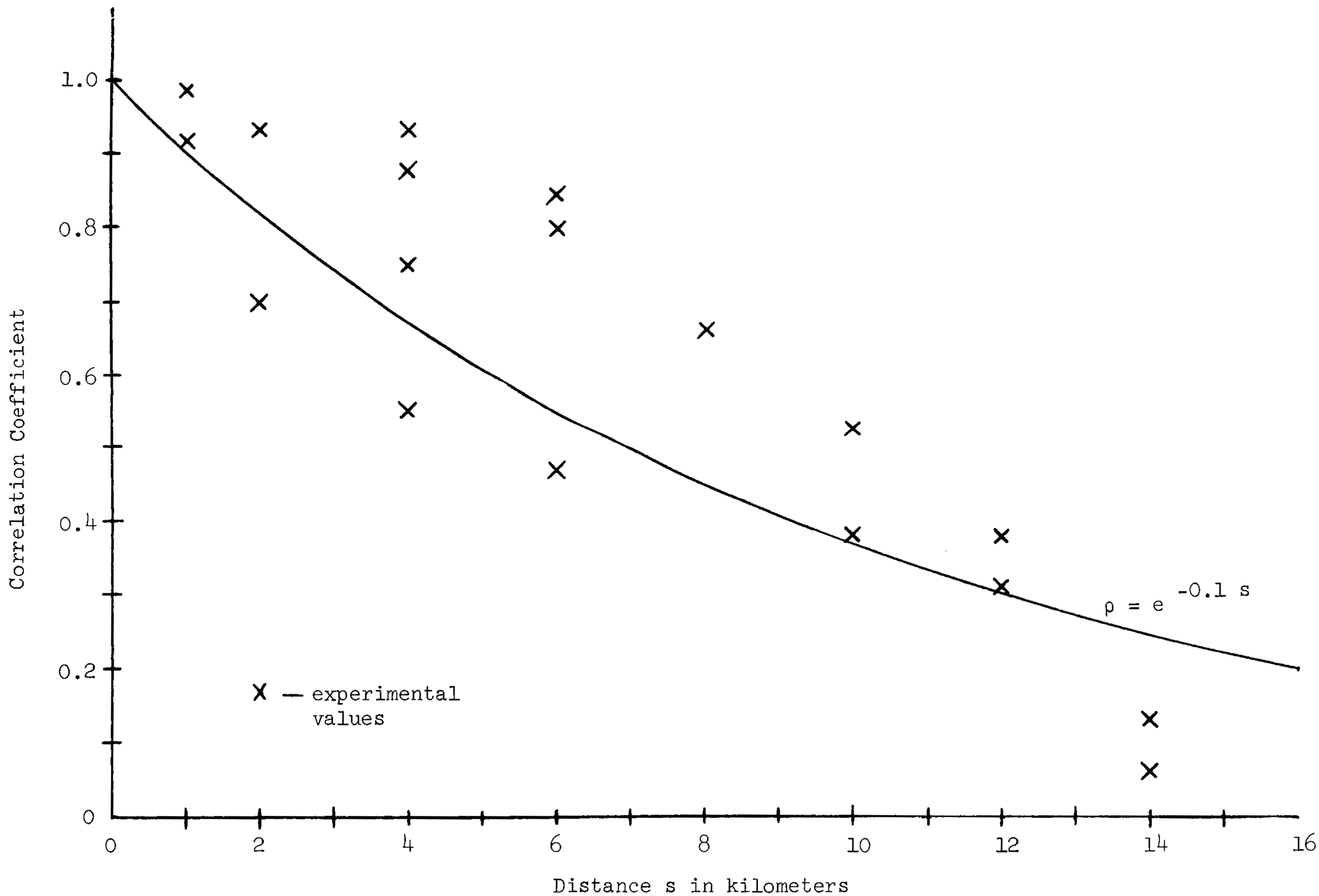

Figure 5.4 Correlation Coefficient of the Horizontal Component w_x of Wind Velocity.

An inspection of (5.19) shows that the parameter values of $\alpha = 0.1$ and $\beta = 0$ have been selected for the correlation coefficient of (5.3).

An analog computer network was built to mechanize the mathematical model for the example presented in this chapter. The parameter values given in (5.19) were used for the mathematical model.

To provide an indication of the effectiveness of the simulation, wind velocity profiles as a function of altitude y were generated by the computer network. To accomplish this, the sensing element was assigned a constant vertical velocity v(t) as it moved from y = 0 to y = 20 kilometers. Several typical profiles generated by the analog computer network are shown in Figure 5.5. For comparison, Figure 5.6 presents several experimentally measured wind velocity profiles as given in Reference 7 of the Bibliography.

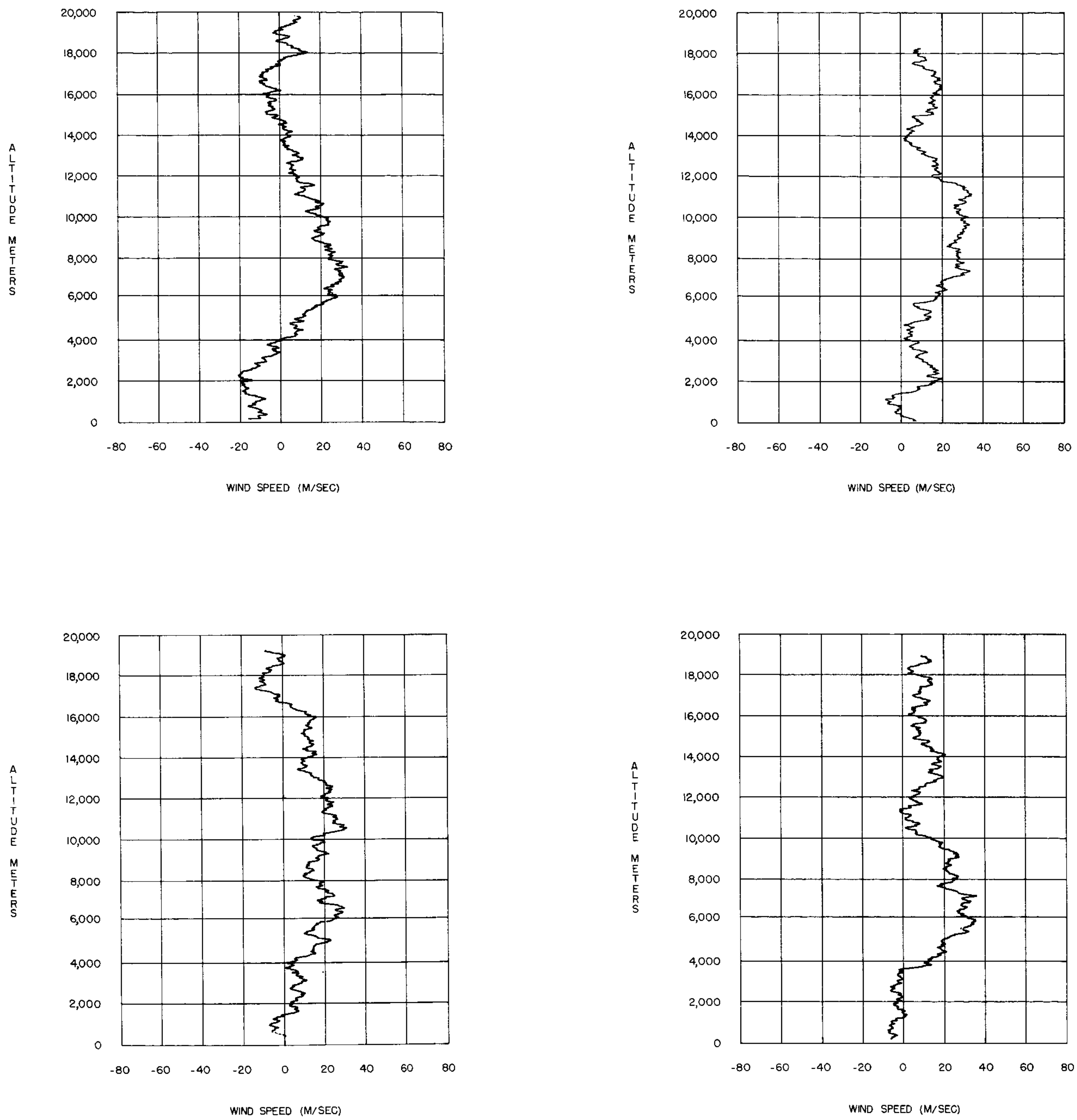

Figure 5.5. Wind Velocity Profiles Generated by the Analog Computer Network of Figure 5.1.

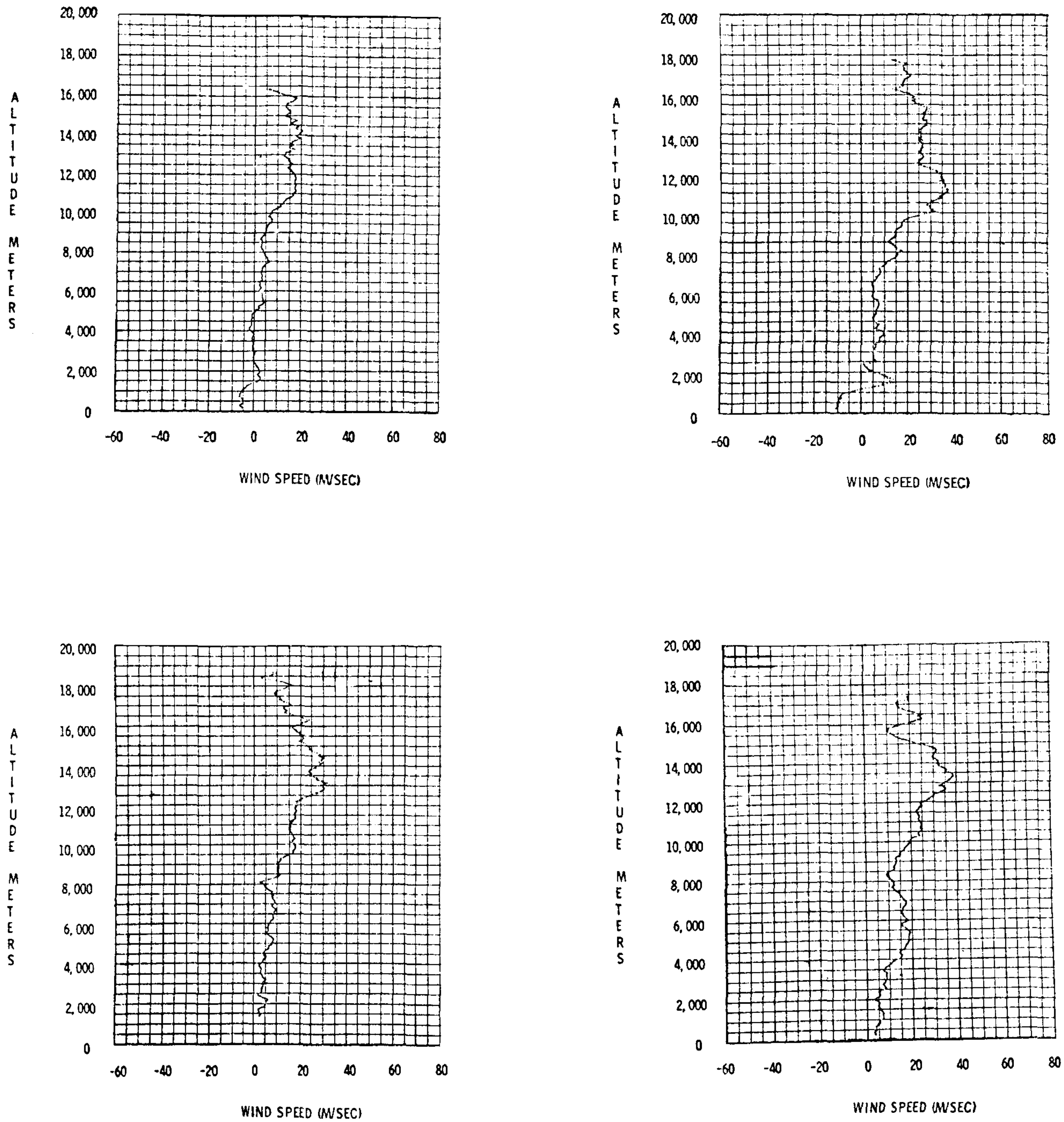

Figure 5.6. Experimental Wind Velocity Profiles.

VI. CONCLUSIONS

This report presents a summary of the results obtained over the past several years in research on the analog generation of nonstationary random processes. The basic synthesis procedure, called the covariance-expansion method, developed during the research was presented in Technical Note No. 3. This basic procedure permitted the synthesis of a network to generate an arbitrary, nonstationary, Gaussian random process. However, it requires the specification of the statistical moments of the process as a function of time. Thus, in applying the method of Technical Note No. 3 to the simulation of wind disturbances affecting a moving vehicle, it is necessary to construct a different network for each different flight path. In subsequent research the synthesis method was adapted to permit a single network to be used for the approximation of random disturbances affecting a vehicle following an arbitrary path of motion. Here, time functions describing the motion of the vehicle are used as inputs to the single network.

An outline of the covariance-expansion synthesis method is included in this technical note. Also, a mathematical model for random wind turbulence is included. This model has been constructed using assumptions selected to simplify the experimental determination of parameters of the model and also to simplify the physical mechanization of the model.

Chapter V of this technical note discusses the mechanization of a simple mathematical model of one horizontal component of random wind velocity. The mechanization permits the simulation of the wind disturbance that affects a vehicle moving on an arbitrary path in three-dimensional space.

Consistent with the restricted amount of experimental wind data that has been studied in the present project, the research that has been summarized indicates that the covariance-expansion synthesis method can be used to construct an analog computer network that approximates the salient statistical characteristics of wind turbulence.

BIBLIOGRAPHY

1. Finn, D. L., Notes on a Class of Stochastic Processes for Use in Analog Simulation Studies, Technical Note No. 5 on Research Project A-588 (Contract NAS8-2473), Georgia Institute of Technology, 6 December 1964.

2. Finn, D. L., and W. A. Yates, The Generation of a Gaussian Random Process in a Position Parameter, Technical Note No. 10 on Research Project A-588 (Contract NAS8-2473), Georgia Institute of Technology, 14 December 1965

3. Finn, D. L., Generation of Nonstationary Random Processes Dependent on Time and Position Parameters, Technical Note No. 11 on Research Project A-588 (Contract NAS8-2473), Georgia Institute of Technology, 15 August 1966.

4. Johnson, R. S., and R. D. Loftin, Introduction to Nonstationary Stochastic Processes for Analog Monte Carlo Studies, Technical Note No. 1 on Research Project A-588 (Contract NAS8-2473), Georgia Institute of Technology, 12 November 1962.

5. Levy, P., "A Special Problem of Brownian Motion and a General Theory of Gaussian Random Functions," Proceedings, Third Berkeley Symposium on Probability and Statistics, Vol. II, 1956, p. 133.

6. Lumley, John L., and Hans A. Panofsky, The Structure of Atmospheric Turbulence, John Wiley and Sons, New York, 1964.

7. Scoggins, James R., and Michael Susko, FPS-16 Radar/Jimsphere Wind Data Measured at the Eastern Test Range, NASA TMX-53290, July 9, 1965.

8. Webb, R. P., Synthesis of Measurement Systems, Ph.D. Thesis, Georgia Institute of Technology, July 1963.

9. Webb, R. P., J. L. Hammond, Jr., R. E. Bryan, and T. M. White, Jr., Generation and Measurement of Nonstationary Random Processes, Technical Note No. 3 on Research Project A-588 (Contract NAS8-2473), Georgia Institute of Technology, 22 July 1964.