February, 1968 USCEE REPORT 250

UNIVERSITY OF SOUTHERN CALIFORNIA

Technical Report

FUEL OPTIMUM STOCHASTIC ATTITUDE CONTROL

Robert B. McGhee
H. J. Payne
William H. Spuck III

ELECTRONIC SCIENCES LABORATORY

USC Engineering

February, 1968 USCEE Report 250

Technical Report

FUEL OPTIMUM STOCHASTIC ATTITUDE CONTROL

Robert B. McGhee
H. J. Payne
William H. Spuck III

Electronics Sciences Laboratory
University of Southern California
Los Angeles, California

This research was supported by the National Aeronautics and Space Administration under Grant NGR05-018-044.

CONTENTS

ABSTRACT

The design of fuel optimum, position and thrust constrained, randomly disturbed spacecraft attitude control systems is a continuing engineering problem. A closely related problem is one in which velocity is directly controllable (here called an antenna steering problem). In their simplest form these problems are easily reduced to those of minimizing the fuel expended to maintain a given integral position error squared for randomly disturbed double integrator and single integrator systems, respectively.

Among the many potentially useful methods of approaching these optimum control problems, that based on Hamilton-Jacobi Theory is pursued here. The Hamilton-Jacobi type equation for the disturbance free system is well known. A similar equation for the stochastic case is presented in this study.

Solutions to these equations are unavailable in general. The deterministic Hamilton-Jacobi equations for the problems under study here are solved analytically and are valid for all states and times to go. A steady state analytic solution for a stochastic case is presented in this study, but the time dependent problem is found to be analytically intractable. As a result, the stochastic cases are solved approximately using numerical techniques.

Chapter 1

INTRODUCTION

1.1 Optimum Spacecraft Attitude Control

There exists a continuing need for investigations into methods of improving the design and performance of spacecraft attitude control systems. Evidence of this need may be seen regularly in the pertinent literature.[1,2] Further evidence may be seen in the requirements of future space missions. The trend towards long duration missions is placing increasingly greater demands on attitude control systems in terms of minimum weight and power, and maximum reliability. The performance requirements in terms of attitude accuracy, speed of response to disturbances, and dynamic range are simultaneously becoming more stringent due to the requirements of some spacecraft devices, such as highly directional scientific instruments, of greater navigational accuracies, and of the environmental factors of manned spaceflight.

---

[1] A reading list and complete references appear in Appendix A.

[2] Recent references include:

E. I. Ergin, "Current Status of Progress in Attitude Control," AIAA Progress in Astronautics and Aeronautics, XIII (June, 1964), 7-36;

J. S. Meditch, "On Minimum-Fuel Satellite Attitude Control," IEEE Transactions on Applications and Industry, LXXXIII, 120-128;

M. Athans, "On the Uniqueness of the Extremal Controls for a Class of Minimum Fuel Problems," IEEE Transactions on Automatic Control, Vol. AC-XI, 660-668; and

J. B. Plant, "An Iterative Procedure for the Computation of Fixed-Time Fuel Optimal Controls," IEEE Transactions on Automatic Control, Vol. AC-XI, 652-660.

The investigation leading to the results that are described in the sequel was concerned with a more theoretical than practical attitude control problem. Of interest is a quantitative evaluation of the best (optimum) performance available from a given class of control systems. Once this result is available, it may be used as a "bench mark" against which the performance of actual systems may be compared. It may also be used as a model for the design of approximate systems. Several interesting techniques for designing such approximately optimum systems have recently been developed.[3] These will not be discussed further here.

Attention here will be directed to rather idealized attitude control situations. The spacecraft kinematics will be considered to be a mass rotating symmetrically about a single axis. Attitude will be maintained by an active controller (thruster, jet) imparting a restoring torque to correct any attitude error.

With respect to figures of merit for system performance Ergin points ou that ".... selection of valid criteria is just as important as the optimization process."[4] Although Ergin would consider them "constraints

---

[3] Three are described in the following:

P. M. DeRusso, R. J. Roy, R. W. Miller, and B. W. Nutting Adaptive-Predictive Modeling of Nonlinear Processes, National Aeronautics and Space Administration Report No. CR-86;

J. C. Bowers, Optimum Analogue and Digital Attitude Control Systems for Space Vehicles (unpublished Sc.D. Dissertation, Washington University, June, 1964); and

C. R. Walli, Finite State Attitude Control, Department of Electrical Engineering Report No. USCEE 158, University of Southern California.

[4] Ergin, loc. cit., p. 8.

of the particular mission", the attitude accuracy and amount of fuel used for attitude control are clearly important design parameters. These are the performance criteria used in this investigation. They will be treated in terms of a minimum fuel problem: minimize the fuel expended over a mission while maintaining a given bound on attitude error.

1.2 An Optimum Spacecraft Attitude Control Problem

1.2.1 A Physical Characterization of the Basic Problem

The Plant. Consider the simplified spacecraft attitude control situation depicted in Figure 1-1.[5] The figure represents the kinematics

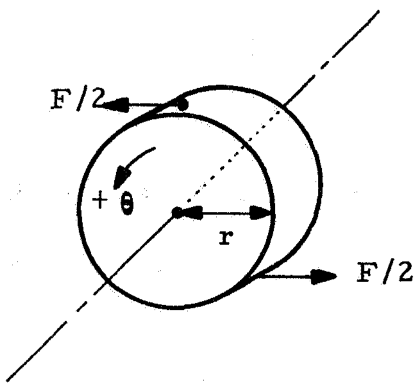


Figure 1-1 Simple single axis altitude control situation

of a rigid body spacecraft about one of its axes. The motion of the center of mass is not of concern, and motions about other axes are assumed to be uncoupled with those about this axis.

[5] For a more detailed presentation of attitude control dynamics, see D. B. DeBra, "The Large Attitude Motions and Stability, Due to Gravity, of a Satellite With Passive Damping in an Orbit of Arbitrary Eccentricity About an Oblate Body," Report No. SUDAER 126, Stanford University, May 1962.

The dynamics of this situation can be described by

$$I \frac{d^2\theta}{dt^2} = I\ddot{\theta} = \frac{1}{2} F(t)\, 2r \tag{1.1}$$

where θ is the (instantaneous) spacecraft attitude ($\ddot{\theta}$ is its angular acceleration), I is its moment of inertia, and $F(t)$ is a net disturbing force applied symmetrically at a radius, r, about the axis. The periodicity of attitude will not be considered in the sequel.

The Control and Disturbances. The force $F(t)$ consists of both random disturbances and the thrust of the attitude control mechanism. The random disturbance sources may be such items as moments imparted during spacecraft separation from the boost vehicle, the movement of men and equipment aboard the vehicle, or the starting and stopping aboard the vehicle of rotating devices (e.g., tape recorders and gyroscopes). These disturbances are all characterized by time durations considerably shorter than those of the system dynamics of interest. This characteristic is precisely that which characterizes Brownian motion and, hence, white noise. Thus, the disturbances will be summarized and represented by a zero mean, white, gaussian random process.

The attitude control mechanism will be assumed to be a variable thrust device subject to a saturation (magnitude) constraint. This assumption is a little more general than that of the on-off type jets commonly used for attitude control because of their simplicity and reliability; the assumed form will turn out to give results identical to the on-off-on case, because of the well known result that attitude constrained fuel optimization problems, at least in the noiseless case, lead to on-off-on type solutions.[6]

Thus, let

$$r\,F(t) = m(t) + n(t)\ , \tag{1.2}$$

where $m(t)$ is the control torque and subject to

$$|m(t)| \leq M = \text{constant}\ , \tag{1.3}$$

and $n(t)$ is the random disturbance process having mean zero and spectral density N_0. It should be noted that the presence of the random input to the system causes (1.1) to become a stochastic differential equation.

The Performance Objectives. The objective is to derive a policy of control for the thrusters such that the expected total fuel expended for attitude control during a mission is made a minimum. At the same time the expected average attitude error must be maintained below some bound. A mean square error definition will be used here as is common for this type problem. It is well known that this definition produces properties which are both physically meaningful and mathematically convenient. This measure also is known to provide identical results to other measures for a wide class of problems, at least for the noise free case.[7]

---

[6] M. Athanassiades, "Optimal Control for Linear Time-Invariant Plants with Time, Fuel, and Energy Constraints," AIEE Transactions, Part II, on Applications and Industry, LXXXI, 321-325.

[7] S. Sherman, "Non-Mean-Square Error Criteria," IRE Transactions on Information Theory, Vol. IT-IV, 125-126.

Thus, it is required that

$$E_{n(t)} \left\{ \frac{1}{(t_f - t_0)} \int_{t_0}^{t_f} [\theta(t) - \rho(t)]^2 dt \,|\, \theta(t_0), \dot{\theta}(t_0) \right\} \le K , \qquad (1.4)$$

where $t_f - t_0$ is the specified length of the mission, and $\rho(t)$ is the specified, desired attitude.[8]

It is typical for space vehicles that the desired attitude, except for a few isolated attitude maneuvers, is either constant or changes very slowly compared to the vehicle attitude dynamics. Thus, it will be assumed that $\rho(t) = \rho$, a constant.

The fuel criterion may be written in integral form:

$$\text{Expected Fuel Used} = E_{n(t)} \left[\int_{t_0}^{t_f} |m(t)| \, dt \,|\, \theta(t_0), \dot{\theta}(t_0) \right] \qquad (1.5)$$

The criterion and constraint as stated here may be inbedded into a more general family of optimization problems by applying the method of Lagrange multipliers.[9] This leads to the expression

$$E_{n(t)} \left\{ \int_{t_0}^{t_f} [\, |m(t)| + \lambda \{\theta(t) - \rho\}^2] dt \,|\, \theta(t_0), \dot{\theta}(t_0) \right\} \qquad (1.6)$$

which is to be minimized. The coefficient λ is the Lagrange multiplier and it includes

---

[8] The notation $E_{n(t)}[g(\theta, t) \,|\, \theta(t_0), \dot{\theta}(t_0)]$ will be used to represent the conditional expectation, given $\theta(t_0)$ and $\dot{\theta}(t_0)$, of the quantity $g(\theta, t)$ with respect to the random process $n(t)$.

[9] See, e.g., Kaplan, Advanced Calculus, 128-129.

the term $\frac{1}{t_f - t_0}$ from (1.4). It is evaluated from the minimum value of (1.6) and the constraint equation, (1.4).

The steady state behavior of the system is of particular interest. In this case, of course, $t_f - t_0 \to \infty$. Normally t_0 will be assumed zero, but it will be useful in some places to carry the more general notation. It is obvious, of course, that the performance of the optimum system must be stable or at least have a stable (and small) limit cycle.

Problem Statement. The problem having been rather carefully developed, it may now be succinctly stated in mathematical terms as follows.

PROBLEM 1-1. MINIMUM FUEL ATTITUDE CONTROL PROBLEM: Given

A. A linear, time-invariant, dynamic system described by the differential equation

$$I \ddot{\theta}(t) = m(t) + n(t), \tag{1.7}$$

where $n(t)$ is a white gaussian random process having mean zero and spectral density N_0;

B. The restriction that

$$m(t) \in \mathcal{M}, \tag{1.8}$$

where

$$\mathcal{M} = \{m(y): m(\theta, \dot{\theta}, t) \text{ a measurable function on } \theta, \dot{\theta}, \text{ and } t \in [0, \infty); \; |m(t)| \leq M\} \tag{1.9}$$

is the set of admissible controls; and

C. The boundary conditions

1. $\theta(t_0) = \theta_{01}$ (1.10)

and

2. $\dot{\theta}(t_0) = \theta_{02}$. (1.11)

Then, find the feedback control, $m_o(\theta(t), \dot{\theta}(t), t)$ that

1. Transfers the system from $\begin{pmatrix} \theta_{01} \\ \theta_{02} \end{pmatrix}$ according to (1.1),

2. Satisfies (1.7) and (1.8), and

3. Causes the quantity

$$P = E_{n(t)}\left[\int_{t_0}^{t_f} \{|m(t)| + \lambda[\theta(t) - \rho]^2\} \, dt \,\Big|\, \theta(t_0), \dot{\theta}(t_0)\right] \quad (1.12)$$

for given λ and ρ to be a minimum.

1.2.2 A Related Problem

There is a problem closely related to Problem 1-1, which is of interest in the sequel. This is the problem of maintaining the position of a system for which the velocity rather than the acceleration of the plant is controlled. Because these dynamics are typical of certain types of tracking antennas such as those used for earth tracking of space vehicles, this problem will be called an antenna steering problem[10]. Maintaining as much as possible the notation of the attitude control problem, this problem becomes the following.

[10] Such an antenna, rotating at sidereal rate and a faster slewing rate, is described in K. W. Linnes, W. D. Merrick, and R. Stevens, "Ground Antenna for Space Communication", IRE Transactions on Space Electronics and Telemetry, Vol. SET-VI.

PROBLEM 1-2. MINIMUM FUEL ANTENNA STEERING PROBLEM: Given

A. A linear, time invariant, dynamical system described by the differential equation

$$I \dot{\theta} = r F(t), \qquad (1.13)$$

where

$$r F(t) = m(t) + n(t), \qquad (1.2)$$

and where $n(t)$ is a white gaussian random process having mean zero and spectral density N_0;

B. The restriction that

$$m(t) \in \mathcal{M}, \qquad (1.8)$$

where

$$\mathcal{M} = \{m(t): m(\theta, t) \text{ a measurable function on } \theta \text{ and } t \in [0, \infty];$$

$$|m(t)| \leq M\} \qquad (1.14)$$

is the set of admissible controls: and

C. The boundary condition

$$\theta(t_0) = \theta_{01}. \qquad (1.10)$$

Then, find the feedback control, $m_o(\theta(t), t)$ that

1. Transfers the system from θ_{01} according to (1.13) and (1.2),
2. Satisfies (1.8) and (1.14), and
3. Causes the quantity

$$P = E_{n(t)}\left\{\int_{t_0}^{t_f}\{|m(t)|+\lambda[\theta(t)-\rho]^2\}dt\,|\,\theta(t_0)\right\} \tag{1.15}$$

<u>for given</u> λ <u>and</u> ρ <u>to be a minimum.</u>

1.3 Normalization

It is useful to investigate the dimensionality of the relations found in Problems 1-1 and 1-2. This not only will allow the relations to be somewhat simplified, but also will permit the results of a single optimizing analysis to be applied in a known way to several physical cases. In the first section below the preliminary step of removing the reference input, ρ, from the equations is performed; in the second, dimensionality is used to determine the minimum set of necessary parameters in the equations.

1.3.1 The Reference Input

Let a new position variable, $\varphi(t)$, be defined by the relation

$$\theta(t) = \varphi(t) + \rho \,. \tag{1.16}$$

Upon substitution for the occurrences of $\theta(t)$ in Problems 1-1 and 1-2, there results from (1.1), (1.10) (1.11), and (1.12), respectively,

$$I\,\ddot{\theta}(t) = I\,\ddot{\varphi}(t) = r\,F(t),$$

$$\theta(t_0) = \varphi(t_0) + \rho \qquad \text{or} \qquad \varphi_{01} = \theta_{01} - \rho,$$

$$\dot{\theta}(t_0) = \dot{\varphi}(t_0),$$

and

$$P = E_n\left[\int_{t_0}^{t_f}\{|m(t)|+\lambda\varphi^2(t)\}dt\,|\,\varphi(t_0), \dot{\varphi}(t_0)\right].$$

Thus, the reference input ρ may always be taken as zero. For arbitrary reference input, the results of the zero input case hold with a simple change of variable of the form (1.16).

1.3.2 Non-Dimensionalizing

Problems with dimensionless variables may be derived from Problems 1-1 and 1-2. To this end the following relations are defined. In each relation the capital letter represents an undetermined characteristic dimension of the problem, and the new lower case letter represents a non-dimensional problem variable.

$$t^* = Tt \tag{1.17}$$

$$x(t^*) = \Theta\,\theta(t) \tag{1.18}$$

$$u(t^*) = Lm(t) \tag{1.19}$$

Also for convenience let

$$y = \dot{x} = \frac{dx}{dt^*}\,. \tag{1.20}$$

Note that then

$$x_0 \triangleq x(t_0) = \Theta\,\theta(t_0)\,, \tag{1.21}$$

and

$$y_0 \triangleq y(t_0) = \frac{\Theta}{T}\,\dot{\theta}(t_0)\,. \tag{1.22}$$

Substitution into (1.1) and (1.12) of Problems 1-1 and 1-2 yields

$$I\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{T^2}{\Theta}\,\frac{d^2x}{dt^{*2}} = \frac{1}{L}u + T^{\frac{1}{2}}n(t^*) \tag{1.23}$$

and

$$P = E_n\left[\int_{t_0}^{t_f}\left\{\left|\frac{u}{L}\right| + \frac{\lambda}{\Theta^2}x^2\right\}d\left(\frac{t^*}{T}\right)\,\Big|\,x_0, y_0\right].^{11} \tag{1.24}$$

11. The noise term must be scaled in amplitude when the time scale is changed. See L. Levine, Methods for Solving Engineering Problems, 350.

The selection of values for T, Θ, and L may be made arbitrarily. It will be useful to require that

$$|u| \leq 1 . \tag{1.25}$$

Then,

$$L = \frac{1}{M} . \tag{1.26}$$

Substitution into (1.23) and (1.24) and rearranging leads to

$$\frac{d^2x}{dt^{*2}} = \frac{\Theta M}{IT^2} u + \frac{\Theta}{IT^{3/2}} n , \tag{1.27}$$

and

$$\frac{TP}{M} = E_n\left[\int_{t_0^*}^{t_f^*} \{ |u| + \frac{\lambda}{M\,\Theta^2} x^2 \} dt^* \;|x_0, y_0 \right] . \tag{1.28}$$

Now set

$$\frac{\lambda}{M\Theta^2} = 1 , \tag{1.29}$$

and

$$\frac{\Theta M}{IT^2} = 1 . \tag{1.30}$$

These yield

$$\Theta = \left(\frac{\lambda}{M} \right)^{1/2} \tag{1.31}$$

and

$$T = \left(\frac{\lambda M}{I^2} \right)^{1/4} . \tag{1.32}$$

After introducing a normalized cost,

$$J = \min_{u \in U} P\left(\frac{T}{M}\right) = \min_{u \in U} P\left(\frac{\lambda}{M^3 I^2} \right)^{1/4} \tag{1.33}$$

which includes the required minimization operator, (1.27) and (1.28) become

$$\frac{d^2x}{dt^{*2}} = u + \left(\frac{T}{M}\right)^{\frac{1}{2}} n \,, \tag{1.34}$$

and

$$J = \min_{u \in U} E_n \left[\int_{t_0^*}^{t_f^*} \{ |u| + x^2 \} \, dt^* \, | x_0 , y_0 \right] . \tag{1.35}$$

Here $U = \{u: u = Lm, \ m \in \mathcal{M}\}$.

Finally, introduce the parameter

$$d = \frac{N_0 T}{4\pi M^2} \,. \tag{1.36}$$

The reasons for this particular choice will be clear in the sequel, but cannot readily be justified at this time. It will turn out that d is the only required parameter for the normalized versions of the basic problems. This substitution leads to the following expression for (1.34):

$$\frac{d^2x}{dt^{*2}} = u + \left(\frac{4\pi M d}{N_0 T^{\frac{1}{2}}}\right) n \,. \tag{1.37}$$

Problems 1-1 and 1-2 will now be restated in normalized forms. This will serve both to summarize the present section and to provide a convenient form of problem statement for reference in the sequel. The star on the new time variable will now be dropped. Also, the position variable for the antenna steering problem will be y rather than x in order to make the two problems look as similar as possible.

PROBLEM 1-3. ATTITUDE CONTROL PROBLEM: Given

A. A linear, time invariant, dynamical system described by the differential equations

$$\dot{x} = y(t) \tag{1.38}$$

and

$$\dot{y} = u(t) + \left(\frac{4\pi md}{N_0 T^{\frac{1}{2}}}\right) n(t) \quad , \tag{1.39}$$

where n(t) is a white, gaussian, stochastic process having mean zero and spectral density N_0;

B. The restriction that

$$u(t) \in U \; , \tag{1.40}$$

where

$$u = \{u(t) : u(x, y, t) \text{ a measurable function on } x, y,, \text{ and } t \in [0, \infty);$$
$$|u(t)| \leq 1\} \tag{1.41}$$

is the set of admissible controls; and

C. The boundary conditions

1. $x(t_0) = x_0$ (1.42)

and

2. $y(t_0) = y_0$. (1.43)

Then, find the feedback control, $u_o(x, y, t)$, that

1. Transfers the system from $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ according to (1.38) and (1.39),

2. Satisifes (1.40) and (1.41), and

3. Causes the quantity

$$J = \min_{u \in U} E_n \left[\int_{t_0}^{t_f} \{ |u| + x^2 \} dt \,|\, x_0, y_0 \right] \tag{1.35}$$

to assume the indicated minimum.

PROBLEM 1.4. ANTENNA STEERING PROBLEM: Given

A. A linear, time invariant, dynamical system described by the differential equation (1.39), where n(t) is a white, gaussian, stochastic process having mean zero and spectral density N_0;

B. The restriction B of Problem 1-3; and

C. The boundary condition (1.43).

Then, find the feedback control, $u_o(x, y, t)$, that

1. Transfers the system from y_0 according to (1.39),

2. Satisfies (1.40) and (1.41), and

3. Causes the quantity

$$J = \min_{u \in U} E_n \left[\int_{t_0}^{t_f} \{ |u| + y^2 \} dt \,|\, y_0 \right] \tag{1.44}$$

to assume the indicated minimum.

1.4 Available Methods of Analysis and Related Problems

1.4.1 Available Methods of Analysis

Except for a few special methods there are three basic approaches to the solution of the stated problems. These may be roughly identified by their association with (1) the Maximum Principle of Pontryagin, (2) Hamilton-Jacobi theory, or (3) approximation methods. The first two classes of methods are more or less analytical, while the last tends to be numerical. As might be expected, the available techniques are much more highly developed for the deterministic cases ($n(t) \equiv 0$) than for the stochastic situations.

Since the work described in the sequel concentrates on Hamilton-Jacobi techniques, these methods will not be described here. The other two approaches will be discussed in the following paragraphs, both for deterministic and stochastic cases.

The Pontryagin Maximum Principle for deterministic systems is a highly developed tool.[12] The primary obstacle to the application of this technique is the open loop nature of the resulting optimum control. When a feedback control is required, it must be synthesized by special techniques that usually involve some form of trajectory tracing. When switching curves are involved in the control law, and when the switching curves do not fall along trajectories, the control may involve an infinite number of switchings and rather special methods must then be applied to identify the switching curves.

[12] L. S. Pontryagin, et. al., The Mathematical Theory of Optimal Processes.

Such is the case for Problem -13. A solution to this (deterministic) problem by the maximum principle is straightforward (although numerical computation may be required to solve the resulting two point boundary value problem).[13] The resulting optimum control has the form of a time function prescribed for each initial state. Then, synthesis of a feedback control for a given time interval of operation (i.e., a given time to go) essentially involves locating a set of switching curves. This may be done by recording the optimum switching point locations for several different initial states and fitting a curve (such as a least squares polynomial) through these locations. If the switching points converge to particular curves as the given time interval grows indefinitely, these curves are the optimum steady state switching curves for the system.

A maximum principle for stochastic, continuous-time systems apparently has not been developed. This is not to surprising, because an optimum stochastic system is of necessity a feedback system, while the maximum principle leads naturally to non-feedback solutions. Some development of feedback maximum principle approaches to discrete time

---

[13] Typical treatments of these computations may be found in C. W. Merrian III, Optimization Theory and the Design of Feedback Control Systems, Chapter 10

and

M. D. Anderson and S. C. Gupta, "Backward Time Analog Computer Solutions of Optimum Control Problems", Proceedings AFIP Spring Joint Computer Conference 1967, 133-139.

systems has been accomplished.[14, 15] A cursory investigation of these methods to discrete time approximations to Problems 1-3 and 1-4 has revealed that the methods do yield results with some of the properties one might expect for these problems, but the calculations involved are very lengthy. In addition, there exist some problems in verifying the properties required to establish the applicability of the methods.

There are several approximation techniques applicable to deterministic cases of the given problems. These include dynamic programming, steepest descent in function space, and parameter optimization techniques.[16, 17] In the latter technique some form for the switching curves (i.e., the feedback control law) must be assumed, after which its parameters may be optimally selected. Because straightforward analytical solutions are available for the given problems, it seems superfluous to give very much attention to approximation methods for deterministic cases.

There are two primary approximation approaches to the stochastic problems that warrant consideration. The first of these is dynamic

---

[14] H. D. Kushner and F. C. Schweppe, "A Maximum Principle for Stochastic Control Systems," Journal of Mathematical Analysis and Applications, VIII, 287-302.

[15] D. D. Sworder, "On the Control of Discrete Time Stochastic Systems," Department of Electrical Engineering Report No. USCEE 145, University of Southern California.

[16] R. E. Bellman, Dynamic Programming, or
R. E. Bellman and S. E. Dreyfus, Applied Dynamic Programming.

[17] A. E. Bryson and W. F. Denham, "A Steepest-Ascent Method for Solving Optimum Programming Problems," ASME Journal of Applied Mechanics, LXXXIV, 247-257.

programming.[15] Dynamic programming may be applied directly to discrete approximations to the given problems. While it is not obvious that all of the properties required to prove convergence can be verified, this is not so great a drawback as is the well known "curse of dimensionality". This curse manifests itself in excessively long computation times.[18]

The second approximation approach requires the form of the control law to be assumed. This reduces the optimization problem to a parameter optimization problem. The parameters may be estimated using such techniques as the stochastic approximation of Robbins and Monro.[19] It is very likely that a formulation based on stochastic approximation will fall outside the presently verified sufficiency conditions of convergence, but this approach should be considered only as a last resort anyway. This is because it is essentially a method which delivers an answer without any real concomitant insight into the problem.

An approximation approach that should also be noted is that of

---

18 The computation times, using a dynamic programming formulation of the attitude control problem implemented early in this study, were two orders of magnitude higher than those using the numerical techniques of Chapter 5 as indicated by a coding analysis.

19 H. Robbins and S. Monro, "A Stochastic Approximation Method", Annals of Mathematical Statistics, XXII,

A good introduction to this technique is given in D. J. Wilde, Optimum Seeking Methods.

A good recent survey of this technique appears in N. V. Loginov, "Methods of Stochastic Approximation", Automation and Remote Control, XXVII, 185-204.

simulating the system on, say, an analog computer and making repeated runs with various parameter settings. It should not take too long with such a technique to find a feedback control that yields a relatively low cost. The problem here is that one can never be sure that an absolute minimum has been found. In many respects this simulation approach shares this difficulty with the previous stochastic approximation approach.

1.4.2 Related Problems

The previous section has reviewed methods potentially applicable to solving the given problems. These methods may or may not have been applied to problems as specific as the given problems. This section is intended to briefly survey some of the results available for problems whose formulations closely resemble Problems 1-3 and 1-4, but which differ in some significant way.

Unconstrained Fuel, Quadratic Cost Problems. For both the deterministic and stochastic cases, solutions are available for the problems arising from 1-3 and 1-4 when (1.3) is not required and when (1.35) and (1.44) are replaced respectively by

$$J = \min_{u \in U} E_n \left[\int_{t_0}^{t_f} \{ u^2 + x^2 \} dt \,|\, x_0, y_0 \right] \tag{1.45}$$

and

$$J = \min_{u \in U} E_n \left[\int_{t_0}^{t_f} \{ u^2 + y^2 \} dt \,|\, y_0 \right] . \tag{1.46}$$ [18]

The control law for these problems turns out to be linear feedback, and the coefficients are readily computed, being the solutions to matrix Ricatti differential equations. The computation must usually be numerical.[20]

<u>Constrained Fuel, Quadratic Cost Problems.</u> The introduction of a fuel constraint into the quadratic cost problem eliminates the possibility of linear feedback control laws (unless the position errors are quite small).[21] The deterministic problems are essentially solved by Athans in the previously cited reference. The stochastic problems have not been solved, apparently, and they can be shown to possess nonlinearity difficulties very similar to those encountered in the sequel.

<u>Constrained Fuel, Quadratic Position Error Problems.</u> When the control energy or fuel term is dropped completely from (1.45) and (1.46), another class of problems develops. This class is really a limiting case of Problems 1-3 and 1-4: the case when $\lambda \rightarrow \infty$ in (1.6). These problems have been investigated thoroughly in the deterministic case by Fuller and his associates at Cambridge. Some of the results and references appear in Section 3.1 below. Stochastic problems in this class apparently have not been treated to date.

<u>Fuel Optimum Problems.</u> One last class of problems will be

---

[20] For a discussion of the deterministic case, see
M. Athans and P. L. Falb, <u>Optimal Control</u>, Chapter 9.

The stochastic case is discussed in
J. J. Florentin, "Optimal Control of Continuous Time, Markov Stochastic Systems," <u>Journal of Electronics and Control</u>, X, 473-488.

[21] See, e.g., M. Athanassiades, <u>loc. cit.</u>

considered. In this class the integral position error term is dropped from (1.45) and (1.46). (This corresponds to letting $\lambda \to 0$ in (1.6).) In its place a target set, or enforced terminal state, is imposed, say $x(t_f) = y(t_f) = 0$. Of course, this problem makes sense only in the deterministic case, and it has been treated rather fully.[22]

The remainder of this report will be concerned with the principal topic of this effort:namely, Hamilton-Jacobi related approaches to the solution of Problems 1-3 and 1-4.

---

[22] See, e.g., M. Athans and P. L. Falb, op. cit., Chapters 6 and 8.

Chapter 2

A HAMILTON-JACOBI EQUATION TYPE FORMULATION OF THE PROBLEMS

In this chapter partial differential equations of the Hamilton-Jacobi type will be derived from the properties of Problems 1-3 and 1-4. The basic properties will also be shown to imply certain properties of the cost surface and switching curves, which will be needed later for solving the equations. The first section below treats the noise-free case; the second treats the stochastic problem.

2.1 Deterministic Case

2.1.1 Derivation of a Hamilton-Jacobi Type Equation

In this section Problems 1-3 and 1-4 will be treated in the noise free case, that is, with

$$n(t) \equiv 0. \tag{2.1}$$

In this case the expectation may be dropped from the cost relations, (1.35) and (1.44) because of the fact that the integral is no longer a random variable.[1]

There are two basic methods which permit the stated problems to be converted into problems involving the solutions to partial differential equations of the Hamilton-Jacobi type. These methods will be

[1] This follows L.I. Rozonoér, "L.S. Pontryagin Maximum Principle in the Theory of Optimum Systems", Automation and Remote Control, XX, 1288-1302, 1405-1421, 1517-1532.

called the methods of Bellman and Kalman for obvious reasons.[2] Following Kalman the results may be written down immediately.

Problem 1-3.

For Problem 1-3 the Hamilton-Jacobi type equation is

$$-\frac{\partial J}{\partial t} = \min_{u \in U} \{u \frac{\partial J}{\partial y} + y \frac{\partial J}{\partial x} + x^2 + |u|\}. \tag{2.2}$$

Problem 1-4.

For Problem 1-4 the Hamilton-Jacobi type equation is

$$-\frac{\partial J}{\partial t} = \min_{u \in U} \{u \frac{\partial J}{\partial y} + y^2 + |u|\} \tag{2.3}$$

If (2.2) and (2.3) are interpreted strictly, they are applicable only at the initial time, t_0; but if the initial time is considered variable, then these equations may be interpreted as applicable at any time. It will be convenient to make a change of variable and treat (2.2) and (2.3) in terms of time to go, τ. This is possible because of the autonomy of the system. Let

---

[2] For Kalman's approach see:
R.E. Kalman, "The Theory of Optimal Control and the Calculus of Variations," ed. R. Bellman, Mathematical Optimization Techniques, 309-331.
or
M. Athans and P.L. Falb, Optimal Control, 355-363.
For Bellman's approach, see for example:
R.E. Bellman and S.E. Dreyfus, Applied Dynamic Programming, Chapter 5.
or
C.W. Merriam III, Optimization Theory and the Design of Feedback Control Systems, Chapter 5.

$$\tau = t_f - t \tag{2.4}$$

and

$$\tau_0 = t_f - t_0 . \tag{2.5}$$

Then (2.2) and (2.3) may be replaced by the equations,

$$\frac{\partial J}{\partial \tau} = \min_{u \in U} \{u \frac{\partial J}{\partial y} + y \frac{\partial J}{\partial x} + X^2 |u| \} \tag{2.6}$$

and

$$\frac{\partial J}{\partial \tau} = \min_{u \in U} \{u \frac{\partial J}{\partial y} + y^2 + |u| \} , \tag{2.7}$$

respectively. In the sequel the following terminology with respect to τ will be adopted: the condition $\tau = 0$ rather than the system initial condition, $\tau = \tau_0$, will be called the initial condition for the partial differential equation. The latter will be called the terminal condition.

2.1.2. Properties of the Cost Surfaces and Switching Curves

Several properties of the cost surfaces and switching surfaces associated with Problems 1-3 and 1-4 or equations (2.2) and (2.3) are stated and justified in the paragraphs that follow. The justifications fall somewhat short of being mathematical proofs, although in general they do sketch out possible proofs. The properties are stated in Problem 1-3. The corresponding properties for Problem 1-4 follow in an obvious way.

PROPERTY 2. 1-1. INITIAL CONDITION: For all values of the state variables, the initial cost ($\tau = 0$) is zero.

This property is a direct result of the equation defining the cost: namely (1.6). Whenever $\tau = 0$ (i.e., $t_f = t_0$) in this equation, it is clear that the corresponding cost, J, is zero, regardless of the initial state, $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

PROPERTY 2.1-2. SYMMETRY: Problem 1-3 is unchanged under a change of variables of the form

$$\left.\begin{aligned} x &\rightarrow -x \\ y &\rightarrow -y \\ u &\rightarrow -u. \end{aligned}\right\} \qquad (2.8)$$

This property is easily verified by reference to the basic problem equations with $n(t) \equiv 0$-- in particular (1.38), (1.39), (1.41), and (1.35). It may also be verified in the Hamilton-Jacobi equation, (2.2). This property is one of symmetry about the origin of state space.

PROPERTY 2.1-3. EXISTENCE OF CHARACTERISTIC CURVES AND TRAJECTORIES There is a family of curves in phase space, $(X \times Y \times T)$, intrinsicly defined by the Hamilton-Jacobi equation. These curves are the trajectories of the system. Their projections onto the cost surface are called characteristic curves.

Consider the general total derivative of the cost surface along some curve, s:

$$\frac{dJ}{ds} = \frac{\partial J}{\partial \tau}\frac{d\tau}{ds} + \frac{\partial J}{\partial x}\frac{dx}{ds} + \frac{\partial J}{\partial y}\frac{dy}{ds}. \tag{2.9}$$

Comparison of this equation with (2.6) after substitution of the optimum control, u_o, indicates that the parametric curve described by

$$\frac{d\tau}{ds} = 1 \tag{2.10}$$

$$\frac{dx}{ds} = -y \tag{2.11}$$

$$\frac{dy}{ds} = -u_o \tag{2.12}$$

is characteristic of the equation. In fact, after allowing for (2.4) and (2.10), (2.11) and (2.12) become the equations for the trajectories of the system, (1-38) and (1-39). The projections of these trajectories onto the cost surface, J, are the curves commonly called characteristic (or integral) curves of the optimized equation, (2.2) with u.[3]

PROPERTY 2.1-4. CONTINUITY OF THE COST SURFACE: Over the entire phase space the cost surface is continuous.

This is a fact deducible from the basic problem statement. It follows by noting properties of integrals of solutions to differential equations.[4] Since this is discussed by Fuller, further justification will not be presented here. This property is an assumption in the derivation

---

[3] Integral curves are discussed in I. N. Sneddon, Elements of Partial Differential Equations, chaps. 1 and 2.

[4] A. T. Fuller, "Optimization of Some Non-Linear Control Systems by means of Bellman's Equation and Dimensional Analysis," International Journal of Control, III, 362-363.

of the Hamilton-Jacoby type equations.[5]

PROPERTY 2.1-5. OPTIMALITY OF ON-OFF-ON CONTROLS: The optimum cc u_o, is of the on-off-on type. In particular, the condition

$$u_o = \begin{cases} -1 & \frac{\partial J}{\partial y} \geq 1 \\ 0 & \left|\frac{\partial J}{\partial y}\right| \leq 1 \\ 1 & \frac{\partial J}{\partial y} \leq -1 \end{cases} \tag{2.13}$$

$$\triangleq - \text{dez} \left(\frac{\partial J}{\partial y}\right)$$

is necessary.[6]

Performing the indicated minimum operation of (2.2) at each time instant leads to the necessary condition for the optimum control, (2.13). This is a three state, bang-coast-bang, or on-off-on type control.

PROPERTY 2.1-6. CONTINUITY OF DERIVATIVES. In every region of phase space for which the optimum control, $u_o(x, y, \tau)$, is continuous in all of its arguments, all of the phase derivatives of the cost surface are continuous.

This property is made evident by referring to (1.35) with

---

[5] See the references of footnote 1.

[6] Values for u_o at $\left|\frac{\partial J}{\partial y}\right| = 1$ are ambiguous, but they have no effect on the results.

$n(t) \equiv 0$ and applying Leibnitz's rule along with the continuity properties of the solution to differential equations.[7,8]

NOTE: The three properties which immediately follow (Properties 2.1-7, 2.1-8, and 2.1-9) are justified in rather intuitive terms. The justifications are reasonable if the various derivatives involved behave well enough, but the difficulties that may arise when this is not true (as, for example, when a derivative becomes unbounded somewhere) are not considered. Nevertheless, if these three properties are instrumental in leading to a solution of the Hamilton-Jacobi equation, (2.6), which meets the initial boundary condition, Property 2.1-1, then the resulting solution is in no sense suboptimum. This is because the Hamilton-Jacobi equation is known to be both a necessary and sufficient condition for a solution to the basic problem.[9] If the three properties lead to a solution, then this solution will be at least as good as any other possible solution.

PROPERTY 2.1-7. EXISTENCE AND SMOOTHNESS OF SWITCHING SURFACES: The loci of switching points form continuous surfaces. These surfaces are called switching surfaces, and they are smooth in the phase variables, x, y, and τ.

The idea behind justifying this property is that the switching surfaces are the intersections of regions of constant control. The cost surface within each of these regions is smooth (its derivatives are

---

[7] For Leibnitz's rule see J.M.H. Olmsted, Real Variables, 416-418.

[8] For continuity properties see E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, chaps. 1 and 2.

[9] See, eg., Kalman, loc. cit.

continuous--see Properties 2.1-4 and 2.1-6) thus guaranteeing the existence and continuity of the gradients of each surface at an intersection. The two gradients along the intersection must have the same projection into phase space (by the very nature of an intersection), and this projection is precisely the gradient of the switching surface. Now, a projection is a linear transformation, which preserves continuity so that the switching surface has a continuous gradient.

PROPERTY 2.1-8. CONTINUITY OF DERIVATIVES OF J: Over the entire phase space all of the first partial derivatives of the cost with respect to the phase variables are continuous.

Consider a derivative of the cost surface in the direction of any curve, s, in the switching surface. In particular, calling on symmetry, consider only the intersection of the cost surfaces for regions where, rexpectively, $u_o = 0$ and $u_o = -1$ are the optimum controls. Then,

$$\frac{dJ_0}{ds} = \frac{dJ_{-1}}{ds} \tag{2.14}$$

because of continuity, Property 2.1-4. Equation (2.14) may also be written

$$\nabla J_0^t \cdot \alpha = \nabla J_{-1}^t \cdot \alpha, \tag{2.15}$$

where ∇J_u is the gradient of the cost surface corresponding to the optimum control u evaluated at the switching surface (perhaps as a limit), and $\alpha = (\alpha_T, \alpha_x, \alpha_y)^t$ is the tangent vector to the curve s. If

$\nabla S = (1, \beta_x, \beta_y)^t$ is a normal to the switching surface, then it is clear that

$$\alpha^t \cdot \nabla S = 0 \tag{2.16}$$

or

$$\alpha_\tau = \alpha_x \beta_x - \alpha_y \beta_y. \tag{2.17}$$

After substituting the value of the optimum control, (2.6) becomes

$$\frac{\partial J_u}{\partial \tau} = \frac{\partial J_u}{\partial x} y + \frac{\partial J_u}{\partial y} u + |u| + x^2, \qquad u = 0, -1. \tag{2.18}$$

Substituting (2.18) and (2.17) into (2.15) and rearranging terms leads to the relation

$$\alpha_x \left[(1-\beta_x y)\left(\frac{\partial J_0}{\partial x} - \frac{\partial J_{-1}}{\partial x}\right) - \beta_x \frac{\partial J_{-1}}{\partial y} + \beta_x \right]$$

$$+ \alpha_y \left[-\beta_y y \left(\frac{\partial J_0}{\partial x} - \frac{\partial J_{-1}}{\partial x}\right) + \left(\frac{\partial J_0}{\partial y} - \frac{\partial J_{-1}}{\partial y}\right) - \beta_y \frac{\partial J_{-1}}{\partial y} + \beta_y \right] = 0 \tag{2.19}$$

Now, after accounting for (2.17), equation (2.16) is valid for any α_x and α_y. Thus, (2.19) can be true only if the coefficients of α_x and α_y therein are simultaneously zero. That is

$$(1-\beta_x y)\left(\frac{\partial J_0}{\partial x} - \frac{\partial J_{-1}}{\partial x}\right) - \beta_x \frac{\partial J_{-1}}{\partial y} + \beta_x = 0, \tag{2.20}$$

and

$$-\beta_y y \frac{\partial J_0}{\partial x} - \frac{\partial J_{-1}}{\partial x} + \frac{\partial J_0}{\partial y} - (1+\beta_y)\frac{\partial J_{-1}}{\partial y} + \beta_y = 0, \tag{2.21}$$

simultaneously. Eliminating the term $\left(\frac{\partial J_0}{\partial x} - \frac{\partial J_{-1}}{\partial x}\right)$ between (2.20) and (2.21) yields

$$\frac{\partial J_0}{\partial y} = \frac{(1-\beta_x y+\beta_y)}{(1-\beta_x y)} \frac{\partial J_{-1}}{\partial y} - \frac{\beta_y}{(1-\beta_x y)} \quad . \tag{2.22}$$

Now, Property 2.1-5 indicates that

$$\frac{\partial J_0}{\partial y} \leq 1, \tag{2.23}$$

and

$$\frac{\partial J_{-1}}{\partial y} \geq 1. \tag{2.24}$$

Furthermore, it is reasonable to require that any trajectory intersecting the switching surface continue in the new region in a direction away from the old region. This is equivalent to saying that the projection on the gradient vector for the switching surface tangent vector to each of the trajectories will be required to yield identical signs. By reference to Property 2.1-3, the tangent vector to the trajectories is

$$S_u = \left(\frac{d\tau}{dS_u} \frac{dx}{dS_u}, \frac{dy}{dS_u} \right)^t = (1, -y, -u), \qquad u = 0, -1. \tag{2.25}$$

Then, the requirement is that

$$(\nabla S^t \cdot S_0)(\nabla S^t \cdot S_{-1}) = (1-\beta_x y)(1-\beta_x y + \beta_y) \geq 0. \tag{2.26}$$

Requirement (2.26) guarantees that the coefficient of $\frac{\partial J_{-1}}{\partial y}$ in (2.22) is positive. Applying (2.23) to (2.22) then yields

$$1 \geq \frac{(1-\beta_x y + \beta_y)}{(1-\beta_x y)} \frac{\partial J_{-1}}{\partial y} - \frac{\beta_y}{(1-\beta_x y)} \tag{2.27}$$

or

$$\frac{(1-\beta_x y + \beta_y)}{(1-\beta_x y)} \geq \frac{(1-\beta_x y+\beta_y)}{(1-\beta_x y)} \frac{\partial J_{-1}}{\partial y} , \tag{2.28}$$

or

$$1 \geq \frac{\partial J_{-1}}{\partial y} . \tag{2.29}$$

Considering (2.24), then,

$$\frac{\partial J_{-1}}{\partial y} = 1 \tag{2.30}$$

at the switching surface.

Solving (2.22) for $\frac{\partial J_{-1}}{\partial y}$ yields

$$\frac{\partial J_{-1}}{\partial y} = \frac{(1-\beta_x y)}{(1-\beta_x y+\beta_y)} \frac{\partial J_0}{\partial y} + \frac{\beta_y}{(1-\beta_x y+\beta_y)} . \tag{2.31}$$

Inequality (2.26) indicates that the coefficient of $\frac{\partial J_0}{\partial y}$ is positive. Applying (2.24) to (2.31) yields

$$1 \leq \frac{(1-\beta_x y)}{(1-\beta_x y+\beta_y)} \frac{\partial J_0}{\partial y} + \frac{\beta_y}{(1-\beta_x y+\beta_y)} , \tag{2.32}$$

or

$$\frac{(1-\beta_x y)}{(1-\beta_x y+\beta_y)} \leq \frac{(1-\beta_x y)}{(1-\beta_x y+\beta_y)} \frac{\partial J_0}{\partial y} , \tag{2.33}$$

or

$$1 \leq \frac{\partial J_0}{\partial y} . \tag{2.34}$$

Considering (2.23), then,

$$\frac{\partial J_0}{\partial y} = 1$$

at the switching surface.

It is now clear that $\frac{\partial J}{\partial y}$ is continuous across the switching surface as long as one of the trajectories does not lie in the switching surface. The effect in the latter case is to make, for example, the coefficient $(1-\beta_x y)$ in (2.20) to be zero. Then (2.30) is a clear result from (2.20) anyway, and the knowledge that the $u_o = 0$ trajectory is on the switching surface more than compensates for the unavailability of condition (2.35).

Once it is known that $\frac{\partial J}{\partial y}$ is continuous across a switching surface, the same result for $\frac{\partial J}{\partial x}$ follows directly from (2.20) or (2.21). Furthermore, the continuity of $\frac{\partial J}{\partial \tau}$ is then directly deducible from (2.14) or (2.15).

PROPERTY 2.-10. A NECESSARY CONDITION AT A SWITCHING SURFACE: At a switching surface separating regions where $u_o = 0$ and $u_o = u$ are the optimum controls

$$\frac{\partial J}{\partial y} = -u , \qquad u \in \{-1, 1\} . \tag{2.36}$$

There are no switches between $u_o = -1$ and $u_o = +1$ or vice versa.

Condition (2.36) follows directly from (2.30) and (2.35) for $u = -1$ and by symmetry, Property 2.1-2, for $u = +1$.

To justify the last part of the property consider the equivalent of (2.22) for a $u_o = 1$ to $u_o = -1$ switch. (This is easily derived.)

$$\frac{\partial J_{-1}}{\partial y} = \frac{(1-\beta_x y-\beta_y)}{(1-\beta_x y+\beta_y)} \frac{\partial J_1}{\partial y} + \frac{2\beta_y}{(1-\beta_x y+\beta_y)} \tag{2.37}$$

Then, the equivalent to (2.27) is

$$1 \leq \frac{(1-\beta_x y-\beta_y)}{(1-\beta_x y+\beta_y)} \frac{\partial J_1}{\partial y} + \frac{2\beta_y}{(1-\beta_x y+\beta_y)} , \tag{2.38}$$

and the equivalent to (2.29) becomes

$$1 \leq \frac{\partial J_1}{\partial y} . \tag{2.39}$$

By Property 2.1-5 this is clearly impossible. An equivalent result holds for $\frac{\partial J_{-1}}{\partial y}$.

PROPERTY 2.1-1-. ANOTHER BOUNDARY CONDITION: <u>The minimum cost at the origin of state space is zero regardless of time to go.</u>

Consider the control $u_o \equiv 0$. Then, (1-35) together with (1-38) and (1-39) imply $J = 0$ when $n(t) \equiv 0$. This is clearly minimum, since $J \geq 0$ everywhere.

2.2 Stochastic Case

This section will repeat the developments of Section 2.1 under the assumption that (2.1) does not hold. That is, the full stochastic problem will be considered. Whenever possible, results will be stated based on the development of the previous section.

2.2.1 Derivation of a Hamilton-Jacobi Type Equation

2.2.1.1 Background

There currently exists a considerable garner of methodology applicable to systems governed by stochastic differential equations. This store stems from the basic work of Einstein and Wiener on Brownian motion and more or less reaches maturity in the Chapman-Kolmogorov and Fokker-Planck relations. It is from these efforts also that the concepts and properties of white gaussian noise have derived. In this respect it should be pointed out that the disturbance process n(t) of Problems 1-3 and 1-4 is defined precisely as the formal derivative of the Wiener process.[10]

Problems 1-3 and 1-4 may be reformulated such that the performance criterion becomes that of minimizing the sum of the expected values of the state variables of an augmented system.[11] This is a common technique employed in relation to the Pontryagin Maximum Principle. Then, in the reformulated form either the Fokker-Planck or the Chapman-Kolmogorov equations for the augmented system may be used to derive a partial differential equation whose solution also yields a solution to the basic problem. This equation closely resembles the partial differential equation of the Hamilton-Jacobi type derived previously for the deterministic problems. For this reason it will be called a Hamilton-Jacobi "type" partial differential equation.

The derivation of the Hamilton-Jacobi type equations from the Chapman-Kolmogorov equation is described in a paper by .

---

10 J. L. Doob, Stochastic Processes, 96-98.

11 A good explanation of this reformulation appears in L. I. Rozonoer, "L. S. Pontryagin Maximum Principle in the Theory of Optimum Systems," Part I, Automation and Remote Control, XX, 1282-1302.

Florentin.[12] Derivation from the Fokker-Planck equation for cases where the control input, u(t), is specified is well known.[13] For cases where the cost is a functional on u(t), the derivation from the Fokker-Planck equation appears straightforward, but no derivation has been found in the literature. The derivation which follows is based on Florentin's work, leaving a derivation from the Fokker-Planck equation to some future report.

2.2.1.2 The problems reformulated

Following Rozonoér let the vector

$$z = \begin{pmatrix} x \\ y \\ v \end{pmatrix}, \tag{2.40}$$

where

$$J = \min_{u \in U} E_n[v|x_0, y_0] \tag{2.41}$$

or

$$v = \int_{t_0}^{t_f} [|u| + x^2]dt, \tag{2.42}$$

all from Problem 1-3.[14] Then the performance criterion (1-35) may be restated as

$$J = \min_{u \in U} E_n[b^t z|z_0], \tag{2.43}$$

---

12 J.J. Florentin, "Optimal Control of Continuous Time, Markov, Stochastic Systems," Journal of Electronics and Control, X, 473-488.

13 A good summary appears in T.K. Caughey, "Derivation and Application. . . ." Journal of the Acoustical Society of America, XXXV, 1683-1692.

14 This follows Rozonoér, loc. cit.

where

$$b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.44}$$

$$z_0 = \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix}, \tag{2.45}$$

and the t denotes the transpose vector. For Problem 1-4, (2.40) becomes

$$z = \begin{pmatrix} y \\ v \end{pmatrix}, \tag{2.46}$$

(2.42) becomes

$$v = \int_{t_0}^{t_f} [|u| + y^2]dt, \tag{2.47}$$

(2.44) becomes

$$b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.48}$$

and (2.45) becomes

$$z_0 = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}. \tag{2.49}$$

2.2.1.3 The Derivation

Consider the following form of the Chapman-Kolmogorov equation used by Florentin for the intermediate phase $(z_0+\Delta z_0, t_0+\Delta t_0) \triangleq (z(t_0+\Delta t_0), t_0 + \Delta t_0)$ near (z_0, t_0) :

$$p(z, t_f | z_0, t_0) = \int_{z_0+\Delta z_0} p(z, t_f | z_0+\Delta z_0, t_0+\Delta t_0) \tag{2.50}$$

$$p(z_0+\Delta z_0, t_0+\Delta t_0 | z_0, t_0) d(z_0+\Delta z_0)$$

$$= \int_{\Delta z_0} p(z, t_f | z_0+\Delta z_0, t_0+\Delta t_0) p(\Delta z_0, \Delta t_0 | z_0, t_0) d\Delta z_0 ,$$

where $t_0 < t_0 + \Delta t_0 < t_f$, $p(z, t_f | z_0, t_0)dz$ is the probability that the system state is in the neighborhood dz of state z at the t_f given that it was in state z_0 at t_0, and $p(\Delta z_0, \Delta t_0 | z_0, t_0)$ is the probability density function of changes in the system position up to and including Δz_0 when the state is z_0 at t_0.[15] From (2.43), clearly J depends on z_0 and t_0. Then, for either Problem 1-3 or 1-4,

$$J(z_0, t_0) = \min_{u \in U} \int_{-\infty}^{\infty} b' \cdot zp(z, t_f | z_0, t_0)dz. \tag{2.51}$$

Applying the Chapman-Kolmogorov equation and changing the order of integration yields

$$J(z_0, t_0) = \min_{u \in U} \int_{\Delta z_0} \int_{-\infty}^{\infty} b'zp(z, t_f) z_0 + \Delta z_0 \cdot t_0 + \Delta t_0)dz \cdot p(\Delta z_0, \Delta t_0 | z_0 t_0) d\Delta z_0 \tag{2.52}$$

Careful inspection of (2.52) reveals that the inner integral must be a minimum over $u \in U$ in order for the whole expression on the right to be minimum. Then, comparing (2.51) with the minimum of the inner integral of (2.52) leads to the relation

$$J(z_0, t_0) = \min_{u \in U} \int_{\Delta z_0} J(z_0 + \Delta z_0, t_0 + \Delta t_0) p(\Delta z_0, \Delta z_0 | z_0, t_0) d\Delta z_0. \tag{2.53}$$

---

[15] Florentin, loc. cit.

Now, because probability densities integrate to one, the following is an identity:

$$J(z_0, t_0 + \Delta t_0) = \int_{\Delta z_0} J(z_0, t_0 + \Delta t_0) p(\Delta z_0, \Delta t_0 | z_0, t_0) d\Delta z_0. \tag{2.54}$$

Further, from (2.40), (2.41), (2.42), and (2.43) it is clear that

$$\begin{aligned} J(z_0, t_0) &= \min_{u \varepsilon U} E_n \{ \int_{t_0}^{t_0 + \Delta t_0} [|u| + x^2] dt + \int_{t_0 + \Delta t_0}^{t_f} [|u| + x^2] dt \} \\ &= \min_{u \varepsilon U} \{ E_n [\int_{t_0}^{t_0 + \Delta t_0} \{|u| + x^2\} dt | z_0] + E_n [J(z_0 + \Delta z_0, t_0 + \Delta t_0) | z_0] \}. \end{aligned} \tag{2.55}$$

Then, subtracting the constant (2.54) from (2.55), considering Δt_0 small and u(t) constant over Δt_0 almost everywhere (at least in the limit as $\Delta t_0 \to 0$), and expanding the factor J in (2.54) and (2.55) appropriately in a Taylor's series leads to

$$\begin{aligned} J(z_0, t_0) - J(z_0, t_0 + \Delta t_0) = \min_{u \varepsilon U} \{ &[|u(t_0)| + x_0^2] \Delta t_0 \\ &+ (\nabla J)^t m_{\Delta z_0} + \tfrac{1}{2} ms_{\Delta z_0} \frac{\partial^2 J}{\partial z^2} (z_0, t_0) \\ &+ \Delta t_0 (\frac{\partial}{\partial t} \nabla J)^t \; m_{\Delta z_0} + o(\Delta t) \}. \end{aligned} \tag{2.56}$$ [16]

The following notation is used in this expression:

$$m_{\Delta z_0} = E_n [\Delta z_0 | z_0, t_0] \tag{2.57}$$

and

$$ms_{\Delta z_0} \frac{\partial^2 J}{\partial z^2} (z_0, t_0) = \sum_{p \varepsilon \{x, y, v\}} \sum_{q \varepsilon \{x, y, v\}} \frac{\partial^2 J}{\partial p \, \partial q} (z_0, t_0) \cdot E_n [\Delta p_0 \, \Delta q_0 | z_0, t_0]. \tag{2.58}$$

---

[16] $o(\Delta t_0)$ denotes terms with the property $\lim_{\Delta t_0 \to 0} o(\Delta t_0)/\Delta t_0 = 0$.

Let

$$(\sigma^2_{\Delta z_0} + m^2_{\Delta z_0}) = E_n[\Delta z_0 \cdot (\Delta z_0)^t \mid z_0, t_0]. \tag{2.59}$$

This is the mean square matrix of Δz_0.

It is now necessary to evaluate the various expectation terms of (2.56). For this purpose it should be noted that (2.57) is merely the vector mean of Δz_0, $m_{\Delta z_0}$. Similarly, the expectation terms in (2.58)are the components of the mean square matrix of Δz_0. It is a lengthy computation, but easily verified using the standard techniques of random noise theory that

$$m_{\Delta z_0} = \begin{bmatrix} y_0 \Delta t_0 + \frac{1}{2} \Delta^2 t_0 u(t_0) \\ \Delta t_0 u(t_0) \\ [\,|u(t_0)| + x_0^2\,] \Delta t_0 \end{bmatrix} \tag{2.60}$$

and

$$(\sigma^2_{\Delta z_0} + m^2_{\Delta z_0}) = \begin{bmatrix} o(\Delta t & o(\Delta t_0) & o(\Delta t_0) \\ o(\Delta t_0) & \dfrac{N_0 T}{2\pi M^2} \Delta t_0 + o(\Delta t) & o(\Delta t_0) \\ o(\Delta t_0) & o(\Delta t_0) & o(\Delta t_0) \end{bmatrix}^{17} \tag{2.61}$$

Now substituting (2.60) and (2.61) back into (2.56) and dividing by Δt_0 leads to

$$\frac{1}{\Delta t_0} [J(z_0, t_0) - J(z_0, t_0 + \Delta t_0)] = \min_{u \varepsilon U} \{ |u(t_0)| + x_0^2 + y_0 \frac{\partial J}{\partial x}(z_0, t_0)$$

$$+ u(t_0) \frac{\partial J}{\partial y}(z_0, t_0) + \frac{1}{2} \frac{N_0 T}{2\pi M^2} \frac{\partial^2 J}{\partial y^2}(z_0, t_0) + o(\Delta t_0) \} . \tag{2.62}$$

[17] See, e.g. W. B. Davenport, Jr. and W. L. Root, Random Signals and Noise.

In the limit as $\Delta t_0 \to 0$ after applying the definition of the partial derivative this becomes

$$-\frac{\partial J}{\partial t}(z_0, t_0) = \min_{u(t_0)\varepsilon[-1,1]} \{ y_0 \frac{\partial J}{\partial x}(z_0, t_0) + u(t_0)\frac{\partial J}{\partial y}(z_0, t_0) + d\frac{\partial^2 J}{\partial y^2}(z_0, t_0) + |u(t_0)| + x_0^2 \} , \tag{2.63}$$

where

$$d = \frac{N_0 T}{4\pi M^2} \tag{2.64}$$

as previously assumed in (1.36). On the right hand side the limit operation and minimization operation are commutative because the minimum is understood to be performed for each instant of time independently.

It is convenient to treat (2.63) in terms of the time to go rather than time from t_0. This is possible because the system is autonomous. Thus, let

$$\tau = t_f - t \tag{2.65}$$

and

$$\tau_0 = t_f - t_0. \tag{2.66}$$

Then (2.63) becomes

$$\frac{\partial J}{\partial \tau}(z_0, \tau_0) = \min_{u(t_0)\varepsilon[-1,1]} \{ y_0 \frac{\partial J}{\partial x}(z_0, \tau_0) + u(t_0)\frac{\partial J}{\partial y}(z_0, \tau_0) + d\frac{\partial^2 J}{\partial y^2}(z_0, \tau_0) + |u(t_0)| + x_0^2 \} . \tag{2.67}$$

For the one dimensional case an equation equivalent to (2.67) may be similarly derived, and the result is

$$\frac{\partial J}{\partial \tau}(z_0, \tau_0) = \min_{u(t_0)\varepsilon[-1,1]} \{u \frac{\partial J}{\partial y}(z_0, t_0) + d \frac{\partial^2 J}{\partial y^2}(z_0, t_0) + |u(t_0)| + y_0^2\}. \tag{2.68}$$

2.2.1.4 Observations

It is worthwhile to observe a few facts about (2.67) and (2.68). First, the similarity to the deterministic Hamilton-Jacobi equation should be noted. In fact, (2.67) and (2.68) formally reduce to (2.2) and (2.3) as the disturbance term, d, and in turn the noise spectral density, N_0 [See (2.64)] are reduced to zero.

It is also worthwhile to note that (2.67) and (2.68) are equations in the initial phase (state and time). Thus, the derived control, u_0 and resulting cost, $J(z_0, \tau_0)$, represent the control to be used and the cost expected to be incurred starting from x_0 and y_0 at time t_0. However, the initial time may be considered variable, and then the results derived for τ_0 may be applied for any state, $\binom{x}{y}$, and time to go, τ. For this reason the zero subscript will be dropped in the sequel.

Finally, it should be observed that (2.67) and (2.68) constitute both necessary and sufficient conditions for solution of the basic problems. Certain continuity properties and boundary conditions are applicable as discussed in the next section, but no basic assumptions are required, such as, for example, the assumption of the existence of an optimum control.

2.2.2 Properties of the Cost Surfaces and Switching Curves

Properties similar to those of Section 2.1.2 may be stated for the stochastic attitude control and antenna steering problems. As before, the properties will be stated for Problem 1-3, and the corresponding properties for Problem 1-4 may be deduced.

The justification for most of the properties in the stochastic case follows that for the deterministic case almost directly once one basic fact is understood. This fact is that almost all sample functions (trajectories, loosely speaking) of the disturbed system are continuous.[18] Thus, the stochastic system has essentially the same properties almost surely as the deterministic system. The cost equation, (2.51), in the stochastic case involves a double integral, of course, but the expectation operator is always independent of the variables of differentiation that are of interest. The properties stated below, then, will be justified only where some additional essential factor need be noted.

Property 2.2-1. Initial Condition: For all values of the state variables, the initial cost is zero.

---

[18] J. L. Doob, *op. cit.*, chap. VI, sec. 3.

Property 2.2-2. Symmetry: The symmetry property of Property 2.1-2 is applicable in the stochastic as well as the deterministic case.

Property 2.2-3. Existence of Characteristic Curves and Trajectories: There is a family of curves in phase space intrinsically defined by the Hamilton-Jacobi type equation. These curves are the trajectories of the system in the limit as the random disturbance is reduced to zero. These curves will be called congeneric trajectories. The projections of the congeneric trajectories onto the expected cost surface will be called congeneric characteristic curves.

The congeneric trajectories coincide with the trajectories of the corresponding deterministic problem. See Property 2.1-3. These curves are not related to the so called characteristic curves of the theory of partial differential equations.[19]

Property 2.2-4. Continuity of the Cost Surface: Over the entire phase space the cost surface is continuous.

Property 2.2-5. Optimality of On-off-on Controls: The optimul control is of the on-off-on type, and (2.13) holds.

Note here that the presence of the second derivative term in (2.67) does not affect the result in making the right hand side a minimum.

Property 2.2-6. Continuity of Derivatives: In every region of phase space for which the optimum control, $u_o(x, y, \tau)$, is continuous, all of the phase

19 Sneddon, op. cit., chap. 3, sec. 6.

derivatives of the optimum cost surface are continuous.

Note: Analogs to Properties 2.2-7 through 2.2-9 will be stated at this point. These properties are proposed on the basis that one expects the stochastic optimum cost surface to be smoother than the corresponding deterministic surface, because, intuitively speaking, any sharp bends at switching points are rounded off by the uncertainty of the disturbed trajectories. Mathematical techniques for treating the stochastic case are not as simple as those for deterministic problems, if they are even available, and the arguments used before have little advantage. Consequently, Properties 2.2-7 through 2.2-9 will be assumed in the sequel without further justification.

Property 2.2-7. **Existence** and Smoothness of Switching Surfaces: The loci of switching points are assumed to form continuous surfaces. These surfaces are called the switching surfaces, and they are assumed smooth in the phase variables.

Property 2.2-8. Continuity of Derivatives of J: Over the entire phase space all of the first partial derivatives of the optimum cost surface with respect to the phase variables are assumed continuous.

Property 2.2-9. A necessary condition at a switching surface: At a switching surface between regions where $u_o = 0$ and $u_o = u$ are the optimum controls

$$\frac{\partial J}{\partial y} = -u, \qquad u \in \{-1, 1\}. \tag{2.69}$$

There are no switches between $u_o = -1$ and $u_o = +1$ or vice versa.

Note: It is clear that Property 2.1-10 is not valid in the stochastic case. The closest available condition is one resulting from symmetry: namely, that $\frac{\partial J}{\partial y}(0, 0, \tau) = \frac{\partial J}{\partial x}(0, 0, \tau) = 0$, or in one dimension $\frac{\partial J}{\partial y}(0, \tau) = 0$. This supplementary condition is clearly also true in the deterministic cases.

Note: Properties 2.2-6 and 2.2-8 together imply the continuity of the term $\frac{\partial^2 J}{\partial y^2}$ in the Hamilton-Jacobi type equation.

Chapter 3

SOLUTIONS TO THE DETERMINISTIC HAMILTON-JACOBI EQUATIONS

This chapter presents solutions to the deterministic Hamilton-Jacobi problems as reformulated in Section 2.1 from the basic problems of Chapter 1. The results here are apparently the first to be derived directly from the Hamilton-Jacobi formulation and the first to describe the cost function over the entire phase space. Problem 1-4 will be treated first due to its simplicity.

3.1. Previously known results

Analytical methods for solving various facets of the deterministic problems have received considerable attention in the International Journal of Control (and its predecessor the Journal of Electronics and Control) for the past several years. This attention stems from the early work of Fuller on relay control of single and double integrator systems.[1] This work treats performance criteria of the form

$$J_n = \min_{u \in U} \int_0^\infty |x|^n dt \qquad n \geq 0 \tag{3.1}$$

subject to a saturation constraint,

$$|u(t)| \leq 1. \tag{3.2}$$

[1] A. T. Fuller, "Relay Control Systems Optimized for Various Performance Criteria," Proceedings First International Congress IFAC, 1960, ol. I, 510-519.

The case $n = 2$ is of particular interest here because it corresponds to a limiting case of (1.6): the case when $\lambda \to \infty$. For $n = 2$ Fuller showed that the optimum control is bang-bang; that the optimum cost for initial states with the property $y_0 = 0$ is given by

$$J_2 = \frac{23}{30} a|x_0|^{5/2}, \tag{3.3}$$

where $a \cong .9965$; and that the switching curve is given by

$$x + by|y| = 0, \tag{3.4}$$

where $b \cong 0.4446$. These results hold as long as

$$t_f > \frac{2\{|x_0|(1+k)\}^{\frac{1}{2}}}{1 - k^{\frac{1}{2}}} \cong 1.36\, t_f^*, \tag{3.5}$$

where $k \cong .05862$ and t_f^* is the well known minimum settling time.[2]

By 1963 this same limiting case had been approached in two distinct ways using the Bellman-Hamilton-Jacobi equation. Fuller showed that the results previously derived satisfied this equation.[3] For initial states not on the x axis the minimum cost surface is shown to be

---

[2] For a minimum settling time discussion see M. Athans and P. L. Falb, Optimal Control.

[3] A. T. Fuller, "Further Study of an Optimum Non-linear Control System," Journal of Electronics and Control, XVII, 283-300.

$$J = \begin{cases} x^2 y + \frac{2}{3} xy^3 + \frac{2}{15} y^5 + a(x + \frac{1}{2} y^2)^{5/2} & x \geq by|y| \\ -x^2 y + \frac{2}{3} xy^3 - \frac{2}{15} y^5 + a(-x + \frac{1}{2} y^2)^{5/2} & x \leq by|y|, \end{cases} \tag{3.6}$$

where a and b are as before and a is known more precisely to be

$$a = \frac{1}{20} (222 + 2\sqrt{33})^{\frac{1}{2}}. \tag{3.7}$$

About the same time Wonham derived the same results directly from the partial differential equation using invariant scaling methods.[4] He thus, in effect, reduced the number of independent variables of the problem. Fuller subsequently rederived the results directly from the Hamilton-Jacobi equation using the pi-theorem of the theory of dimensions, and showed that Wonham's method was essentially the same.[5] It should be noted that all of these later results are constrained by (3.5). The optimum solution is not presented for phases in time constrained situations where the origin of state space is reached either in the interval $1.36 t_f^* \geq \tilde{t}_f \geq t_f^*$ or is not reached at all.

---

[4] W. M. Wonham, "Note on a Problem in Optimal Non-linear Control," Journal of Electronics and Control, XV, 59-62.

[5] A. T. Fuller, "Optimization of some Non-Linear Control Systems by means of Bellman's Equation and Dimensional Analysis," International Journal of Control, III, 359-394.

There appear to be no published results of similar treatments of minimum fuel to target set problems. These problems constitute approximately the alternative limiting case of (1-6): those for which $\lambda \to 0$. The true limiting case, of course, is trivial, for without a target set constraint no fuel need be expended.

The above discussion relates to the double integral plant. The single integral plant is of little interest in the steady state case (t_f large enough), because, as Fuller shows, the optimum feedback control merely assumes its maximum magnitude and a sign opposite to that of the position error.[6] Again, the time constrained regions of phase space are not considered.

3.2. The one-dimensional deterministic equation

The Hamilton-Jacobi type partial differential equation for the antenna steering problem, (2-7), is a linear, first order partial differential equation.[7] Such equations are rather easily solved by the method of characteristic (integral) curves.[7] It is only the presence of the necessary condition, Property 2.1-9, that causes the solution in this case to be non-trivial.

3.2.1 The Solution

Comparing the initial condition (Property 2.1-1) with the control

[6] Fuller, "Relay Control Systems...," loc. cit.

[7] See, e.g., I. N. Sneddon, Elements of Partial Differential Equations, 44-85

property (Property 2.1-5) shows that the initial control ($\tau = 0$) is always zero. This is because $J(x, y, 0) \equiv 0$ implies $\frac{\partial J}{\partial y}(x, y, 0) = 0$. Furthermore, from Property 2.1-8 it is clear that $\frac{\partial J}{\partial \tau}$ is finite near $\tau = 0$ for finite y. Thus $\frac{\partial J}{\partial y}$ near $\tau = 0$ cannot differ much from its value at $\tau = 0$, and the control $u_o = 0$ must be optimum in some neighborhood of $\tau = 0$.

For the region where $u_o = 0$ is optimum (hereinafter called R_0), (2.7) becomes

$$\frac{\partial J}{\partial \tau} = y^2, \tag{3.8}$$

so,

$$J = y^2 \tau + c. \tag{3.9}$$

Since $J = 0$ when $\tau = 0$, $c = 0$. Now,

$$\frac{\partial J}{\partial y} = 2y\,\tau\,. \tag{3.10}$$

Then

$$|y\tau| = \tfrac{1}{2} \tag{3.11}$$

when Property 2.1-9 is satisfied.

Because of symmetry, it is only necessary to consider the region $y \geq 0$ in the sequel. For $y \geq 0$ (3.11) becomes

$$y\,\tau = \tfrac{1}{2}\,, \tag{3.12}$$

and this is necessarily a switching curve. R_0 is bounded by this curve.

It is of interest to note that along the curve $y = 0$ (3.9) gives $J = 0$. This is seen to agree with Property 2.1-9. Along $y = 0$, $\frac{\partial J}{\partial y}$ is also zero.

This will be of interest in the stochastic cases.

Along (3.12), (3.10) becomes

$$\frac{\partial J}{\partial y} = 1 . \tag{3.13}$$

The switch at this curve must be to a region where $u_o = -1$. For such a region (2-7) is

$$\frac{\partial J}{\partial \tau} + \frac{\partial J}{\partial y} = y^2 + 1. \tag{3.14}$$

The characteristic curves for (3.14) are described by

$$\tau = y + c_1 \tag{3.15}$$

and

$$J = y + \frac{1}{3} y^3 + c_2 . \tag{3.16}$$

Now, along $y\tau = \frac{1}{2}$,

$$J = y^2 \tau$$

$$= \frac{1}{2} y ; \tag{3.17}$$

this is the boundary condition for R_{-1} (the region where $u_o = -1$ is the optimum control). Solving (3.12) and (3.15) simultaneously by eliminating τ yields

$$y = \frac{1}{2} (-c_1 \pm \sqrt{c_1^2 + 2}). \tag{3.18}$$

To pick the proper sign in (3.18), note that the line $y = \tau$ intersects (3.12) in such a way that $c_1 = 0$. Then, since in R_{-1} $y \geq 0$, the plus sign must be used. Substituting this result simultaneously into (3.16) and

(3.17) leads to

$$c_2 = \frac{1}{6}(c_1^3 + 3c_1) - \frac{1}{6}\left(\sqrt{c_1^2 + 2}\right)^3. \tag{3.19}$$

By resubstituting (3.15) and (3.16) back this means that in R_{-1}

$$J = \frac{1}{6}(\tau + y)^3 + \tfrac{1}{2}(\tau + y) - y\tau^2 - \frac{1}{6}\left[(\tau - y)^2 + 2\right]^{3/2}. \tag{3.20}$$

It is easy to verify that

$$\frac{\partial J}{\partial y} = \tfrac{1}{2}\left[(\tau + y)^2 + 1\right] - \tau^2 + \tfrac{1}{2}\left[(t-y)^2 + 2\right]^{\frac{1}{2}}(\tau - y), \tag{3.21}$$

$$\frac{\partial J}{\partial y}\left(y, \frac{1}{2y}\right) = 1. \tag{3.22}$$

and

$$\frac{\partial J}{\partial \tau}\left(y, \frac{1}{2y}\right) = y^2. \tag{3.23}$$

This verifies the continuity of the derivatives across the switching curve. Conversely, it can be verified that when $\frac{\partial J}{\partial y} = 1$, $\tau = \frac{1}{2y}$ is the only solution to (3.21). Continuity of the cost surface at $y\tau = \frac{1}{2}$ is also easily verified.

Although it may be mathematically verified that there exist no boundaries for a $u_o = +1$ region for $y \geq 0$, this same result should be intuitively evident from the physical problem. This fact will be assumed with no further justification.

The solution for the entire phase space is now available. Recalling the symmetry condition (Property 2.1-2), the feedback control satisfies

$$
u_o = \begin{cases} +1 \text{ for } y < -\frac{1}{2\tau} \\ 0 \text{ for } -\frac{2}{2\tau} \leq y \leq \frac{1}{2\tau} \\ -1 \text{ for } y > \frac{1}{2\tau} \end{cases}
$$

$$
= \text{dez } 2y\tau \ . \tag{3.24}
$$

The minimum cost surface (as a function of initial condition) is

$$
J = \begin{cases} \frac{1}{6}(\tau - y^3) + \frac{1}{2}(\tau - y) + \tau^2 y - \frac{1}{6}[(\tau+y)^2 + 2]^{3/2} \\ \qquad\qquad\qquad\qquad\qquad \text{for } y < -\frac{1}{2\tau} \\ y^2\tau \quad \text{for} \quad -\frac{1}{2\tau} \leq y \leq \frac{1}{2\tau} \\ \frac{1}{6}(\tau+y)^3 + \frac{1}{2}(\tau+y) - \tau^2 y - \frac{1}{6}[(\tau-y)^2 + 2]^{3/2} \quad \text{for } y > \frac{1}{2}\tau \, . \end{cases} \tag{3.25}
$$

The switching surfaces (here curves) are plotted in Figure 3-1. The cost surface as seen from behind the $\tau = 0$ plane is sketched in Figure 3-2.

3.2.2 Remarks

The following remarks are worthy of note:

a. The boundary condition, Property 2.1-10, was not used to solve the problem; it was verified as an additional property of the solution satisfying the initial condition, Property 2.1-1. This is fortuitous, because it is evident that no simple corresponding boundary condition is available for the stochastic case (see note following Property 2.2-7).

b. The solution of this section agrees with the known solution

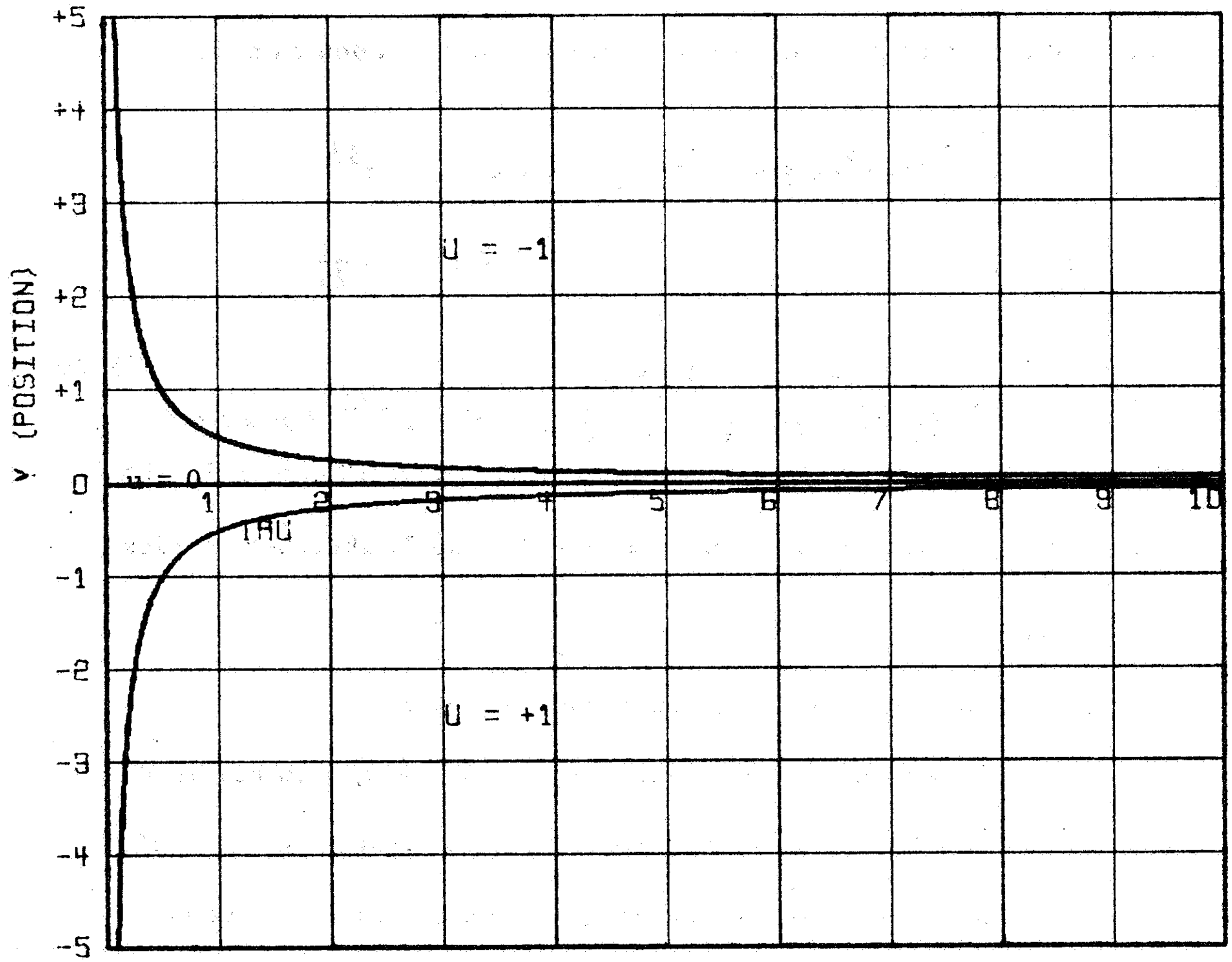


Figure 3-1.

Switching Curves in Phase Space for Problem 1-4 (deterministic case).

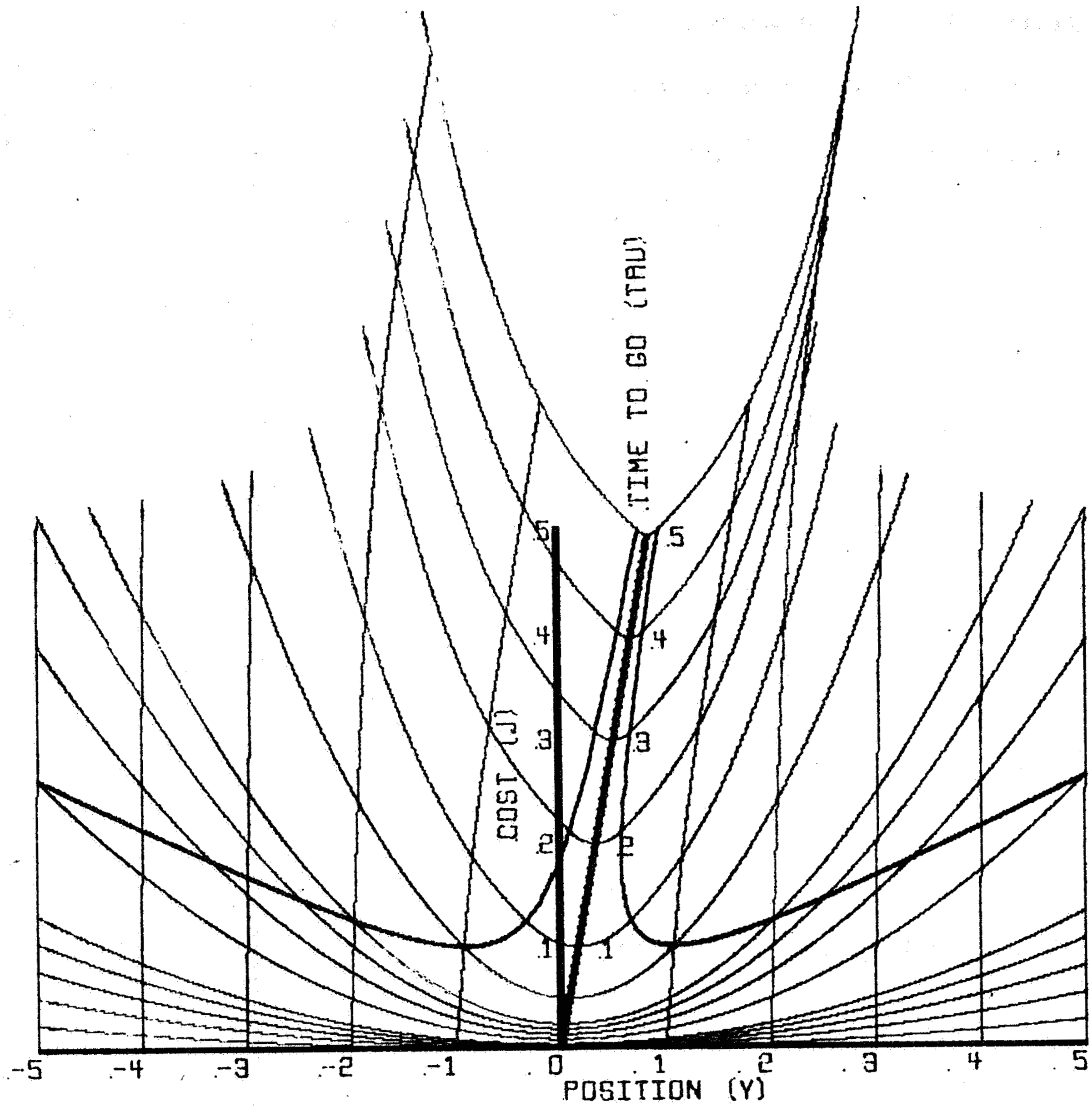


Figure 3-2.

Cost Surface for Problem 1-4 (deterministic case).

to the limiting case of Problem 1-4 for which no cost emphasis is placed on fuel. This limiting case corresponds to letting λ grow indefinitely. The corresponding cost must, of course, be divided by λ in order to compensate for the artifice of an infinite weight on integral error squared.

Consider the switching curve, (3.12), first. The results of Section 3.1 correspond to Problem 1-4 when in the latter $n(t) \equiv 0$, $\lambda \to \infty$, and $t \to \infty$. To denormalize the results of the previous section, apply the one dimensional equivalents of (1-20), (1-31), (1-32), and (1-33) to (3.12) with $M = I = 1$. These equivalents are, respectively,

$$y = \Theta\,\theta \tag{3.26}$$

$$\Theta = \left(\frac{\lambda}{M}\right)^{\frac{1}{2}} = \lambda^{\frac{1}{2}}, \tag{3.27}$$

$$T = \left(\frac{\lambda M}{I^2}\right)^{\frac{1}{2}} = \lambda^{\frac{1}{2}}, \tag{3.28}$$

and

$$J = \min_{u \in U} P\left(\frac{T}{M}\right) = \min_{u \in U} P\left(\frac{\lambda}{I^2 M}\right)^{\frac{1}{2}} = \min_{u \in U} P\lambda^{\frac{1}{2}}. \tag{3.29}$$

This gives as the denormalized switching curve

$$\theta t = \frac{1}{2\lambda} \tag{3.30}$$

Taking the limit as $\lambda \to \infty$ yields

$$\theta t = 0\,. \tag{3.31}$$

The limit $t \to \infty$ then leads to a switching curve described by

$$\theta = 0 \; . \tag{3.32}$$

This is the obvious result referred to in Section 3.1.

The cost surface from Section 3.1 may be easily calculated for t_f large enough:

$$\begin{aligned}
\text{cost} &= \min_{u \in U} \int_0^{t_f} \chi_0(\theta)\theta^2(t)dt \\
&= \min_{u \in U} \int_{\theta_0}^{0} \theta^2(t) \frac{d\theta}{\dot{\theta}} \\
&= \int_{\theta_0}^{0} \theta^2 \frac{d\theta}{-1} \\
&= \frac{1}{3}\theta_0^3 \; .^{8}
\end{aligned} \tag{3.33}$$

The corresponding denormalization of (3.25) to check against (3.33) follows the denormalization process used for (3.32), viz.:

$$J = \frac{1}{6}(\tau+y)^3 + \tfrac{1}{2}(\tau+y) - y\tau^2 - \frac{1}{6}[(\tau-y)^2 + 2]^{3/2}, \tag{3.34}$$

so that

---

[8] The characteristic function, $\chi_0(y)$, is defined as

$$\chi_0(y) = \begin{cases} 0 & y = 0 \\ 1 & y \neq 0. \end{cases} \tag{29}$$

$$\lim_{\lambda \to \infty} \frac{P_{opt.}}{\lambda} = \lim_{\lambda \to \infty} = \frac{J\lambda^{-\frac{1}{2}}}{\lambda}$$

$$= \lim_{\lambda \to \infty} \lambda^{-3/2} \Big\{ \frac{1}{6} (\lambda^{\frac{1}{2}} 6 + \lambda^{\frac{1}{2}} \theta)^3 + \tfrac{1}{2} (\lambda^{\frac{1}{2}} t + \lambda^{\frac{1}{2}} \theta) - \lambda^{3/2} \theta t^2 - \frac{1}{6}$$

$$[(\lambda^{\frac{1}{2}} t - \lambda^{\frac{1}{2}} \theta)^2 + 2]^{3/2} \}$$

$$= \lim_{\lambda \to \infty} \Big\{ \frac{1}{6} (t+\theta)^3 + \frac{1}{2\lambda} (t+\theta) - \theta t^2 - \frac{1}{6} [(t-\theta)^2 + \frac{2}{\lambda}]^{3/2} \Big\}$$

$$= \frac{1}{6} (t+\theta)^3 - \theta t^2 - \frac{1}{6} [(t-\theta)^2]^{3/2}$$

$$= \begin{cases} 1/3\ \theta^3 & \text{for } t - \theta \geq 0 \\ 1/3\ t^3 - t^2\theta + t\theta^2 & \text{for } t - \theta \leq 0. \end{cases} \qquad (3.35)$$

c. The case where fuel receives the entire emphasis may also be verified. This is the case where $\lambda \to 0$. Although it may be mathematically confirmed, the following physical argument leads to the optimum solution: if no emphasis is placed on position, the best use of fuel is to do nothing at all (switching curve at infinity), and the optimum cost for no expended fuel is zero.

Now, consider the solution of the previous section. The switching curve may, again, be determined by denormalizing and taking $I = M = 1$, $\lim_{\lambda \to 0}$, and $\lim_{t \to \infty}$. This leads first to (3.26). Then,

$$\lim_{\lambda \to 0} \theta t = \infty \qquad (3.36)$$

for all t, but if t is finite then θ must be infinite. Thus, regardless of t the switching curve is

$$\theta = \infty . \tag{3.37}$$

By similar substitutions

$$\begin{aligned} J &= y^2 \tau \\ &= \lambda^{3/2} \theta^2 t \end{aligned} \tag{3.38}$$

and

$$\lim_{\lambda \to 0} J = 0, \tag{3.39}$$

the expected result.

3.3 The two dimensional deterministic equation

The Hamilton-Jacobi type equation for the attitude control problem, (2.2), has properties very much like the one dimensional equation. The method of characteristic curves is again applicable. Strong parallels will be drawn to the one dimensional case.

3.3.1 The Solution

Properties 2.1-1 and 2.1-5 [equation (2.13)] indicate that the initial control is always zero as before. The derivative $\frac{\partial J}{\partial y}$ near $\tau = 0$ cannot differ much from zero or else Property 2.1-10 would be violated. Therefore, $u_o = 0$ is the optimum control for some neighborhood of the plane $\tau = 0$.

For the u_o region (2.2) becomes

$$\frac{\partial J}{\partial \tau} - \frac{\partial J}{\partial x} y = x^2, \tag{3.40}$$

and the characteristic curves for this equation are

$$y = c_1, \tag{3.41}$$

$$\tau = -\frac{x}{c_1} + c_2 = -\frac{x}{y} + c_2, \tag{3.42}$$

and

$$J = -\frac{1}{c_1}\,\frac{1}{3}x^3 + c_3 = -\frac{1}{3}\,\frac{x^3}{y} + c_3. \tag{3.43}$$

Since $J = 0$ when $\tau = 0$, (3.42) and (3.43) become

$$0 = \frac{x}{c_1} + c_2 \tag{3.44}$$

and

$$0 = -\frac{1}{c_1}\,\frac{1}{3}x^3 + c_3. \tag{3.45}$$

These imply

$$c_3 = \frac{1}{3}c_1^2 c_2^3 . \tag{3.46}$$

Resubstituting (3.41), (3.42), and (3.43) yields

$$J = \frac{1}{3}\tau^3 y^2 + \tau^2 xy + \tau x^2. \tag{3.47}$$

This is the cost in the $u_o = 0$ region.

Now

$$\frac{\partial J}{\partial y} = \frac{2}{3}\tau^3 y + \tau^2 x, \tag{3.48}$$

and $\left|\frac{\partial J}{\partial y}\right| = 1$ when

$$x = -\frac{2}{3} y\tau \pm \frac{1}{\tau^2}. \tag{3.49}$$

This is necessarily a switching surface.

The plus (minus) sign corresponds to a switch to a $u_o = 1$ (+1) control. Symmetry, Property 2.1-7, again allows consideration only of a switch to $u_o = -1$; the switch to $u_o = +1$ follows by symmetry.

It is interesting to note that the curve $x = y = 0$ is within the $u_o = 0$ region, and that (3.47) gives the value $J = 0$. This is seen to agree with Property 2.1-10.

For a region of $u_o = -1$ control, (2.2) is

$$\frac{\partial J}{\partial \tau} + \frac{\partial J}{\partial y} - \frac{\partial J}{\partial x} y = 1 + x^2. \tag{3.50}$$

The corresponding characteristic curves are described by

$$\tau = y + c_1, \tag{3.51}$$

$$x = -\tfrac{1}{2} y^2 + c_2, \tag{3.52}$$

and

$$\begin{aligned} J &= \frac{1}{20} y^5 - \frac{1}{3} c_2 y^3 + (c_2^2 + 1)y + c_3 \\ &= \frac{2}{15} y^5 + \frac{2}{3} xy^3 + (x^2+1)y + c_3. \end{aligned} \tag{3.53}$$

The boundary condition for these curves is the value of cost on the switching surface. Equating (3.52) and (3.49), and (3.53) and (3.47), and substituting(3.51) leads to

$$-\frac{1}{2}y^2 + c_2 = -\frac{2}{3}y(y+c_1) + \frac{1}{(y+c_1{}^2)} \tag{3.54}$$

and

$$\frac{1}{20}y^5 - \frac{1}{3}c_2y^3 + (c_2^2+1)y + c_3 = \frac{1}{3}(y+c_1)^3y^2 + (y+c_1)^2xy + (y+c_1)x^2. \tag{3.55}$$

Equation (3.54) may be rearranged to give

$$y^4 + Ay^3 + By^2 + Cy + D = 0, \tag{3.56}$$

where

$$A = 6c_1, \tag{3.57}$$

$$B = 9c_1^2 + 6c_2, \tag{3.58}$$

$$C = 4c_1^3 + 12c_1c_2, \tag{3.59}$$

and

$$D = 6(c_1^2c_2-1). \tag{3.60}$$

To solve (3.49) for y, first apply Ferrari's method, giving the following resolvent cubic equation:

$$z^3 + Ez^2 + Fz + G = 0, \tag{3.61}$$

where

$$E = -B = -(9c_1^2 + 6c_2), \tag{3.62}$$

$$F = AC - 4D = 24c_1^4 + 48c_1^2c_2 + 24, \tag{3.63}$$

and

$$G = 4BD - A^2D - C^2 \tag{3.64}$$

$$= -16c_1^6 - 96c_1^4c_2 - 144c_2,$$

and where z is a number to be specified later.[9]

Now, following Tartaglia's method for (3.61), let

$$p = F - \frac{1}{3}E^2 \tag{3.65}$$

$$= -3c_1^4 + 12\, c_1^2 c_2 - 12c_2^2 + 24,$$

$$q = G - \frac{1}{3}\,EF + \frac{2}{27}\,E^3 \tag{3.66}$$

$$= 2c_1^6 - 12c_1^4\, c_2 + 24c_1^2 c_2^2 - 16c_2^3 + 72c_1^2 - 96c_2,$$

and

$$R = \frac{1}{27}\,p^3 + \frac{1}{4}\,q^2$$

$$= 96c_1^8 - 720c_1^6 c_2 + 2016\, c_1^4 c_2^2 - 2496c_1^2 c_2^3 + 1152\, c_2^4 \tag{3.67}$$

$$+ 1104\, c_1^4 - 2688\, c_1^2\, c_2 + 1536\, c_2^2 + 512.$$ [10]

Then, the roots of (3.61) are

$$z_i = w_i - \frac{p}{3w_i} - \frac{E}{3}, \qquad i = 1, 2, 3, \tag{3.68}$$

where the w_i are the three roots of

$$w^3 = -\tfrac{1}{2}\,q + \sqrt{R}\,, \tag{3.69}$$

or of

$$w^3 = -\tfrac{1}{2}\,q - \sqrt{R}\,. \tag{3.70}$$

---

9. See, e.g., W. L. Hart, Brief College Algebra, 174-175

10. See, e.g., Ibid, 173-174.

It should be clear that (3.69) and (3.70) have at most one real root. Call the root for (3.69) w and that for (3.70) w^*. Then, by substitution of the indicated quantities back into (3.68) the following relationship can be derived

$$z = w + w^* - 3c_1^2 - 2c_2. \tag{3.71}$$

Solution of (3.69) for w in terms of c_1 and c_2 involves a lengthy algebraic excercize. It also requires careful analysis like than done for (3.18) in order to select algebraic signs at certain points in the reduction. These manipulations have not been completed, and will have to be left for a future report.

3.3.2 Remarks

The following points may be noted in the results of the previous section.

a. As for the one dimensional case, Property 2.1-10 is not required to solve the Hamilton-Jacobi equation. It is satisfied by the solution that satisfies the initial condition.

b. The solution of this section entails solving a quartic equation. If an equivalent three dimensional problem (i.e., a three integrator system with an integral error squared cost term) were to be considered it would lead to a sixth order equation. The well known theorem of Abel indicates that there is no hope of achieving an analytical solution to the Hamilton-Jacobi equation using the method of this chapter.[11]

---

[11] See, e.g., G. Birkhoff and S. MacLane, A Survey of Modern Algebra, Revised edition (1953), 452-455.

Furthermore, increasing the order of the position error term in the cost expression also increases the order of the algebraic equation that must be solved. For example, substitution of x^4 for x^2 in (1.35) leads to a problem of solving an eigth order equation. This is clearly intractable.

Thus, the methods of this chapter are applicable to only a small class of problems, and even for these problems the algebra involved in solution tends to become very complex.

c. The solution of this section should agree with the known solution to the limiting case of Problem 1-3 for which no cost emphasis is placed on fuel ($\lambda \to \infty$). This cannot be displayed until the algebra associated with equations (3.69) and (3.71) is completed.

d. The solution of this section also should agree with the fuel only ($\lambda \to 0$) case of Problem 1-3. Verification here also depends on the uncompleted algebra.

Chapter 4

ANALYTICAL SOLUTIONS OF THE STOCHASTIC HAMILTON-JACOBI TYPE PROBLEMS

4.1 One-Dimensional Problem

In Chapter 2, the problem of the optimal control of a disturbed integrator was reduced to the study of equation (2.68):

$$\frac{\partial J}{\partial \tau} = \min_{u \in U} \left[d \frac{\partial^2 J}{\partial x^2} + u \frac{\partial J}{\partial x} + |u| + x^2 \right], \tag{4.1}$$

or, after applying Property 2.2-8,of

$$\begin{cases} \dfrac{\partial J}{\partial \tau} = d \dfrac{\partial^2 J}{\partial x^2} + x^2 , \text{ when } \left|\dfrac{\partial J}{\partial x}\right| \leq 1 \quad (u = 0) & (4.2) \\[2ex] \dfrac{\partial J}{\partial \tau} = d \dfrac{\partial^2 J}{\partial x^2} - \dfrac{\partial J}{\partial x} + 1 + x^2, \text{ when } \dfrac{\partial J}{\partial x} > 1 \quad (u = -1) & (4.3) \\[2ex] \dfrac{\partial J}{\partial \tau} = d \dfrac{\partial^2 J}{\partial x^2} + \dfrac{\partial J}{\partial x} + 1 + x^2, \text{ when } \dfrac{\partial J}{\partial x} < 1 \quad (u = +1) & (4.4) \end{cases}$$

with the initial condition, $J(x, 0) = 0$. (4.5)

The investigation of the properties of the switching curves revealed that on the switching curve separating regions of $u = 0$ and $u = +1$, $(u = -1)$, $\frac{\partial J}{\partial x} = -1$, $(\frac{\partial J}{\partial x} = +1)$ and $\frac{\partial^2 J}{\partial x^2}$ is continuous everywhere.

The study of this equation can be simplified if certain additional assumptions are made. First, it is clear that if $J_1(x, \tau)$ is a solution of the problem as stated, i.e., satisfies (4.1) and the initial condition, (4.5) etc, then $J_2(x, \tau) \equiv J_1(-x, \tau)$ does also. It follows that if a solution exists, then a symmetric solution also exists. On an intuitive basis, it is clear that the optimal cost is symmetric, so it will in the future be restricted to such a form. Specifically, it will be assumed that $J(x, \tau)$ satisfies the symmetry condition

$$J(x, \tau) \equiv J(-x, \tau) \tag{4.6}$$

Note that if it could be established that the solution is unique, it would follow that the solution is symmetric in x.

By examining the equation for the cost in the form

$$J(x, \tau) = E\left[\int_0^\tau [|u| + x^2(s)]ds\right],$$

it would appear as though J would increase in magnitude whenever x did, that is

$$\operatorname{sgn}(x) \frac{\partial J}{\partial x} \geq 0. \tag{4.7}$$

This will be a further requirement on the solution. (This may be a redundant requirement).

$\frac{\partial J}{\partial x}$ is everywhere continuous. Since the solutions under consideration are symmetric,

$$\frac{\partial J}{\partial x}(0, \tau) = - \frac{\partial J}{\partial x}(0, \tau)$$

so that

$$\frac{\partial J}{\partial x}(0, \tau) \equiv 0 \tag{4.8}$$

These assumptions lead to a study of the restricted problem

$$\frac{\partial J}{\partial \tau} = d \frac{\partial^2 J}{\partial x^2} - \frac{\partial J}{\partial x} + x^2 + \min\left(1, \frac{\partial J}{\partial x}\right), \quad x > 0, \quad \tau > 0, \tag{4.9}$$

with initial condition

$$J(x, 0) = 0, \tag{4.10}$$

and boundary condition

$$\lim_{x \to 0^+} \frac{\partial J}{\partial x}(x, \tau) = 0. \tag{4.11}$$

4.1.1 Steady-State Solutions

A simple solution based on an assumed control law. If the system is to operate for a long time (τ, time to go, is large), a reasonable control law should not depend upon the time to go. This idea can be used to obtain a steady-state control law quite easily.

It is apparent that in this case the switching curve is in fact just a pair of symmetric points, i.e., the optimal control law will be of the form

$$\begin{aligned} u &= +1 \qquad \text{if } x < -a \\ u &= 0 \qquad \text{if } -a < x < a \\ u &= -1 \qquad \text{if } x > a \end{aligned} \tag{4.12}$$

with a a positive constant. The problem of optimal control in the steady-state is then simply reduced to the determination of a single parameter, a.

To solve the problem in this form, reconsider the basic formulation of the problem. The system is described by the stochastic differential equation

$$\dot{x} = u + \dot{w}(t) \tag{4.13}$$

where w(t) is a Wiener process with $E[w(t)w(s)] = d|t - s|$. The steady-state probability density then satisfies the steady-state Fokker-Planck equation

$$d \frac{d^2 p_s}{dx^2} - u \frac{d(up_s)}{dx} = 0.$$ [1] (4.14)

Noting the form for u(and its symmetry) leads to

$$d \frac{d^2 p_s}{dx^2} = 0, \quad 0 < x < a, \tag{4.15}$$

[1] H. J. Payne, The Response of Nonlinear Systems to Stochastic Excitation, (unpublished Ph. D. Dissertation, California Institute of Technology, 1967).

$$d \frac{d^2 p_s}{dx^2} - \frac{dp_s}{dx} = 0, \quad a < x, \tag{4.16}$$

and

$$p_s(x) = p_{x_s}(-x) \tag{4.17}$$

These equation are to be solved under the conditions

$$p_s \text{ continuous}, \tag{4.18}$$

$$d\frac{dp_s}{dx} - up_s \text{ continuous (continuity of probability flux)}, \tag{4.19}$$

and

$$\int_{-\infty}^{\infty} p_s(x)dx = 1.^2 \tag{4.20}$$

Equations (4.15) and (4.16) are easily solved, and, with the application of the conditions (4.18) through (4.20) one finds

$$p_s(x) = \frac{1}{2d} \frac{1}{1 + \frac{a}{d}}, \quad 0 < x < a, \tag{4.21}$$

and

$$p_s(x) = \frac{1}{2d} \frac{\exp[(a-x)/d]}{1 + \frac{a}{d}}, \quad a < x. \tag{4.22}$$

---

[2] J. D. Atkinson, Spectral Density of First Order Piecewise Linear Systems Excited by White Noise, (unpublished Ph. D. Dissertation, California Institute of Technology, 1967).

The performance index,

$$J = E\left[\int_0^T [|u| + x^2]\, ds\right], \tag{4.23}$$

needs some modification for the time independent situation. In the steady-state, the integrand,

$$|u| + x^2, \tag{4.24}$$

will have an expected value which is some constant. It is clear that this expected value should be made a minimum. Having $p_s(x)$, it is easy to determine

$$K \triangleq E\,[\,|u| + x^2] \tag{4.25}$$

as a function of the parameter a and then to minimize. The details are as follows:

$$\begin{aligned} K &= 2\int_0^\infty [|u| + x^2]\, p_s(x)dx \\ &= 2\int_a^\infty p_s(x)dx + 2\int_0^a x^2 p_s(x)dx + 2\int_a^\infty x^2 p_s(x)dx \\ &= 2\left[\frac{d}{2d} \cdot \frac{1}{1+\frac{a}{d}}\right] + 2\cdot \frac{a^3}{3} \cdot \frac{1}{2d} \cdot \frac{1}{1+\frac{a}{d}} + 2 \cdot \frac{d^3}{2d} \frac{[1+(1+a/d)^2]}{1+\frac{a}{d}}. \end{aligned} \tag{4.26}$$

Introducing the parameter $\mu = 1 + a/d$,

$$K(\mu) = \frac{1}{\mu} + d^2\left[\frac{1}{3}\, \frac{(\mu-1)^3}{\mu} + \frac{(1+\mu^2)}{\mu}\right], \tag{4.27}$$

or

$$\mu K(\mu) = 1 + \frac{d^2}{3}\,[\mu^3 + 3\mu + 2]. \tag{4.28}$$

Now, $\frac{d}{d\mu}[\mu K(\mu)] = \frac{d^2}{3}[3\mu^2+3] = d^2[\mu^2+1] = K(\mu) + \mu\frac{dK}{d\mu}$. (4.29)

For an extremum $\frac{dK}{d\mu} = 0$, then,

$$u^*K(\mu^*) = \mu^* d^2[\mu^{*2}+1] = 1+\frac{d^2}{3}[\mu^{*3}+3\mu^*+2]; \tag{4.30}$$

so μ^* satisfies

$$\frac{2}{3}d^2\mu^{*3} = 1+\frac{2d^2}{3}, \tag{4.31}$$

and

$$\mu^* = \left(1+\frac{3}{2d^2}\right)^{1/3}. \tag{4.32}$$

Then,

$$a^* = d\left[\left(1+\frac{3}{2d^2}\right)^{1/3}-1\right]. \tag{4.33}$$

Note that

$$\mu^*K(\mu^*) = 1+\frac{d^2}{3}\left[1+\frac{3}{2d^2}+3\mu^*+2\right]$$

$$= d^2\left[\frac{3}{2d^2}+1+\mu^*\right] = d^2[\mu^{*3}+\mu^*], \tag{4.34}$$

so that

$$K(\mu^*) = d^2[\mu^{*2}+1] = d^2\left[\left(1+\frac{3}{2d^2}\right)^{2/3}+1\right]. \tag{4.35}$$

Since a is restricted to the interval $0 \leq a < \infty$, extrema may occur at the endpoints. First, note that

$$\lim_{a\to\infty} K(\mu) = \infty, \tag{4.36}$$

so this point is ruled out as a minimum. When $a = 0$, $\mu = 1$, and

$$K(1) = 1 + 2d^2. \tag{4.37}$$

It is now necessary only to show $K(1) > K(\mu^*)$ to conclude that μ^* does in fact achieve the minimum . Now,

$$K(\mu^*) = d^2 + d^2 \mu^{*2}, \tag{4.38}$$

so it is only required to show that

$$d^2 \mu^{*2} < 1 + d^2, \tag{4.39}$$

or

$$\mu^{*2} < 1 + \frac{1}{d^2}, \tag{4.40}$$

or

$$\mu^{*6} < 1 + \frac{3}{d^2} + \frac{3}{d^4} + \frac{1}{d^6}; \tag{4.41}$$

but

$$\mu^{*6} = \left(1 + \frac{3}{2d^2}\right)^2 = 1 + \frac{3}{d^2} + \frac{9}{4d^4}, \tag{4.42}$$

and, since

$$\frac{9}{4d^4} < \frac{3}{d^4} + \frac{1}{d^6} \tag{4.43}$$

for any d, the desired result is attained.

Another solution based on an assumed form for minimum cost.

Reconsidering the performance index,

$$J = E\left[\int_0^\tau (|u| + x^2)\,ds\right], \tag{4.44}$$

it should be clear that for large τ

$$J \sim K(\mu^*)\tau.$$

This observation provides a second method of determining the steady-state behavior. Notice that the asymptotic behavior of J, to the first approximation, does not depend upon the initial state of the system. The value of J, in full detail, would of course depend upon the initial state. To reflect this "residual" effect of the initial state, a performance index of the form

$$J = \lambda \tau + f(x) \tag{4.45}$$

is proposed in the steady-state. One expects to identify λ as $K(\mu^*)$, but at this stage this is not required. Rather, it will be deduced from the ensuing analysis.

Inserting the assumed form,(4.45)for $J(x, \tau)$ into equation (4.9), a problem involving only ordinary differential equations is obtained; namely,

$$\lambda = d \frac{d^2 f}{dx^2} - \frac{df}{dx} + x^2 + \min\left(1, \frac{df}{dx}\right), \quad x > 0, \tag{4.46}$$

with the boundary condition

$$\frac{df}{dx}(0) = 0. \tag{4.47}$$

(The initial condition has, of course, been dropped.). Since this problem does not involve f(x) explicitly, introduce

$$g(x) \triangleq \frac{df}{dx} \tag{4.48}$$

and obtain the problem

$$\lambda = d \frac{dg}{dx} - g + x^2 + \min(1, g), \quad x > 0, \tag{4.49}$$

$$g(0) = 0. \tag{4.50}$$

Because of the initial condition, $g < 1$ for some region near $x = 0$. Solving for g in this region first yields

$$\lambda = d \frac{dg}{dx} + x^2. \tag{4.51}$$

Then,

$$gd = \frac{x^3}{3} + \lambda x. \tag{4.52}$$

This solution is valid up to the point, say $x = a$, when $g = 1$. Then

$$d = -\frac{a^3}{3} + \lambda a. \tag{4.53}$$

Now solving for g in the next interval over which $g > 1$ leads to

$$\lambda = d\,\frac{dg}{dx} - g + x^2 + 1, \tag{4.54}$$

and

$$g(a) = 1. \tag{4.55}$$

It is easy to determine a particular solution of this differential equation having the form

$$g_P(x) = \alpha x^2 + \beta x + \gamma \tag{4.56}$$

Inserting this form, and equating coefficients of like powers of x, yields

$$g_P(x) = x^2 + 2dx + 1 - \lambda + 2d^2. \tag{4.57}$$

The general solution of the homogeneous equation is

$$g(x) = k\ \exp[x/d] \tag{4.58}$$

The presence of such a term in the residual would seem to be unreasonable, particularly in view of the $1/d$ factor in the exponential.[3] Therefore take $k = 0$, so that the solution for $g > 1$ is just $g_P(x)$.

Condition (4.55) on $g_P(a)$ requires that

$$0 = a^2 + 2ad - \lambda + 2d^2 \tag{4.59}$$

---

[3] In the limiting case, $d \to 0$, the optimal control law is obviously $u = \mathrm{sgn}\,x$; it follows that $\lambda = 0$ and $g(x) = x^2$.

This equation, together with(4.53) serves to determine both a and λ :

$$a = d\left[\left(1+\frac{3}{2d^2}\right)^{1/3} - 1\right] \tag{4.60}$$

$$\lambda = d^2\left[\,(1+a/d)^2+1\right] = d^2\left[\mu*^2+1\right] = K(\mu*). \tag{4.61}$$

These results are identical to those obtained previously.

To complete the analysis, note that

$$f(x) = \int_0^x g(\xi)d\xi, \tag{4.62}$$

and

$$f(x) = f_0(x) = \frac{1}{2d}\left(\lambda x^2 - \frac{1}{6}x^4\right), \quad 0 < x < a, \tag{4.63}$$

and

$$f(x) = f_1(x) = f_0(a) + \frac{1}{3}(x^3-a^3) + d(x^2-a^2) + (1-\lambda+2d^2)(x-a),$$

$$a < x. \tag{4.64}$$

$f(0) = 0$ is assumed. It can be verified that $g(x) < 1$ for $0 \leq x < a$, so no switching can occur in this interval. Finally, note that $g(x) > 1$ for all $x > a$, so there are no more switching points.

4.1.2 Time-dependent problem

Formulation as a "free-boundary" problem.

Now consider again the full equations (4.9) through (4.11). Equation (4.9) is clearly nonlinear. There is another way to state the problem which is of some interest. (4.9)can be broken up into two equations:

$$\frac{\partial J}{\partial \tau} = d\,\frac{\partial^2 J}{dx^2} + x^2, \quad \frac{\partial J}{\partial x} < 1 \tag{4.65}$$

$$\frac{\partial J}{\partial \tau} = d\,\frac{\partial^2 J}{\partial x^2} - \frac{\partial J}{\partial x} + x^2 + 1, \quad \frac{\partial J}{\partial x} > 1 \tag{4.66}$$

If the solution were at hand, $\frac{\partial J}{\partial x} = 1$ would describe a curve in the x-τ plane. Conversely, if this curve were known, these two equations could be solved from the knowledge of the initial and boundary conditions. This type of problem in which a boundary curve must be found along with the solutions is known as a "free-boundary" problem[4].

A traditional problem of this sort is that of determining the temperature in a one-dimensional material which is part liquid, part solid--a melting or freezing problem. The free boundary in this case is the interface between liquid and solid[5]. Unfortunately, an examination of the methods employed to solve melting problems reveals that they are inadequate for treating the problem above.

A Nonlinear integral equation. Equation (4.9) can be rewritten as

$$LJ \triangleq \left(\frac{\partial}{\partial \tau} - d\,\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\right) J = x^2 + \min\left(1, \frac{\partial J}{\partial x}\right),$$

$$x > 0 \qquad (4.67)$$

Suppose one can find a function $G(x, \tau | \xi)$ which has the properties

$$LG = 0, \quad x > 0, \quad \tau > 0,$$

$$\lim_{x \to 0^+} \frac{\partial G}{\partial x} (\xi, \tau | x) = 0, \qquad (4.68)$$

and

$$\lim_{\tau \to 0} \int_0^\infty G(\xi, \tau | x) f(\xi)\, d\xi = f(x) \quad \text{for any } x > 0. \qquad (4.69)$$

---

[4] A. Friedman, Partial Differential Equations of Parabolic Type, Chapter 8.

[5] Ibid.

Then $J(x, \tau)$ can be represented as

$$J(x, \tau) = \int_0^\tau \int_0^\infty G(\xi, \tau - s | x)\left[x^2 + \min(1, \frac{\partial J}{\partial \xi}(\xi, s))\right] d\xi \, ds. \tag{4.70}$$

To verify this, first note that

$$LJ(x, \tau) = \lim_{s \to \tau} \int_0^\infty G(\xi, \tau - s | x)\left[\xi^2 + \min(1, \frac{\partial J}{\partial \xi}(\xi, s))\right] d\xi \tag{4.71}$$

since $LG = 0$. Then, because of (4.69) this reduces to

$$LJ(x, \tau) = x^2 + \min\left(1, \frac{\partial J}{\partial x}(x, \tau)\right), \quad x > 0, \tag{4.72}$$

which is just (4.67). Clearly, (4.70) implies that $J(x, 0) \equiv 0$, and $\lim_{x \to 0^+} \frac{\partial J}{\partial x}(x, \tau) = 0$ because of (4.68) Then (4.70) provides a representation which meets all conditions required of $J(x, \tau)$. This is a nonlinear integral equation for J.

Fortunately, the function $G(x, \tau | \xi)$ is available.[6] Caughey and Dienes studied the following one-dimensional stochastic differential equation:

$$\dot{x} = -\operatorname{sgn} x + \dot{w}(t), \tag{4.73}$$

with $w(t)$ the Wiener process having the property $E[w(t) w(s)] = d|t-s|$. The Fokker-Planck equation associated with (4.73) is

$$\frac{\partial T}{\partial t} = + \operatorname{sgn} \xi \frac{\partial T}{\partial \xi} + d \frac{\partial^2 T}{\partial \xi^2}, \tag{4.74}$$

[6] T.K. Caughey and J.K. Dienes, "Analysis of a Non-linear First Order System with a White Noise Input", Journal of Applied Physics, XXXII, 2476-2479.

and the backward equation is

$$\frac{\partial T}{\partial \tau} = - \operatorname{sgn} x \frac{\partial T}{\partial x} + d \frac{\partial^2 T}{\partial x^2} . \tag{4.75}$$

Caughey and Dienes found a function, T, which satisfies both of these equations:

$$T(\xi, \tau | x) = \frac{\exp[-|\xi|/d]}{2\pi^{1/2} d} \int_w^\infty e^{-u^2} du$$

$$+ \frac{\exp\{\frac{1}{2d}|x| - \frac{1}{2d}|\xi| - \frac{1}{4d\tau}[(\xi - x)^2 + \tau^2]\}}{(4\pi dt)^{1/2}} , \tag{4.76}$$

where

$$\omega = \frac{|x| + |\xi| - t}{2(dt)^{1/2}} . \tag{4.77}$$

(Actually, in their paper only the solution valid for $x > 0$ is given; the expression above is the valid expression for any x). This function is not quite $G(x, \tau | \xi)$. It satisfies (4.67) and (4.69) as is evident from the reference cited but does not satisfy (4.68) Consider $G(\xi, \tau \mid x) \triangleq T(\xi, \tau \mid x) + T(-\xi, \tau \mid x)$. (4.78)

Clearly LG + 0, and

$$\lim_{\tau \to 0} G(\xi, \tau \mid x) = \delta(x - \xi) + \delta(x + \xi) . \tag{4.79}$$

However,

$$\lim_{\tau \to 0} \int_0^\infty G(\xi, \tau \mid x) f(\xi) d\xi = f(x) \quad \text{for any } x > 0 , \tag{4.80}$$

i.e., G satisfies (4.69).

Now it is evident that T has the property that

$$T(-\xi,\tau \mid x) = T(\xi, \tau \mid -x). \tag{4.81}$$

Then,

$$\frac{\partial T}{\partial x}(-\xi,\tau \mid x) = -\frac{\partial T}{\partial x}(\xi, \tau \mid -x). \tag{4.82}$$

It follows that

$$\frac{\partial G}{\partial x}(\xi, \tau \mid x) = \frac{\partial T}{\partial x}(\xi, \tau \mid x) - \frac{\partial T}{\partial x}(\xi, \tau \mid -x), \tag{4.83}$$

and in particular

$$\frac{\partial G}{\partial x}(\xi, \tau \mid 0) = 0; \tag{4.84}$$

so G satisfies (4.68)

Using (4.78) the represenatation for J by means of the function $T(\xi, \tau \mid x)$ becomes

$$J(x, \tau) = \int_0^T \int_0^\infty [T(\xi, \tau - s \mid x) + T(-\xi, \tau - s \mid x) \; [\xi^2 + \min(1, \frac{\partial J}{\partial \xi}(\xi, s)] \, d\xi ds. \tag{4.85}$$

Some simplification is possible. Note that

$$\int_0^\infty T(-\xi,\tau \mid x)\xi^2 d\xi = \int_{-\infty}^0 T(\xi, \tau \mid x)\xi^2 d\xi. \tag{4.86}$$

Then,

$$J(x, \tau) = \int_0^T \int_{-\infty}^\infty T(\xi, \tau - s \mid x) \xi^2 d\xi d\tau + \int_0^T \int_0^\infty [T(\xi, \tau - s \mid x) + T(-\xi, \tau - s \mid x)] \min(1, \frac{\partial J}{\partial \xi}) d\xi \, ds, \tag{4.87}$$

A slightly simpler looking equation is obtained by differentiating with respect to x and introducing

$$p(x, \tau) \triangleq \frac{\partial}{\partial x} J(x, \tau): \tag{4.88}$$

$$p(x, \tau) = \Gamma(x, \tau) + \int_0^\tau \int_0^\infty \left[\frac{\partial T}{\partial x}(\xi, \tau - s \mid x) + \frac{\partial T}{\partial x}(-\xi, \tau - s \mid x) \right] \min(1, p(\xi, s)) d\xi\, ds, \tag{4.89}$$

where

$$\Gamma(x, \tau) \triangleq \int_0^\tau \int_{-\infty}^\infty \frac{\partial T}{\partial x}(x, \tau - s \mid \xi)\, \xi^2 d\xi ds \tag{4.90}$$

Evaluation of an integral:

$T(x, \tau)$ does not involve $J(x, \tau)$ and so can be evaluated separately. Because of the complex nature of T, a somewhat indirect method will be employed.

First, note that

$$\Gamma(x, t) = \int_0^t \frac{\partial}{\partial x} \int_{-\infty}^\infty T(\xi, s \mid x)\, \xi^2 d\xi \; ds. \tag{4.91}$$

The quantity

$$\int_{-\infty}^\infty T(\xi, s \mid x)\, \xi^2 d\xi \triangleq m(x, s) \tag{4.92}$$

has the interpretation as the mean-square displacement given the initial state x, s units of time earlier, for the problem considered by Caughey and Dienes. Since m(x, s) also satisfies the backward equation:[7]

---

[7] Payne, op. cit.

$$\frac{\partial m(x,s)}{\partial s} = \int_{-\infty}^{\infty} \frac{\partial T}{\partial s} (\xi, s|x)\, \xi^2 d\xi$$

$$= \int_{-\infty}^{\infty} (-\mathrm{sgn} x \frac{\partial}{\partial x} + d \frac{\partial^2}{\partial x^2})\, T(\xi, s|x)\, \xi^2 d\xi$$

$$= (-\mathrm{sgn} x \frac{\partial}{\partial x} + d \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} T(\xi, s|x)\, \xi^2 d\xi$$

$$= (-\mathrm{sgn} x \frac{\partial}{\partial x} + d \frac{\partial^2}{\partial x^2})\, m(x,s). \tag{4.93}$$

The initial condition is easily obtained:

$$\lim_{s\to 0} m(x,s) = \lim_{s\to 0} \int_{-\infty}^{\infty} T(\xi, s|x)\, \xi^2 d\xi = x^2. \tag{4.94}$$

Equation (4.93) may be solved by means of a Laplace transform. Let

$$\overline{m}(x,\lambda) \triangleq \int_0^{\infty} e^{-\lambda t} m(x,t) dt. \tag{4.95}$$

Then,

$$\lambda \overline{m} - x^2 = -\mathrm{sgn} x \frac{\partial \overline{m}}{\partial x} + d \frac{\partial^2 \overline{m}}{\partial x^2}. \tag{4.96}$$

It is easily verified that

$$\overline{m}_p = \frac{1}{\lambda} x^2 - \frac{2}{\lambda^2}|x| + \frac{2}{\lambda^3}(1 + d\lambda) \tag{4.97}$$

is a particular solution. Then $\overline{m}$ can be represented as

$$\overline{m} = \overline{m}_p + \overline{m}_0, \tag{4.98}$$

where $\overline{m}_o$ satisfies

$$\lambda \overline{m}_o = -\mathrm{sgn} x \frac{\partial \overline{m}_o}{\partial x} + d \frac{\partial^2 \overline{m}_o}{\partial x^2}. \tag{4.99}$$

The general solution of this equation is

$$\overline{m}_o = A \exp [-\alpha_1 |x|] + B \exp [-\alpha_2 |x|] , \qquad (4.100)$$

where

$$\alpha_{1,2} = \frac{1}{2d} \pm \frac{1}{\sqrt{d}} \sqrt{\lambda + 1/4d} \ . \qquad (4.101)$$

(The symmetry of $\overline{m}_o$ has been used to identify coefficients from the region $x < 0$ with those for $x > 0$.) To determine the coefficients A and B two conditions are applied:

(1) $\overline{m}(x, \lambda)$ is analytic for all sufficiently large λ.[8] (4.102)

(2) $$\frac{\partial m}{\partial x}\Big|_{x=0} = 0 . \qquad (4.103)$$

(This derives from properties of $T(x, \tau \,|\, \xi)$.) The first of these conditions requires taking $B = 0$.

From the second of these conditions

$$A = \frac{-2}{\lambda^2(\sqrt{\lambda + 1/4} - 1/2)} , \qquad (4.104)$$

so the total solution is

$$\overline{m} = \frac{1}{\lambda} x^2 - \frac{2}{\lambda^2} |x| + \frac{2}{\lambda^3}(1 + d\lambda) - \frac{2}{\lambda^2(\frac{1}{\sqrt{d}}\sqrt{\lambda + 1/4d} - 1/2d)} \exp\left[\left(\frac{1}{2d} + \frac{1}{\sqrt{d}}\sqrt{\lambda + 1/4d}\right)\right]$$

(4.105)

[8] This follows by assuming $m(x, t) = 0(e^{ct})$ for some constant C as $t \to \infty$. $\overline{m}(x, \lambda)$ is then analytic for any λ such that $\mathrm{Re}(\lambda) > C$.

One can verify that the initial condition,

$$\lim_{\lambda \to \infty} \lambda\, \overline{m}(x, \lambda) = \lim_{t \to 0} m(x, t) = x^2, \qquad (4.106)$$

is satisfied.

Referring to equation (4.91),

$$\frac{\partial \overline{m}(x, \lambda)}{\partial x} = \frac{2x}{\lambda} - \frac{2\,\mathrm{sgn}x}{\lambda^2} + \frac{2\,\mathrm{sgn}x}{\lambda^2} \exp\left[\left(\frac{1}{2d} + \frac{1}{\sqrt{d}}\sqrt{\lambda + 1/4d}\right)|x|\right].$$

(4.107)

Finally, one can obtain

$$\overline{\Gamma}(x, \lambda) \equiv \int_0^\infty e^{-\lambda t}\, \Gamma(x, t)dt, \qquad (4.108)$$

by noting that the time integration corresponds to division by λ:

$$\overline{\Gamma}(x, \lambda) = \frac{2x}{\lambda^2} - \frac{2\,\mathrm{sgn}x}{\lambda^3} + \frac{\mathrm{sgn}x}{\lambda^3} \exp\left[\left(\frac{1}{2d} + \frac{1}{\sqrt{d}}\sqrt{\lambda + 1/4d}\right)|x|\right].$$

(4.109)

The inversion of this expression can be accomplished by means of the formula

$$\exp\left[-r\sqrt{\lambda}\right]\lambda^{-k-2} \leftrightarrow (4t)^{k+1}\, i^{2k+2}\,\mathrm{erfc}\left[\frac{r}{2\sqrt{t}}\right], \quad r > 0, \qquad (4.110)$$

where i^n erfc is the n^{th} integral of the complementary error function.[9,10] (The left side is the Laplace transform in the variable λ of the expression on the right.)

---

[9] M. Abramovitz and I. A. Stegin, Handbook of Mathematical Functions, 1026, formula (293.86).

[10] Ibid, 299.

Noting that

$$f(\lambda) \longleftrightarrow g(t) \tag{4.111}$$

implies that

$$f(\lambda + a) \longleftrightarrow e^{-at} g(t), \tag{4.112}$$

it is evident that if

$$\frac{\exp\left[-\sqrt{\lambda + 1/4d}\,\nu\right]}{\lambda^3} \longleftrightarrow g(t), \tag{4.113}$$

then

$$\frac{\exp\left[-\sqrt{\lambda}\,\nu\right]}{(\lambda - 1/4d)^3} \longleftrightarrow e^{t/4d} g(t). \tag{4.114}$$

the denominator of (4.114) can be expanded as follows

$$\frac{1}{(\lambda - 1/4d)^3} = \frac{1}{\lambda^3(1 - 1/4\lambda d)} = \frac{1}{\lambda^3} \sum_{n=0}^{\infty} + \left(\frac{1}{4\lambda d}\right)^{\nu} \frac{(n+1)(n+2)}{2}. \tag{4.115}$$

Then,

$$\Gamma(x,t) = 2xt - t^2 \operatorname{sgn} x + 2 \operatorname{sgn} x \exp\left[\frac{1}{2d}|x| - \frac{t}{4d}\right] \cdot \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} \left(\frac{1}{4d}\right)^k (4t)^{k+2} i^{2k+4} \operatorname{erfc}\left[\frac{|x|}{2\sqrt{dt}}\right]$$

$$= 2xt - t^2 \operatorname{sgn} x + 16\, t^2 \operatorname{sgn} x \exp\left[\frac{1}{2d}|x| - \frac{t}{4d}\right] \cdot \sum_{k=0}^{\infty} \left(\frac{t}{d}\right)^k (k+1)(k+2)\, i^{2k+4} \operatorname{erfc}\left[\frac{|x|}{2\sqrt{dt}}\right]. \tag{4.116}$$

(This expression can be checked for $x = 0$. For this purpose note that

$$i^n \operatorname{erfc}(0) = \frac{1}{2^n \Gamma\left(\frac{n}{2} + 1\right)} \; ; \; i^{2k+4} \operatorname{erfc}(0) = \frac{1}{4^{k+2} \Gamma(k+3)}$$

$$= \frac{4^{-k-2}}{(k+2)!}. \tag{4.117}$$

[11]

---

[11] Ibid, p. 300.

Then

$$\Gamma(0^+,t) = -t^2 + 16t^2 \exp\left[-\frac{t}{4d}\right] \cdot \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{16} \cdot \left(\frac{t}{4}\right)^k = 0.) \tag{4.118}$$

To simplify the expression somewhat, introduce the normalized integrals of the error function:

$$\overline{i^n \operatorname{erfc}(x)} \equiv \frac{1}{2^n \Gamma\left(\frac{n}{2}+1\right)} i^n \operatorname{erfc}(x) \tag{4.119}$$

Then,

$$\overline{i^n \operatorname{erfc}(0)} = 1, \tag{4.120}$$

and for all $x > 0$, $\overline{i^n \operatorname{erfc}(x)} < 1.$[12] Then,

$$\Gamma(x,t) = 2xt - \operatorname{sgn}x \cdot t^2 + \operatorname{sgn}x\, t^2 \exp\left[\frac{1}{2d}|x| - \frac{t}{4d}\right] \sum_{k=0}^{\infty} \left(\frac{t}{4d}\right)^k \frac{1}{k!} \overline{i^{2k+4}} \operatorname{erfc}\left[\frac{|x|}{2\sqrt{dt}}\right] \tag{4.121}$$

For purposes of numerical computation, it is useful to introduce the variables

$$\eta = x/d$$
$$\theta = t/d$$

Then

$$\frac{\Gamma(x,t)}{d^2} = 2\eta\theta - \theta^2 \operatorname{sgn}(\eta) + \theta^2 \operatorname{sgn}(\eta) \exp\left[\frac{|\eta|}{2} - \frac{\theta}{4}\right] \cdot \sum_{k=0}^{\infty} \left(\frac{\theta}{4}\right)^k \frac{1}{k!} i^{2k+4} \operatorname{erfc}\left[\frac{|\eta|}{2\sqrt{\theta}}\right]. \tag{4.122}$$

The appealing aspect of this expression is that the right hand side is

---

[12] *Ibid*, p. 300.

independent of d, so that numerical computations need only be made for pairs of variables (η, θ) rather than for sets of three variables (x, t, d) as suggested by (4.121).

Discussion of the integral equation. The complexity of the integral equation (4.89) would seem to preclude the possibility of determining an exact analytical solution. In principle, one possible means of obtaining a solution is as follows. Set

$$p_0(x, \tau) = \Gamma(x, \tau) \tag{4.123}$$

and then successively calculate

$$p_n(x, \tau) = \Gamma(x, \tau) + \int_0^\tau \int_0^\infty \left[\frac{\partial T}{\partial x}(\xi, \tau - s \mid x) + \frac{\partial T}{\partial x}(-\xi, \tau - s \mid x)\right] \min\left[1, p_{n-1}(\xi, s)\right] d\xi\, ds \tag{4.124}$$

Due to the nonlinear character of this formula, it is not even evident that such a scheme would converge.

At this point some basic questions remain. Does the integral equation have a solution? If so, is it unique? Does the iterative scheme suggested above converge to the solution? Are there any useful approximations based on the integral equation, say, for the noise parameter, d, very small or very large?

4.2 Two-Dimensional Problem

Unfortunately, very little of the analysis which could be carried out for the one-dimensional problem can be extended to the two-dimensional problem. Recall that the Hamilton-Jacobi formulation resulted in the requirement that the performance index, J, satisfy

$$\frac{\partial J}{\partial \tau} = \min \left[d \frac{\partial^2 J}{\partial y^2} + y \frac{\partial J}{\partial x} + u \frac{\partial J}{\partial y} + |u| + x^2 . \right] \tag{2.31}$$

This problem has symmetry properties but these are not as simple as for the one-dimensional problem. If the indicated minimization is performed, there remains

$$\frac{\partial J}{\partial \tau} = d \frac{\partial^2 J}{\partial y^2} + y \frac{\partial J}{\partial x} - \left| \frac{\partial J}{\partial y} \right| + x^2 + \min \left(1, \left| \frac{\partial J}{\partial y} \right| \right) \tag{4.125}$$

From this it is clear that

$$J(x, y) = J(-x, -y). \tag{4.126}$$

The direct method employed to study the steady-state of the one-dimensional problem is not applicable to the two-dimensional problem because the form of the control law is no longer obvious. It entails a switching curve rather than a switching point. However, again considering the second method employed previously, namely assuimg a steady-state solution of the form

$$J = \lambda \tau + f(x, y). \tag{4.127}$$

leads to f(x, y) satisfying

$$\lambda = d \frac{\partial^2 f}{\partial y^2} + y \frac{\partial f}{\partial x} - \left| \frac{\partial f}{\partial y} \right| + x^2 + \min \left(1, \left|\frac{\partial f}{\partial y}\right| \right). \tag{4.128}$$

There are two significant differences between this formulation of the steady-state problem and that for the one-dimensional problem. First, (4.128) is still a partial differential equation [cf. eqn. (4.46)] Second, no boundary conditions are presented here. The first step to a successful solution of (4.128) is the determination of proper boundary conditions.

The study of the time-dependent problem is plagued by even more difficulties. There is no apparently convenient way to formulate this problem as a nonlinear integral equation. This difficulty derives from 1) the lack of a useful reformulation of the problem by making use of symmetry and other intuitive properties of J (cf. eqns. (4.9) through (4.11), and 2) the lack of a two-dimensional "bang-bang" solution corresponding to (4.76).

It appears that any practical attempt to solve these problems has to be a numerical approach.

Chapter 5

NUMERICAL SOLUTIONS OF THE STOCHASTIC HAMILTON-JACOBI TYPE PROBLEMS

This chapter presents numerical analyses and results for the two Hamilton-Jacobi type partial differential equations derived in Section 2.2. The approach of this chapter is to use the simplest numerical techniques that are consistent with reasonably accurate results--even at the expense of computation time, if necessary. While the techniques applied involve digital computer operations, discussion of these operations will be minimized in favor of problem related matters. The actual computer programs will be presented, however, because they are quite simple and pinpoint the methods used.

5.1 The One-Dimensional Problem

5.1.1 Numerical Formulation

The one-dimensional Hamilton-Jacobi type equation for the antenna steering problem is, as was mentioned before, a parabolic partial differential equation. Typically such equations require boundary conditions of the form

$$J(y_1, \tau) = J_1(\tau) \qquad y = y_1, \ \tau \geq 0 \tag{5.1}$$

$$J(y_2, \tau) = J_2(\tau) \qquad y = y_2, \ \tau \geq 0 \tag{5.2}$$

$$J(y, 0) = J_3(y) \qquad y_1 \leq y \leq y_2, \ \tau = 0 \tag{5.3}$$

The condition (5.3) is clearly available here; it is, by Property 2.2-1,

$$J(y, 0) = J_3(y) \equiv 0. \tag{5.4}$$

Conditions corresponding to (5.1) and (5.2) are not so readily available. The only relevent property is the symmetry condition, Property 2.2-2. It was remarked in Section 3.2.2 that a condition corresponding to Property 2.1-is not available for the stochastic problem, because the expected cost for initially zero position errors is not known without somehow solving the basic problem in the first place. It is also true that the range of definition of the state variable, y, is unrestricted. The net result is that the only reasonable boundaries corresponding to (5.1) and (5.2) are what might he called the natural boundaries consistent with symmetry and (5.3). This is, then, a pure initial value problem.

To reduce (2.68) to a numerical relationship, a rectangular grid is placed over a region, R, of phase space that includes part of the line $\tau = 0$ and the state domain of interest, say, $0 \leq y \leq b$. (The negative region will follow by summetry.) The grid spacing in the y direction is Δ, and that in the τ direction is δ. At a mesh point, y_i, τ_n,

$$J(y_i, \tau_n) = J(i\Delta, n\delta) \overset{\Delta}{=} J_{i,n}. \tag{5.5}$$

If $i_{max}-2$ is the number of interval grid points in a single row of y in R, then Δ is defined such that

$$(i_{max}-1)\,\Delta = b. \tag{5.6}$$

A bound on δ will be dictated by convergence properties of the numerical procedure and by the time domain of interest.

The simplest convergent finite difference representation for (2.68) uses a central difference formula for J_{yy} and J_y and a forward difference for J_τ.[1] Thus, let

$$J_\tau \approx \frac{J_{i,n+1} - J_{i,n}}{\delta}, \tag{5.7}$$

$$J_y \approx \frac{J_{i+1,n} - J_{i-1,n}}{2\Delta}, \tag{5.8}$$

and

$$J_{yy} \approx \frac{J_{i+1,n} - 2J_{i,n} + J_{i-1,n}}{\Delta^2} \tag{5.9}$$

Substituting in (2 68) and rearranging terms leads to

$$J_{i,n+1} = (c - u_o a)\, J_{i-1,n} + (1-2c)\, J_{i,n}$$

$$+ (c + u_o a)\, J_{i+1,n} + \delta[(i-1)^2\Delta^2 + |u_o|], \tag{5.10}$$

where

$$c = \frac{\delta d}{\Delta^2}, \tag{5.11}$$

$$a = \frac{\delta}{2\Delta}, \tag{5.12}$$

and u_o is the optimum control selected in accordance with Property 2.2-9. Actually all that will be required for u_o by way of assumptions is to make some reasonable assumptions about its regions of constant state.

It can be shown that (5.10) is a useful approximation to (2.68) as long as

---

[1] The usual subscript notation will be used here to represent partial differentiation. Context should prevent confusion with the grid notation.

$$c \leq \frac{1}{2}\ .^{2}$$

That this result is also valid for piecewise constant u_o is assumed. Formula (5 has errors of $o(\Delta^2)$ for constant u_o.[3] It will be assumed that equivalent errors apply for the present problem. For the particular case in which $c = \frac{1}{2}$ (5.10) reduces to a form similar to the familiar Schmidt formula of heat transfer:

$$J_{i,n+1} = (\tfrac{1}{2} - u_o a)\, J_{i-1,n} + (\tfrac{1}{2} + u_o a)\, J_{i+1,n}$$
$$+ \delta[\,(i-1)^2 \Delta^2 + |u_o|\,]^3. \qquad (5.13)$$

The boundary and initial conditions become

$$J_{i,1} = 0, \qquad i = 1, 2, \ldots\ i_{max}, \qquad (5.14)$$

$$J_{0,n} = J_{2,n}, \qquad n = 1, 2, \ldots, n_{max}, \qquad (5.15)$$

and

$$J_{i_{max},n} = 3(J_{i_{max}-1,n} - J_{i_{max}-2,n}) + J_{i_{max}-3,n}, \qquad (5.16)$$
$$n = 1, 2, \ \ldots,\ n_{max}.$$

Equation (5.14) is just a numerical transliteration of (5.4). Equation (5.15) is an application of the symmetry property. Equation (5.16) results from a proces of naturally extrapolating the cost surface at each time, τ, using Newton's

---

[2] See, eg., G. Forsythe and W. R. Wasow, Finite Difference Methods for Partial Differential Equations, sec. 14.

[3] H. B. Keller, "The numerical solution of parabolic partial differential equatio ed. A. Ralston and H. S. Will, Mathematical Methods for Digital Computers, 135-143.

backward interpolation formula over three points.[4]

The manner of solving (5.10) and (5.14) through (5.16) for $J_{i,n}$ at the mesh points is quite straight forward. The initial and boundary conditions determine the values of $J_{i,n}$ at all the boundary points. Since $J_y(y, 0) = 0$, $u_o = 0$ along the initial boundary ($\tau = 0$). Then formula (5.10) with $u_o = 0$ may be used to determine $J_{i,n}$ for all interior points, i, and successive values of n as long as n is small. For each n as soon as the interior points are treated, the boundary conditions, (5.15) and (5.16), may be used for the end points. Once values of cost are obtained for an entire column (constant n), the magnitude of the first central difference derivative,

$$J_y \approx \frac{J_{i+1,n} - J_{i-1,n}}{2\Delta} \tag{5.17}$$

may be compared to unity (corresponding to Property 2.2-8). From the results of Chapters 3 and 4 it is to be expected that the location of this derivative condition will start at a maximum y and gradually decrease with n. Once the derivative reaches unity, points above the unity derivative point are computed using (5.10) with $u_o = -1$ and those below using (5.10) with $u_o = 0$. The location where the derivative is unity is the switching curve. No region of $u = +1$ control is expected in the region, R, selected, that is, for $y \geq 0$.

5.1.2 Computational Algorithm and Results

The Fortran coded computer program realizing the above algorithm is shown in Figure 5-1. It should be noted that the particular derivative

[4] See, eg., L. Lapidus, Digital Computation for Chemical Engineers, 21-23.

```
C   NUMERICAL INTEGRATION OF NOISEY ONE DIMENSIONAL HAMILTON - JACOBI
C        TYPE EQUATION
       DIMENSION F(101)
       IDAY = 22
       MO = 11
       IYR = 1967
       D = 3.
       DEL = .2
       DELT = .005
       N1 = 1000
       IPUNCH = 2
C        NO = 1   YES = 2.
       TDEL = 2. * DEL
       C = DELT * D / DEL**2
       E = 1. - 2. * C
       R = DELT / 2. / DEL
       G = C + R
       P = C - R
       GO TO (1,2), IPUNCH
     2 PAUSE 1
       WRITE (7,16) D, DEL, DELT
     1 WRITE (6,8) IDAY, MO, IYR, D, DEL, DELT, (I, I = 1, 101, 10)
       DO 14  I = 1, 101
    14 F(I) = 0.
       N = 1
       I1 = 100
    17 A = F(2)
       B = F(1)
       DO 3 I = 1, I1
       FIM1 = I - 1
       F(I) = (A + F(I+1)) * C + B * E + DELT * (FIM1 * DEL)**2
       A = B
       B = F(I + 1)
       IF (I - 100) 3, 6, 6
     3 CONTINUE
       I1P1 = I1 + 1
       DO 4 I = I1P1, 100
       FIM1 = I - 1
       F(I) = A * G + B * E + F(I+1) * P + DELT * (FIM1 * DEL)**2 + DELT
       A = B
     4 B = F(I + 1)
     6 F(101) = 3. * (F(100) - F(99)) + F(98)
    15 I11 = I1
     5 IF (F(I1 + 1) - F(I1 - 1) - TDEL) 9, 10, 10
    10 I1 = I1 - 1
       GO TO 5
     9 IF (I1 - I11) 12, 13, 12
    12 FI = I1 - 1
       Y = FI * DEL
       FN = N
       T = (FN - 1.) * DELT
       WRITE (6,7) T, (F(I), I = 1, 101, 10), Y
       GO TO (13,18), IPUNCH
    18 WRITE (7,16) T, Y
    13 N = N + 1
       IF (N - N1) 17, 17, 11
    11 FN = N
       T = (FN - 1.) * DELT
       WRITE (6,7) T, (F(I), I = 1, 101, 10), Y
       GO TO (19,20), IPUNCH
    20 WRITE (7,16) T, Y
    19 CALL EXIT
     7 FORMAT (1H 1X 13E9.2)
     8 FORMAT (1H1 19X 79HNUMERICAL INTEGRATION OF NOISEY ONE DIMENSIONAL
      1 HAMILTON - JACOBI TYPE EQUATION / 20X 11HW. H. SPUCK 46X 22HINITI
      2ATED 18 AUG. 1967 / 78X 9HRUN DATE I2, I5, I5 /// 34X 3HD = E9.2,
      35X 5HDEL = E9.2, 5X 6HDELT = E9.2 /// 4X 3H T 3X 11(2X 2HF( I3, 1H
      4) 1X), 2X 7HSW. PT. /)
    16 FORMAT (8E9.2)
       END
```

Figure 5-1. Fortran Coded Program for Stochastic Problem 1-4.

comparison scheme corresponding to (5. 14) coded in Figure 5-1 results in a value for the switching curve that is slightly below the actual curve. In fact, the actual curve (within the validity of the numerical approximation) will fall somewhere between the indicated point and Δ units above this point.

Table 5-1 is a summary listing of the switching points produced by the computational algorithm. Figure 5-2 shows a plot of this data. Both give results for various values of the noise coefficient, d.

It is interesting to note that for small times to go (small τ) the switching curves closely approximate the switching curve for the noiseless case. At the opposite extreme, for large times to go the switching point is very near the steady state value obtained analytically in Chapter 4. The steady state results are compared in Figure 5-3 for various d.

The results presented here must be considered preliminary in that the numerical conditions under which they were completed are not thoroughly understood. Thus, for instance, the effects of the choice of grid size are not known nor are the effects of the extrapolated upper boundary condition.

5.2 The Two-Dimensional Problem

Results for the numerical solution of the two-dimensional Hamilton-Jacobi type equation, (2.31), have not been obtained as of the date of this report. The following formulation of a numerical algorithm is a direct extension of that used for the one-dimensional problem and should yield equivalent results if the computation times do not become excessive.

5.2.1 Numerical Formulation

The two-dimensional equation for the altitude control problem, (2.67),

T* \ d**	.001	.01	.1	.3	1.	3.
1.		0.48	0.40	0.40	0.60	0.60
2.		0.28	0.40	0.40	0.40	0.40
3.	0.18	0.24	0.40	0.40	0.40	0.20
4.	0.15	0.22	0.40	0.40	0.40	0.20
5.	0.13	0.22	0.40	0.40	0.20	0.20
6.	0.12	0.22	0.40	0.40	0.20	0.20
7.	0.11	0.22	0.40	0.40	0.20	0.20
8.	0.11	0.22	0.40	0.40	0.20	0.20
9.	0.11	0.22	0.40	0.40	0.20	0.20
10.	0.11	0.22	0.40	0.40	0.20	0.20

*See equation (2.65)

**See equation (2.64)

Table 5-1. Location of Switching Points for Stochastic Problem 1-4.

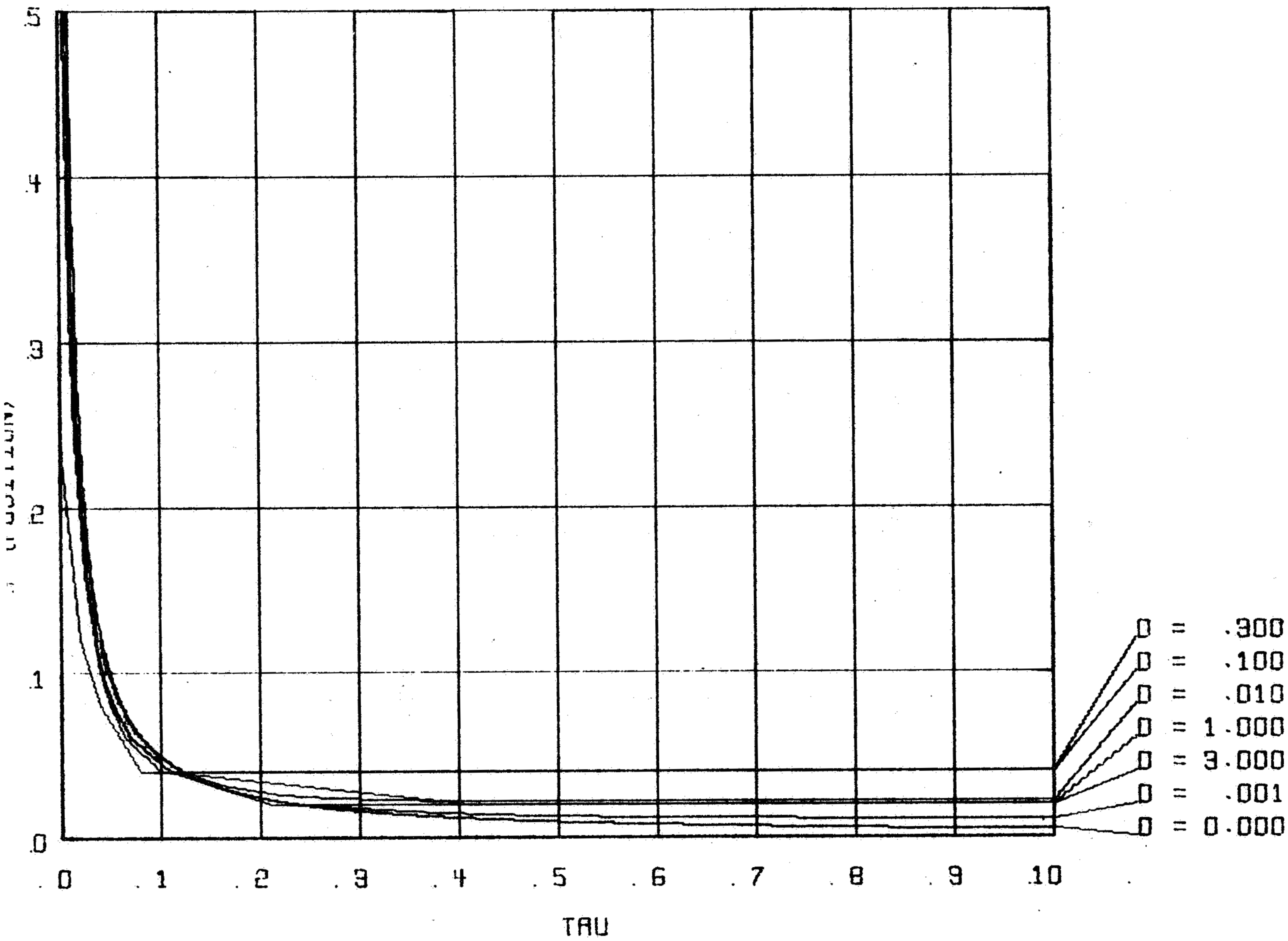
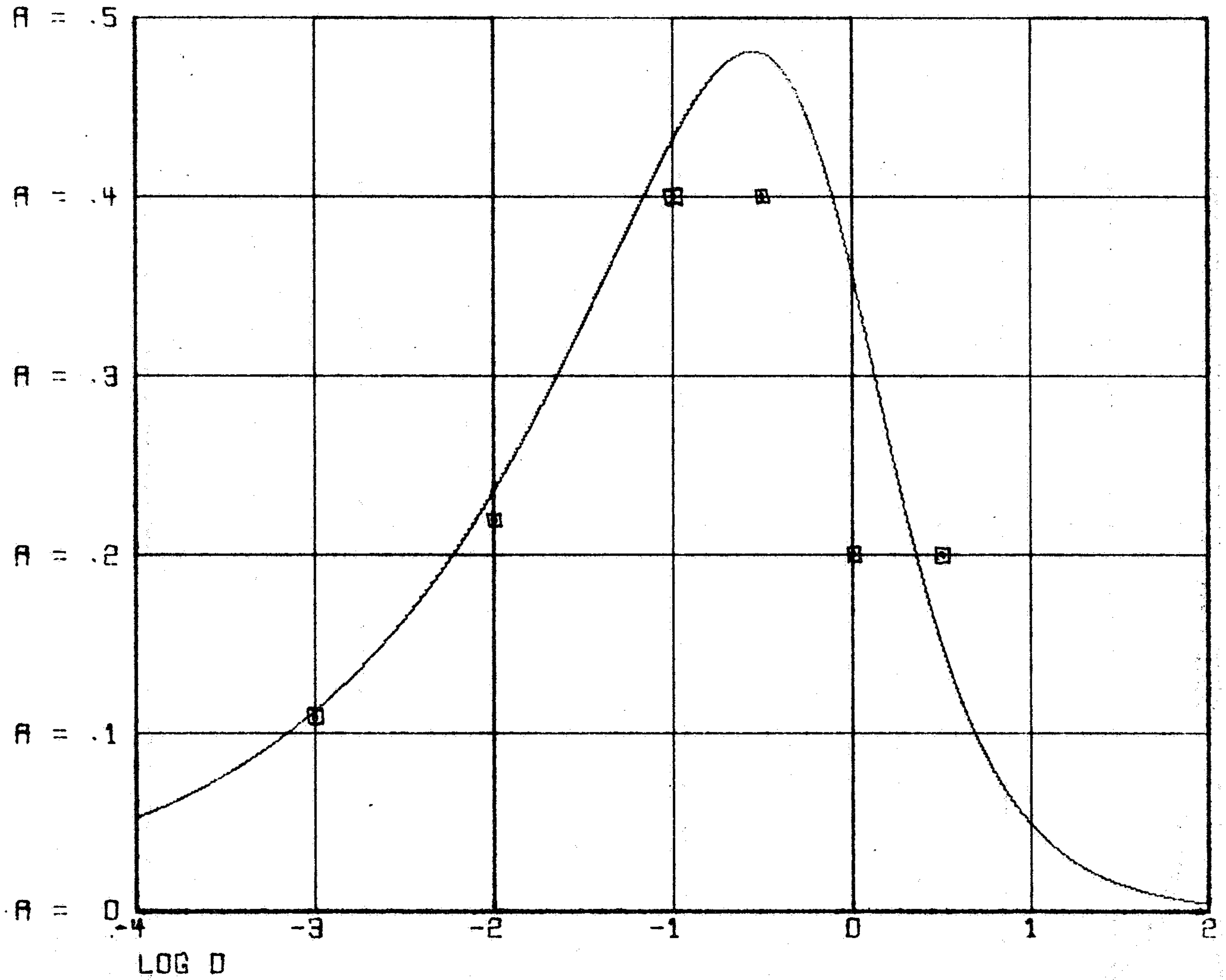


Figure 5-2. Switching Curves for Stochastic Problem 1-4 (for various values of d).



Note: ⊡ From figure 5-2
-From formula (4.33)

Figure 5-3. Steady State Switching Points for Stochastic Problem 1-4.

is a parabolic partial differential equation as is the one-dimensional equation. Typically these equations require boundary conditions of the form

$$J(x_1, y, \tau) = J_1(y, \tau), x = x_1, y_1 \leq y \leq y_2, 0 \leq \tau; \tag{5.18}$$

$$J(x_2, y, \tau) = J_2(y, \tau), x = x_2, y_1 \leq y \leq y_2, 0 \leq \tau; \tag{5.19}$$

$$J(x, y_1, \tau) = J_3 x, \tau), x_1 \leq x \leq x_2, y = y_1, 0 \leq \tau; \tag{5.20}$$

$$J(x, y_2, \tau) = J_4(x, \tau), x_1 \leq x \leq x_2, y = y_2, 0 \leq \tau; \tag{5.21}$$

and

$$J(x, y, 0) = J_5(x, y), x_1 \leq x \leq x_2, y_1 \leq y \leq y_2, \tau = 0. \tag{5.22}$$

Condition (5.22) is clearly available by Property 2.2-1:

$$J(x, y, 0) = J_5(x, y) = 0. \tag{5.23}$$

The other conditions are not readily available. This will again lead to the assumption of "natural" boundary conditions. A degree of symmetry is also applicable to the two-dimensional problem, and this allows the x_1 of (5.21) through (5.23) to be selected as zero; that is, only half the state plane need be considered. The cost at negative x values is then the corresponding value at the point image through the origin.

Equation (2.67) is now reduced to a numerical relationship by placing a rectangular grid in a region, R, of phase space. This region is bounded by the surfaces $\tau = 0$ and $x = 0$, and extends in other directions to cover the regions of interest. ($\tau \geq 0$ and $x \geq 0$, of course.) The grid spacing in the x direction is Δ_x, in the y direction Δ_y, and in the τ direction δ. At a mesh point, x_i, y_k, τ_n,

$$J(x_i, y_k, \tau_n) = J(i\Delta_x, k\Delta_y - b_2, n\delta) \overset{\Delta}{=} J_{i,k,n}. \qquad (5.24)$$

If $i_{max}-2$ and $k_{max}-2$ are the number of internal grid points in a single row of x and y, respectively, in R, then Δ_x and Δ_y are defined such that

$$(i_{max}-1)\,\Delta_x = b_1, \qquad (5.25)$$

and

$$(k_{max}-1)\,\Delta_y = 2b_2. \qquad (5.26)$$

A bound on δ will be determined in the sequel

The simplest convergent finite difference representation for (2. 67) uses a central difference formula for J_{yy}, J_y, and J_x and a forward difference for J_τ. (See (5. 7) through (5.9)Thus leads to the recursion relationship

$$\begin{aligned} J_{i,k,n+1} = {} & (c-u_o a)\, J_{i,k-1,n} + (1-2c)\, J_{i,k,n} \\ & + (c+u_o a)\, J_{i,k+1,n} + e(k-1)\,(J_{i+1,k,n} - J_{i-1,k,n}) \\ & + \delta\,[\,|u_o| + (i-1)^2\,\Delta_x^2\,], \end{aligned} \qquad (5.27)$$

where

$$c = \frac{\delta d}{\Delta_y^2}, \qquad (5.28)$$

$$a = \frac{\delta}{2\Delta_x}$$

$$e = \frac{\delta \Delta_y}{2\Delta_x} , \tag{5.30}$$

and u_o is the optimum control as before. It can be shown that (5.27) is a useful approximation to (2.67) for constant u_o as long as

$$d\delta\left(\frac{1}{\Delta_x^2} + \frac{1}{\Delta_y^2}\right) \leq \frac{1}{2}.^{5} \tag{5.31}$$

The validity of this result in the present case is assumed. It will be assumed that (5.27) has errors of $o(\Delta_y) + o(\Delta_x)$. (The errors probably are of $o(\Delta_y^2) + o(\Delta_x^2)$, but the assumption is sufficient. The particular case in which the coefficient of $J_{i,k,n}$ vanishes is unstable by (5.31).

The boundary and initial conditions here are

$$J_{i,k,1} = 0, \quad i = 1, 2, \ldots, i_{max}, \quad k = 1, 2, \ldots, k_{max}; \tag{5.32}$$

$$J_{0,k,n} = J_{2,k_{max}+1-k,n}, \quad k = 1, 2, \ldots, k_{max}, n=1, 2, \ldots \tag{5.33}$$

$$J_{i_{max},k,n} = 3(J_{i_{max}-1,k,n} - J_{i_{max}-2,k,n}) + J_{i_{max}-3,k,n},$$

$$k = 2, 3, \ldots, k_{max}-1, \; n=1, 2, \ldots; \tag{5.34}$$

$$J_{i,1,n} = 3(J_{i,2,n} - J_{i,3,n}) + J_{i,4,n}, \tag{5.35}$$

$$i = 2, 3, \ldots, i_{max}-1, \; n = 1, 2, \ldots;$$

---

[5] See, eg., G. D. Smith, Numerical Solution of Partial Differential Equations, 41-42.

$$J_{i,k_{max},n} = 3(J_{i,k_{max}-1,n} - J_{i,k_{max}-2,n})\ J_{i,k_{max}-3,n},$$

$$i = 2, 3, \ldots, k_{max}-1, n = 1, 2, \ldots; \qquad (5.3$$

$$J_{1,1,n} = \frac{1}{2}[3(J_{1,2,n} - J_{1,3,n}) + J_{1,4,n}]$$

$$+ \frac{1}{2}[3(J_{2,1,n} - J_{3,1,n}) + J_{4,1,n}],$$

$$n = 1, 2, \ldots, \qquad (5.3$$

$$J_{1,k_{max},n} = \frac{1}{2}[3(J_{1,k_{max}-1,n} - J_{1,k_{max}-2,n}) + J_{1,k_{max}-3,n}]$$

$$+ \frac{1}{2}[3(J_{2,k_{max},n} - J_{3,k_{max},n}) + J_{4,k_{max},n}]$$

$$n = 1, 2, \ldots; \qquad (5.3$$

$$J_{i_{max},1,n} = \frac{1}{2}[3(J_{i_{max}-1,1,n} - J_{i_{max}-2,1,n}) + J_{i_{max}-3,1,n}]$$

$$+ \frac{1}{2}[3(J_{i_{max},2,n} - J_{i_{max},3,n}) + J_{i_{max},4,n}],$$

$$n = 1, 2, \ldots; \qquad (5.3$$

and

$$J_{i_{max},k_{max},n} = \frac{1}{2}[3(J_{i_{max}-1,k_{max},n} - J_{i_{max}-2,k_{max},n})$$

$$+ J_{i_{max}-3,k_{max},n}]$$

$$+ \frac{1}{2}[3(J_{i_{max}, k_{max}-1, n} - J_{i_{max}, k_{max}-2, n}) + J_{i_{max}, k_{max}-3, n}],$$

$$n = 1, 2, \ldots \tag{5.40}$$

Equation (5.32) is just the numerical equivalent of (5.23). Equation (5.33) represents the symmetry property. Equations (5.34) through (5.36) are extrapolations based on Newton's forward or backward interpolation formulae. Finally, (5.37) through (5.40) are averages of Newton's formula extrapolations in the x and y directions at the corners.

The manner of solving these numerical relations for $J_{i,k,n}$ at the mesh points closely parallels the one-dimensional method. The initial and boundary conditions determine the values of $J_{i,k,n}$ at all boundary points. Since $J_y(x, y, 0) = 0$, $u_o = 0$ on the initial boundary ($\tau=0$). Then formula (5.27) with $u_o = 0$ may be used to determine $J_{i,k,n}$ for all interior points, i, of successive columns, k, and for successive but small values of n. The values of $J_{i,k,n}$ for column k-1 must be preserved for use when computing $J_{i,k,n+1}$ because of the particular form of (5.27). For each n the boundary conditions (5.34) through (5.40) may be applied once all of the interior points have been treated. When the cost over the entire region, R, at time $n\delta$ has been calculated, the value of the first central difference

$$J_y \approx \frac{J_{i,k+1,n} - J_{i,k-1,n}}{2\Delta_y} \tag{5.41}$$

may be compared to plus and minus one for each interior point of R. The results of Chapter 3 suggest that this partial derivative will pass through each of +1 and -1 at only one point per column (per k). The locations of these two points form two switching curves by Property 2.2-9. Above the $J_y = +1$ curve (i.e., for larger algebraic values of Y) the cost should be computed using (5.27) with u_o -1. Below the $J_y = -1$ line (i.e., smaller algebraic y), (5.27) should be used with $u_o = +1$. In between, of course, $u_o = 0$ should be used. In each case the calculation procedure should be as described for the $u_o = 0$ case for small n.

5.2.2 Computational Algorithm and Results

As noted previously, numerical results are not presently available for the two dimensional case. These results will appear in a future report.

Chapter 6

CONCLUSIONS

It is the purpose of this chapter to briefly summarize the results presented in this report and to indicate the work that remains to be done before this study can be considered complete. It should be emphasized that this report is primarily intended to indicate the status of the study effort. It should also be made evident that the completion of the study amounts to reaching certain arbitrarily defined goals, certainly not exhaustion of the study area.

6.1 Summary of Results

6.1.1 Narrative Summary

Problem Formulation. The study begins by heuristically deriving mathematical relations governing a class of attitude control and antenna steering problems. A scaling study of these relations reveals that there are no necessary parameters in the deterministic case and only one in the stochastic case.

Hamilton-Jacobi Equation and Property Derivations. A Hamilton-Jacobi equation for the deterministic case is presented without elaboration. This equation is drawn directly from the control system literature. Solution of this equation is known to yield necessary and sufficient conditions for the solution of the original problem.

For the stochastic case a Hamilton-Jacobi type equation is derived from the Chapman-Kolmogorov equation. This derivation is presented in some detail, because the technique is not so well known.

A list of properties pertaining to the cost surface and switching surfaces is presented for both the deterministic and stochastic cases. These properties serve as the boundary conditions for the Hamilton-Jacobi partial differential equations. Description of these properties represents a substantial step towards solving the equations.

Solution of Deterministic Equations. The method of characteristic curves is used to solve the linear, first order Hamilton-Jacobi equations of the deterministic case. Some of these results have been presented before using similarity methods, but others were previously unknown. All of the results presented here are derived from the basic boundary conditions represented by the selected list of properties. The algebraic manipulations involved in these solutions becomes immense, and it is shown that the classes of problems amenable to the methods used here is quite restricted.

Analytic Treatment of Stochastic Equations. The success of analytical treatment of the stochastic cases was far smaller than in the deterministic cases. Nevertheless, for the antenna steering problem, analytical treatment of the steady state situation was possible in a couple of ways. A similar treatment of the two-dimensional problem was unsuccessful. Some progress on the time dependent one-dimensional problem is reported. This leads to a nonlinear integral equation involving the known solution to a simpler one-dimensional problem.

Numerical Treatment of Stochastic Equations. A useful technique for the numerical solution of the one-dimensional equation is presented. This technique involves the simplest explicit-type numerical approximation available for the equation. The computational results provide the first available set of switching curves and cost surfaces for this problem. For the two-dimensional case only a computational algorithm is presented; the computational results are still to be achieved.

6.1.2 Unique Contributions.

It is of interest to note the elements of the study as reported here that do not appear elsewhere. These are as follows:

1) The investigation of the essential problem parameters. This is primarily an engineering result.

2) The statement of a list of properties which are sufficient for solving the Hamilton-Jacobi type equations. These are of both theoretical and practical interest.

3) The complete analytical solution of the deterministic Hamilton-Jacobi equation. This results in a complete set of switching surfaces and a complete definition of the optimum cost surface. These have theoretical and engineering value.

4) The analytical derivation of steady state switching points for the stochastic one-dimensional case. This is of theoretical and practical interest.

5) The derivation of a nonlinear integral equation for the time dependent one-dimensional problem. This result is primarily of theoretical interest at present.

6) The demonstration that analytical solution of even the steady state two-dimensional problem involves (presently) insurmountable difficulties. This result directed the efforts away from analytical treatment of the stochastic case.

7) A numerical solution to the stochastic problems. This has resulted, so far, in the first presentation of the switching curves and optimum expected cost surface for the one-dimensional problem. These loci are of considerable practical interest.

6.2 Work Remaining To Be Done

It is obvious that the present study is incomplete. This section lists and briefly describes the efforts required to complete the investigation. Some of these efforts are listed for reference only, since they might well require more effort than the entire study to date. Topics in this latter category are appropriately identified.

Investigate the Validity of the Stochastic Hamilton-Jacobi Type Equation. It is more or less assured that the derived Stochastic Hamilton-Jacobi type equation represents both necessary and sufficient conditions for the solution of the stochastic problems as did the equation for the deterministic case. A careful investigation of this facet of the equation is warranted. (For reference only)

Derive the Hamilton-Jacobi Type Equation from Fokker-Planck Theory. Since there is a direct relationship between the Fokker-Planck equation and the Chapman-Kolmogorov equation, it should be possible to derive the Hamilton-Jacobi type equation directly from the former. This would tend to place the techniques used here more into the mainstream of modern stochastic system analysis.

Derive the Property Lists Rigorously. The lists of properties presented in Chapter 2 have been justified rather than proved from first principles. It is of some theoretical interest to put these properties on a more mathematical basis. This is particularly true because the available results tend to link the Pontryagin Maximum Principle and Hamilton-Jacobi approaches more than was heretofore realized. (For reference only)

Complete the Algebra for the Two-Dimensional Deterministic Problem. This set of details should be completed before the study can be considered finished.

Compare the Results of the Two-Dimensional Deterministic Problem with the Known Limiting Case Results. This is a complimentary activity to the previous one.

Further Investigate the Unsolved Simpler Two-Dimensional Stochastic Steady State Problem. A two-dimensional stochastic problem involving bang-bang controls, which is simpler than Problem 1-3, was identified in Chapter 4. This problem is presently unsolved. It appears that until this simpler problem is solved analytically one cannot expect to solve the more difficult problems posed here. (For reference only)

Further Investigate the Integral Equation for the Time Dependent Stochastic Problem. This integral equation could be used to iteratively approximate the solution of the time dependent Hamilton-Jacobi equation. Some of the properties of this equation are known, but several more are required before a solution can be hoped for. (For reference only)

Obtain Numerical Results for the Two-Dimensional Stochastic Problem. The algorithm for the two-dimensional case as derived in Chapter 5 has not been tried. This needs to be done.

Investigate the Validity of the Numerical Methods. The numerical methods used to solve the stochastic Hamilton-Jacobi type equations appear valid, but their sensitivity to parameter values and boundary condition treatment is presently largely unknown. Some investigation of these properties is essential to achieving confidence in the numerical results.

REFERENCES

Books

Abramowitz, M., and I. A. Stegin (eds.). Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series 55.

Athans, Michael, and Peter L. Falb. Optional Control: An Introduction To the Theory and Its Applications. New York: McGraw-Hill Book Company, Inc., 1966.

Bellman, R. E. Dynamic Programming. Princeton, New Jersey: Princeton University Press, 1957.

Bellman, R. E., and Dreyfus, S. E. Applied Dynamic Programming. Princeton, New Jersey: Princeton University Press, 1962.

Birkhoff, Garrett, and Mac Lane, Saunders. A Survey of Modern Algebra, (rev. ed.). New York: The MacMillan Company, 1953.

Coddington, Earl A. and Levinson, Norman. Theory of Ordinary Differential Equations. New York: McGraw-Hill Book Company, Inc., 1955.

Davenport, Wilber B., Jr., and Root, William L. An Introduction To The Theory of Random Signals and Noise. New York: McGraw-Hill Book Company, Inc., 1958.

Doob, J. L. Stochastic Processes. New York: John Wiley & Sons, Inc., 1953.

Forsythe, George and Wasow, Wolfgang R. Finite Difference Methods for Partial Differential Equations. New York: John Wiley & Sons, Inc., 1960.

Friedman, Avner. Partial Differential Equations of Parabolic Type. Edgewood Cliffs, New Jersey: Prentice-Hall, Inc., 1964.

Hart, William L. Brief College Algebra. Boston: D. C. Heath and Company, 1932.

Kaplan, Wilfred. Advanced Calculus. Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1952.

Lapidus, Leon. Digital Computation for Chemical Engineers. New York: McGraw-Hill Book Company, Inc., 1962.

Levine, Leon. Methods for Solving Engineering Problems Using Analog Computers. New York: McGraw-Hill Book Company, Inc., 1964.

Merriam, C. W., III. Optimization Theory and the Design of Feedback Control Systems. New York: McGraw-Hill Book Company, Inc., 1964.

Olmsted, J. M. H. Real Variables. New York: Appleton-Century-Crofts, Inc., 1959.

Pontryagin, L. S., et. al. The Mathematical Theory of Optimal Processes. New York: Interscience Publishers Division, John Wiley & Sons, Inc., 1962.

Smith, G. D. Numerical Solution of Partial Differential Equations. London: Oxford University Press, 1965.

Sneddon, Ian N. Elements of Partial Differential Equations, New York: McGraw-Hill Book Company, Inc., 1957.

Wilde, D. J. Optimum Seeking Methods. Edgewood Cliffs, New Jersey,: Prentice-Hall, Inc., 1964.

Articles and Periodicals

Arhanassiades, Michael. "Optimal Control for Linear Time-Invariant Plants with Time, Fuel, and Energy Constraints", AIEE, Transactions, Part II, on Applications and Industry, LXXXI (January, 1963), 321-325.

Athans, Michael. "On the Uniqueness of the Extremal Controls for a Class of Minimum Fuel Problems", IEEE Transactions on Automatic Control, Vol. AC-11, No. 4, (October, 1966), 660-668.

Bryson, A. E., and Denham, W. F., "A Steepest-Ascent Method for Solving Optimum Programming Problems", Transactions of the ASME, Journal of Applied Mechanics, LXXIV (June, 1962), 247-257.

Caughey, T. K. "Derivation and Application of Fokker-Planck Equation to Discrete Nonlinear Dynamical Systems Subject to White Random Excitation", Journal of the Acoustical Society of America, XXXV, No. 11 (November, 1963), 1678-1721.

Caughey, T. K., and Dienes, J. K., "Analysis of a Nonlinear First-Order System With a White Noise Input", Journal of Applied Physics. XXXII, No. 11 (November, 1961), 2476-2479.

Florentin, J. J. "Optimal Control of Continuous Time, Markov Stochastic Systems", Journal of Electronics and Control, X (June, 1961), 473-488.

Fuller, A. T. "Further Study of an Optimum Non-Linear Control System", Journal of Electronics and Control, XVII (1964), 283-300.

Fuller, A. T. "Optimization of Some Non-Linear Control Systems By Means of Bellman's Equation and Dimensional Analysis", International Journal of Control, III, No. 4 (1966), 359-394.

Kushner, Harold J. and Schweppe, Fred C., "A Maximum Principle for Stochastic Control Systems", Journal of Mathematical Analysis and Applications, VIII (1964), 287-302.

Linnes, K. W., Merrick, W. D., and Stevens, R. "Ground System for Space Communication", IRE Transactions on Space Electronics, Vol. SET-6.

Loginov, N. V. "Methods of Stochastic Approximation", Automation and Remote Control, XXVII, No. 4 (April, 1966), 706-728.

Meditch, J. S. "On Mininal-Fuel Satellite Attitude Control", IEEE Transactions on Applications and Industry, LXXXIII (March, 1964), 120-128.

Plant, J. B. "An Iterative Procedure for the Computation of Fixed-Time Fuel Optimal Controls", IEEE Transactions on Automatic Control, Vol. AC-11, No. 4 (October, 1966), 652-660.

Robbins, H. and Monro, S. "A Stochastic Approximation Method", Annals of Mathematical Statistics, XXII, No. 1 (1951), 400-407.

Rozoner, L. I. "L. S. Pontryagin Maximum Principle in the Theory of Optimum Systems", Automation and Remote Control, XX (1960), 1288-1302, 1405-1421, and 1517-1532.

Sherman, Seymour. "Non-Mean-Square Error Criteria", IRE Transactions on Information Theory, Vol. IT-4, (September, 1958), 125-126.

Wonham, W. M. "Note on a Problem in Optimal Non-Linear Control", Journal of Electronics and Control, XV (1963), 59-62.

Conference Proceedings and Collections

Anderson, Max D., and Gupta, Someshwar C., "Backward Time Analog Computer Solutions of Optimum Control Problems", Proceedings of the Spring Joint Computer Conference, 1967, 133-139.

Ergin, E. I., "Current Status of Progress In Attitude Control", Progress in Astronautics and Aeronautics, XIII (June, 1964), 7-36. (Paper presented at the AIAA Guidance and Control Conference-II, at Cambridge, Mass., August 12-14, 1963).

Fuller, A. T. "Relay Control Systems Optimized for Various Performance Criteria", Automatic and Remote Control, Vol. I. London: Butterworths, 1961, 510-519. (Proceedings of the First International Congress of the IFAC at Moscow, 1961).

Kalman, R. E. "The Theory of Optimal Control and the Calcus of Variations", in Mathematical Optimization Techniques, ed. R. Bellman, Berkeley and Los Angeles: University of California Press, 1963, 309-331.

Keller, Herbert B., "The Numerical Solution of Parabolic Partial Differential Equations", Chap. 12 in Mathematical Methods for Digital Computers, ed. Anthony Ralston and Herbert S. Wilf, New York: John Wiley and Sons, Inc., August, 1965, 135-143.

Reports

DeBra, D. B. The Large Attitude Motions and Stability, Due to Gravity, of a Satellite with Passive Damping in an Orbit of Arbitrary Ecentricity about an Oblate Body. Department of Aeronautical Engineering Report No. SUDAER 126, Stanford University, May 1962.

Derusso, P. M., et. al. Adaptive-Predictive Modeling of Nonlinear Processes, NASA Contractor Report, NASA CR-86, N-64-30046. Washington, D. C.: National Aeronautics and Space Administration, September, 1964.

Sworder, D. D. On The Control of Discrete-Time Stochastic Systems, Department of Electrical Engineering Report No. USCEE 145, University of Southern California, October, 1965.

Walli, C. E. Finite State Attitude Control, Department of Electrical Engineering Report No. USCEE 158, University of Southern California, January, 1966.

Unpublished Material

Atkinson, J. D. Spectral Density of First Order Piecewise Linear Systems Excited by White Noise. Unpublished Ph. D. Dissertation, Department of Applied Mechanics, California Institute of Technology, 1967.

Bowers, James Clark, Optimum Analogue and Digital Attitude Control Systems For Space Vehicles. Unpublished Sc. D. dissertation, Sever Institute of Technology, Washington University, St. Louis, Missouri, June, 1964.

Payne, H. J., The Response of Nonlinear Systems to Stochastic Excitation. Unpublished Ph. D. dissertation, Department of Applied Mathematics, California Institute of Technology, 1967.