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Restoration of the Original Synchrotron Radiation Formula $P = 2e^4 H_\perp^2 \gamma^2/3m_o{}^2c^3$ *

by

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ABSTRACT

Under most conditions, such as cases where the radiating electrons are confined to a fixed or slowly expanding region, recent results amending the synchrotron radiation rate equation to replace $H_{\perp}^2$ by H^2 are not applicable; the original form of the equation prevails.

The formula usually given[1] for the rate of radiation from an ultrarelativistic electron of energy $\gamma m_o c^2$ spiraling with pitch angle α in a uniform magnetic field $\vec{H}$, namely

$$P(\gamma,\alpha) = 2e^4 H^2 \gamma^2 \sin^2 \alpha/3m_o{}^2 c^2 \quad (1)$$

has recently been criticized.[2,3] Specifically, it was pointed out that to see the radiation at any appreciable intensity, a distant observer must be located so that a line joining him to the electron makes an angle α with $\vec{H}$. Therefore, he sees the gyro frequency, the power spectrum, and the total radiation rate increased by the Doppler factor

$$\nu'/\nu = [1 - (v/c)\cos^2\alpha]^{-1} \approx (\sin \alpha)^{-2} \quad (2)$$

where v = electron velocity. As a result, it is proposed that the factor $\sin^2\alpha$ be omitted from (1) and that the resulting formula

$$P(\gamma,\alpha) = 2e^4 H^2 \gamma^2/3m_o{}^2 c^2 \quad (3)$$

be used in all calculations of synchrotron emission.[2,3] Equation (3) would imply a nonzero radiation rate when $\alpha = 0$, an impossible result that suggests further scrutiny; however, this could conceivably be a singular case of no physical importance.

The new calculations,[2,3] in fact, are quite rigorous for the case to which they are applied, namely an electron that spirals <u>forever</u> along a <u>perfectly</u> uniform magnetic field. In most cases, however, one deals with electrons that are forced to occupy a bounded region of slowly changing size and position, for one reason or another. For example, the field lines may be rather disordered, or may gradually curve so as to wind through a bounded region. Scattering by magnetic irregularities, mirroring, scattering by protons, or the two-stream instability may prevent the electrons from proceeding indefinitely along $\vec{H}$. If there is some residual streaming along $\vec{H}$, or a bulk motion, or expansion of the electron cloud, it will be clear presently that this should be treated by standard methods,[4,5] not by the introduction of the factor $(\sin \alpha)^{-2}$. The reason is, that the new effect being introduced through (2) is valid only so long as the electron travels toward the observer; during any return trip, one must pay for the time borrowed through the time contraction implicit in (2). To see this, consider a simple model in which $\vec{H}$ is uniform, but end-plates which reflect the electron are placed at the origin and some point a distance L from the origin toward the observer. Practically, this would induce a pulse of radiation as the electron is reflected, but this could be reduced to as small a proportion as desired of the total radiation by increasing L. Actually, the abrupt reflection at the ends will be seen to be irrelevant, and the model serves as well for cases where the field lines gradually bend back on themselves, or where the electron is mirrored or scattered so as to make its mean vector velocity, averaged over a suitably long time, much less than c in magnitude.

In this end-wall model, the electron travels toward the observer a time $T = L/v \cos\alpha$ and an equal time away, as measured in a local rest frame. However, the observer, due to the finite velocity of light, sees the electron approach and recede during consecutive time intervals T^- and T^+, respectively, where

$$T^- = T - L(\cos\alpha)/c \text{ (approach)} \tag{4a}$$

and

$$T^+ = T + L(\cos\alpha)/c \text{ (recession).} \tag{4b}$$

The fraction of observer time during which the electron approaches is

$$f^- = T^-/(T^+ + T^-) \approx \frac{1}{2} \sin^2\alpha \tag{5}$$

for $v \approx c$. Without the relative time-dilatation and contraction given by (4), one would have just $f^- = \frac{1}{2}$. Thus, exactly the same time transformation given by (2) leads to a <u>reduction</u> of f^- by a factor $\sin^2\alpha$. During periods of recession, the ultrarelativistic particle cannot be seen. Therefore, if one has an ensemble of such particles radiating, all having started with the same α but being at different points along their trajectories, only the fraction f^-, which contains the factor $\sin^2\alpha$, can be seen at one time, and the total radiation is reduced to its former value (1). The spectral distribution $P(\nu,\gamma,\alpha)$ "fortuitously"[2] was supposed to be modified only by the same factor (2) as appeared in $P(\gamma,\alpha)$; hence the foregoing argument restores $P(\nu,\gamma,\alpha)$ to its original value.[6]

To be sure that the same kind of argument can be developed for a rather different model geometry, consider the model of Komesaroff[7] for a non-thermal emitting region in the Southern Milky Way. In this model, an isotropic distribution of electrons is trapped in a toroidal magnetic field.[8] Takakura and Uchida[5] assert that the factor $(\sin\alpha)^{-2}$ invalidates the analysis of Komesaroff. Yet electrons that approach the observer must recirculate around the cylinder before they can again approach. Thus, if one described their orbits in terms of the retarded time as seen by the observer, they spend only a fraction f' of the time in parts of their trajectory where they can be seen, where f' again contains the factor $\sin^2\alpha$. Rather than calculating this in detail, one can derive it in a more fundamental way. The electrons having the correct pitch angle α to be seen have an apparent velocity (taking $v \approx c$)

$$v_{ap} = c/\sin^2\alpha \tag{6}$$

which exceeds c, as is allowable because it is only an apparent velocity in terms of the observer's time. (The latter has a varying retardation relative to the time in the rest frame of the particles' guiding center.) Now, the usual law of conservation of number density for a fluid in steady motion

$$\mathrm{div}(n\,\vec{v}) = 0 \tag{7}$$

is a kinematic equation that applies in terms of time, number, and velocity as measured consistently in any frame or manner whatever. In a one-dimensional case, such as this circulation around a ring, (7) implied that number density and velocity are reciprocal, where these quantities may be defined either as seen by local Lorentz observers or by the distant observer, with his prejudices due to retardation effects. Using the latter definitions of n and v, we see that when v is increased by $(\sin\alpha)^{-2}$ according to (6), n must go down by the reciprocal factor! These effects then compensate so as to restore Komesaroff's analysis!

From the foregoing examples, it is clear that the only correction that may be properly applied to (1) is one for bulk motion, when an entire cloud or electrons streams toward or away from the observer, or undergoes some bulk motion. If a long time average over the particle orbits is taken, all time-dilatation effects must be reducible to ones that can be calculated from the mean motion of the material. Similarly, the relaxation of the particle distribution must be calculated in a Lorentz-invariant manner in the local rest frame, and then converted to the observer's frame according to any possible bulk motion. This contrasts with the suggestion of Epstein and Feldman[2] that relaxation effects should be treated through the use of (3).

REFERENCES

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[2] R. I. Epstein and P. A. Feldman, Astrophys. J. Letters 150, L109 (1967).

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[5] M. J. Rees, Nature 211, 468 (1966).

[6] K. C. Westfold, Astrophys. J. 130, 241 (1959).

[7] M. M. Komesaroff, Australian J. Phys. 19, 75 (1966).

[8] The lines run around the torus, which has the cross-section of a rectangle very long along the axis of symmetry. Thus, it is a cylindrical shell.

NOTE: Since preparing this report, the author has found that a very similar discussion reaching the same conclusions has been published by P. A. G. Scheuer, Ap. J. Letters 151, L139 (1968).