TECHNICAL MEMORANDUM

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APPLICATION OF CONTINUOUS FORM OF DYNAMIC PROGRAMMING AND INACCESSIBLE STATE VARIABLE THEORY TO THE K-8 ACCELEROMETER

by

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I. INTRODUCTION

Although a large amount of interest has been shown in the theoretical aspects of optimal control over the last decade, there has only been a minor attempt to apply this somewhat sophisticated theory. It is true that, in general, large scale computers would be required. But, if the study is limited to linear systems certain simplifications result.

These simplifications lead to a simple design technique which has, apparently, been overlooked or underestimated. The effectiveness of the technique for a linear system with accessible state variables has already been discussed [1, 2] and this memorandum reports a preliminary study of the case where some of the state variables are inaccessible.

II. THEORETICAL BACKGROUND

The requisite theory is readily available in the modern control theory textbooks [3, 4] and only the necessary results are presented here.

Consider a system described by the equations

$$\dot{\underline{X}} = B\underline{X} + C\underline{U} + \underline{d} \quad , \qquad \text{(II-1)}$$

where $\underline{X}$ is an mth order column vector, $\underline{U}$ is an sth order column vector, and $\underline{d}$ is a disturbance vector. It is desired to minimize the performance criterion E, where

$$E = \begin{matrix}\text{Expected}\\ \text{Value of}\end{matrix} \int_0^T \left\{ \underline{X}' A \underline{X} + \underline{U}' H \underline{U} \right\} dt \quad , \qquad \text{(II-2)}$$

and where the primes denote the transpose and A and H are positive definite weighting matrices.

The optimal forcing function has been shown to be,

$$\underline{U}^* = -D\underline{X} \quad , \qquad \text{(II-3)}$$

where

$$D = H^{-1} C' K_2 \quad , \qquad \text{(II-4)}$$

and where

$$\dot{K}_2(t) = -A - B'K_2 - K_2B + K_2CH^{-1}C'K_2 \quad . \tag{II-5}$$

If all of the state variables $\underline{X}$ cannot be measured, then the optimal control has been shown to be,

$$\underline{U}^* = -D\hat{\underline{X}} \quad , \tag{II-6}$$

where $\hat{\underline{X}}$ is the best estimate of $\underline{X}$ in a linear or least squares sense. This estimate is given by

$$\dot{\hat{\underline{X}}} = -Q\hat{\underline{X}} + A_1\underline{Y} \quad , \tag{II-7}$$

where $\underline{Y}$ is a vector of the measured outputs contaminated by some noise ω and Q is given by

$$Q = A_1M - B + CD \quad . \tag{II-8}$$

The matrix M is a measurement matrix, i.e.,

$$\underline{Y} = M\underline{X} + \underline{\omega} \tag{II-9}$$

and A_1 is given by

$$A_1 = KM'W^{-1} , \tag{II-10}$$

where W is formed from the measurement noise vector $\underline{\omega}$. Finally K is given by the steady state solution of

$$\dot{K} = CVC' + BK + KB' - KM'W^{-1}MK , \tag{II-11}$$

where CVC' is a matrix formed from the disturbances in the system, d.

It should be noticed that equations (II-5) and (II-11) are very similar and the same computer program can be used to compute the K_2 and K matrices.

III. APPLICATION OF CONTINUOUS FORM OF DYNAMIC PROGRAMMING

The state equations for the K-8 accelerometer, as shown in Figure 1, are

$$
\begin{aligned}
\dot{X}_1 &= X_2 \,, \\
\dot{X}_2 &= (H/J_\beta \; T_\beta) + T_\beta / J_\beta \,, \\
\dot{X}_3 &= X_4 \,, \\
\dot{X}_4 &= (-H/J_\alpha) X_2 + T_\alpha / J_\alpha \,,
\end{aligned}
\qquad \text{(III-1)}
$$

where

$X_1 = \beta$ (radians)

$X_2 = \dot{\beta}$ (radians/sec)

$X_3 = \alpha$ (radians)

$X_4 = \dot{\alpha}$ (radian/sec)

and

$H = 93.0$ gm - cm - sec

$J_\alpha = 2.66$ gm - cm - sec^2

$J_\beta = 0.123$ gm - cm - sec^2

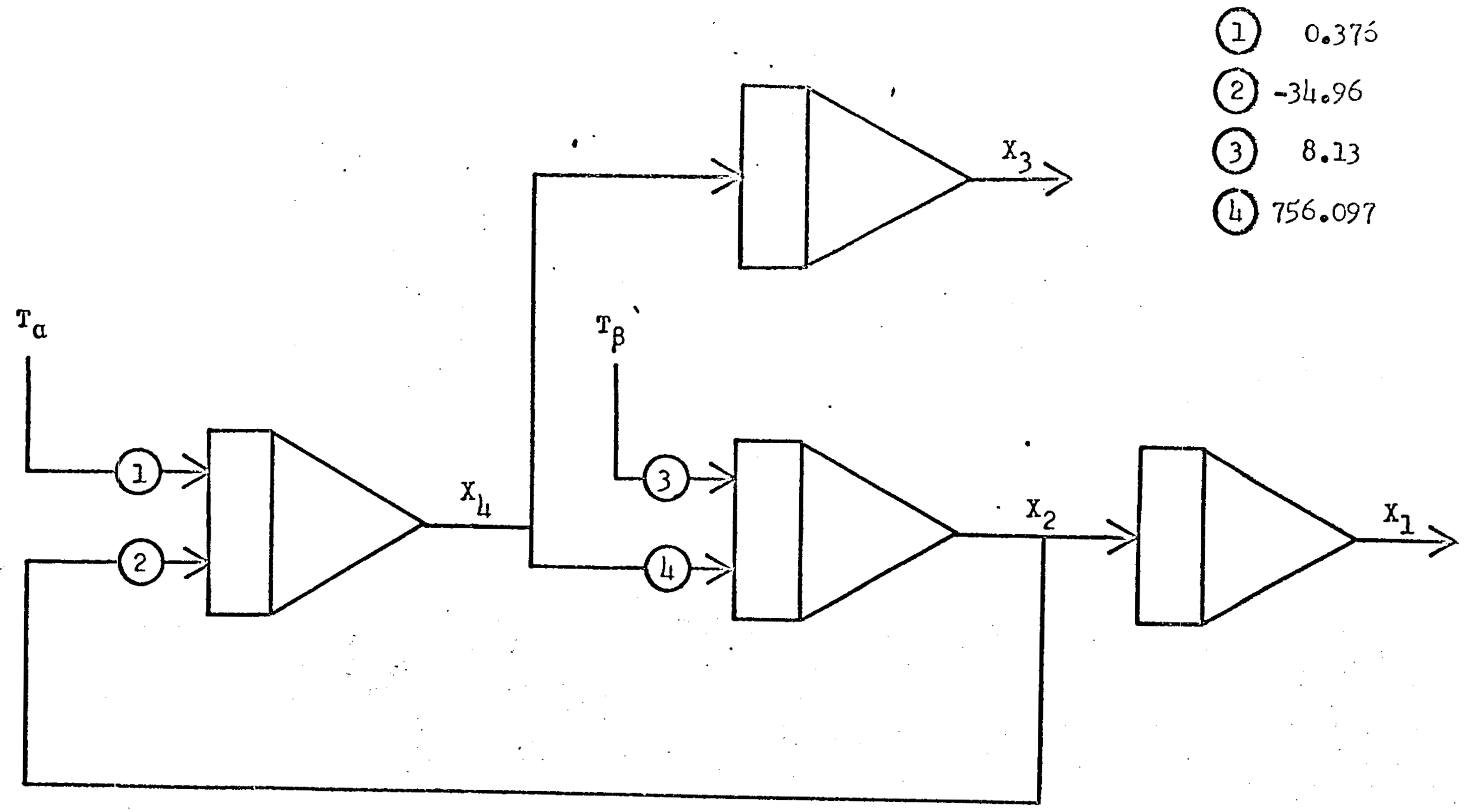

Figure 1.--Simplified Analog Diagram of the K-8 Accelorometer

and

$$T_\beta = \text{(input torque)} \quad \text{gm - cm}$$

$$T_\alpha = \text{(feedback torque)} \quad \text{gm - cm}$$

Limits are imposed upon T_α and X_1 and are

$$|T_\alpha| \leq 1440.18 \text{ gm - cm},$$

and

$$|X_1| \leq 0.05236 \text{ radians}.$$

The obvious choice of a function to minimize T_α and X_1 is

$$\int_0^T (X_1^2 + \lambda T_\alpha)dt. \tag{III-2}$$

Rearranging the above equations to fit the form of (II-1) and (II-2), the following matrices are obtained.

$$A = \begin{bmatrix} a(1,1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{III-3}$$

$$B = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 756.097 \\ 0 & 0 & 0 & 1.0 \\ 0 & -34.96 & 0 & 0 \end{bmatrix} \tag{III-4}$$

$$C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.376 \end{bmatrix} \qquad H = [\lambda] = [1.0] \tag{III-5}$$

Equations (III-3), (III-4), and (III-5) may be inserted into (II-5) to form the Ricatti equation

$$\dot{K} = -A - B^T K - KB + KCH^{-1}C^T K \tag{III-6}$$

with the final conditions $K(T) = 0$. Using time reversal, $\tau = T - t$, gives the following Ricatti equation

$$\dot{K} = A + B^T K + KB - KCH^{-1}C^T K \tag{III-7}$$

with initial conditions $K(o) = 0$. The solution of (III-7) for the steady-state value of K may be used to obtain the feedback necessary for the optimum control subject to the constraints of (II-2). Therefore

$$T_\alpha = -H^{-1}C^T K\underline{X} = -D\underline{X} . \tag{III-8}$$

Figure 2 is the block diagram of the plant with the optimal control. Table 1 is a tabulation of the different values of a(1,1) and the D matrix as well as the poles of the controlled system. Figure 3 is a plot of the poles with a(1,1) as a parameter.

Unfortunately, in this case, the state variables X_2, X_3, and X_4 are inaccessible and so have to be estimated.

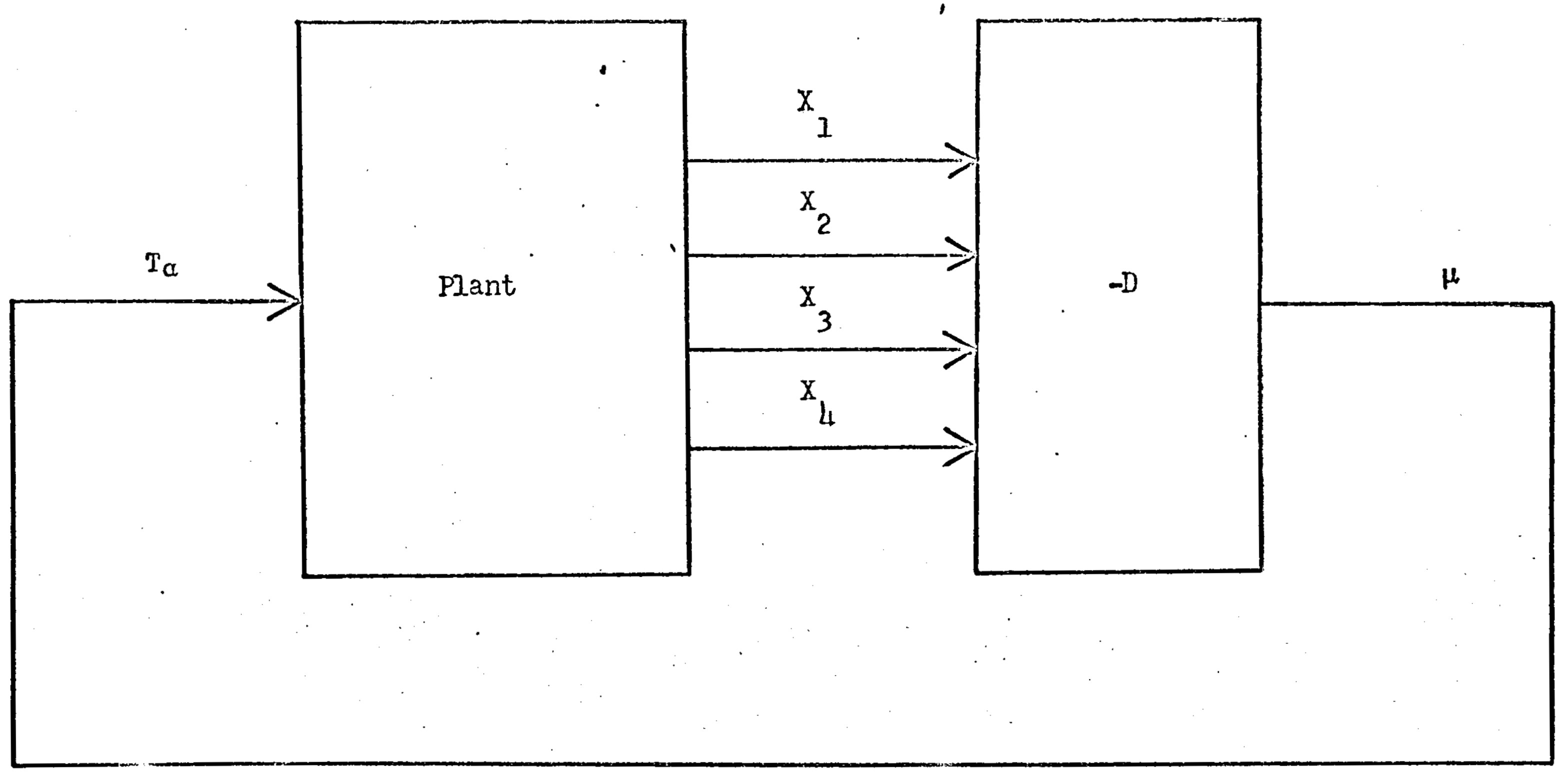

Figure 2.--Block Diagram of the K-8 Accelerometer with Optimal Control Feedback.

Table 1.--Tabulation of the Values of the D-matrix and Eigenvalues as a Function of a (1,1).

Code	a(1, 1)	-D(1, 1)	-D(2, 1)	-D(3, 1)	-D(4, 1)	Eigen-Values 1	Eigenvalues 2 & 3
1	$.1 \times 10^{9}$	-10,000.00	-50.38	0	-450.15	-84.61	$-42.31 \pm j178.33$
2	$.1 \times 10^{10}$	-31,622.78	-194.45	0	-884.44	-166.24	$-83.12 \pm j217.16$
3	$.1 \times 10^{11}$	-100,000.00	-537.58	0	-1,470.50	-276.41	$-138.20 \pm j289.37$
4	$.1 \times 10^{12}$	-316,227.78	-1,290.69	0	-2,278.53	-428.30	$-214.15 \pm j404.98$
5	$.1 \times 10^{13}$	-1,000,000.00	-2,919.09	0	-3,426.64	-644.10	$-322.05 \pm j581.02$
6	$.1 \times 10^{14}$	-3,162,277.80	-6,429.97	0	-5,085.68	-955.95	$-477.98 \pm j843.69$

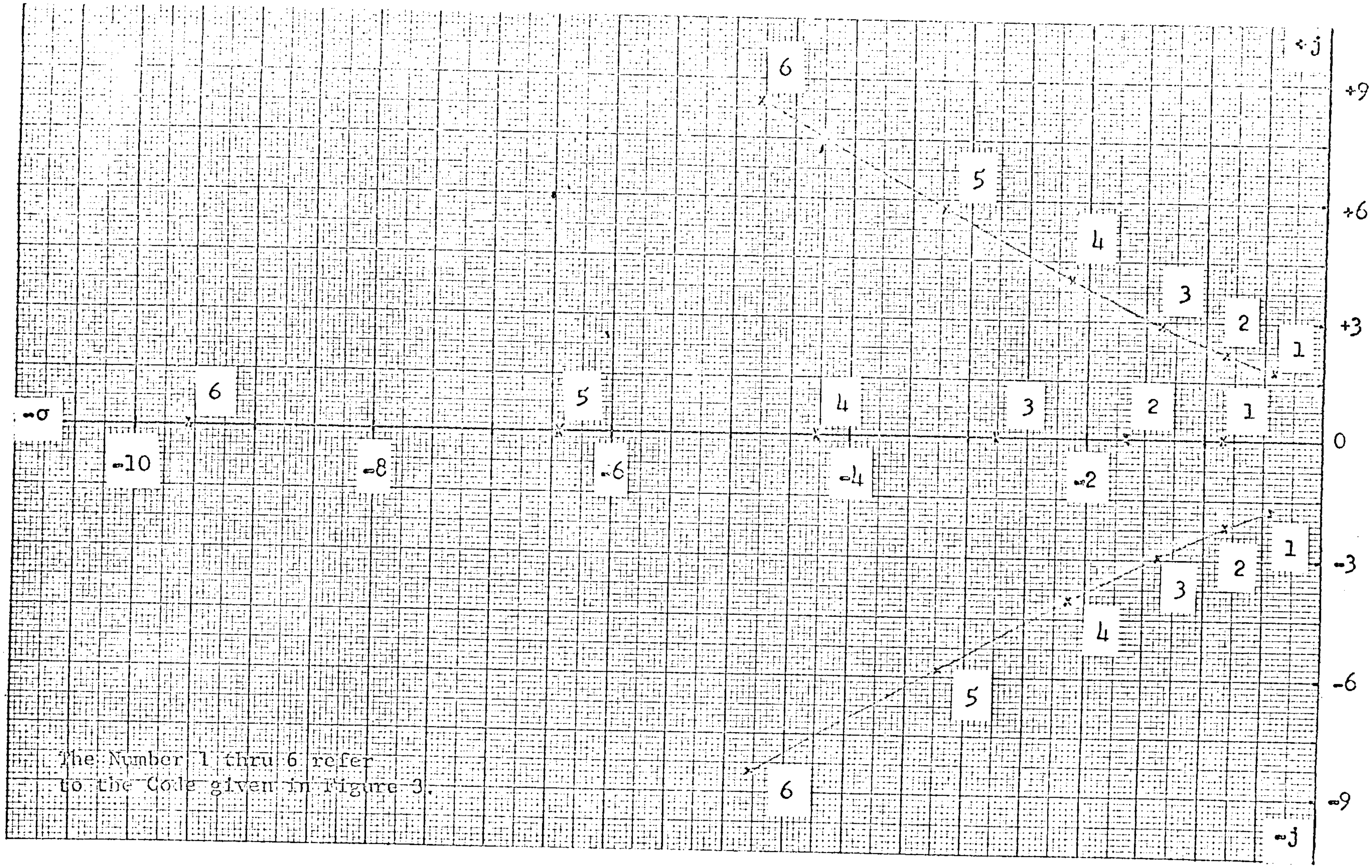

Figure 3.--Plot of the Eigenvalues with a (1,1) as a Parameter.

IV. APPLICATION OF INACCESSIBLE STATE VARIABLE THEORY

The state equations may be rewritten to include the effects of noise; in this case a disturbance, d_α, and measurement noise, ω_1. Refer to Figure 4.

$$\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= H/J_\beta + T_\beta/J_\beta \\ \dot{X}_3 &= X_4 \\ \dot{X}_4 &= -H/J_\alpha X_2 + T_\alpha/J_\alpha + d_\alpha . \end{aligned} \qquad \text{(IV-1)}$$

The output, Y_1, is given by

$$Y_1 = X_1 + \omega_1 . \qquad \text{(IV-2)}$$

The computation of the D matrix proceeds as before and the D matrix will be unchanged. The additional matrices required for the computation of Q and A_1 are,

$$M = [1 \quad 0 \quad 0 \quad 0]$$

$$CVC' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & sd_\alpha \end{bmatrix} \qquad \text{(IV-3)}$$

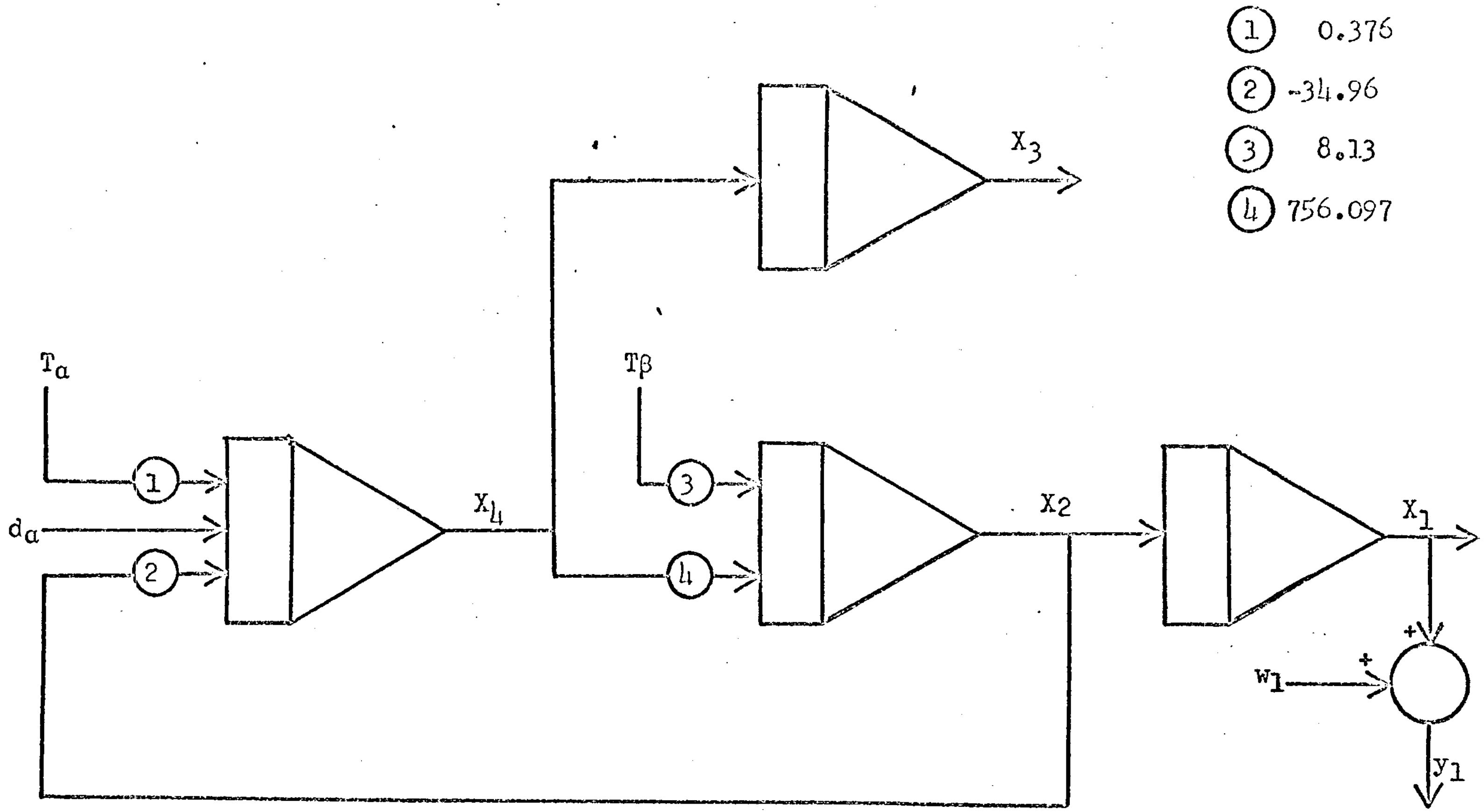

Figure 4.--Simplified Analog Diagram of the K-8 Accelerometer Showing the Introduction of Disturbance, d_{α}, and Measurement Noise, ω_1.

$$W = [sd_{\omega}] \quad ,$$

where sd_{α} is the spectral density of the disturbance d_{α} and ds_{ω} is the spectral density of the measurement noise ω_1.

Employing (II-11), (II-10), (II-8) and (II-7) enables the formation of the best estimate of $\underline{X}$, $\hat{\underline{X}}$.

As there was little information available on the noise present in the system, somewhat arbitrary values were used. These corresponded to noise of 1 g amplitude and 20 KC/s bandwidth for d_{α} and .005 radians amplitude and a similar bandwidth for ω_1. These figures were then multiplied by 0.2, 2, 5 and 10 to give four sets of values.

The matrices A_1 and Q were computed for each of these values and are given in Tables 2 and 3. The estimator (II-7) may now be formed. Alternatively take the Laplace transform of (II-7) and use capital letters to denote Laplace Transforms of X and Y:

$$[sI + Q]X = A_1Y \tag{IV-4}$$

where s is Laplace's operator and I, of course, the unit matrix.

$$X = (sI + Q)^{-1}A_1Y \tag{IV-5}$$

and since

$$U = -DX \quad , \tag{IV-6}$$

Noise Code	$a_1(1,1)$	$a_1(2,1)$	$a_1(3,1)$	$a_1(4,1)$
0.1	7.87	31.24	- .034	- 275.33
1.0	77.25	2,984.16	- 3.55	- 2,700.83
5.0	294.34	43,319.11	-21.71	-10,290.26
10.0	459.01	105,344.61	-27.95	-16,046.95

Table 2.--Tabulation of the a_1 Matrix with the Disturbance, d_α, and Measurement Noise, ω_1, as a Parameter. The Noise Code Refers to those Noise Values in the Text.

Table 3--Tabulation of the Q Matrix with the A Matrix and Noise Code as a Parameter.

Noise Code 10			
a(1,1)	q(4,1)	q(4,2)	q(4,4)
$.1 \times 10^{10}$	-4,158.69	108.06	332.48
$.1 \times 10^{11}$	21,547.03	237.06	552.82
$.1 \times 10^{12}$	102,835.67	520.18	856.59
$.1 \times 10^{13}$	359,892.92	1,132.36	1,288.21
$.1 \times 10^{14}$	1,172,779.20	2,452.24	1,911.91

Noise Code 5			
a(1,1)	q(4,1)	q(4,2)	q(4,4)
$.1 \times 10^{10}$	1,598.00	108.06	332.48
$.1 \times 10^{11}$	27,303.73	237.06	552.82
$.1 \times 10^{12}$	108,592.36	520.18	856.59
$.1 \times 10^{13}$	365,649.61	1,132.36	1,288.21
$.1 \times 10^{14}$	1,178,535.90	2,452.24	1,911.91

Noise Code 1			
a(1,1)	q(4,1)	q(4,2)	q(4,4)
$.1 \times 10^{10}$	9,187.43	108.06	332.48
$.1 \times 10^{11}$	34,893.15	237.06	552.82
$.1 \times 10^{12}$	116,181.79	520.18	856.59
$.1 \times 10^{13}$	373,239.04	1,132.36	1,288.21
$.1 \times 10^{14}$	1,186,125.4	2,452.24	1,911.91

Noise Code .1			
a(1,1)	q(4,1)	q(4,2)	q(4,4)
$.1 \times 10^{10}$	11,612.93	108.06	332.48
$.1 \times 10^{11}$	37,318.65	237.06	552.82
$.1 \times 10^{12}$	118,607.29	520.18	856.59
$.1 \times 10^{13}$	375,664.54	1,132.36	1,288.21
$.1 \times 10^{14}$	1,188,550.90	2,452.24	1,911.91

$$U = -D(sI + Q)^{-1}A_1Y \ . \tag{IV-7}$$

Thus the transfer function of the filter is

$$U/Y = -D(sI + Q)^{-1}A_1Y \ . \tag{IV-8}$$

The poles and zeros of this filter, for different values of the D matrix, and different noise levels, are given in Table 4. The computation involved the inversion of a $\{4 \times 4\}$ matrix and it is felt that there might be some error in the computation of the zeros of the function. However the poles given are correct.

Table 4.--Tabulation of the Poles and Zeros of the Transfer Function of the Filter as given in (IV-8) with the A Matrix and Noise Code as Parameters.

Noise Code 10		
a(1,1)	Zeros	Poles
$.1 \times 10^{10}$	0, + 110.	- 378., -206 ± j 375.
$.1 \times 10^{11}$	0, + 18.	- 551., -230 ± j 481.
$.1 \times 10^{12}$	0, + 38.	- 746., -284 ± j 621.
$.1 \times 10^{13}$	0, - 80.	- 995., -375 ± j 814.
$.1 \times 10^{14}$	0, - 115.	-1,334., -518 ± j1,088.

Noise Code 5		
a(1,1)	Zeros	Poles
$.1 \times 10^{10}$	0, + 172.	- 318., -154.± j 317.
$.1 \times 10^{11}$	0, 55.	- 468., -189.± j 415.
$.1 \times 10^{12}$	0, 2.	- 648., -251.± j 547.
$.1 \times 10^{13}$	0, - 40.	- 883., -349.± j 734.
$.1 \times 10^{14}$	0, - 69.	-1,211., -498.± j1,004.

Noise Code 1		
a(1,1)	Zeros	Poles
$.1 \times 10^{10}$	0, + 622.	- 208., -100.± j 241.
$.1 \times 10^{11}$	0, + 178.	- 332., -149.± j 322.
$.1 \times 10^{12}$	0, + 87.	- 492., -220.± j 443.
$.1 \times 10^{13}$	0, + 43.	- 713., -326.± j 622.
$.1 \times 10^{14}$	0, + 17.	-1,028., -480.± j 887.

Noise Code .1		
a(1,1)	Zeros	Poles
$.1 \times 10^{10}$	0, +3,482.	- 170., - 85.± j 219.
$.1 \times 10^{11}$	0, + 276.	- 282., -139.± j 293.
$.1 \times 10^{12}$	0, + 137.	- 435., -214.± j 408.
$.1 \times 10^{13}$	0, 82.	- 651., -322.± j 585.
$.1 \times 10^{14}$	0, 52.	- 963., -478.± j 848.

V. EVALUATION

The usefulness of the procedure outlined above depends upon the ease with which it may be applied and the effectiveness of the optimal control. Once a computer program has been written the design procedure is relatively simple so that the performance is the determining factor.

The performance of the system discussed in this memorandum is now being evaluated at Huntsville.

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