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A SECOND ORDER THEORY OF MOTION IN THE VICINITY OF THE EARTH-MOON LIBRATION POINT L_2, WITH THE EFFECT OF SOLAR PERTURBATION

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A SECOND ORDER THEORY OF MOTION IN THE VICINITY OF THE EARTH-MOON LIBRATION POINT L_2, WITH THE EFFECT OF SOLAR PERTURBATION

by

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Prepared Under Contract No. NAS5-9870

by

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for

Mission and Trajectory Analysis Division

Goddard Space Flight Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

ABSTRACT

This paper presents the development of a second order theory for trajectories in the vicinity of the lunar libration point L_2. This development is based on a four-body model including the sun, earth, moon, and a satellite, all assumed to move in the same plane.

As a result we obtain a system of two simultaneous second order differential equations with time dependent coefficients. Some selected solutions of interest are derived here, including a first order periodic solution and a second order quasi-periodic solution. Various trajectories around L_2 are plotted and computations of the velocity, acceleration, range, range rate and flight path angle are presented.

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TABLE OF SYMBOLS

E	=	earth
M	=	moon
S	=	sun
P	=	satellite
M_E	=	mass of earth
M_M	=	mass of moon
M_S	=	mass of sun
μ_E	=	gravitational potential of earth
μ_M	=	gravitational potential of moon
μ_S	=	gravitational potential of sun
ω	=	angular velocity of earth-moon system
ϕ	=	angular velocity of sun w.r.t. inertial system
K	=	gravitational constant
$\vec{r}_{EM}$	=	position vector of moon w.r.t. earth
$\vec{r}_{PE}$	=	position vector of satellite w.r.t. earth
$\vec{r}_{ES}$	=	position vector of sun w.r.t. earth
$\vec{r}_{MP}$	=	position vector of satellite w.r.t. moon
$\vec{r}_{MS}$	=	position vector of sun w.r.t. moon
$\vec{r}_{PS}$	=	position vector of sun w.r.t. satellite
$\vec{r}_{L_2E}$	=	position vector of L_2 w.r.t. earth
o	=	distance between moon and L_2 divided by r_{EM}
t	=	unit of time (days)

1. INTRODUCTION

The general three body problem is known to admit only one solution, the one found by Lagrange. This solution is satisfied at the five libration points of the earth-moon system. The libration point, L_2, located at the far side of the moon is of special interest to scientists lately because it could provide an "anchor" for a communications satellite behind the moon.

However, the three body model yields only a rough first order approximation to the motion of a satellite in the vicinity of L_2. Even this solution demonstrates that L_2 is an unstable "anchor" or, in other words, a satellite will not stay in orbit around L_2 unless forced to by an onboard variable thrust engine.

It is precisely for this reason that we are interested in finding out an improved approximation to the motion about L_2. The relevant factors that do not appear in this earth-moon model include the gravitational field of the sun, the oblateness of the earth, the eccentricity of the moon's orbit, and the inclination of the moon's orbit to the earth's equatorial plane. Two other factors that are also excluded are the pressure of solar radiation and meteoroid disturbances. Of all of these external perturbative forces, the gravitational field of the sun is the most important.

In this analysis we will consider first and second order effects of the sun on the motion of a body around L_2; these effects introduce a non-homogeneous forcing function. We construct a four body model consisting of the sun, earth, moon and a satellite stationed initially at L_2 or its immediate vicinity. The sun and the moon are assumed to move in circular coplanar orbits with respective constant angular velocities ϕ and ω. By assuming ω to be constant, we clearly neglect the eccentricity of the moon's orbit. An expression for a generalized acceleration, including solar perturbation, is then developed. Its components are the force functions of the equations of motion.

The complete solution is given in terms of first and second order solutions derived by the method of regular perturbations. Furthermore, the initial conditions are chosen in such a way as to eliminate the dominant unstable contribution. This effect can be implemented in practice during the injection of the satellite into its orbit around L_2. One has to specify an injection location and then the injection velocity. Although the second order solution is unbounded in time, its rate of growth is small and can be corrected by an onboard engine. Various computations such as velocity, acceleration, perturbative acceleration, range, range rate, etc., are presented here as related to the trajectory. These computations, which include the effect of solar perturbation on the motion, give sufficient information to design a mechanical correction for the unstable effect of the motion.

It should be noted that a first order analysis with solar perturbation has been done before for a satellite near the collinear libration points, but, as our paper will show, second order effects in solar perturbation are of great importance since L_2 is an unstable point. Thus, these effects must be included to give a good analytic approximation to the motion, as was done here.

2. DERIVATION OF THE EQUATIONS OF MOTION

The following derivation is for a 3-body system in 2 dimensions, describing a planar motion. The effect of solar perturbation will be added later.

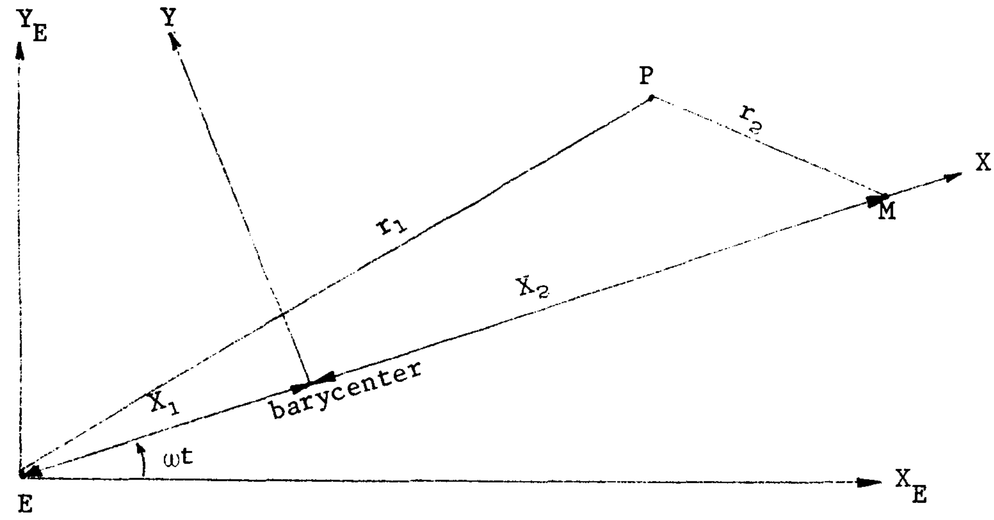

Figure 1. Coordinate Systems

Let (X_E, Y_E) be a set of inertial coordinates referenced at the earth's center. In terms of the rotating coordinates (X, Y), centered at the earth-moon barycenter, the equations of motion for a satellite, P, due to the gravity field of the earth and the moon are:

$$\ddot{X} - 2\omega\dot{Y} = \omega^2 X - \mu_E \frac{(X-X_1)}{r_1^3} - \mu_M \frac{(X-X_2)}{r_2^3} \tag{1a}$$

$$\ddot{Y} + 2\omega\dot{X} = \omega^2 Y - \mu_E \frac{Y}{r_1^3} - \mu_M \frac{Y}{r_2^3} \tag{1b}$$

where ω denotes the rate of rotation of the earth-moon system, and the terms $2\omega\dot{X}$, $2\omega\dot{Y}$ are the Coriolis accelerations, whereas the terms $\omega^2 X$, $\omega^2 Y$ are the centrifugal accelerations.

At the five "libration points" the right hand side of equations (1a) and (1b) is identically zero, and thus the solutions at these points are X = constant, Y = constant. Let the coordinates of the L_2 libration point be given by $X = X_c$, $Y = Y_c$. Then in order to study the small motion near (X_c, Y_c), let $X = X_c + x$ and $Y = Y_c + y$ in (1a) and (1b), and expand the r.h.s. about (X_c, Y_c). If we expand only up to and including the first two terms of the Taylor series, we will obtain the following set of linear differential equations in (x, y), centered at L_2.

$$\ddot{x} - 2\omega\dot{y} - (1 + 2A)\ \omega^2 x = 0 \qquad (2a)$$

$$\ddot{y} + 2\omega\dot{x} - (1 - A)\ \omega^2 y = 0 \qquad (2b)$$

$$\text{where } A = \frac{1}{\mu_E + \mu_M}\left[\frac{\mu_E}{(1 + \rho)^3} + \frac{\mu_M}{\rho^3}\right] \qquad (3)$$

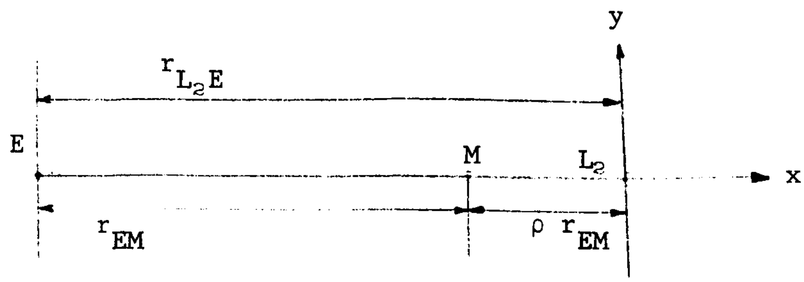

Figure 2. Location of L_2

To include solar perturbation we must develop an expression for the acceleration of the satellite P, relative to the acceleration of the libration point L_2. That is:

$$\ddot{\bar{r}}_{PL_2} = \ddot{\bar{r}}_{PE} - \ddot{\bar{r}}_{L_2E} \qquad (4)$$

Then the equations of motion will read:

$$\ddot{x} - 2\omega\dot{y} - (1 + 2A)\ \omega^2 x = a_x \qquad (5a)$$

$$\ddot{y} + 2\omega\dot{x} - (1 - A)\ \omega^2 y = a_y \qquad (5b)$$

where a_x, a_y denote respectively the x and y components of $\ddot{\bar{r}}_{PL_2}$.

So we proceed to develop expressions for $\ddot{\vec{r}}_{PE}$ and $\ddot{\vec{r}}_{L_2E}$ in terms of known parameters.

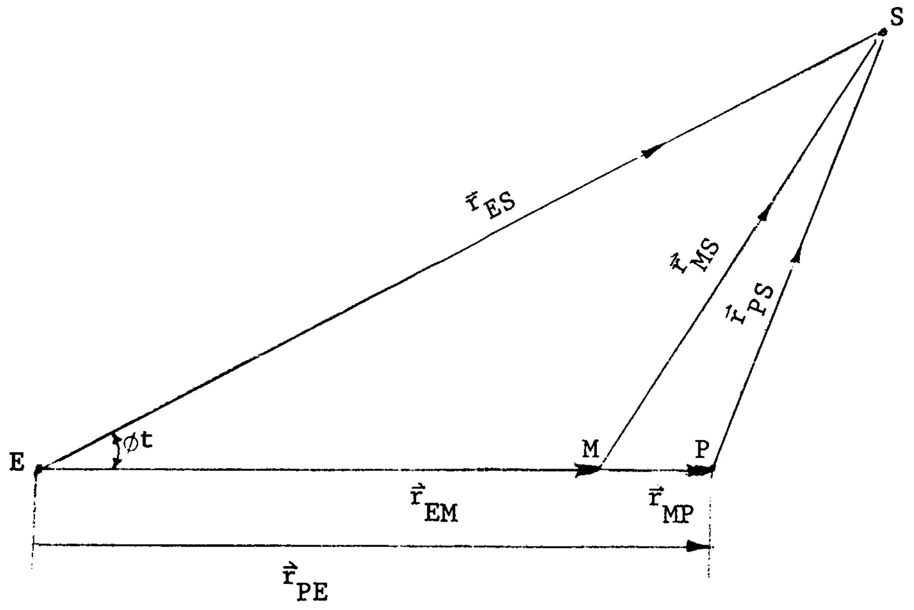

Figure 3. 4-Body Configuration

The acceleration with respect to earth, of the moon, M, due to the gravitational attraction of the earth, E, and the sun, S, is given by:

$$\ddot{\vec{r}}_{EM} = -\frac{\mu_E}{r_{EM}^3}\vec{r}_{EM} - \frac{\mu_M}{r_{EM}^3}\vec{r}_{EM} + \frac{\mu_S}{r_{MS}^3}\vec{r}_{MS} - \frac{\mu_S}{r_{ES}^3}\vec{r}_{ES} \tag{6}$$

The only vector in (6) that is not known directly is $\vec{r}_{MS}$.

$$\text{But } \vec{r}_{MS} = \vec{r}_{ES} - \vec{r}_{EM} \tag{7}$$

$$\text{and } r_{MS}^2 = r_{ES}^2 - 2\vec{r}_{ES}\cdot\vec{r}_{EM} + r_{EM}^2 = r_{ES}^2\left[1 - \frac{2\vec{r}_{ES}\cdot\vec{r}_{EM}}{r_{ES}^2} + \frac{r_{EM}^2}{r_{ES}^2}\right]$$

Using the binomial expansion and retaining only second order terms in $\frac{1}{r_{ES}}$, we obtain:

$$r_{MS}^{-3} \approx r_{ES}^{-3}\left[1 + 3\frac{\vec{r}_{ES}\cdot\vec{r}_{EM}}{r_{ES}^2} - \frac{3}{2}\frac{r_{EM}^2}{r_{ES}^2} + \frac{15}{2}\frac{(\vec{r}_{ES}\cdot\vec{r}_{EM})^2}{r_{ES}^4}\right] \tag{8}$$

Substituting (7) and (8) into (6) and rearranging terms, retaining only fourth order terms in $\frac{1}{r_{ES}}$:

$$\ddot{\vec{r}}_{EM} \approx - \frac{\mu_E + \mu_M}{r_{EM}^3} \vec{r}_{EM} + \frac{\mu_S}{r_{ES}^3} \left[- \vec{r}_{EM} + \frac{3(\vec{r}_{ES} \cdot \vec{r}_{EM})}{r_{ES}^2} \vec{r}_{ES} - \frac{3(\vec{r}_{ES} \cdot \vec{r}_{EM})}{r_{ES}^2} \vec{r}_{EM} - \frac{3}{2} \frac{r_{EM}^2}{r_{ES}^2} \vec{r}_{ES} + \frac{15}{2} \frac{(\vec{r}_{ES} \cdot \vec{r}_{EM})^2}{r_{ES}^4} \vec{r}_{ES} \right] \tag{9}$$

Now, $\vec{r}_{L_2E} = (1 + \rho) \vec{r}_{EM}$ as can be seen from Figure 2,

then $\ddot{\vec{r}}_{L_2E} = (1 + \rho) \ddot{\vec{r}}_{EM}$ (10)

and:

$$\ddot{\vec{r}}_{L_2E} \approx - (1 + \rho) \frac{\mu_E + \mu_M}{r_{EM}^3} \vec{r}_{EM} + (1 + \rho) \frac{\mu_S}{r_{ES}^3} \left[- \vec{r}_{EM} + \frac{3(\vec{r}_{ES} \cdot \vec{r}_{EM})}{r_{ES}^2} \vec{r}_{ES} - \frac{3(\vec{r}_{ES} \cdot \vec{r}_{EM})}{r_{ES}^2} \vec{r}_{EM} - \frac{3}{2} \frac{r_{EM}^2}{r_{ES}^2} \vec{r}_{ES} + \frac{15}{2} \frac{(\vec{r}_{ES} \cdot \vec{r}_{EM})^2}{r_{ES}^4} \vec{r}_{ES} \right] \tag{11}$$

The acceleration of the satellite, P, with respect to the earth, due to the attraction of the earth, moon, and sun, is:

$$\ddot{\vec{r}}_{PE} = - \frac{\mu_E}{r_{PE}^3} \vec{r}_{PE} + \frac{\mu_M}{r_{MP}^3} \vec{r}_{MP} - \frac{\mu_M}{r_{EM}^3} \vec{r}_{EM} + \frac{\mu_S}{r_{PS}^3} \vec{r}_{PS} - \frac{\mu_S}{r_{ES}^3} \vec{r}_{ES} \tag{12}$$

where $\vec{r}_{PE}$ is the distance vector from the earth to the satellite. To obtain a good approximation for $\vec{r}_{PE}$ we perturb the satellite along the earth-moon axis by a distance x measured from L_2. The motion is not perturbed in the normal direction since the dominant gravity effect is along the earth-moon axis. Since the motion is confined to the neighborhood of L_2, due to the linearization of the equations of motion, x is small compared to r_{EM}; but the inclusion of x gives rise to a few large terms in the equation of motion as will be seen now.

Thus let $\vec{r}_{PE} = (1 + \rho) \vec{r}_{EM} + x\vec{i}$ (13)

where $\vec{i}$ denotes a unit vector along the x-axis.

It then follows that:

$$\vec{r}_{MP} = \rho\vec{r}_{EM} + x\vec{i} \tag{14}$$

and $\vec{r}_{PS} = \vec{r}_{ES} - \vec{r}_{PE}$ (15)

From (13), (14) and (15) we now obtain the corresponding scalars:

$$r_{PE}^2 = (1+\rho)^2 \; r_{EM}^2 \left[1 + \frac{2}{1+\rho}\frac{x}{r_{EM}} + \frac{1}{(1+\rho)^2}\frac{x^2}{r_{EM}^2}\right]$$

$$r_{MP}^2 = \rho^2 \; r_{EM}^2 \left[1 + \frac{2}{\rho}\frac{x}{r_{EM}} + \frac{1}{\rho^2}\frac{x^2}{r_{EM}^2}\right]$$

$$r_{PS}^2 = r_{ES}^2 \left[1 + \frac{r_{PE}^2}{r_{ES}^2} - \frac{2(\vec{r}_{ES}\cdot\vec{r}_{PE})}{r_{ES}^2}\right]$$

Applying the binomial expansion and neglecting second-order and higher terms in x/r_{EM}, we obtain:

$$r_{PE}^{-3} \approx (1+\rho)^{-3} \; r_{EM}^{-3}\left[1 - \frac{3}{1+\rho}\frac{x}{r_{EM}}\right] \tag{16}$$

$$r_{MP}^{-3} \approx \rho^{-3} \; r_{EM}^{-3}\left[1 - \frac{3}{\rho}\frac{x}{r_{EM}}\right] \tag{17}$$

$$r_{PS}^{-3} = r_{ES}^{-3}\left[1 - \frac{3}{2}\frac{r_{PE}^2}{r_{ES}^2} + \frac{3(\vec{r}_{ES}\cdot\vec{r}_{PE})}{r_{ES}^2} + \frac{15}{2}\frac{(\vec{r}_{ES}\cdot\vec{r}_{PE})^2}{r_{ES}^4}\right] \tag{18}$$

Substituting (18) in (12) and neglecting fifth-order and higher terms in $1/r_{ES}$ we obtain for $\ddot{\vec{r}}_{PE}$:

$$\ddot{\vec{r}}_{PE} = -\frac{\mu_E}{r_{PE}^3}\vec{r}_{PE} + \frac{\mu_M}{r_{MP}^3}\vec{r}_{MP} - \frac{\mu_M}{r_{EM}^3}\vec{r}_{EM} + \frac{\mu_S}{r_{ES}^3}\left[-\vec{r}_{PE} - \frac{3}{2}\frac{r_{PE}^2}{r_{ES}^2}\vec{r}_{ES} + \frac{3(\vec{r}_{ES}\cdot\vec{r}_{PE})}{r_{ES}^2}\vec{r}_{ES} - \frac{3(\vec{r}_{ES}\cdot\vec{r}_{PE})}{r_{ES}^2}\vec{r}_{PE} + \frac{15}{2}\frac{(\vec{r}_{ES}\cdot\vec{r}_{PE})}{r_{ES}^4}\vec{r}_{ES}\right] \tag{19}$$

Substituting (11) and (19) in (4), we get the expression for the net acceleration of the satellite P with respect to L_2:

$$\ddot{\vec{r}}_{PL_2} \approx -\frac{\mu_E}{r_{PE}^3}\vec{r}_{PE} + \frac{\mu_M}{r_{MP}^3}\vec{r}_{MP} - \frac{\mu_M}{r_{EM}^3}\vec{r}_{EM} + (1+\rho)\frac{\mu_E+\mu_M}{r_{EM}^3}\vec{r}_{EM} + \frac{\mu_S}{r_{ES}^3}\Big[-\vec{r}_{PE}$$

$$+ (1+\rho)\,\vec{r}_{EM} - \frac{3}{2}\frac{r_{PE}^2}{r_{ES}^2}\vec{r}_{ES} + \frac{3}{2}(1+\rho)\frac{r_{EM}^2}{r_{ES}^2}\vec{r}_{ES} + \frac{3(\vec{r}_{ES}\cdot\vec{r}_{PE})}{r_{ES}^2}\vec{r}_{ES}$$

$$- 3(1+\rho)\frac{(\vec{r}_{ES}\cdot\vec{r}_{EM})}{r_{ES}^2}\vec{r}_{ES} - \frac{3(\vec{r}_{ES}\cdot\vec{r}_{PE})}{r_{ES}^2}\vec{r}_{PE} + 3(1+\rho)\frac{(\vec{r}_{ES}\cdot\vec{r}_{PE})}{r_{ES}^2}\vec{r}_{EM}$$

$$+ \frac{15}{2} \frac{(\vec{r}_{ES} \cdot \vec{r}_{PE})^2}{r_{ES}^4} \vec{r}_{ES} - \frac{15}{2} (1+\rho) \frac{(\vec{r}_{ES} \cdot \vec{r}_{EM})^2}{r_{ES}^4} \vec{r}_{ES} \Big] \tag{20}$$

Let $\vec{r}_{ES} = x_{ES} \vec{i} + y_{ES} \vec{j}$

$\vec{r}_{EM} = r_{EM} \vec{i}$

then $(\vec{r}_{ES} \cdot \vec{r}_{EM}) = x_{ES} r_{EM}$

$(\vec{r}_{ES} \cdot \vec{r}_{PE}) = (1+\rho)\, x_{ES} r_{EM} + x_{ES} x$

Using these relations in addition to (13), (14), (16) and (17), we obtain a simplified version of (20):

$$\begin{aligned}
\ddot{\vec{r}}_{PL_2} \approx & - \frac{\mu_E}{(1+\rho)^3 \, r_{EM}^3} \left[1 - \frac{3}{1+\rho} \frac{x}{r_{EM}}\right] \left[(1+\rho)\, \vec{r}_{EM} + x\vec{i}\right] + \frac{\mu_M}{\rho^3 \, r_{EM}^3} \Big[1 \\
& - \frac{3}{\rho} \frac{x}{r_{EM}} \Big] \left[\rho \, \vec{r}_{EM} + x\vec{i}\right] - \frac{\mu_M}{r_{EM}^3} \vec{r}_{EM} + (1+\rho) \frac{\mu_E + \mu_M}{r_{EM}^3} \vec{r}_{EM} \\
& + \frac{\mu_S}{r_{ES}^3} \Big\{ - x\vec{i} - \frac{3}{2} \frac{\vec{r}_{ES}}{r_{ES}^2} r_{EM}^2 \left[\rho\, (1+\rho) + 2\, (1+\rho) \frac{x}{r_{EM}}\right] + 3 \frac{\vec{r}_{ES}}{r_{ES}^2} x_{ES} x \\
& + \frac{3}{r_{ES}^2} \left[-\rho\, (1+\rho)\, x_{ES} r_{EM} \vec{r}_{EM} - (1+\rho) x_{ES} r_{EM} x \vec{i} + (1+\rho) x_{ES} x \vec{r}_{EM}\right] \\
& + \frac{15}{2} \frac{\vec{r}_{ES}}{r_{ES}^4} \left[\rho\, (1+\rho)\, x_{ES}^2 r_{EM}^2 + 2\, (1+\rho)\, x_{ES}^2 r_{EM} x\right] \Big\}
\end{aligned} \tag{21}$$

At x = 0 the contribution to $\ddot{\vec{r}}_{PL_2}$ of the moon and the earth is zero since a libration point in the three body system is a point where the sum of the forces is zero. That is:

$$\text{at} \quad x = 0 \quad \ddot{\vec{r}}_{PL_2} \Big|_{\substack{\text{earth} \\ \text{moon}}} = - \frac{\mu_E}{(1+\rho)^2 \, r_{EM}^3} \vec{r}_{EM} + \frac{\mu_M}{\rho^2 \, r_{EM}^3} \vec{r}_{EM} - \frac{\mu_M}{r_{EM}^3} \vec{r}_{EM} + (1+\rho) \frac{\mu_E + \mu_M}{r_{EM}^3} \vec{r}_{EM} = 0 \tag{22}$$

With (22) taken into account, (21) now becomes:

$$\ddot{\vec{r}}_{PL_2} \approx \frac{1}{r_{EM}^3}\left[\frac{\mu_M}{\rho^3} - \frac{\mu_E}{(1+\rho)^3}\right]\left[\vec{i} - 3\frac{\vec{r}_{EM}}{r_{EM}}\right] x + \frac{\mu_S}{r_{ES}^3}$$

$$\left\{-x\vec{i} - \frac{3}{2}\frac{\vec{r}_{ES}}{r_{ES}^2} r_{EM}^2\left[\rho(1+\rho) + 2(1+\rho)\frac{x}{r_{EM}}\right] + 3\frac{\vec{r}_{ES}}{r_{ES}^2} x_{ES} x\right.$$

$$+ \frac{3}{r_{ES}^2}\left[-\rho(1+\rho)\, x_{ES} r_{EM} \vec{r}_{EM} - (1+\rho) x_{ES} r_{EM} x\vec{i} + (1+\rho)\, x_{ES} x \vec{r}_{EM}\right]$$

$$\left. + \frac{15}{2}\frac{\vec{r}_{ES}}{r_{ES}^4}\left[\rho(1+\rho)\, x_{ES}^2 r_{EM}^2 + 2(1+\rho)\, x_{ES}^2 r_{EM} x\right]\right\} \tag{23}$$

With $x = 0$ the above expression for $\ddot{\vec{r}}_{PL_2}$ reduces to that developed by F. T. Nicholson (ref. 4). However, in our case first and second order terms in solar perturbation appear in the acceleration, and the contributions of the earth and the moon are also included.

In accordance with eqs. (5a) and (5b) we are looking for the rectangular components of $\ddot{\vec{r}}_{PL_2}$.

$$\text{Let } \ddot{\vec{r}}_{PL_2} = a_x\vec{i} + a_y\vec{j} \tag{24}$$

and break $\ddot{\vec{r}}_{PL_2}$ into its components to obtain a_x and a_y.

$$a_x \approx \frac{2}{r_{EM}^3}\left[\frac{\mu_E}{(1+\rho)^3} - \frac{\mu_M}{\rho^3}\right] x + \frac{\mu_S}{r_{ES}^3}\left\{-x - \frac{3}{2}\frac{x_{ES}}{r_{ES}^2} r_{EM}^2\left[\rho(1+\rho) + 2(1+\rho)\frac{x}{r_{EM}}\right]\right.$$

$$+ 3\frac{x_{ES}^2}{r_{ES}^2} x + \frac{3}{r_{ES}^2}\left[-\rho(1+\rho)\, x_{ES} r_{EM}^2\right] + \frac{15}{2}\frac{x_{ES}}{r_{ES}^4}\left[\rho(1+\rho)\, x_{ES}^2 r_{EM}^2\right.$$

$$\left.\left. + 2(1+\rho)\, x_{ES}^2 r_{EM} x\right]\right\}$$

Rearranging and neglecting terms in x of order $\frac{x}{r_{ES}}$ and higher, we obtain the final expression for a_x.

$$a_x \approx \frac{2}{r_{EM}^3}\left[\frac{\mu_E}{(1+\rho)^3} - \frac{\mu_M}{\rho^3}\right] x + \frac{3}{2}\rho(1+\rho)\frac{\mu_S}{r_{ES}^4} r_{EM}^2\left[5\left(\frac{x_{ES}}{r_{ES}}\right)^3 - 3\left(\frac{x_{ES}}{r_{ES}}\right)\right]$$

$$+ \frac{\mu_S}{r_{ES}^3}\left[3\left(\frac{x_{ES}}{r_{ES}}\right)^2 - 1\right] x \tag{25}$$

Similarly:

$$a_y \approx \frac{\mu_S}{r_{ES}^3} \left\{ - \frac{3}{2} \frac{y_{ES}}{r_{ES}^2} r_{EM}^2 \left[\rho(1+\rho) + 2 (1+\rho) \frac{x}{r_{EM}} \right] + 3 \frac{y_{ES} x_{ES}}{r_{ES}^2} x \right.$$

$$\left. + \frac{15}{2} \frac{y_{ES}}{r_{ES}^4} \left[\rho(1+\rho) x_{ES}^2 r_{EM}^2 + 2 (1+\rho) x_{ES}^2 r_{EM} \right] \right\}$$

Rearranging as previously:

$$a_y \approx \frac{3}{2} \rho(1+\rho) \frac{\mu_S}{r_{ES}^4} r_{EM}^2 \left[5 \left(\frac{x_{ES}}{r_{ES}} \right)^2 \left(\frac{y_{ES}}{r_{ES}} \right) - \left(\frac{y_{ES}}{r_{ES}} \right) \right]$$

$$+ 3 \frac{\mu_S}{r_{ES}^3} \left(\frac{x_{ES}}{r_{ES}} \right) \left(\frac{y_{ES}}{r_{ES}} \right) x \tag{26}$$

As can be seen, a_y contains only solar terms. This is an expected result since we assumed perturbations only along the x-axis. As stated before, the earth and the moon do not exert large forces on the satellite normal to their axis as long as the satellite is in the neighborhood of L_2. We simplify the equations by introducing the following constants.

$$K_1 = \frac{2}{r_{EM}^3} \left[\frac{\mu_E}{(1+\rho)^3} - \frac{\mu_M}{\rho^3} \right]$$

$$K_2 = \frac{3}{2} \rho (1+\rho) \frac{\mu_S}{r_{ES}^4} r_{EM}^2$$

$$K_3 = \frac{\mu_S}{r_{ES}^3}$$

where

$$\mu_E = K \frac{M_E}{M_E + M_M} = \omega^2 r_{EM}^3 \frac{M_E}{M_E + M_M} \tag{27}$$

$$\mu_M = K \frac{M_E}{M_E + M_M} = \omega^2 r_{EM}^3 \frac{M_E}{M_E + M_M} \tag{28}$$

$$\mu_S = K \frac{M_S}{M_E + M_M} = \omega^2 r_{EM}^3 \frac{M_S}{M_E + M_M} \tag{29}$$

and $\dfrac{r_{ES}}{r_{EM}} = 388.9237$

Thus

$$K_1 = 2\omega^2 \left[\frac{M_E/(M_E+M_M)}{(1+\rho)^3} - \frac{M_M/(M_E+M_M)}{\rho^3} \right] \tag{30a}$$

$$K_2 = \frac{3}{2}\,\rho\,(1+\rho)\,\frac{\omega^2 r_{EM} M_S/(M_E + M_M)}{(388.9237)^4} \tag{30b}$$

$$K_3 = \frac{\omega_2 M_S/(M_E + M_M)}{(388.9237)^3} \tag{30c}$$

and $$A = \frac{M_E/(M_E + M_M)}{(1+\rho)^3} + \frac{M_M/(M_E + M_M)}{\rho^3} \tag{30d}$$

We adopt the units of kilometers and days so that most of our computations will be of the order of one. Furthermore, since we are dealing with long missions, the units of days are quite appropriate. The constants used are defined below:

$M_M/(M_E + M_M) = 0.0121$

$M_E/(M_E + M_M) = 0.9879$

$M_S/(M_E + M_M) = 328{,}430$

$\rho = 0.167832$

$\omega = 0.22997$ radians/day

$\dot{\phi} = -0.2128$ radians/day

$r_{EM} = 384{,}752.7$ kilometers

$A = 3.17979$

$K_1 = -0.20512$ radians^2/day^2

$K_2 = 0.8587319 \times 10^{-1}$ (kilometers)/day^2

$K_3 = 0.29525 \times 10^{-3}$ radians^2/day^2

Now, suppose that the sun moves in a circular orbit, coplanar with the moon's orbit in the (x, y) plane, with an angular velocity $\dot{\phi}$. Then:

$$x_{ES} = r_{ES} \cos\phi t$$
$$y_{ES} = r_{ES} \sin\phi t \tag{31}$$

Substituting (31) into (28) and (29) we obtain expressions for the acceleration in terms of x and t.

$$a_x \approx K_1 x + K_2\,[5\cos^3\phi t - 3\cos\phi t] + K_3\,[3\cos^2\phi t - 1]\,x$$

$$a_y \approx K_2 \left[5 \cos^2\phi t \sin\phi t + \sin\phi t\right] + 3 K_3 \left[\cos\phi t \sin\phi t\right] x$$

A further rearrangement leads to a simpler mathematical form with double and triple angles.

$$a_x \approx K_1 x + \frac{1}{4} K_2 \left[3 \cos\phi t + 5 \cos 3\phi t\right] + \frac{1}{2} K_3 \left[1 + 3 \cos 2\phi t\right] x \tag{32a}$$

$$a_y \approx \frac{1}{4} K_2 \left[\sin\phi t + 5 \sin 3\phi t\right] + \frac{3}{2} K_3 \left[\sin 2\phi t\right] x \tag{32b}$$

The resultant acceleration is:

$$a = a(x,t) = \sqrt{a_x^{\,2} + a_y^{\,2}} \tag{33}$$

at x=0, which coincides with the location of L_2:

$$a = \frac{K_2}{4} \left[26 + 8 \cos^2\phi t + 30 \cos 3\phi t \cos\phi t + 10 \sin 3\phi t \sin\phi t\right]^{1/2} \tag{34}$$

and the maximum acceleration at x=0 is:

$$a_{max} = 2 K_2 = 0.171746 \text{ kilometers/day}^2$$

In Figures 4 and 5 (pages 20 and 21) a (x, t) is plotted as a function of t for fixed values of x.

Substituting (32a) and (32b) in the equations of motion for the satellite (5a) and (5b), we obtain:

$$\ddot{x} - 2\omega\dot{y} - (1+2A)\,\omega^2 x = K_1 x + \frac{1}{4} K_2 \left[3 \cos\phi t + 5 \cos 3\phi t\right] + \frac{1}{2} K_3 \left[1 + 3 \cos 2\phi t\right] x \tag{35a}$$

$$\ddot{y} + 2\omega\dot{x} - (1-A)\,\omega^2 y = \frac{1}{4} K_2 \left[\sin\phi t + 5 \sin 3\phi t\right] + \frac{3}{2} K_3 \left[\sin 2\phi t\right] x \tag{35b}$$

These are the linearized equations of motion for a satellite in the vicinity of L_2, including both first and second order effects of solar perturbation.

Table 1

Perturbative Acceleration of Satellite at L_2

t (days)	a_x (kilometers/day^2)	a_y (kilometers/day^2)	a (kilometers/day^2)
.00	.17175	-.00000	.17175
1.00	.14915	$-.68500 \times 10^{-1}$	.16413
2.00	$.89765 \times 10^{-1}$	-.11160	.14322
3.00	$.15478 \times 10^{-1}$	-.11383	.11488
4.00	$-.46866 \times 10^{-1}$	$-.75687 \times 10^{-1}$	$.89022 \times 10^{-1}$
5.00	$-.75944 \times 10^{-1}$	$-.13361 \times 10^{-1}$	$.77111 \times 10^{-1}$
6.00	$-.64205 \times 10^{-1}$	$.47681 \times 10^{-1}$	$.79973 \times 10^{-1}$
7.00	$-.20666 \times 10^{-1}$	$.82775 \times 10^{-1}$	$.85316 \times 10^{-1}$
8.00	$.32836 \times 10^{-1}$	$.77801 \times 10^{-1}$	$.84446 \times 10^{-1}$
9.00	$.70455 \times 10^{-1}$	$.34758 \times 10^{-1}$	$.78563 \times 10^{-1}$
10.00	$.72738 \times 10^{-1}$	$-.29024 \times 10^{-1}$	$.78315 \times 10^{-1}$
11.00	$.34490 \times 10^{-1}$	$-.87729 \times 10^{-1}$	$.94265 \times 10^{-1}$
12.00	$-.32981 \times 10^{-1}$	-.11725	.12180
13.00	-.10615	-.10474	.14913
14.00	-.15841	$-.53721 \times 10^{-1}$	.16727
15.00	-.17044	$.17252 \times 10^{-1}$	.17131
16.00	-.13777	$.81807 \times 10^{-1}$	.16023
17.00	$-.72519 \times 10^{-1}$	.11609	.13688
18.00	$.13288 \times 10^{-2}$	.10807	.10808
19.00	$.57311 \times 10^{-1}$	$.62245 \times 10^{-1}$	$.84611 \times 10^{-1}$
20.00	$.76783 \times 10^{-1}$	$-.22258 \times 10^{-2}$	$.76815 \times 10^{-1}$

Table 2

Perturbative Acceleration of Satellite Near L_2
($x = 1$ kilometer)

t (days)	a_x (kilometers/day^2)	a_y (kilometers/day^2)	a (kilometers/day^2)
.00	$-.33082 \times 10^{-1}$	-.00000	$.33082 \times 10^{-1}$
1.00	$-.55689 \times 10^{-1}$	$-.68683 \times 10^{-1}$	$.88423 \times 10^{-1}$
2.00	-.11511	-.11193	.16056
3.00	-.18946	-.11426	.22124
4.00	-.25186	$-.76126 \times 10^{-1}$	.26311
5.00	-.28100	$-.13737 \times 10^{-1}$	.28133
6.00	-.26930	$.47435 \times 10^{-1}$	.27345
7.00	-.22579	$.82703 \times 10^{-1}$	.24046
8.00	-.17228	$.77916 \times 10^{-1}$	.18908
9.00	-.13463	$.35040 \times 10^{-1}$	.13912
10.00	-.13230	$-.28627 \times 10^{-1}$	.13536
11.00	-.17049	$-.87286 \times 10^{-1}$	.19154
12.00	-.23790	-.11684	.26505
13.00	-.31102	-.10444	.32809
14.00	-.36325	$-.53580 \times 10^{-1}$	.36718
15.00	-.37527	$.17208 \times 10^{-1}$	.37566
16.00	-.34262	$.81585 \times 10^{-1}$	.35220
17.00	-.27741	.11573	.30058
18.00	-.20362	.10764	.23032
19.00	-.14770	$.61814 \times 10^{-1}$	.16011
20.00	-.12828	$-.25762 \times 10^{-2}$	.12831

Table 3

Perturbative Acceleration of a Satellite Near L_2
(x = -1 kilometer)

t (days)	a_x (kilometers/day^2)	a_y (kilometers/day^2)	a (kilometers/day^2)
.00	.37657	.00000	.37657
1.00	.35399	$-.68317 \times 10^{-1}$	.36053
2.00	.29464	-.11127	.31495
3.00	.22041	-.11341	.24788
4.00	.15813	$-.75248 \times 10^{-1}$	.17512
5.00	.12911	$-.12985 \times 10^{-1}$	.12976
6.00	.14089	$.47927 \times 10^{-1}$	.14882
7.00	.18446	$.82847 \times 10^{-1}$	.20221
8.00	.23795	$.77686 \times 10^{-1}$	.25031
9.00	.27555	$.34477 \times 10^{-1}$	.27769
10.00	.27778	$-.29422 \times 10^{-1}$	.27933
11.00	.23947	$-.88172 \times 10^{-1}$	.25519
12.00	.17194	-.11766	.20834
13.00	$.98716 \times 10^{-1}$	-.10505	.14415
14.00	$.46425 \times 10^{-1}$	$-.53863 \times 10^{-1}$	$.71109 \times 10^{-1}$
15.00	$.34389 \times 10^{-1}$	$.17297 \times 10^{-1}$	$.38494 \times 10^{-1}$
16.00	$.67081 \times 10^{-1}$	$.82030 \times 10^{-1}$	.10597
17.00	.13237	.11645	.17630
18.00	.20628	.10851	.23307
19.00	.26232	$.62676 \times 10^{-1}$	.26970
20.00	.28185	$-.18754 \times 10^{-2}$	.28186

Table 4

Perturbative Acceleration of a Satellite Near L_2
(x = 0.5 kilometers)

t (days)	a_x (kilometers/day^2)	a_y (kilometers/day^2)	a (kilometers/day^2)
.00	$.69322 \times 10^{-1}$	-.00000	$.69332 \times 10^{-1}$
1.00	$.46732 \times 10^{-1}$	$-.68591 \times 10^{-1}$	$.82998 \times 10^{-1}$
2.00	$-.12675 \times 10^{-1}$	-.11177	.11248
3.00	$-.86988 \times 10^{-1}$	-.11404	.14343
4.00	-.14936	$-.75907 \times 10^{-1}$	.16755
5.00	-.17847	$-.13549 \times 10^{-1}$	.17898
6.00	-.16675	$.47558 \times 10^{-1}$	.17340
7.00	-.12323	$.82739 \times 10^{-1}$	.14843
8.00	$-.69724 \times 10^{-1}$	$.77859 \times 10^{-1}$	.10451
9.00	$-.32090 \times 10^{-1}$	$.34899 \times 10^{-1}$	$.47410 \times 10^{-1}$
10.00	$-.29782 \times 10^{-1}$	$-.28825 \times 10^{-1}$	$.41447 \times 10^{-1}$
11.00	$-.68000 \times 10^{-1}$	$-.87508 \times 10^{-1}$	.11082
12.00	-.13544	-.11705	.17901
13.00	-.20859	-.10459	.23334
14.00	-.26083	$-.53651 \times 10^{-1}$	.26629
15.00	-.27285	$.17230 \times 10^{-1}$	.27340
16.00	-.24019	$.81696 \times 10^{-1}$	.25371
17.00	-.17496	.11591	.20987
18.00	-.10114	.10785	.14786
19.00	$-.45194 \times 10^{-1}$	$.62030 \times 10^{-1}$	$.76748 \times 10^{-1}$
20.00	$-.25750 \times 10^{-1}$	$-.24010 \times 10^{-2}$	$.25862 \times 10^{-1}$

Table 5

Perturbative Acceleration of a Satellite Near L_2
($x = -0.5$ kilometers)

t (days)	a_x (kilometers/day^2)	a_y (kilometers/day^2)	a (kilometers/day^2)
.00	.27416	.00000	.27416
1.00	.25157	$-.68409 \times 10^{-1}$	.26071
2.00	.19220	-.11143	.22217
3.00	.11794	-.11362	.16377
4.00	$.55632 \times 10^{-1}$	$-.75468 \times 10^{-1}$	$.93756 \times 10^{-1}$
5.00	$.26583 \times 10^{-1}$	$-.13173 \times 10^{-1}$	$.29668 \times 10^{-1}$
6.00	$.38344 \times 10^{-1}$	$.47804 \times 10^{-1}$	$.61282 \times 10^{-1}$
7.00	$.81895 \times 10^{-1}$	$.82811 \times 10^{-1}$	.11647
8.00	.13539	$.77743 \times 10^{-1}$	.15613
9.00	.17300	$.34617 \times 10^{-1}$	.17643
10.00	.17526	$-.29223 \times 10^{-1}$	.17768
11.00	.13698	$-.87950 \times 10^{-1}$	.16279
12.00	$.69479 \times 10^{-1}$	-.11746	.13647
13.00	$-.37179 \times 10^{-2}$	-.10490	.10496
14.00	$-.55993 \times 10^{-1}$	$-.53792 \times 10^{-1}$	$.77645 \times 10^{-1}$
15.00	$-.68025 \times 10^{-1}$	$.17275 \times 10^{-1}$	$.70184 \times 10^{-1}$
16.00	$-.35343 \times 10^{-1}$	$.81919 \times 10^{-1}$	$.89218 \times 10^{-1}$
17.00	$.29926 \times 10^{-1}$	.11627	.12006
18.00	.10380	.10829	.15000
19.00	.15982	$.62461 \times 10^{-1}$	.17159
20.00	.17932	$-.20506 \times 10^{-2}$	.17933

Table 6

Perturbative Acceleration of a Satellite Near L_2
(x = 50 kilometers)

t (days)	a_x (kilometers/day^2)	a_y (kilometers/day^2)	a (kilometers/day^2)
.00	$-.10070 \times 10^2$	$-.00000$	$.10070 \times 10^2$
1.00	$-.10093 \times 10^2$	$-.77643 \times 10^{-1}$	$.10093 \times 10^2$
2.00	$-.10154 \times 10^2$	$-.12825$	$.10155 \times 10^2$
3.00	$-.10231 \times 10^2$	$-.13503$	$.10232 \times 10^2$
4.00	$-.10297 \times 10^2$	$-.97640 \times 10^{-1}$	$.10297 \times 10^2$
5.00	$-.10329 \times 10^2$	$-.32156 \times 10^{-1}$	$.10329 \times 10^2$
6.00	$-.10319 \times 10^2$	$.35398 \times 10^{-1}$	$.10319 \times 10^2$
7.00	$-.10277 \times 10^2$	$.79195 \times 10^{-1}$	$.10277 \times 10^2$
8.00	$-.10223 \times 10^2$	$.83562 \times 10^{-1}$	$.10223 \times 10^2$
9.00	$-.10184 \times 10^2$	$.48833 \times 10^{-1}$	$.10184 \times 10^2$
10.00	$-.10179 \times 10^2$	$-.91466 \times 10^{-2}$	$.10179 \times 10^2$
11.00	$-.10215 \times 10^2$	$-.65596 \times 10^{-1}$	$.10215 \times 10^2$
12.00	$-.10279 \times 10^2$	$-.96813 \times 10^{-1}$	$.10279 \times 10^2$
13.00	$-.10350 \times 10^2$	$-.89644 \times 10^{-1}$	$.10350 \times 10^2$
14.00	$-.10400 \times 10^2$	$-.46655 \times 10^{-1}$	$.10400 \times 10^2$
15.00	$-.10412 \times 10^2$	$.15024 \times 10^{-1}$	$.10412 \times 10^2$
16.00	$-.10380 \times 10^2$	$.70681 \times 10^{-1}$	$.10380 \times 10^2$
17.00	$-.10317 \times 10^2$	$.98049 \times 10^{-1}$	$.10317 \times 10^2$
18.00	$-.10246 \times 10^2$	$.86339 \times 10^{-1}$	$.10246 \times 10^2$
19.00	$-.10193 \times 10^2$	$.40697 \times 10^{-1}$	$.10193 \times 10^2$
20.00	$-.10177 \times 10^2$	$-.19746 \times 10^{-1}$	$.10177 \times 10^2$

Table 7

Perturbative Acceleration of a Satellite Near L_2
($x = -50$ kilometers)

t (days)	a_x (kilometers/day^2)	a_y (kilometers/day^2)	a (kilometers/day^2)
.00	$.10413 \times 10^2$	$.00000$	$.10413 \times 10^2$
1.00	$.10391 \times 10^2$	$-.59358 \times 10^{-1}$	$.10391 \times 10^2$
2.00	$.10334 \times 10^2$	$-.94946 \times 10^{-1}$	$.10334 \times 10^2$
3.00	$.10262 \times 10^2$	$-.92637 \times 10^{-1}$	$.10263 \times 10^2$
4.00	$.10203 \times 10^2$	$-.53735 \times 10^{-1}$	$.10203 \times 10^2$
5.00	$.10177 \times 10^2$	$.54331 \times 10^{-2}$	$.10177 \times 10^2$
6.00	$.10191 \times 10^2$	$.59964 \times 10^{-1}$	$.10191 \times 10^2$
7.00	$.10235 \times 10^2$	$.86355 \times 10^{-1}$	$.10236 \times 10^2$
8.00	$.10289 \times 10^2$	$.72040 \times 10^{-1}$	$.10289 \times 10^2$
9.00	$.10325 \times 10^2$	$.20683 \times 10^{-1}$	$.10325 \times 10^2$
10.00	$.10325 \times 10^2$	$-.48902 \times 10^{-1}$	$.10325 \times 10^2$
11.00	$.10284 \times 10^2$	$-.10986$	$.10284 \times 10^2$
12.00	$.10213 \times 10^2$	$-.13769$	$.10214 \times 10^2$
13.00	$.10137 \times 10^2$	$-.11985$	$.10138 \times 10^2$
14.00	$.10083 \times 10^2$	$-.60787 \times 10^{-1}$	$.10084 \times 10^2$
15.00	$.10071 \times 10^2$	$.19481 \times 10^{-1}$	$.10071 \times 10^2$
16.00	$.10105 \times 10^2$	$.92933 \times 10^{-1}$	$.10105 \times 10^2$
17.00	$.10172 \times 10^2$	$.13412$	$.10173 \times 10^2$
18.00	$.10249 \times 10^2$	$.12980$	$.10250 \times 10^2$
19.00	$.10308 \times 10^2$	$.83794 \times 10^{-1}$	$.10308 \times 10^2$
20.00	$.10330 \times 10^2$	$.15295 \times 10^{-1}$	$.10330 \times 10^2$

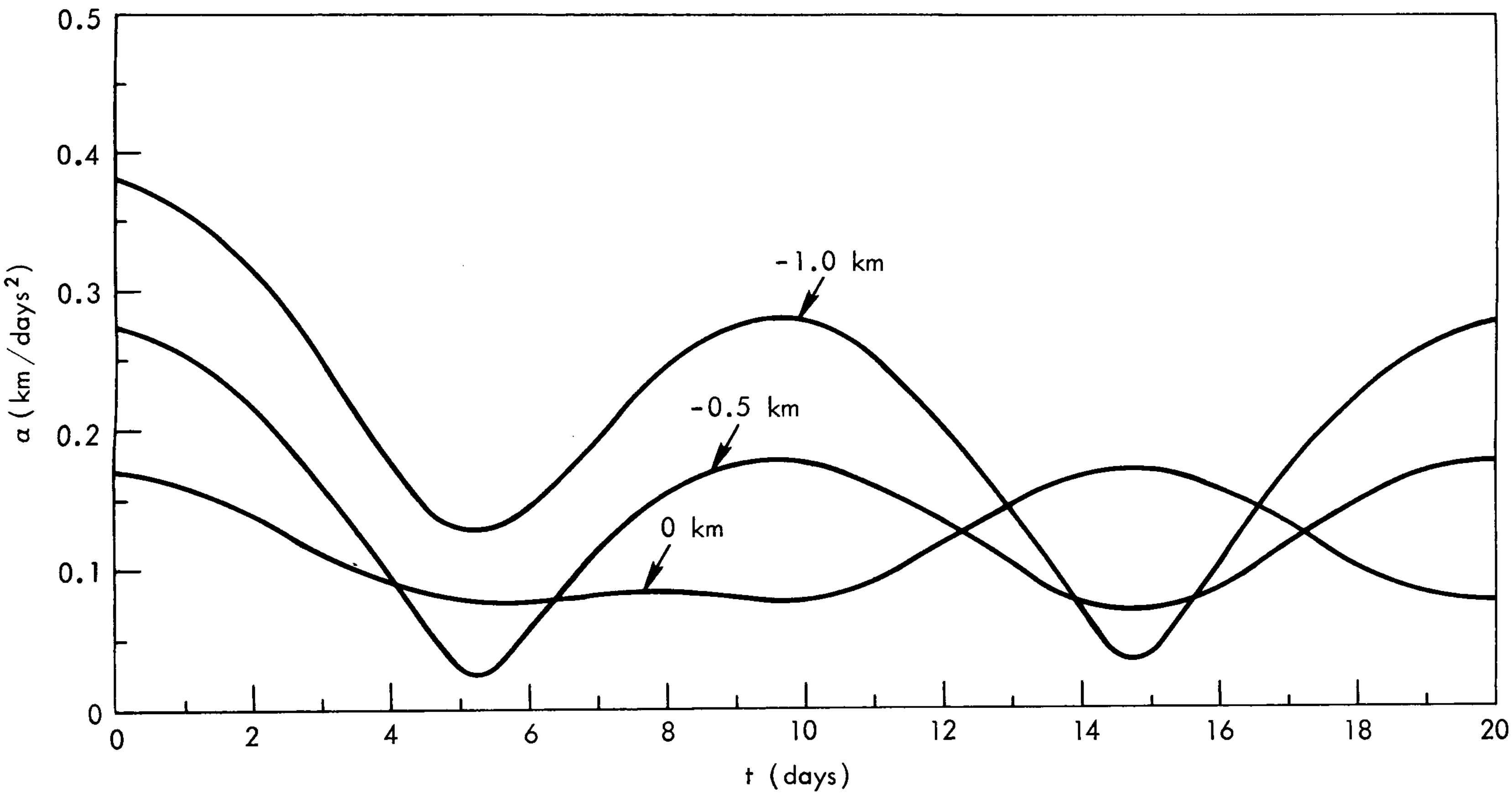

Figure 4. Perturbative Acceleration of Satellite $x \leq 0$

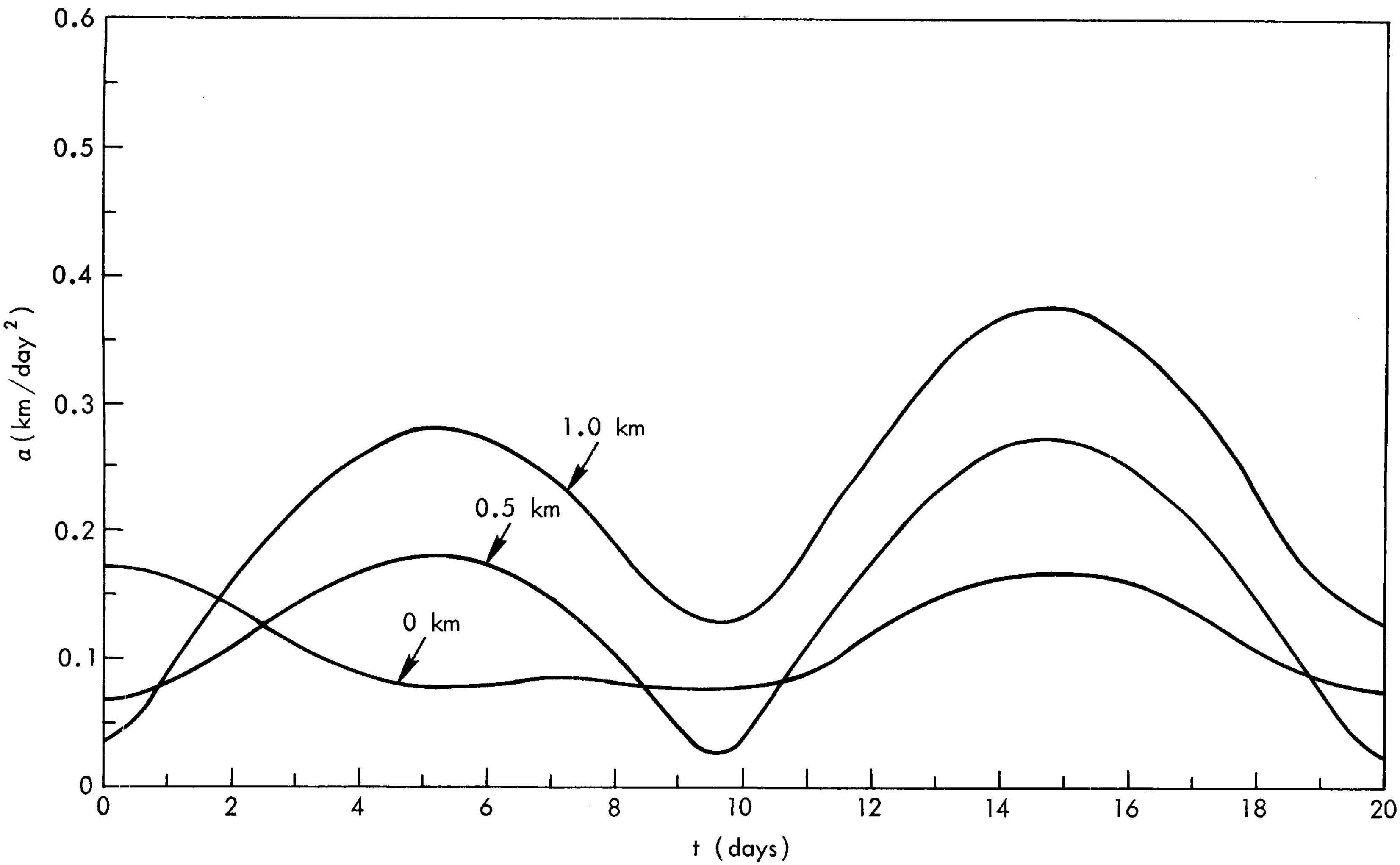

Figure 5. Perturbative Acceleration of Satellite $x \geq 0$

3. ANALYTICAL SOLUTION OF THE EQUATIONS OF MOTION

Since no closed form solution is known to exist for equations (35a) and (35b), we will develop approximate analytic solutions of first and second order. The constant K_3, which appears on the right hand side of eqations (35a) and (35b), is far smaller than K_1 and K_2 as can be seen from the numerical values of these constants given in this report. Therefore, it is logical to express both x and y in powers of K_3 and then solve the equations of motion by the method of regular perturbations.

For conformity, let $\epsilon = K_3$. Then:

$$x = x_1 + \epsilon x_2 + \epsilon^2 x_3 + \ldots \tag{36a}$$

$$y = y_1 + \epsilon y_2 + \epsilon^2 y_3 + \ldots \tag{36b}$$

where the subscripts denote the order of solution.

The initial conditions will be taken as

$$\begin{aligned}
x_1(0) &= x(0) \\
\dot{x}_1(0) &= \dot{x}(0) \\
y_1(0) &= y(0) \\
\dot{y}_1(0) &= \dot{y}(0) \\
x_2(0) &= x_3(0) = \ldots = 0 \\
\dot{x}_2(0) &= \dot{x}_3(0) = \ldots = 0 \\
y_2(0) &= y_3(0) = \ldots = 0 \\
\dot{y}_2(0) &= \dot{y}_3(0) = \ldots = 0
\end{aligned} \tag{37}$$

In this section we will derive the solutions corresponding to ϵ^0 and ϵ^1. There is little point in going to higher order terms since the equations of motion are linearlized, and are valid only in the vicinity of L_2.

Substituting the expansions (36a, b) in (35a, b) we obtain two sets of differential equations corresponding respectively to ϵ^0 and ϵ^1.

For ϵ^0:

$$\ddot{x}_1 - 2\omega\dot{y}_1 - \alpha x_1 = \frac{1}{4} K_2 \; [3 \cos\phi t + 5 \cos 3\phi t] \tag{38a}$$

$$\ddot{y}_1 + 2\omega\dot{x}_1 - \beta y_1 = \frac{1}{4} K_2 \; [\sin\phi t + 5 \sin 3\phi t] \tag{38b}$$

where $\quad \alpha = (1+2A)\omega^2 + K_1 = 0.184100$

and $\quad \beta = (1-A)\omega^2 \quad = -0.115282$

Similarly, for ϵ^1:

$$\ddot{x}_2 - 2\omega\dot{y}_2 - \alpha x_2 = \frac{1}{2} [1 + 3 \cos 2\phi t] x_1 \tag{39a}$$

$$\ddot{y}_2 + 2\omega\dot{x}_2 - \beta y_2 = \frac{3}{2} [\sin 2\phi t] x_1 \tag{39b}$$

The complete second-order solution will then be given by

$$x = x_1 + K_3 x_2 \tag{40a}$$

$$y = y_1 + K_3 y_2 \tag{40b}$$

3.1 FIRST ORDER SOLUTION (ϵ^0)

This section will deal strictly with the system (38a, b). We first solve the set of homogeneous equations, which gives rise to a fourth-order characteristic equation:

$$D^4 + D^2 \; (4\omega^2 - \alpha - \beta) + \alpha\beta = 0 \tag{41}$$

which has two equal and opposite real roots denoted by $\pm$ p , and two equal and opposite imaginary roots denoted by $\pm\, i\Omega$.

$$p = \frac{1}{\sqrt{2}} \left[-(2-A)\omega^2 + K_1 + \sqrt{\omega^4 \; (9A^2-8A) + \omega^2 K_1 \; (6A-8) + K_1{}^2} \; \right]^{1/2} \tag{42}$$

$$= 0.301427$$

$$\Omega = \frac{1}{\sqrt{2}} \left[(2-A)\omega^2 - K_1 + \sqrt{\omega^4 \; (9A^2-8A) + \omega^2 K_1 \; (6A-8) + K_1{}^2} \; \right]^{1/2} \tag{43}$$

$$= 0.483308$$

The homogeneous solution is, therefore:

$$x_{1_H}(t) = A_1 \sin\Omega t + A_2 \cos\Omega t + A_3 \sinh pt + A_4 \cosh pt \tag{44a}$$

$$y_{1_H}(t) = B_1 \sin\Omega t + B_2 \cos\Omega t + B_3 \sinh pt + B_4 \cosh pt \tag{44b}$$

The B's are related to the A's as follows:

$$\begin{aligned} B_1 &= -\gamma A_2 \\ B_2 &= \gamma A_1 \\ B_3 &= \delta A_4 \\ B_4 &= \delta A_3 \end{aligned} \tag{45}$$

where:

$$\gamma = \frac{1}{2\omega\Omega}\left[\Omega^2 + (1+2A)\omega^2 + K_1\right] = 1.878986 \tag{46}$$

and:

$$\delta = \frac{1}{2\omega p}\left[p^2 - (1+2A)\omega^2 + K_1\right] = -3.631650 \tag{47}$$

The particular solution is found to be:

$$x_{1_p}(t) = A_5 \cos\phi t + A_6 \cos 3\phi t \tag{48a}$$

$$y_{1_p}(t) = B_5 \sin\phi t + B_6 \sin 3\phi t \tag{48b}$$

where:

$$A_5 = \frac{K_2}{4} \; \frac{2\omega\phi - 3(\phi^2+\beta)}{(\alpha+\phi^2)(\beta+\phi^2) - 4\omega^2\phi^2} \tag{49a}$$

$$A_6 = \frac{5K_2}{4} \; \frac{6\omega\phi - (9\phi^2+\beta)}{(\alpha+9\phi^2)(\beta+9\phi^2) - 36\omega^2\phi^2} \tag{49b}$$

$$B_5 = -\frac{1}{2\omega\phi}\left[\frac{3}{4} K_2 + (\alpha+\phi^2)A_5\right] \tag{49c}$$

$$B_6 = -\frac{1}{6\omega\phi}\left[\frac{5}{4} K_2 + (\alpha+9\phi^2)A_6\right] \tag{49d}$$

The remaining four constants can be determined in terms of the four given initial conditions. Namely:

$$A_1 = \frac{py(0) - \delta\dot{x}(0)}{p\gamma - \Omega\delta} \tag{50a}$$

$$A_2 = \frac{\delta p\left[x(0) - A_5 - A_6\right] + \phi\left[B_5 + 3 B_6\right] - \dot{y}(0)}{\Omega\gamma + p\delta} \tag{50b}$$

$$A_3 = \frac{1}{p}\left[\dot{x}(0) - \Omega A_1\right] \tag{50c}$$

$$A_4 = x(0) - A_2 - A_5 - A_6 \tag{50d}$$

The general first order solution is given by:

$$x_1\ (t) = A_1\ \sin\Omega t + A_2\ \cos\Omega t + A_3\ \sinh\ pt + A_4\ \cosh\ pt + A_5\ \cos\phi t + A_6\ \cos 3\phi t \tag{51a}$$

$$y_1\ (t) = B_1\ \sin\Omega t + B_2\ \cos\Omega t + B_3\ \sinh\ pt + B_4\ \cosh\ pt + B_5\ \sin\phi t + B_6\ \sin 3\phi t \tag{51b}$$

where all the constants have been defined either explicitly or in terms of the initial conditions. It is evident that this solution is unbounded due to the presence of the hyperbolic functions. Even for small values of t the exponential terms are larger than the sinuosoidal terms. Consequently, this solution demonstrates little, if any, periodic behavior even in the initial phase of the trajectory. The source of the instability can be eliminated, however, by a proper choice of initial conditions. That is, we want the coefficients of the hyperbolic terms A_3, A_4, B_3, B_4, to vanish. As a matter of fact, we must only require that $A_3 = A_4 = 0$ since B_3 and B_4 are multiples of A_4 and A_3 respectively (see (45)). By letting $A_3 = 0$ and $A_4 = 0$ (in (50c) and (50d)) we determine new expressions for A_1 and A_2, namely:

$$A_1 = \frac{\dot{x}(0)}{\Omega} \tag{52a}$$

$$A_2 = x(0) - A_5 - A_6 \tag{52b}$$

Substituting the above in (50a) and (50b) we come up with the necessary relationship among the initial conditions:

$$\dot{x}(0) = \frac{\Omega}{\gamma}\ y(0) \tag{53a}$$

$$\dot{y}(0) = \phi(B_5+3B_6) + \gamma\Omega\ (A_5+A_6) - \gamma\Omega x(0) \tag{53b}$$

These two relations can be implemented in practice during the injection into an orbit around L_2. Thus we have a periodic first order solution, namely:

$$x_1\ (t) = A_1\ \sin\Omega t + A_2\ \cos\Omega t + A_5\ \cos\phi t + A_6\ \cos 3\phi t \tag{54a}$$

$$y_1(t) = B_1 \sin\Omega t + B_2 \cos\Omega t + B_5 \sin\phi t + B_6 \sin 3\phi t \tag{54b}$$

where:

$$A_5 = -.09389$$

$$A_6 = -.72532$$

$$B_5 = 0.43798$$

$$B_6 = -1.09595$$

and:

$$A_1 = y(0)/\gamma = 0.53220\ y(0)$$

$$A_2 = x(0) - A_5 - A_6 = 0.81922 + x(0)$$

$$B_1 = -\gamma A_2 = -1.53930 - 1.87896\ x(0)$$

$$B_2 = y(0)$$

Using the first order solution we can then compute, in addition to the trajectory, the velocity, acceleration, range to L_2, range rate, and flight path angle. Namely:

$$v(t) = \sqrt{\dot{x}_1^{\,2}(t) + \dot{y}_1^{\,2}(t)} \tag{55}$$

$$a(t) = \sqrt{\ddot{x}_1^{\,2}(t) + \ddot{y}_1^{\,2}(t)} \tag{56}$$

$$R(t) = \sqrt{x_1^{\,2}(t) + y_1^{\,2}(t)} \tag{57}$$

$$\dot{R}(t) = \frac{x_1(t)\ \dot{x}_1(t) + y_1(t)\ \dot{y}_1(t)}{R(t)} \tag{58}$$

$$\theta(t) = \tan^{-1}\left[\frac{\dot{y}_1(t)}{\dot{x}_1(t)}\right] \tag{59}$$

3.2 SECOND ORDER SOLUTION (ϵ^1)

Having found the first-order solution, we can proceed to determine the second-order solution using equations (39a, b). Substituting $x_1(t)$ from equation (54a) into equations (39a, b), we obtain the following system:

$$\ddot{x}_2 - 2\omega\dot{y}_2 - \alpha x_2 = \frac{1}{2}\left[1+3\cos 2\phi t\right]\left[A_1 \sin\Omega t + A_2 \cos\Omega t + A_5 \cos\phi t + A_6 \cos 3\phi t\right] \tag{60a}$$

$$\ddot{y}_2 + 2\omega\dot{x}_2 - \beta y_2 = \frac{3}{2}[\sin 2\phi t][A_1 \sin\Omega t + A_2 \cos\Omega t + A_5 \cos\phi t$$
$$+ A_6 \cos 3\phi t] \tag{60b}$$

Expanding the right hand sides of the above equations and separating all cross products of trigonometric functions, we obtain a simplified version which is easily solved.

$$\ddot{x}_2 - 2\omega\dot{y}_2 - \alpha x_2 = \frac{1}{2} A_1 \sin\Omega t + \frac{1}{2} A_2 \cos\Omega t + \frac{1}{4}[5 A_5 + 3 A_6]\cos\phi t$$
$$+ \frac{1}{4}[2 A_6 + 3 A_5]\cos 3\phi t + \frac{3}{4} A_6 \cos 5\phi t$$
$$+ \frac{3}{4} A_1 \sin(2\phi+\Omega)t - \frac{3}{4} A_1 \sin(2\phi-\Omega)t$$
$$+ \frac{3}{4} A_2 \cos(2\phi+\Omega)t + \frac{3}{4} A_2 \cos(2\phi-\Omega)t \tag{61a}$$

$$\ddot{y}_2 + 2\omega\dot{x}_2 - \beta y_2 = \frac{3}{4}[A_5 - A_6]\sin\phi t + \frac{3}{4} A_5 \sin 3\phi t + \frac{3}{4} A_6 \sin 5\phi t$$
$$+ \frac{3}{4} A_1 \cos(2\phi-\Omega)t - \frac{3}{4} A_1 \cos(2\phi+\Omega)t$$
$$+ \frac{3}{4} A_2 \sin(2\phi+\Omega)t + \frac{3}{4} A_2 \sin(2\phi-\Omega)t \tag{61b}$$

The homogeneous solution is unchanged from the first-order solution except for the coefficients. That is:

$$x_{2_H}(t) = A_1' \sin\Omega t + A_2' \cos\Omega t + A_3' \cosh pt + A_4' \sinh pt \tag{62a}$$

$$y_{2_H}(t) = B_1' \sin\Omega t + B_2' \cos\Omega t + B_3' \cosh pt + B_4' \sinh pt \tag{62b}$$

where:

$$\begin{aligned} B_1' &= -\gamma A_2' \\ B_2' &= \gamma A_1' \\ B_3' &= \delta A_4' \\ B_4' &= \delta A_3' \end{aligned} \tag{63}$$

The particular solution is found to be:

$$x_{2_p}(t) = C_1 \cos\phi t + C_2 \cos 3\phi t + C_3 \cos 5\phi t + C_4 \sin(2\phi+\Omega)t$$

$$+ C_5 \sin(2\phi-\Omega)t + C_6 \cos(2\phi+\Omega)t + C_7 \cos(2\phi-\Omega)t$$

$$+ C_8 \sin\Omega t + C_9 \cos\Omega t \tag{64a}$$

$$y_{2_p}(t) = D_1 \sin\phi t + D_2 \sin 3\phi t + D_3 \sin 5\phi t + D_4 \sin(2\phi+\Omega)t$$

$$+ D_5 \sin(2\phi-\Omega)t + D_6 \cos(2\phi+\Omega)t + D_7 \cos(2\phi-\Omega)t$$

$$+ D_8 \sin\Omega t + D_9 \cos\Omega t \tag{64b}$$

It should be noted that the sinuosoidal part of the homogeneous solution appears in the particular solution without causing resonance. This is due to the particular way in which the differential equations are coupled. The constants are defined below.

$$C_1 = \frac{3/2\ \omega\phi\ (A_5 - A_6) - 1/4\ (\phi^2+\beta)(5A_5 + 3A_6)}{(\phi^2+\alpha)(\phi^2+\beta) - 4\omega^2\ \phi^2} \tag{65a}$$

$$C_2 = \frac{9/2\ \omega\phi\ A_5 - 1/4\ (9\phi^2+\beta)(3A_5 + 2A_6)}{(9\phi^2+\alpha)(9\phi^2+\beta) - 36\omega^2\ \phi^2} \tag{65b}$$

$$C_3 = \frac{3/4\ A_6\ (10\omega\phi - 25\phi^2 - \beta)}{(25\phi^2+\alpha)(25\phi^2+\beta) - 100\omega^2\phi^2} \tag{65c}$$

$$C_4 = -\frac{3/4\ A_1[(2\phi+\Omega)^2 - 2\omega\ (2\phi+\Omega) + \beta]}{[(2\phi+\Omega)^2+\alpha][(2\phi+\Omega)^2+\beta] - 4\omega^2(2\phi+\Omega)^2} \tag{65d}$$

$$C_5 = \frac{3/4\ A_1[(2\phi-\Omega)^2 - 2\omega\ (2\phi-\Omega) + \beta]}{[(2\phi+\Omega)^2+\alpha][(2\phi-\Omega)^2+\beta] - 4\omega^2(2\phi-\Omega)^2} \tag{65e}$$

$$C_6 = \frac{3/4\ A_2[(2\phi+\Omega)^2 - 2\omega\ (2\phi+\Omega) + \beta]}{4\omega^2(2\phi+\Omega)^2 - [(2\phi+\Omega)^2+\alpha][(2\phi+\Omega)^2+\beta]} \tag{65f}$$

$$C_7 = \frac{3/4\ A_2[(2\phi-\Omega)^2 - 2\omega\ (2\phi-\Omega) + \beta]}{4\omega^2(2\phi-\Omega)^2 - [(2\phi-\Omega)^2+\alpha][(2\phi-\Omega)^2+\beta]} \tag{65g}$$

$$C_8 = -\frac{1}{2}\ \frac{(\beta+\Omega^2)\ A_1}{[(\alpha+\Omega^2)(\beta+\Omega^2) - 4\omega^2\Omega^2]} \tag{65h}$$

$$C_9 = -\frac{1}{2}\frac{(\beta+\Omega^2)A_2}{[(\alpha+\Omega^2)(\beta+\Omega^2) + 4\omega^2\Omega^2]} \tag{65i}$$

$$D_1 = -\frac{2\omega\phi C_1 + 3/4\ (A_5 - A_6)}{(\phi^2 + \beta)} \tag{66a}$$

$$D_2 = -\frac{6\omega\phi C_2 + 3/4\ A_5}{(9\phi^2 + \beta)} \tag{66b}$$

$$D_3 = -\frac{10\omega\phi C_3 + 3/4\ A_6}{25\phi^2 + \beta} \tag{66c}$$

$$D_4 = -\frac{2\omega(2\phi+\Omega)\ C_6 + 3/4\ A_2}{(2\phi+\Omega)^2 + \beta} \tag{66d}$$

$$D_5 = -\frac{2\omega(2\phi-\Omega)\ C_7 + 3/4\ A_2}{(2\phi-\Omega)^2 + \beta} \tag{66e}$$

$$D_6 = \frac{2\omega\ (2\phi+\Omega)\ C_4 + 3/4\ A_1}{(2\phi+\Omega)^2 + \beta} \tag{66f}$$

$$D_7 = \frac{2\omega\ (2\phi-\Omega)\ C_5 - 3/4\ A_1}{(2\phi-\Omega)^2 + \beta} \tag{66g}$$

$$D_8 = -\frac{1/2\ A_2 + (\alpha+\Omega^2)\ C_9}{2\omega\Omega} \tag{66h}$$

$$D_9 = \frac{1/2\ A_1 + (\alpha+\Omega^2)\ C_8}{2\omega\Omega} \tag{66i}$$

$$A_1' = \frac{1}{\gamma p - \Omega\delta}\left\{\delta[C_4\ (2\phi+\Omega) + C_5\ (2\phi-\Omega) + C_8\Omega] - p\ [D_6 + D_7 + D_9]\right\} \tag{67a}$$

$$A_2' = \frac{1}{\gamma\Omega + p\delta}\left\{D_1\phi + 3\ D_2\phi + 5\ D_3\phi + (2\phi+\Omega)\ D_4 + (2\phi-\Omega)\ D_5 + D_8\Omega \right.$$
$$\left. - \delta p\ [C_1 + C_2 + C_3 + C_6 + C_7 + C_9]\right\} \tag{67b}$$

$$A_3' = -[C_1 + C_2 + C_3 + C_6 + C_7 + C_9] - A_2' \tag{67c}$$

$$A_4' = -\frac{1}{\delta}[D_6 + D_7 + D_9] - \frac{\gamma}{\delta}A_1' \tag{67d}$$

The second order solution is thus given by:

$$x_2(t) = (A_1' + C_8)\sin\Omega t + (A_2' + C_9)\cos\Omega t + A_3'\cosh pt + A_4'\sinh pt$$
$$+ C_1\cos\phi t + C_2\cos 3\phi t + C_3\cos 5\phi t + C_4\sin(2\phi+\Omega)t$$
$$+ C_5\sin(2\phi-\Omega)t + C_6\cos(2\phi+\Omega)t + C_7\cos(2\phi-\Omega)t \tag{68a}$$

$$y_2(t) = (B_1' + D_8)\sin\Omega t + (B_2' + D_9)\cos\Omega t + B_3'\cosh pt + B_4'\sinh pt$$

$$+ D_1 \sin\phi t + D_2 \sin 3\phi t + D_3 \sin 5\phi t + D_4 \sin(2\phi+\Omega)t$$

$$+ D_5 \sin(2\phi-\Omega)t + D_6 \cos(2\phi+\Omega)t + D_7 \cos(2\phi-\Omega)t \qquad (68b)$$

In the case of this solution the hyperbolic terms are left intact to demonstrate the increasing instability of the solution.

3.3 COMPLETE SECOND ORDER SOLUTION

The complete second order solution is given by:

$$x(t) = x_1(t) + K_3 x_2(t) \qquad (69a)$$

$$y(t) = y_1(t) + K_3 y_2(t) \qquad (69b)$$

where $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$ are given by (54a), (54b), (68a) and (68b), respectively.

Thus the complete solution becomes:

$$x(t) = [A_1 + K_3 (A_1'+C_8)]\sin\Omega t + [A_2 + K_3 (A_2'+C_9)]\cos\Omega t$$

$$+ K_3 A_3' \cosh pt + K_3 A_4' \sinh pt + [A_5+K_3 C_1]\cos\phi t + [A_6+K_3 C_2]\cos 3\phi t$$

$$+ K_3 C_3 \cos 5\phi t + K_3 C_4 \sin(2\phi+\Omega)t + K_3 C_5 \sin(2\phi-\Omega)t$$

$$+ K_3 C_6 \cos(2\phi+\Omega)t + K_3 C_7 \cos(2\phi-\Omega)t \qquad (70a)$$

$$y(t) = [B_1 + K_3 (B_1'+D_8)]\sin\Omega t + [B_2+K_3 (B_2'+D_9)]\cos\Omega t$$

$$+ K_3 B_3' \cosh pt + K_3 B_4' \sinh pt + [B_5+K_3 D_1]\sin\phi t + [B_6+K_3 D_2]\sin 3\phi t$$

$$+ K_3 D_3 \sin 5\phi t + K_3 D_4 \sin(2\phi+\Omega)t + K_3 D_5 \sin(2\phi-\Omega)t$$

$$+ K_3 D_6 \cos(2\phi+\Omega)t + K_3 D_7 \cos(2\phi-\Omega)t \qquad (70b)$$

The numerical values of the coefficients will be given here for a typical trajectory originating at L_2.

Since the initial conditions chosen are periodic only as far as the first order solution is concerned, we will rename them quasi-periodic for the complete solution. Thus, the quasi-periodic initial conditions for a start at L_2 are:

$x(0) = 0$

$y(0) = 0$

$\dot{x}(0) = 0$

$\dot{y}(0) = -.13750$

Then:

$A_1 = 0$	$B_2' = 0$	$D_2 = 1.9471$
$A_2 = 0.81922$	$B_3' = 0$	$D_3 = 0.89387$
$A_5 = -0.09389$	$B_4' = 18.2830$	$D_4 = 4.5579$
$A_6 = -0.72532$	$C_1 = 3.6139$	$D_5 = -1.6151$
$B_1 = -1.53929$	$C_2 = 1.6983$	$D_6 = 0$
$B_2 = 0$	$C_3 = 0.74566$	$D_7 = 0$
$B_5 = 0.43798$	$C_4 = 0$	$D_8 = -0.92132$
$B_6 = -1.09595$	$C_5 = 0$	$D_9 = 0$
$A_1' = 0$	$C_6 = -3.9235$	$K_3 = 0.29525\ 10^{-3}$
$A_2' = 4.6669$	$C_7 = -1.2766$	$\Omega = 0.4833081$
$A_3' = -5.0343$	$C_8 = 0$	$\phi = -0.2128$
$A_4' = 0$	$C_9 = -0.49033$	
$B_1' = -8.7691$	$D_1 = 1.7124$	

3.4 TABULATIONS AND GRAPHS OF SAMPLE TRAJECTORIES

The results of our computations will be tabulated and plotted in the following pages. First, the first order solution will be presented with periodic initial conditions for a few different cases. Then the complete second order solution, with quasi-periodic initial conditions, will be given. The instability of the motion will be demonstrated in this solution. However, it may be noted, the divergence of the motion commences only after 23 days for a motion initiating at L_2, with quasi-periodic initial conditions. Thus the rate of growth of the divergence from the periodic motion is rather slow and probably could be corrected with the aid of an onboard engine.

Table 8

First Order Solution with Periodic Initial Conditions

$x(0) = 1.609$ $y(0) = 1.609$ $\dot{x}(0) = .41395$ $\dot{y}(0) = -1.59899$

$\phi = -0.2128000$ $\Omega = 0.4833081$ $\gamma = 0.1878986 \times 10^1$

$A_1 = .85650$ $A_4 = 0$ $B_1 = -4.56323$ $B_4 = 0$

$A_2 = 2.42856$ $A_5 = -.09389$ $B_2 = 1.60934$ $B_5 = .43798$

$A_3 = 0$ $A_6 = -.72532$ $B_3 = 0$ $B_6 = -1.09595$

t (days)	x (kilo-meters)	y (kilo-meters)	v (kilome-ters/day)	a (kilome-ters/day^2)	R (kilo-meters)
.00	1.60934	1.60934	1.65171	.46133	2.27596
1.00	1.87418	- .13498	1.84632	.36746	1.87903
2.00	1.78880	-1.97305	1.79699	.46649	2.66322
3.00	1.31257	-3.56529	1.50523	.70506	3.79923
4.00	.48143	-4.55886	1.13681	.92590	4.58421
5.00	- .57085	-4.66891	1.16181	1.02809	4.70367
6.00	-1.62015	-3.77076	1.68654	.96341	4.10409
7.00	-2.39561	-1.97113	2.22052	.77185	3.10231
8.00	-2.65254	.37504	2.44877	.67048	2.67892
9.00	-2.25358	2.72216	2.26370	.88874	3.53395
10.00	-1.22986	4.46237	1.78951	1.20773	4.62874
11.00	.20367	5.09379	1.52778	1.37872	5.09786
12.00	1.69091	4.37881	1.94940	1.30519	4.69394
13.00	2.82813	2.43878	2.56134	1.04107	3.73443
14.00	3.28208	- .25247	2.84093	.86181	3.29178
15.00	2.89559	-2.98347	2.61752	1.07069	4.15759
16.00	1.74486	-4.99976	2.04372	1.40878	5.29549
17.00	.12692	-5.72150	1.71217	1.56330	5.72291
18.00	-1.52175	-4.91686	2.15506	1.43435	5.14697
19.00	-2.75163	-2.77578	2.79379	1.10449	3.90851
20.00	-3.23215	.14078	3.05435	.88641	3.23521
21.00	-2.84751	3.06123	2.77230	1.08244	4.18085
22.00	-1.72663	5.22411	2.10337	1.39119	5.50206
23.00	- .19865	6.09095	1.59611	1.50476	6.09419
24.00	1.30961	5.48736	1.87657	1.34779	5.64148
25.00	2.40050	3.63307	2.44919	1.00902	4.35449

Table 8 (Continued)

First Order Solution with Periodic Initial Conditions

t (days)	x (kilometers)	y (kilometers)	v (kilometers/day)	a (kilometers/day^2	R (kilometers)
26.00	2.81586	1.05772	2.71378	.75127	3.00796
27.00	2.49833	-1.56455	2.50981	.85012	2.94780
28.00	1.58865	-3.60669	1.94385	1.07436	3.94107
29.00	.36583	-4.64747	1.35445	1.15777	4.66185
30.00	- .84509	-4.55507	1.27090	1.04307	4.63280
31.00	-1.76548	-3.48113	1.62055	.78887	3.90323
32.00	-2.22744	-1.78057	1.87994	.53643	2.85166
33.00	-2.19510	.10675	1.85536	.48331	2.19769
34.00	-1.74473	1.77120	1.56718	.60175	2.48620
35.00	-1.01941	2.91338	1.14939	.69955	3.08658
36.00	- .17958	3.37659	.85377	.71080	3.38136
37.00	.63260	3.13837	.94836	.64069	3.20150
38.00	1.30724	2.28100	1.25961	.51625	2.62904
39.00	1.76317	.96119	1.51959	.37953	2.00814
40.00	1.93890	- .60945	1.61406	.32603	2.03242
41.00	1.79100	-2.17511	1.50322	.44798	2.81758
42.00	1.30638	-3.45034	1.21811	.65258	3.68938
43.00	.52474	-4.15247	.95885	.83016	4.18550
44.00	- .44177	-4.05767	1.13803	.90773	4.08164
45.00	-1.40736	-3.07149	1.67358	.84610	3.37856
46.00	-2.14094	-1.29040	2.16070	.67840	2.49975
47.00	-2.42302	.97530	2.35652	.60089	2.61194
48.00	-2.11655	3.23987	2.16558	.82220	3.86996
49.00	-1.22748	4.94558	1.69013	1.14082	5.09563
50.00	.07055	5.61048	1.40668	1.32909	5.61092

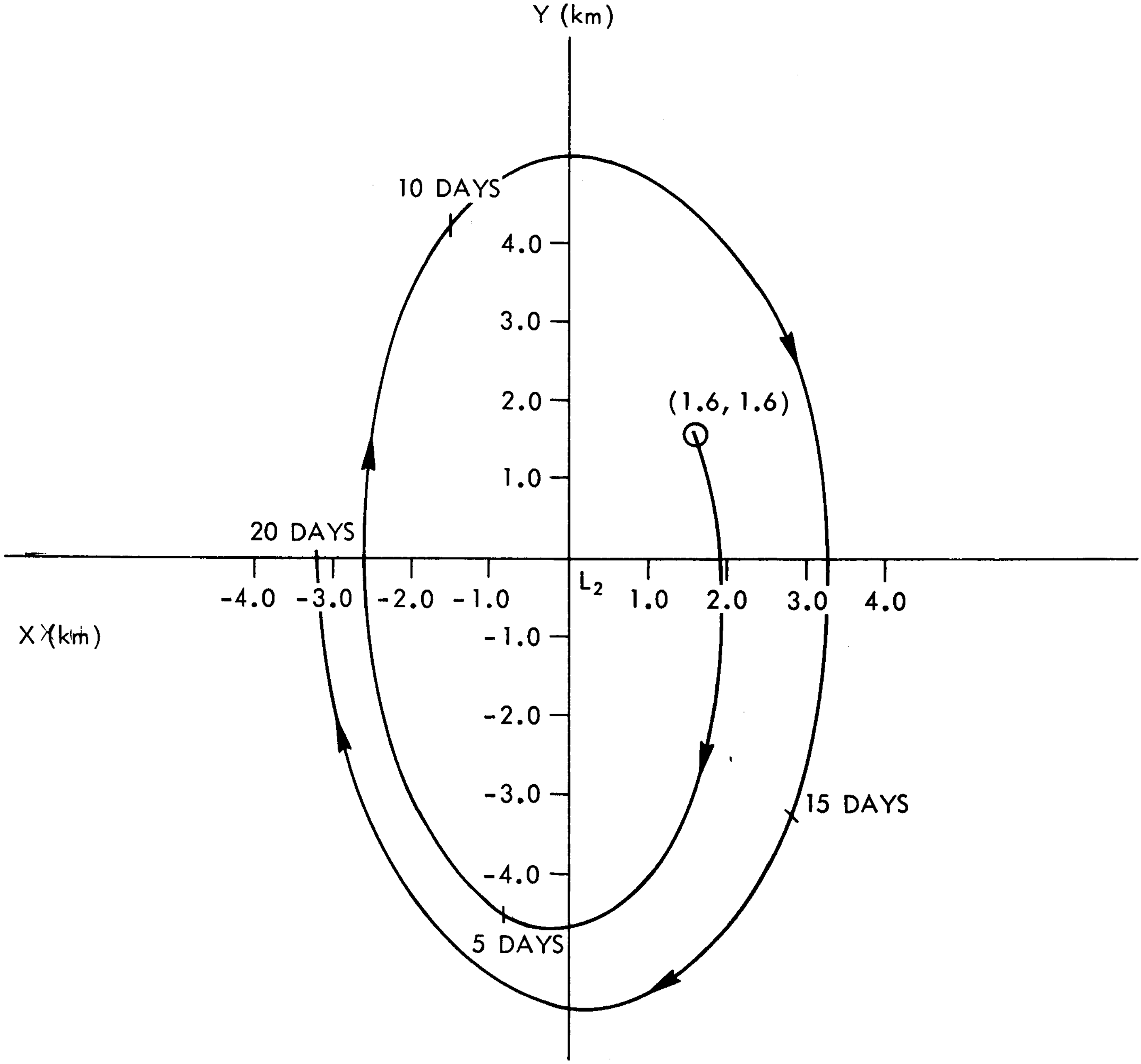

Figure 6. Trajectory of Satellite Around L_2
First Order Solution

Periodic Initial Conditions
$x = 1.61$ km
$y = 1.61$ km
$\dot{x} = 0.414$ km/day
$\dot{y} = -1.60$ km/day

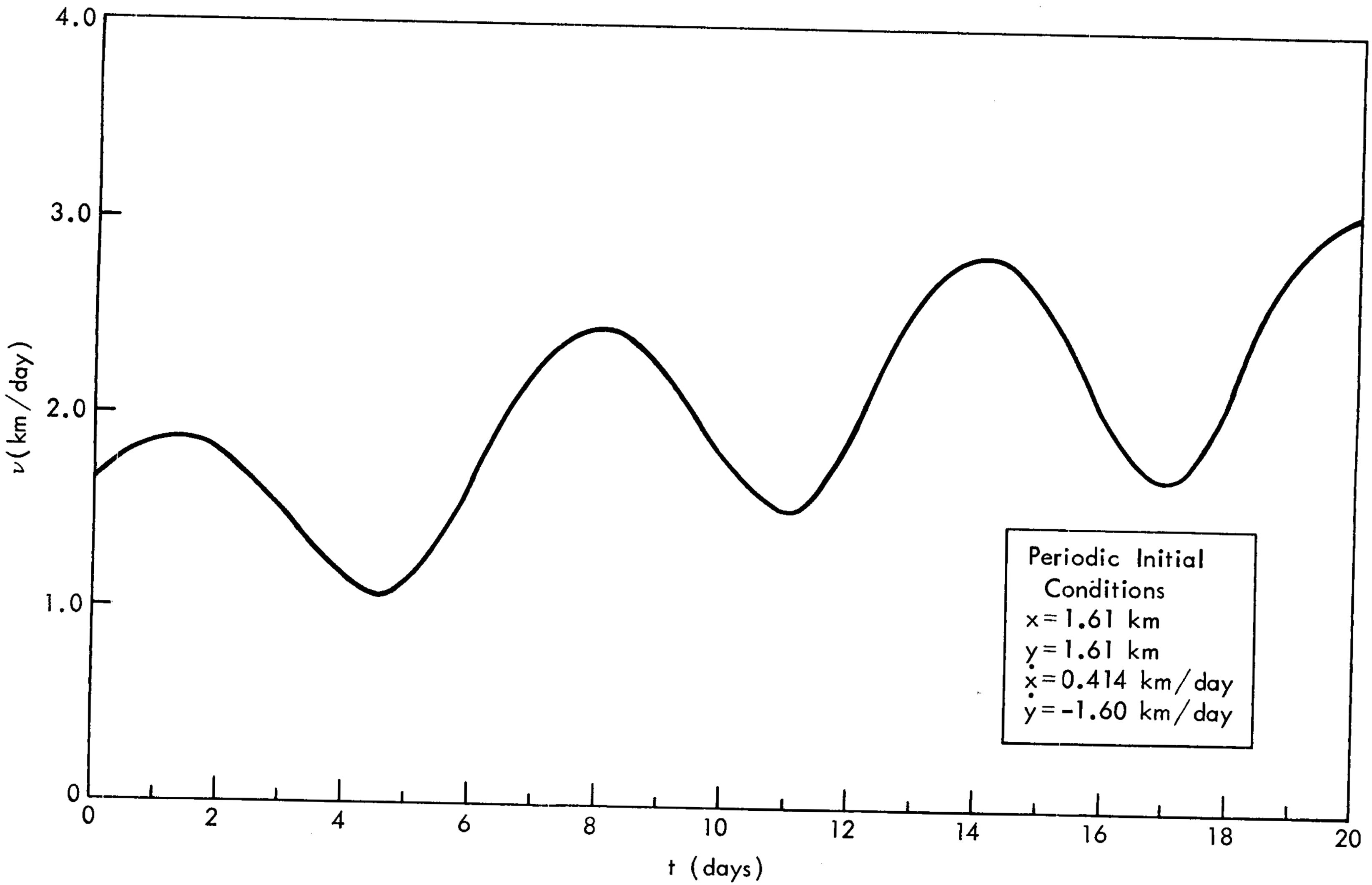

Figure 7. Velocity of Satellite Versus Time
First Order Solution

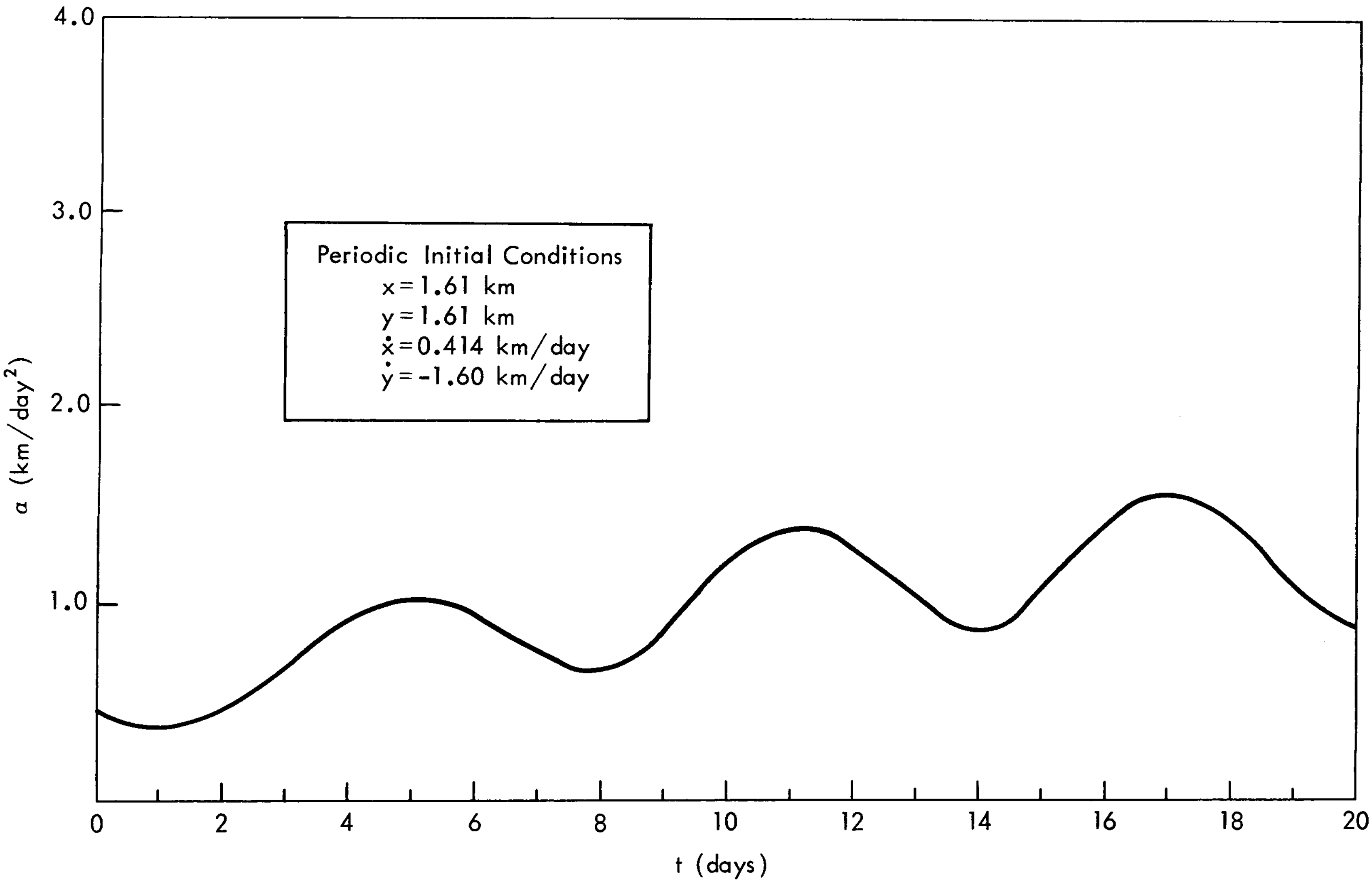

Figure 8. Acceleration of Satellite Versus Time
First Order Solution

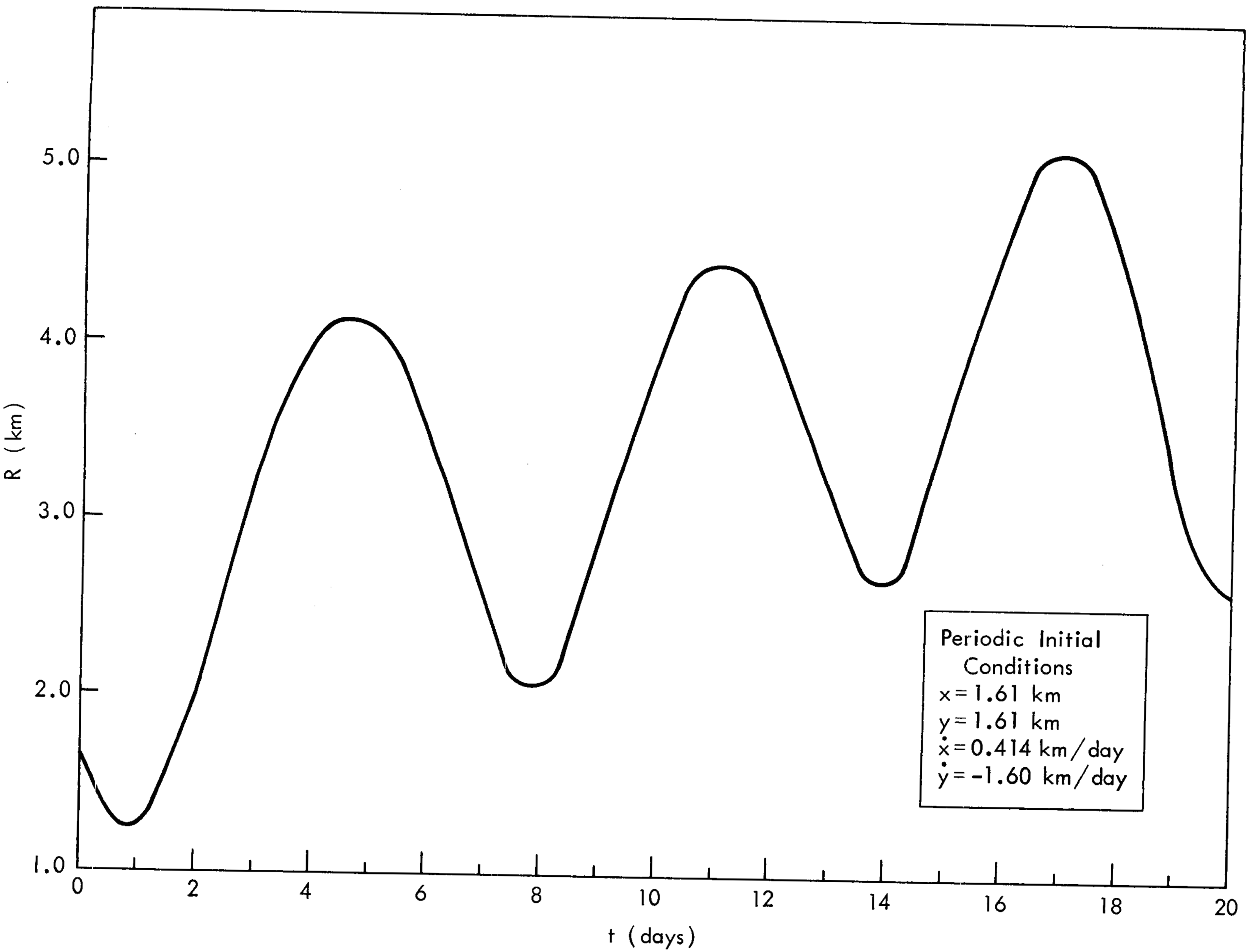

Figure 9. Range of Satellite from L_2 Versus Time First Order Solution

Table 9

First Order Solution with Periodic Initial Conditions

$x(0) = 0.805$ $\qquad y(0) = 0.805$ $\qquad \dot{x}(0) = 0.20698$ $\qquad \dot{y}(0) = -0.86825$

$\phi = -0.2128000$ $\qquad \Omega = 0.4833081$ $\qquad \gamma = 0.1878986 \times 10^{1}$

$A_1 = 0.42825$ $\qquad A_4 = 0$ $\qquad B_1 = -3.05126$ $\qquad B_4 = 0$

$A_2 = 1.62389$ $\qquad A_5 = -0.09389$ $\qquad B_2 = 0.80467$ $\qquad B_5 = 0.43798$

$A_3 = 0$ $\qquad A_6 = -0.72532$ $\qquad B_3 = 0$ $\qquad B_6 = -1.09595$

t (days)	x (kilometers)	y (kilometers)	v (kilometers/day)	a (kilometers/day^2)	R (kilometers)	$\frac{dR}{dt}$ (kilometers/day)
.00	.80467	.80467	.89258	.20407	1.13798	- .46759
1.00	.96266	- .14486	1.02055	.17110	.97350	.24875
2.00	.97925	-1.18587	1.04280	.21688	1.53792	.75294
3.00	.79042	-2.16138	.92705	.36689	2.30138	.71650
4.00	.36630	-2.85982	.72195	.53956	2.88318	.41242
5.00	- .25259	-3.06394	.68883	.65756	3.07433	- .04553
6.00	- .94140	-2.62753	1.01798	.66642	2.79108	- .50238
7.00	-1.51186	-1.55152	1.41941	.56540	2.16632	- .65693
8.00	-1.76623	- .02509	1.64368	.47411	1.76640	.01883
9.00	-1.56774	1.59395	1.58217	.59588	2.23573	.77308
10.00	- .90161	2.86452	1.28945	.83330	3.00306	.64942
11.00	.09914	3.39233	1.09173	.98352	3.39377	.09629
12.00	1.17755	2.96350	1.36418	.95475	3.18888	- .47418
13.00	2.02353	1.63383	1.80370	.77567	2.60079	- .56918
14.00	2.37056	- .26266	2.01323	.64910	2.38507	.26012
15.00	2.08596	-2.19657	1.85589	.80704	3.02922	.86129
16.00	1.22258	-3.59603	1.45555	1.05895	3.79817	.56508
17.00	.01163	-4.02250	1.28265	1.16612	4.02251	- .14592
18.00	-1.20364	-3.31179	1.66236	1.05517	3.52373	- .80091
19.00	-2.07299	-1.63232	2.11732	.80552	2.63851	- .78428
20.00	-2.34843	.56069	2.25231	.68660	2.41444	.46309
21.00	-1.96116	2.66140	1.97603	.87526	3.30594	1.08067
22.00	-1.04068	4.09613	1.45735	1.09440	4.22627	.65375
23.00	.12975	4.49321	1.19248	1.13571	4.49508	- .13727
24.00	1.20525	3.78587	1.51274	.96885	3.97309	- .86555
25.00	1.88728	2.21759	1.90159	.70049	2.91196	-1.14318

Table 9 (Continued)

First Order Solution with Periodic Initial Conditions

t (days)	x (kilometers)	y (kilometers)	v (kilometers/day)	a (kilometers/day^2)	R (kilometers)	$\frac{dR}{dt}$ (kilometers/day)
26.00	2.01134	.25251	1.98983	.58065	2.02713	- .41201
27.00	1.58682	-1.57505	1.72486	.71676	2.23579	.66017
28.00	.77894	-2.82007	1.25316	.85243	2.92567	.57840
29.00	- .15659	-3.24391	.90894	.84123	3.24769	.04030
30.00	- .96054	-2.85610	.99190	.68524	3.01330	- .48283
31.00	-1.44752	-1.87595	1.20561	.47271	2.36949	- .74508
32.00	-1.54891	- .63688	1.25477	.35536	1.67474	- .55376
33.00	-1.31143	.52696	1.09837	.39574	1.41334	.04285
34.00	- .85834	1.37167	.81692	.44668	1.61809	.26816
35.00	- .33336	1.78564	.55359	.42962	1.81649	.09536
36.00	.14898	1.77896	.47656	.35362	1.78519	- .15492
37.00	.52840	1.43684	.57262	.25544	1.53092	- .33390
38.00	.79415	.86535	.68754	.16923	1.17452	- .33781
39.00	.95873	.15570	.76306	.11477	.97129	- .00483
40.00	1.02738	- .62025	.78635	.11321	1.20009	.42044
41.00	.98121	-1.38875	.74290	.18568	1.70041	.53279
42.00	.78383	-2.04696	.62679	.30879	2.19191	.41996
43.00	.40912	-2.45355	.51531	.44260	2.48742	.14846
44.00	- .12397	-2.45238	.64036	.53834	2.45551	- .22037
45.00	- .72894	-1.92756	.99838	.55127	2.06079	- .53923
46.00	-1.25731	- .86989	1.36005	.47432	1.52890	- .37873
47.00	-1.53659	.57607	1.55494	.40598	1.64103	.64110
48.00	-1.43040	2.11235	1.48936	.53099	2.55109	1.02299
49.00	- .89877	3.34806	1.19740	.76856	3.46660	.73227
50.00	- .03349	3.90891	.97707	.93615	3.90906	.12011

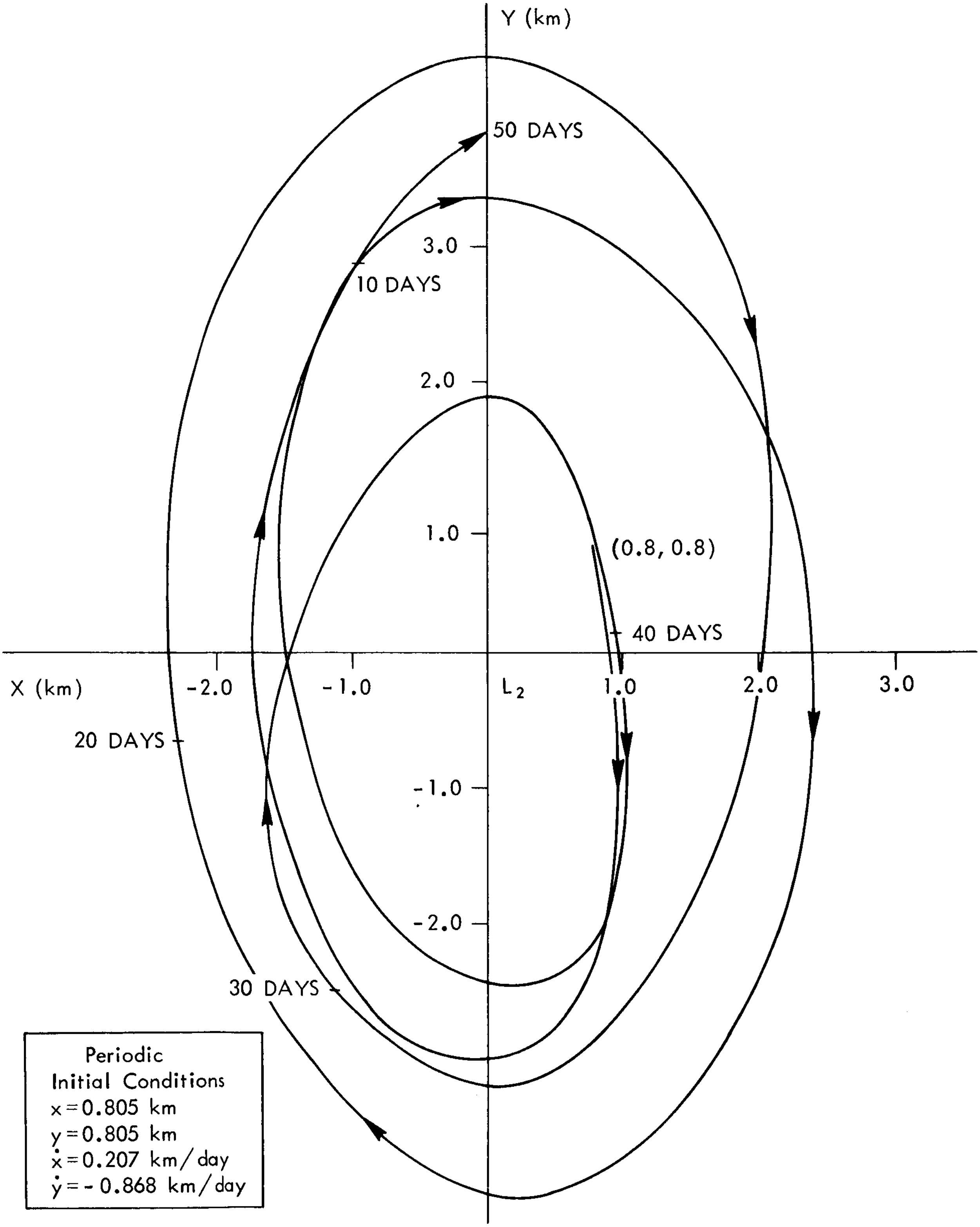

Figure 10. Trajectory of Satellite Around L_2
First Order Solution

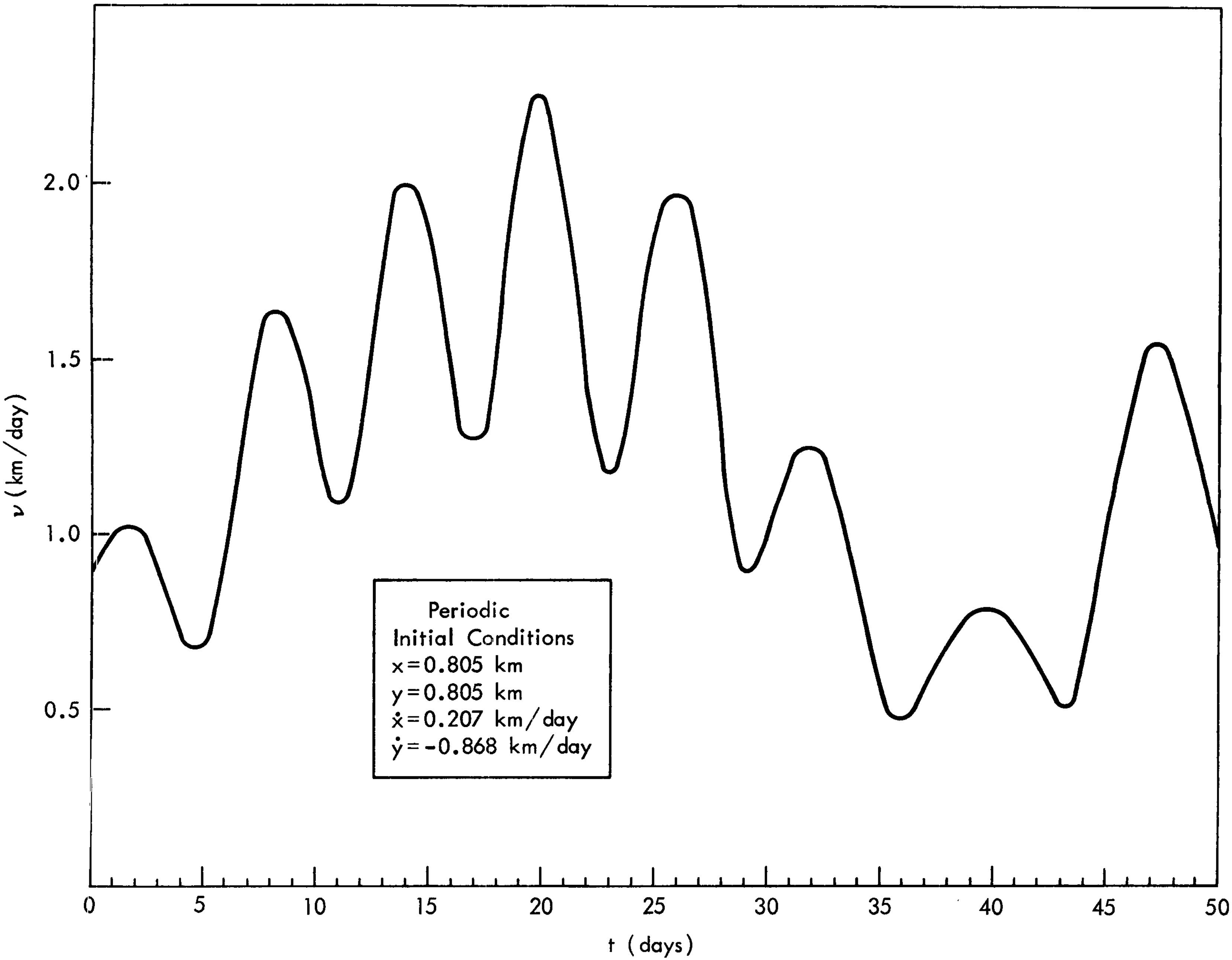

Figure 11. Velocity of Satellite Versus Time
First Order Solution

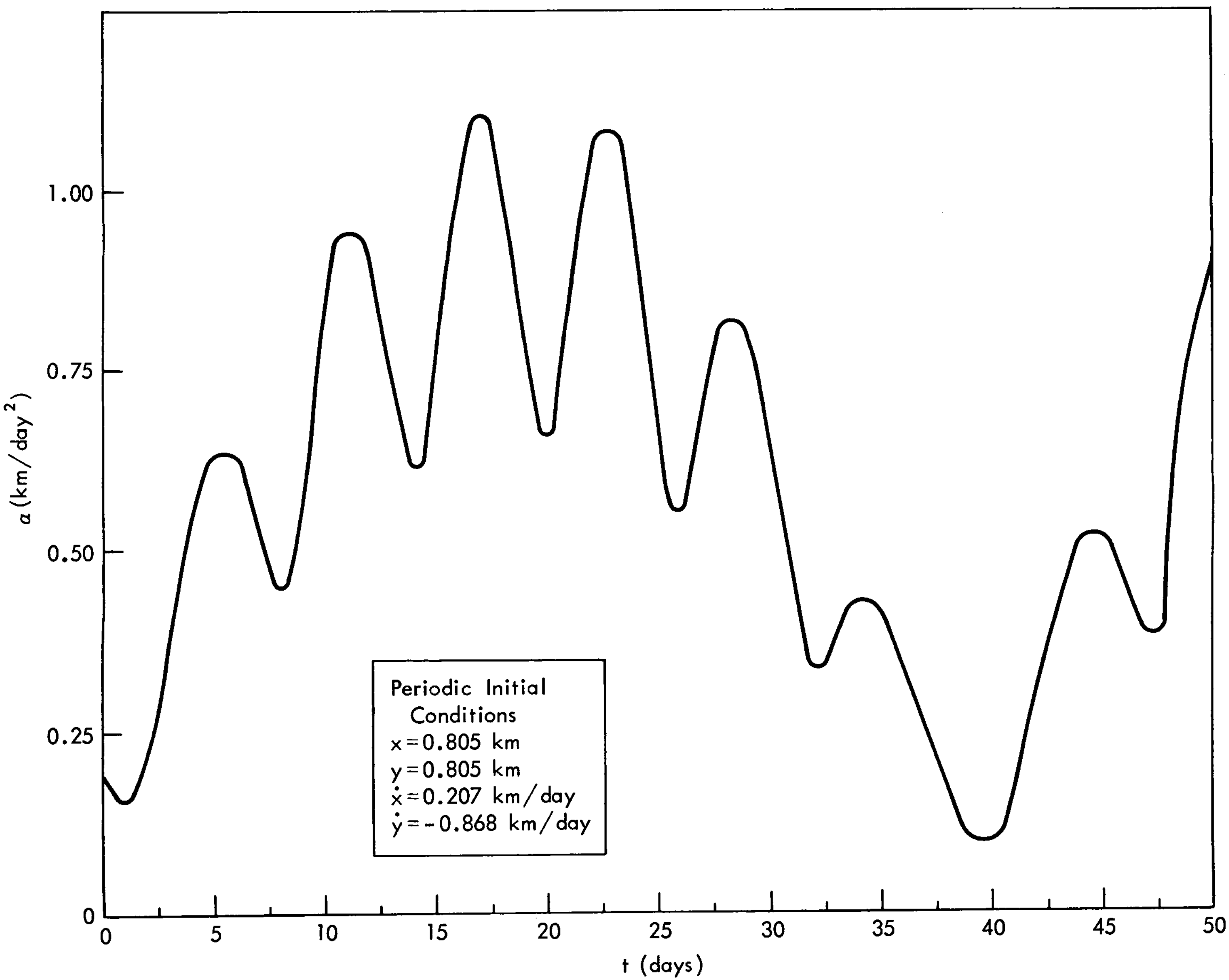

Figure 12. Acceleration of Satellite Versus Time
First Order Solution

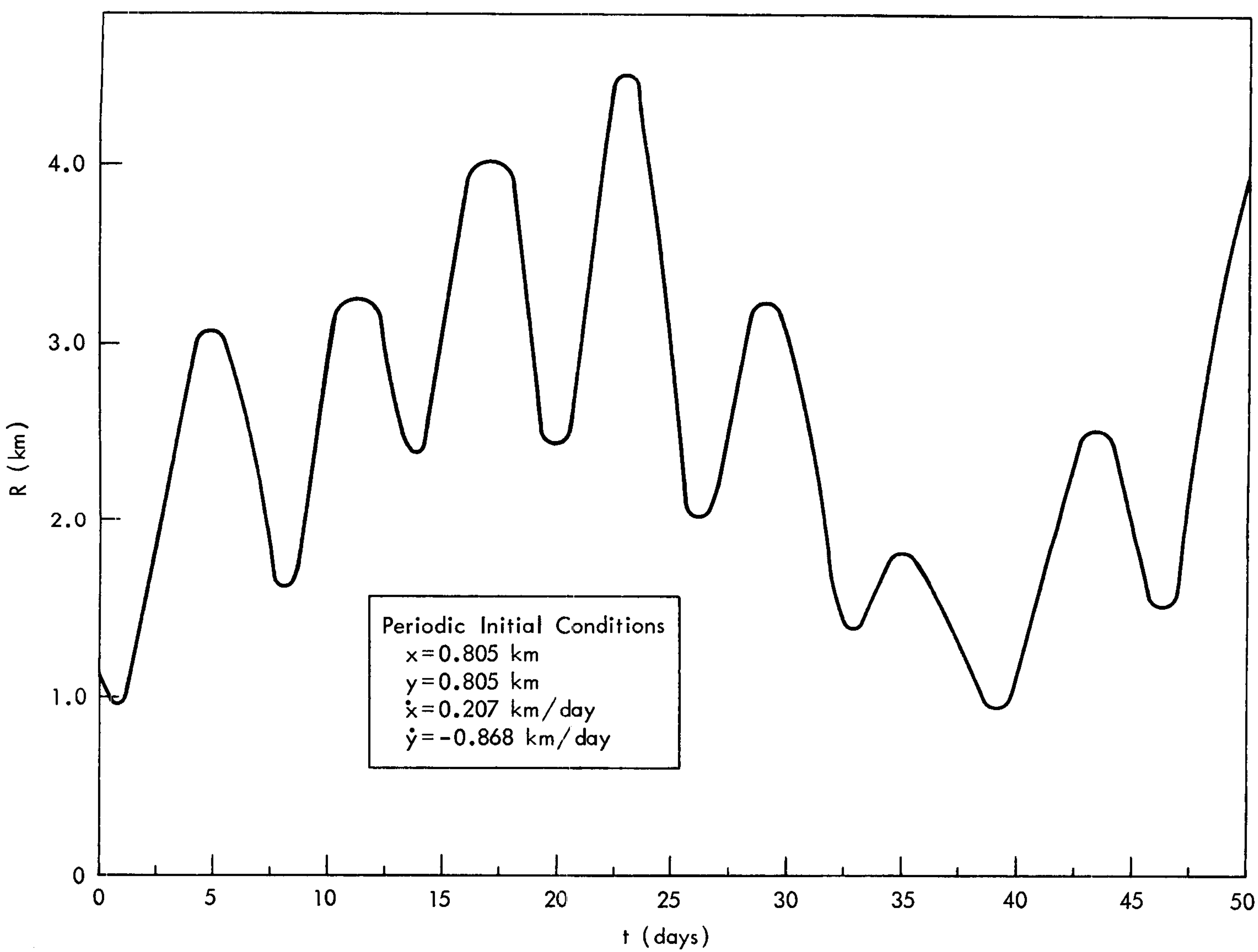

Figure 13. Range of Satellite from L_2 Versus Time
First Order Solution

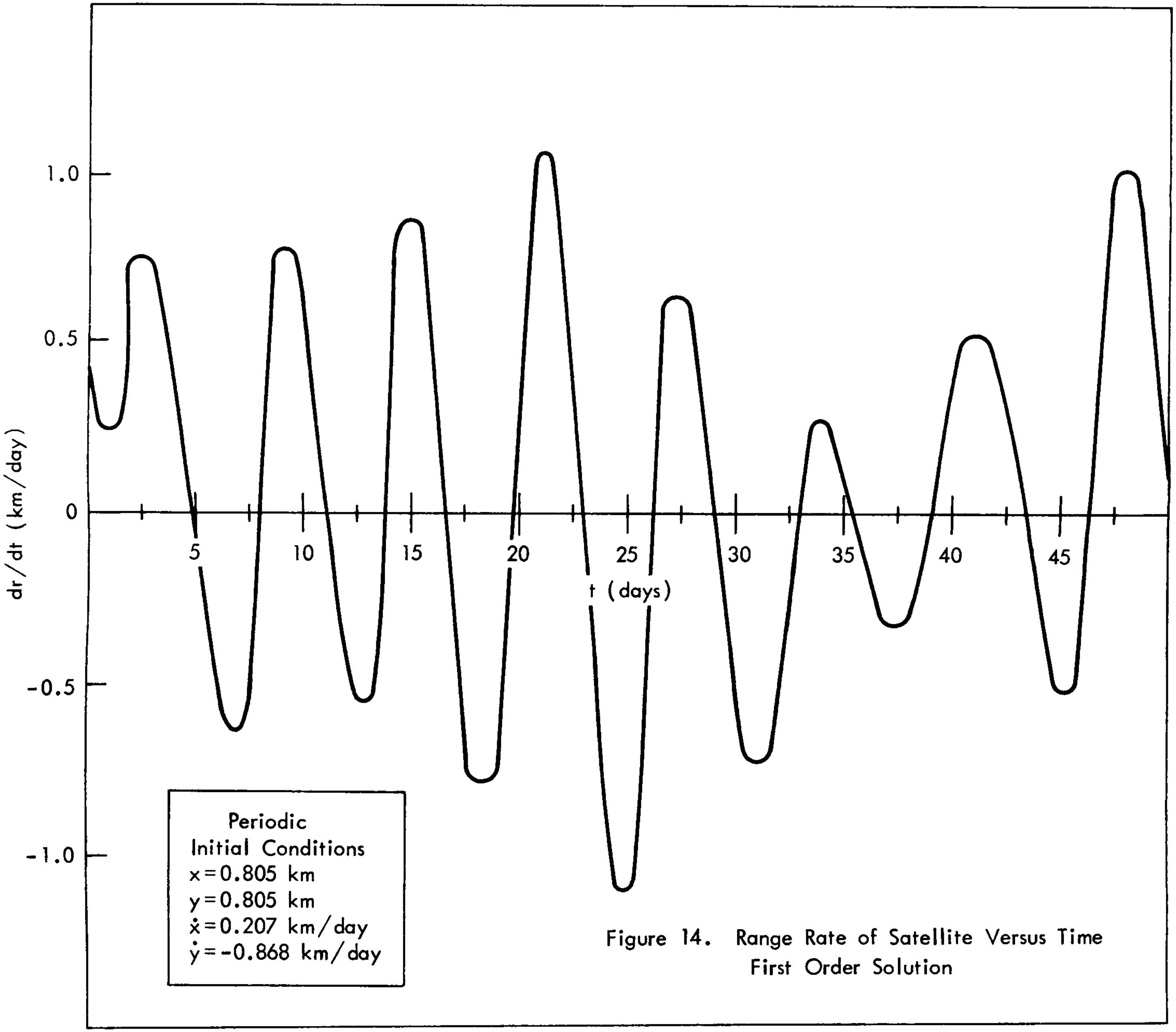

Figure 14. Range Rate of Satellite Versus Time
First Order Solution

Table 10

First Order Solution with Periodic Initial Conditions

$x(0) = 0$ $y(0) = 0$ $\dot{x}(0) = 0$ $\dot{y}(0) = -0.13750$
$\phi = -0.2128000$ $\Omega = 0.4833081$ $\gamma = 0.1878986 \times 10^1$

$A_1 = -0$ $A_4 = 0$ $B_1 = -1.53929$ $B_4 = 0$
$A_2 = 0.81922$ $A_5 = -0.09389$ $B_2 = -0$ $B_5 = 0.43798$
$A_3 = 0$ $A_6 = -0.72532$ $B_3 = 0$ $B_6 = -1.09595$

t (days)	x (kilometers)	y (kilometers)	v (kilometers/day)	a (kilometers/day^2)	R (kilometers)	$\frac{dR}{dt}$ (kilometers/day)	θ (degrees)
.00	.00000	.00000	.13750	.10850	.00000	.00000	370.00000
1.00	.05114	- .15474	.21115	.11918	.16297	.20867	117.09098
2.00	.16969	- .39869	.32964	.12488	.43330	.32961	112.39501
3.00	.26828	- .75747	.40450	.13016	.80358	.39605	97.77431
4.00	.25117	-1.16078	.39233	.20351	1.18764	.35042	104.51576
5.00	.06568	-1.45897	.32725	.31612	1.46045	.17492	145.11177
6.00	- .26265	-1.48430	.40121	.39114	1.50736	-.08834	202.75391
7.00	- .62811	-1.13190	.64049	.38477	1.29450	-.31504	238.49011
8.00	- .87992	- .42522	.85072	.32125	.97727	-.23409	260.17972
9.00	- .88191	.46574	.90911	.33243	.99733	.28427	279.61738
10.00	- .57336	1.26667	.79912	.46962	1.39039	.40916	304.84759
11.00	- .00539	1.69087	.67138	.59233	1.69087	.15733	346.26523
12.00	.66419	1.54819	.78841	.60575	1.68465	-.15452	34.52215
13.00	1.21894	.82889	1.04866	.51072	1.47407	-.18731	66.07344
14.00	1.45905	- .27284	1.18563	.43660	1.48434	.25329	91.74314
15.00	1.27633	-1.40967	1.09561	.54347	1.90162	.49094	111.22027
16.00	.70029	-2.19230	.87814	.70913	2.30143	.24790	145.88709
17.00	- .10366	-2.32349	.87336	.76956	2.32580	-.20935	196.42372
18.00	- .88552	-1.70671	1.17914	.67941	1.92277	-.53900	234.62371
19.00	-1.39435	- .48885	1.44689	.52035	1.47756	-.17870	257.77426
20.00	-1.46472	.98060	1.45881	.51145	1.76266	.66527	276.67000
21.00	-1.07482	2.26157	1.20029	.68091	2.50398	.69280	299.32681
22.00	- .35474	2.96815	.87635	.80636	2.98928	.23281	337.77861
23.00	.45815	2.89547	.90118	.77827	2.93149	-.34366	31.40882
24.00	1.10089	2.08437	1.20676	.61476	2.35724	-.75318	66.46019
25.00	1.37405	.80210	1.39036	.45895	1.59104	-.65532	87.84663

Table 10 (Continued)

First Order Solution with Periodic Initial Conditions

t (days)	x (kilometers)	y (kilometers)	v (kilometers/day)	a (kilometers/day^2)	R (kilometers)	$\frac{dR}{dt}$ (kilometers/day)	θ (degrees)
26.00	1.20682	- .55271	1.30675	.50129	1.32737	.18192	106.60482
27.00	.67530	-1.58555	1.01072	.62887	1.72336	.44949	130.52464
28.00	- .03077	-2.03344	.72493	.66189	2.03367	.12899	189.38277
29.00	- .67901	-1.84036	.72708	.56537	1.96162	-.25875	221.09855
30.00	-1.07600	-1.15714	.86974	.40935	1.58011	-.45734	254.64338
31.00	-1.12956	- .27077	.88799	.34175	1.16156	-.31958	262.38639
32.00	- .87038	.50681	.73972	.39722	1.00718	-.00687	300.74380
33.00	- .42775	.94718	.53086	.42909	1.03928	.01478	334.10055
34.00	.02805	.97213	.44238	.37429	.97254	-.15643	22.36023
35.00	.35269	.65789	.48623	.26443	.74646	-.27033	61.97305
36.00	.47753	.18133	.49101	.18758	.51080	-.15048	87.05333
37.00	.42420	- .26469	.39675	.19271	.50001	.10213	107.04667
38.00	.28106	- .55031	.24122	.19542	.61793	.09843	128.86104
39.00	.15429	- .64979	.09264	.15400	.66786	.00207	165.36021
40.00	.11586	- .63106	.04380	.10008	.64160	-.03844	108.24108
41.00	.17143	- .60240	.08689	.08551	.62631	.02233	179.00860
42.00	.26128	- .64358	.11372	.09774	.69460	.10669	132.35666
43.00	.29350	- .75462	.12519	.13806	.80969	.10508	101.67359
44.00	.19384	- .84709	.18053	.21885	.86898	-.00215	167.79464
45.00	- .05051	- .78364	.35351	.29144	.78527	-.16574	211.64738
46.00	- .37367	- .44938	.58491	.30698	.58444	-.16775	236.41056
47.00	- .65017	.17684	.77436	.26778	.67379	.39442	285.40441
48.00	- .74424	.98484	.83150	.27929	1.23442	.64317	272.25199
49.00	- .57006	1.75055	.72588	.41254	1.84103	.52489	295.65100
50.00	- .13752	2.20735	.57581	.55085	2.21163	.18944	337.22652

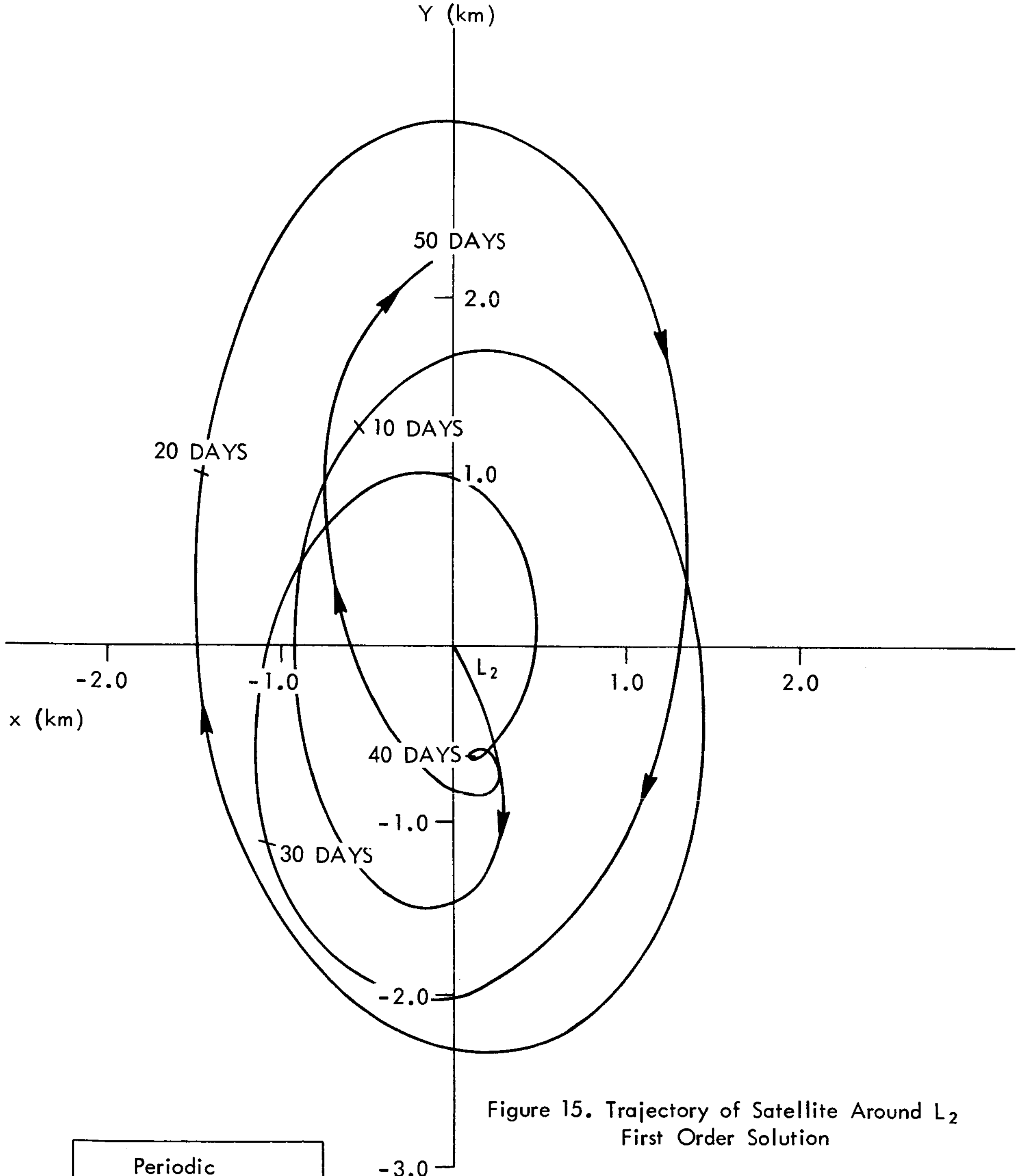

Figure 15. Trajectory of Satellite Around L_2
First Order Solution

Periodic
Initial Conditions
$x = 0$ km
$y = 0$ km
$\dot{x} = 0$ km/day
$\dot{y} = -0.1375$ km/day

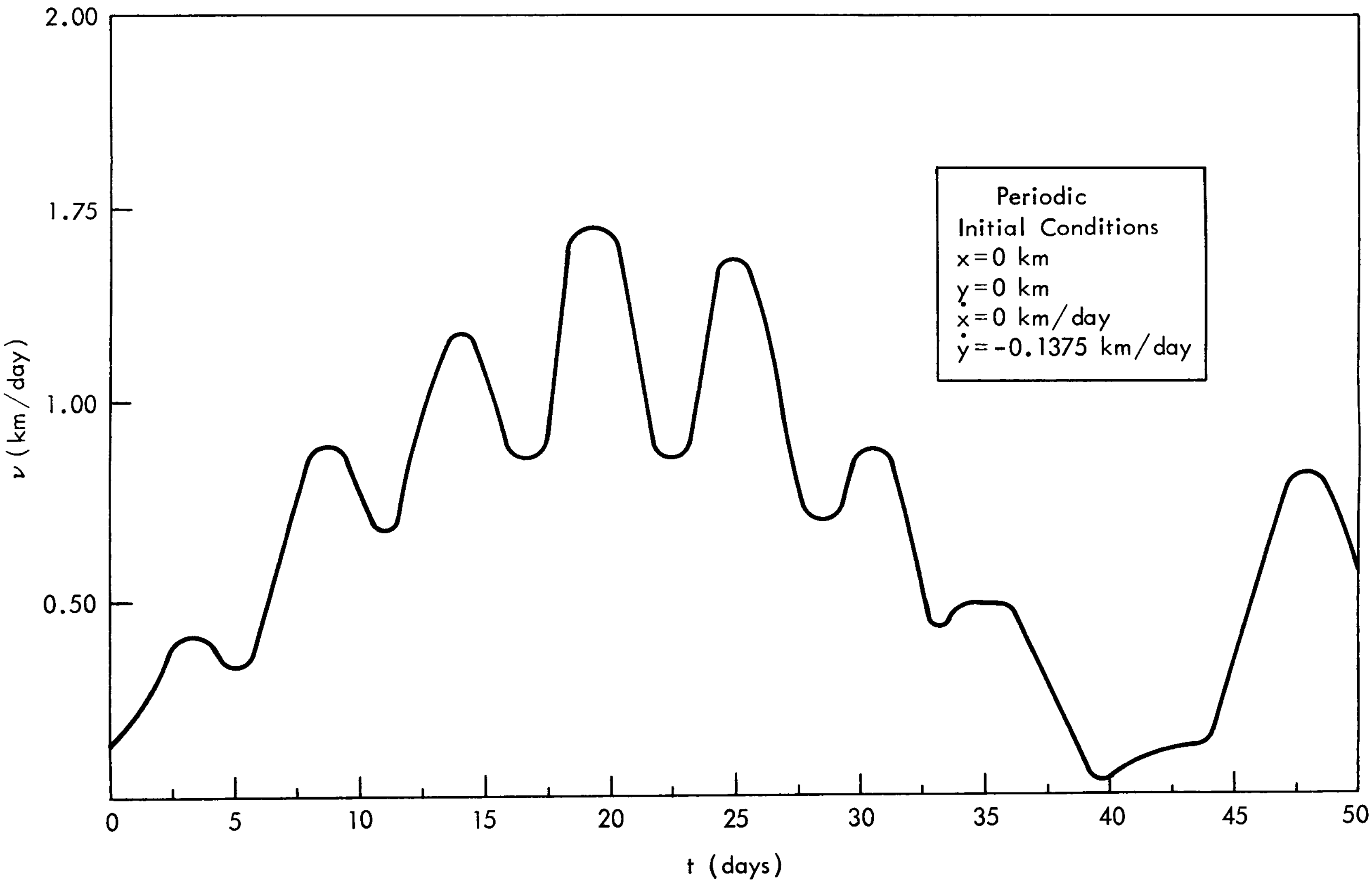

Figure 16. Velocity of Satellite Versus Time
First Order Solution

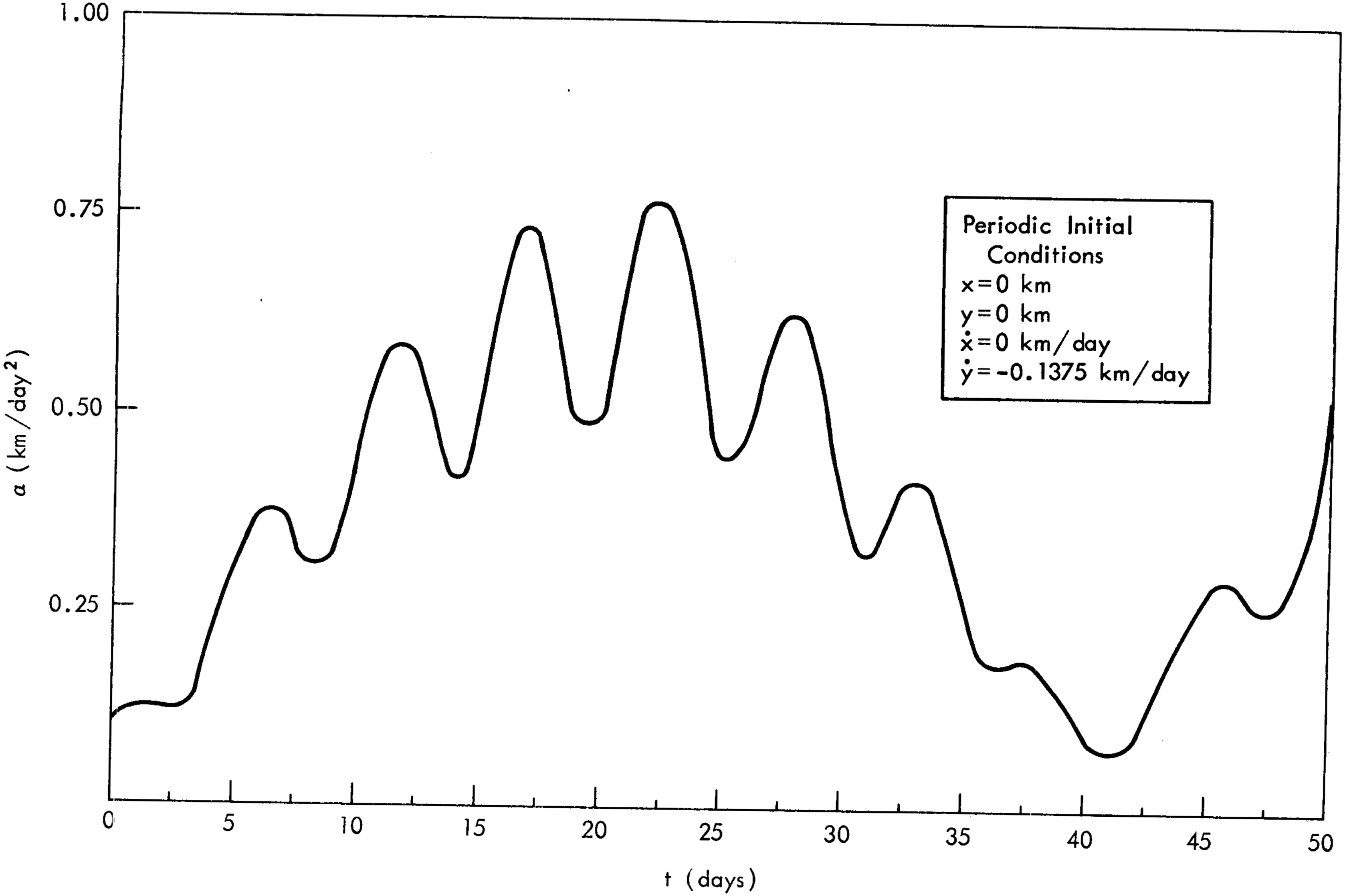

Figure 17. Acceleration of Satellite Versus Time
First Order Solution

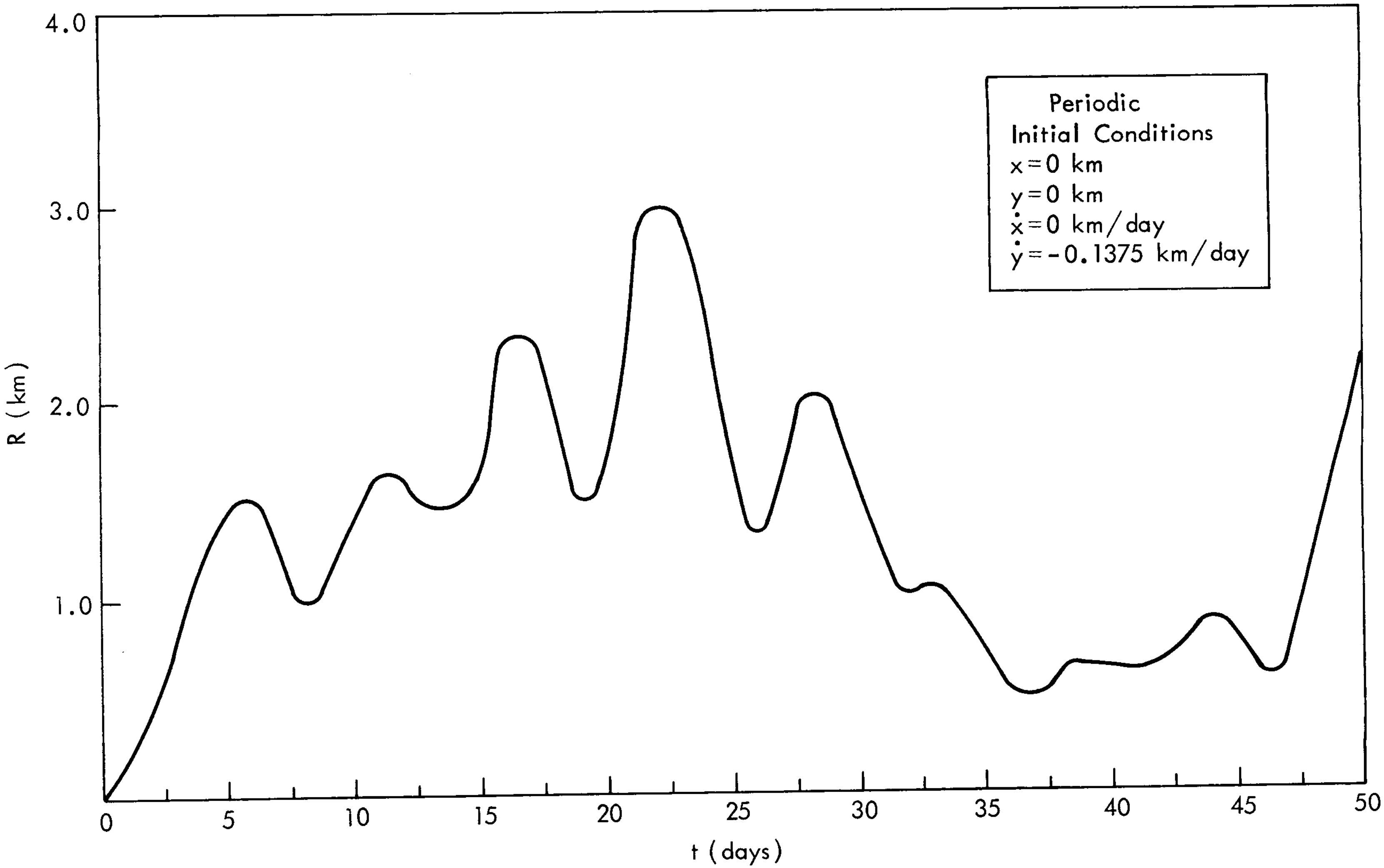

Figure 18. Range of Satellite from L_2 Versus Time First Order Solution

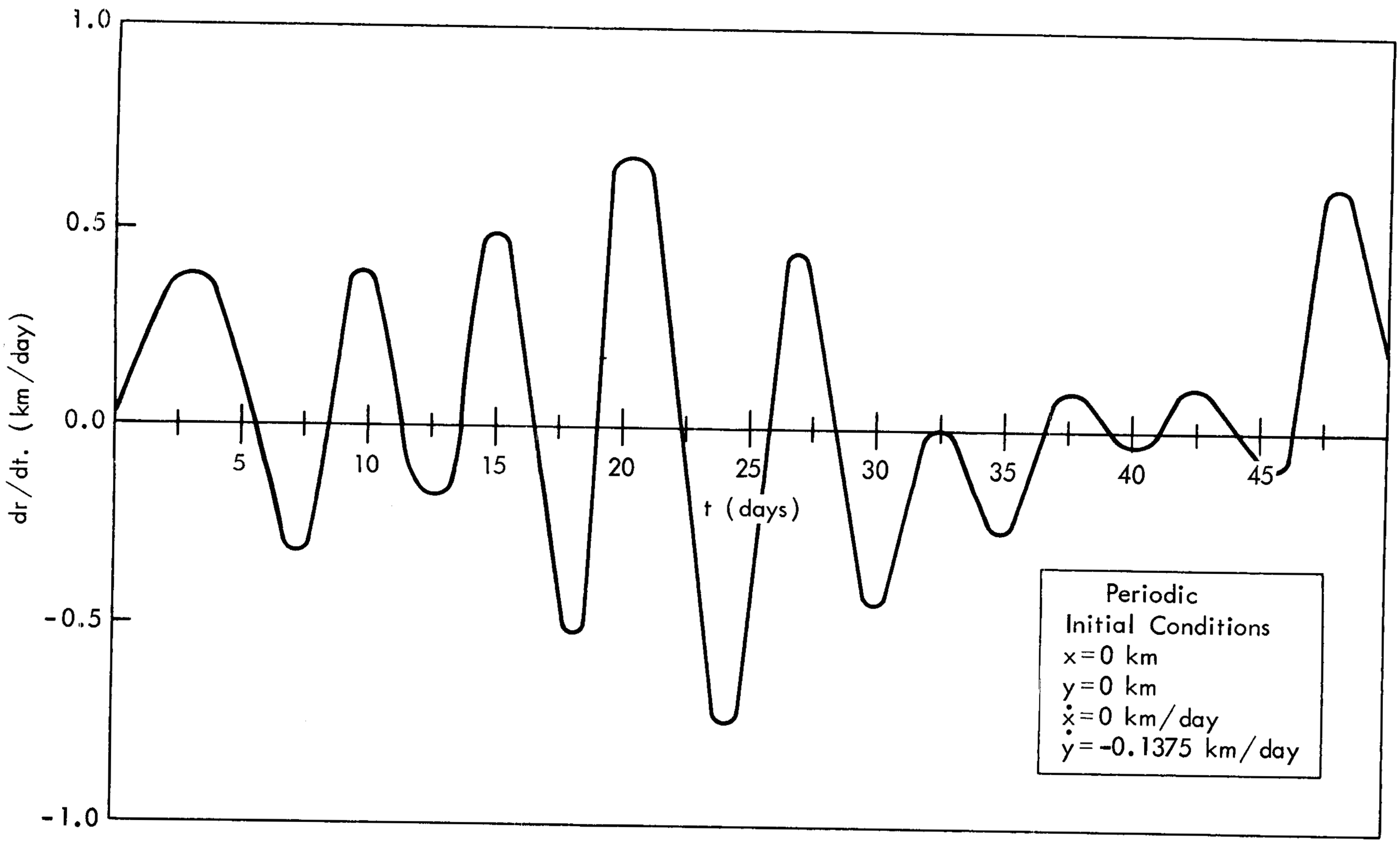

Figure 19. Range Rate of Satellite Versus Time
First Order Solution

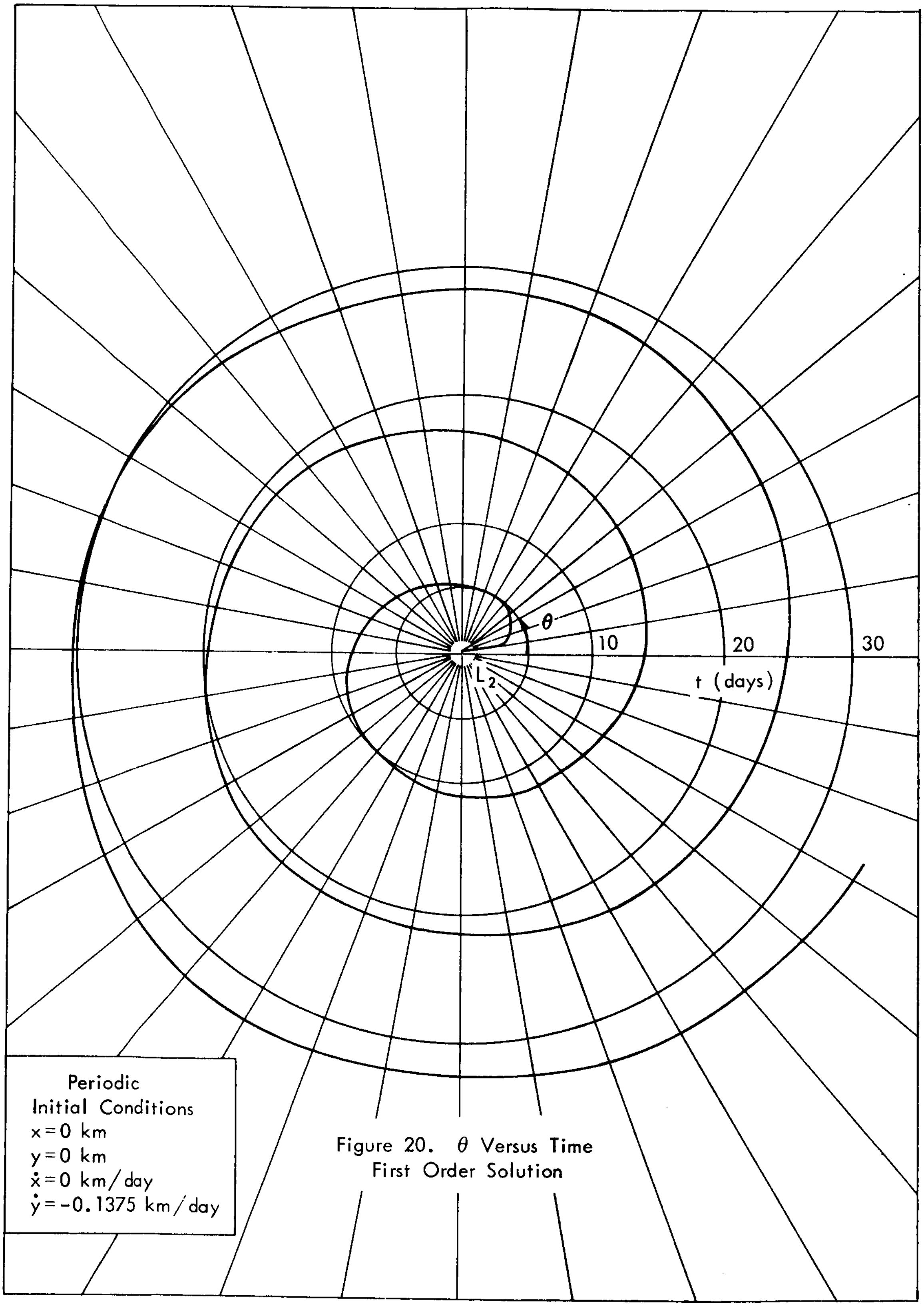

Figure 20. θ Versus Time
First Order Solution

Table 11

Complete Second Order Solution with Quasi-Periodic Initial Conditions

$x(0) = 0$ $y(0) = 0$ $\dot{x}(0) = 0$ $\dot{y}(0) = -0.13750$
$\phi = -0.2128000$ $\Omega = 0.4833081$ $\nu = .1878986 \times 10^{1}$

t (days)	x (kilometers)	y (kilometers)	v (kilometers/day)	a (kilometers/day^2)	R (kilometers)
0.00	0.00000	0.00000	.13783	.10789	0.00000
1.00	$.50863 \times 10^{-1}$	-.15465	.21080	.11843	.16280
2.00	.16860	-.39799	.32814	.12400	.43223
3.00	.26601	-.75522	.40189	.12990	.80069
4.00	.24748	$-.11558 \times 10^{1}$	.37622	.20439	$.11820 \times 10^{1}$
5.00	$.60392 \times 10^{-1}$	$-.14498 \times 10^{1}$	.18240	.31762	$.14510 \times 10^{1}$
6.00	-.26964	$-.14693 \times 10^{1}$	.16156	.39293	$.14938 \times 10^{1}$
7.00	-.63684	$-.11092 \times 10^{1}$	.55446	.38654	$.12791 \times 10^{1}$
8.00	-.89043	-.39308	.84889	.32221	.97334
9.00	-.89452	.50936	.92147	.33089	$.10294 \times 10^{1}$
10.00	-.58899	$.13243 \times 10^{1}$	.81006	.46541	$.14494 \times 10^{1}$
11.00	$-.25808 \times 10^{-1}$	$.17666 \times 10^{1}$	.67082	.58571	$.17667 \times 10^{1}$
12.00	.63647	$.16482 \times 10^{1}$	.76546	.59714	$.17669 \times 10^{1}$
13.00	$.11809 \times 10^{1}$	.96271	$.10077 \times 10^{1}$	.50292	$.15236 \times 10^{1}$
14.00	$.14074 \times 10^{1}$	$-.92276 \times 10^{-1}$	$.11305 \times 10^{1}$	.43950	$.14104 \times 10^{1}$
15.00	$.12068 \times 10^{1}$	$-.11652 \times 10^{1}$	.94712	.56273	$.16775 \times 10^{1}$
16.00	.60722	$-.18614 \times 10^{1}$	.39333	.74010	$.19580 \times 10^{1}$
17.00	-.22856	$-.18760 \times 10^{1}$	.38355	.81009	$.18898 \times 10^{1}$
18.00	$-.10538 \times 10^{1}$	$-.11015 \times 10^{1}$	$.11451 \times 10^{1}$	.72459	$.15244 \times 10^{1}$
19.00	$-.16216 \times 10^{1}$	.33012	$.16632 \times 10^{1}$	.54534	$.16549 \times 10^{1}$
20.00	$-.17715 \times 10^{1}$	$.20889 \times 10^{1}$	$.17861 \times 10^{1}$	.45586	$.27390 \times 10^{1}$
21.00	$-.14886 \times 10^{1}$	$.37608 \times 10^{1}$	$.15695 \times 10^{1}$	.54633	$.40447 \times 10^{1}$
22.00	-.91255	$.49949 \times 10^{1}$	$.11412 \times 10^{1}$	.61968	$.50775 \times 10^{1}$
23.00	-.29443	$.56337 \times 10^{1}$	.64828	.56628	$.56414 \times 10^{1}$
24.00	$.84371 \times 10^{-1}$	$.57835 \times 10^{1}$	.17599	.49617	$.57841 \times 10^{1}$
25.00	$-.20027 \times 10^{-4}$	$.57998 \times 10^{1}$	.37999	.69323	$.57998 \times 10^{1}$

Table 11 (Continued)

Complete Second Order Solution with Quasi-Periodic Initial Conditions

t (days)	x (kilometers)	y (kilometers)	v (kilometers/day)	a (kilometers/day^2)	R (kilometers)
26.00	-.65104	.62011 x 10^1	.96237	.10920 x 10^1	.62352 x 10^1
27.00	-.18366 x 10^1	.75428 x 10^1	.21224 x 10^1	.14891 x 10^1	.77632 x 10^1
28.00	-.34266 x 10^1	.10305 x 10^2	.37431 x 10^1	.17971 x 10^1	.10860 x 10^2
29.00	-.52698 x 10^1	.14838 x 10^2	.56758 x 10^1	.20236 x 10^1	.15746 x 10^2
30.00	-.72826 x 10^1	.21388 x 10^2	.78602 x 10^1	.22641 x 10^1	.22594 x 10^2
31.00	-.95209 x 10^1	.30206 x 10^2	.10352 x 10^2	.26903 x 10^1	.31671 x 10^2
32.00	-.12215 x 10^2	.41706 x 10^2	.13406 x 10^2	.34970 x 10^1	.43458 x 10^2
33.00	-.15764 x 10^2	.56642 x 10^2	.17519 x 10^2	.48333 x 10^1	.58794 x 10^2
34.00	-.20703 x 10^2	.76261 x 10^2	.23271 x 10^2	.67992 x 10^1	.79021 x 10^2
35.00	-.27671 x 10^2	.10243 x 10^3	.31344 x 10^2	.94904 x 10^1	.10610 x 10^3
36.00	-.37403 x 10^2	.13776 x 10^3	.42531 x 10^2	.13049 x 10^2	.14275 x 10^3
37.00	-.50782 x 10^2	.18571 x 10^3	.57775 x 10^2	.17713 x 10^2	.19252 x 10^3
38.00	-.68939 x 10^2	.25084 x 10^3	.78413 x 10^3	.23873 x 10^2	.26014 x 10^3
39.00	-.93417 x 10^2	.33917 x 10^3	.10622 x 10^3	.32118 x 10^2	.35180 x 10^3
40.00	-.12637 x 10^3	.45874 x 10^3	.14366 x 10^3	.43272 x 10^2	.47583 x 10^3
41.00	-.17082 x 10^3	.62036 x 10^3	.19412 x 10^3	.58437 x 10^2	.64345 x 10^3
42.00	-.23088 x 10^3	.83877 x 10^3	.26234 x 10^3	.79048 x 10^2	.86997 x 10^3
43.00	-.31216 x 10^3	.11340 x 10^4	.35465 x 10^3	.10699 x 10^3	.11761 x 10^4
44.00	-.42218 x 10^3	.15330 x 10^4	.47953 x 10^3	.14475 x 10^3	.15901 x 10^4
45.00	-.57100 x 10^3	.20727 x 10^4	.64845 x 10^3	.19570 x 10^3	.21499 x 10^4
46.00	-.77218 x 10^3	.28025 x 10^4	.87679 x 10^3	.26443 x 10^3	.29069 x 10^4
47.00	-.10440 x 10^4	.37891 x 10^4	.11853 x 10^4	.35721 x 10^3	.39303 x 10^4
48.00	-.14111 x 10^4	.51229 x 10^4	.16021 x 10^4	.48257 x 10^3	.53136 x 10^4
49.00	-.19071 x 10^4	.69254 x 10^4	.21652 x 10^4	.65211 x 10^3	.71832 x 10^4
50.00	-.25773 x 10^4	.93615 x 10^4	.29262 x 10^4	.88147 x 10^3	.97098 x 10^4

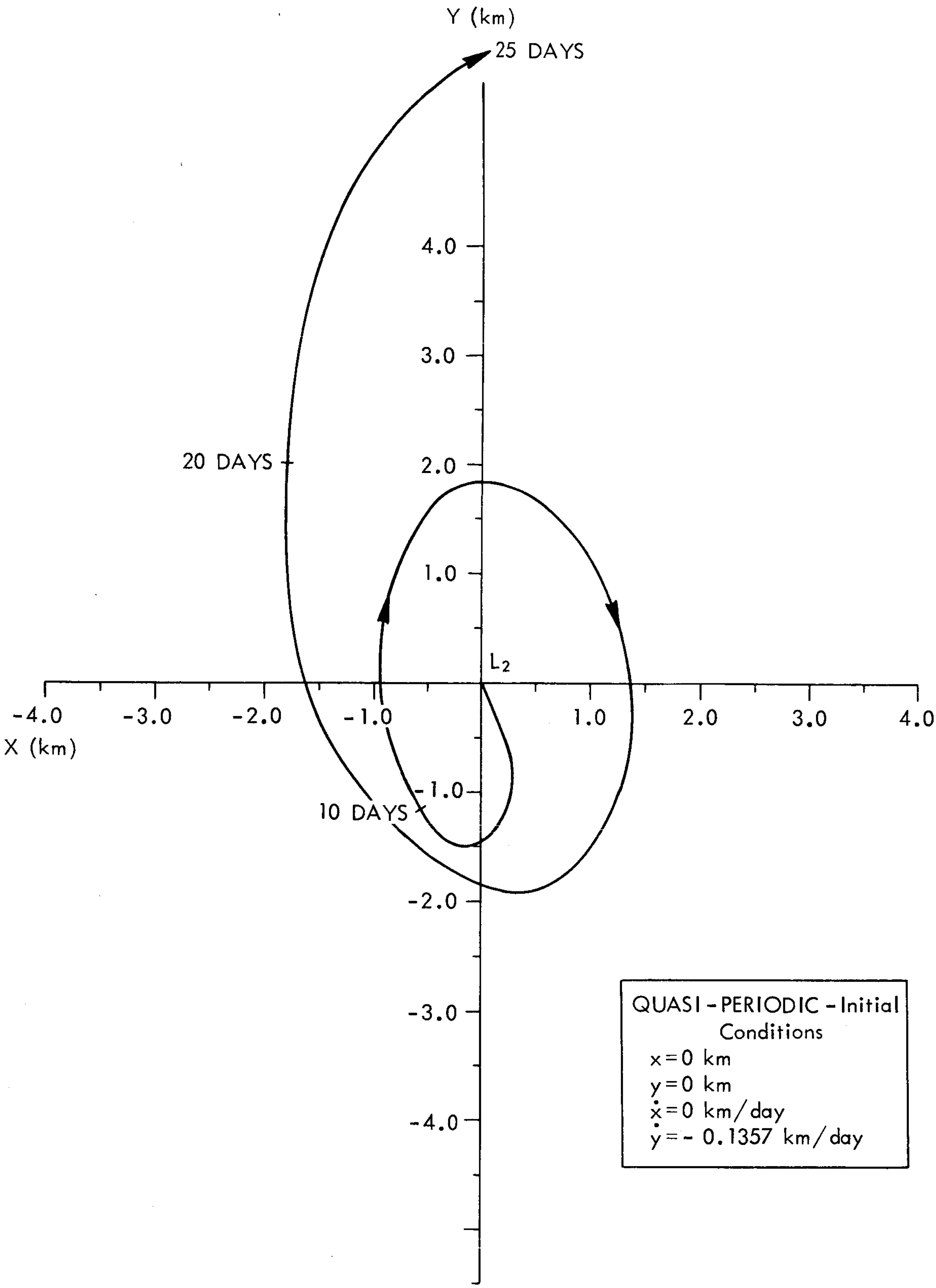

Figure 21. Trajectory of Satellite Around L_2
Complete Second Order Solution

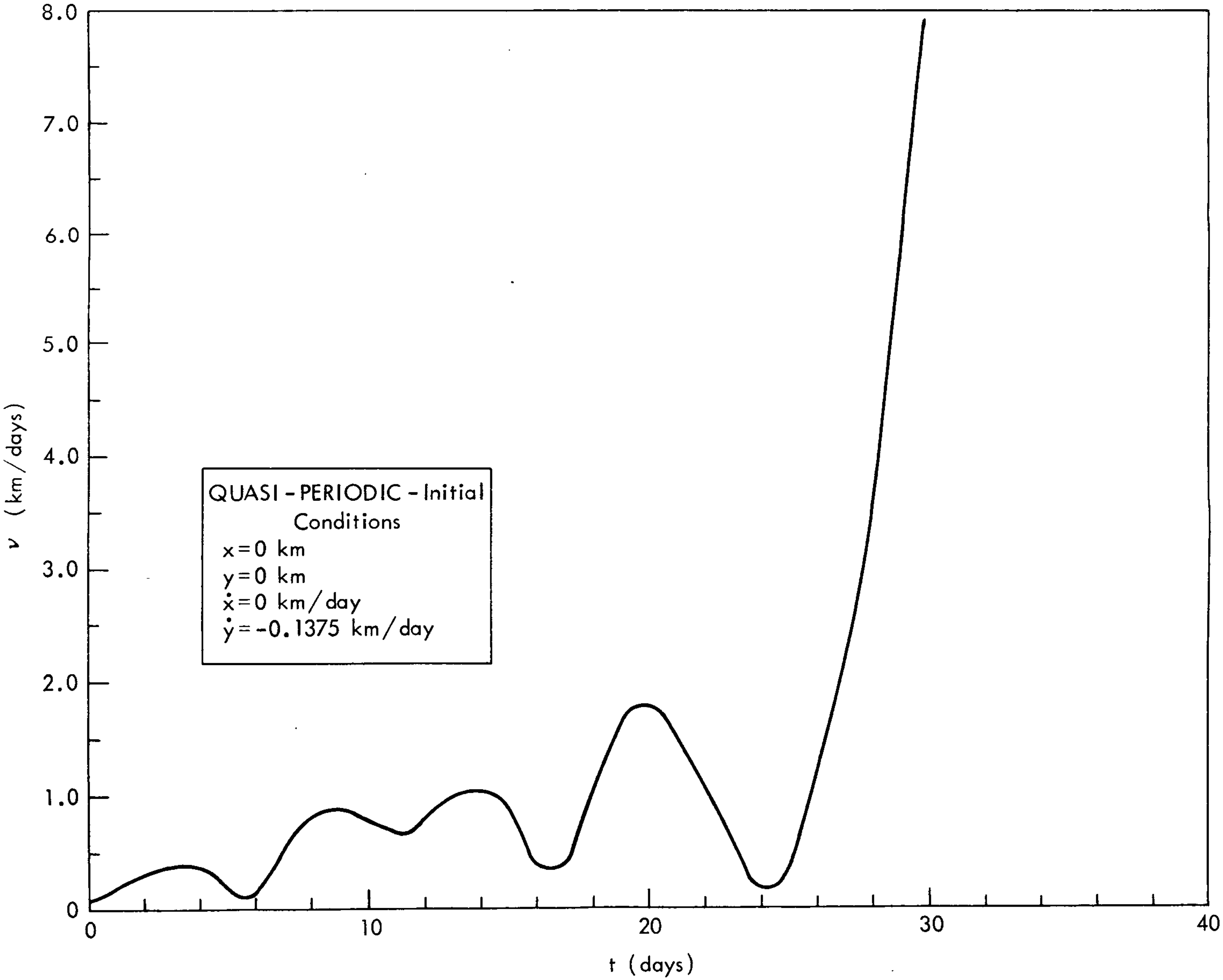

Figure 22. Velocity of Satellite Versus Time
Complete Second Order Solution

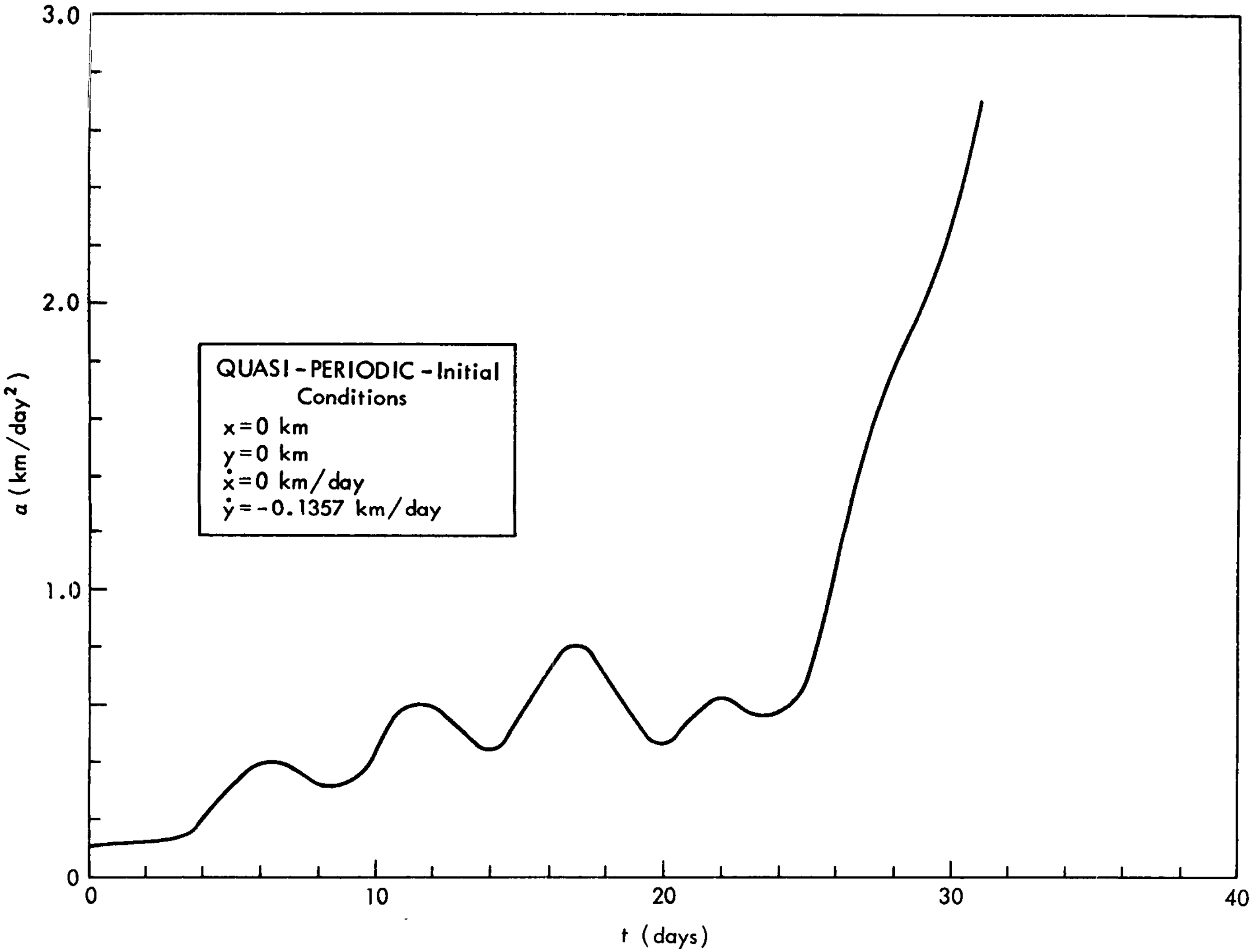

Figure 23. Acceleration of Satellite Versus Time
Complete Second Order Solution

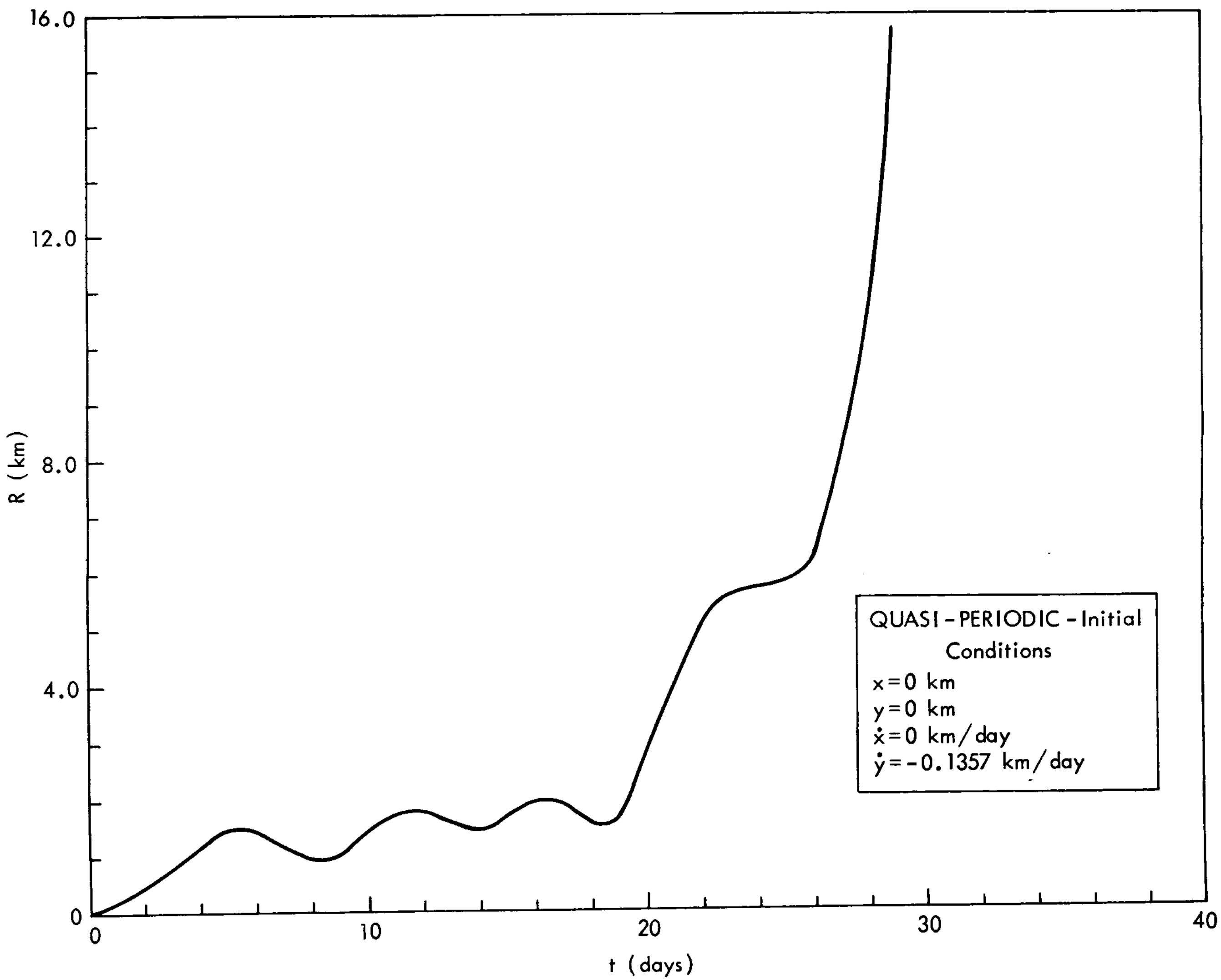

Figure 24. Range of Satellite from L_2 Versus Time
Complete Second Order Solution

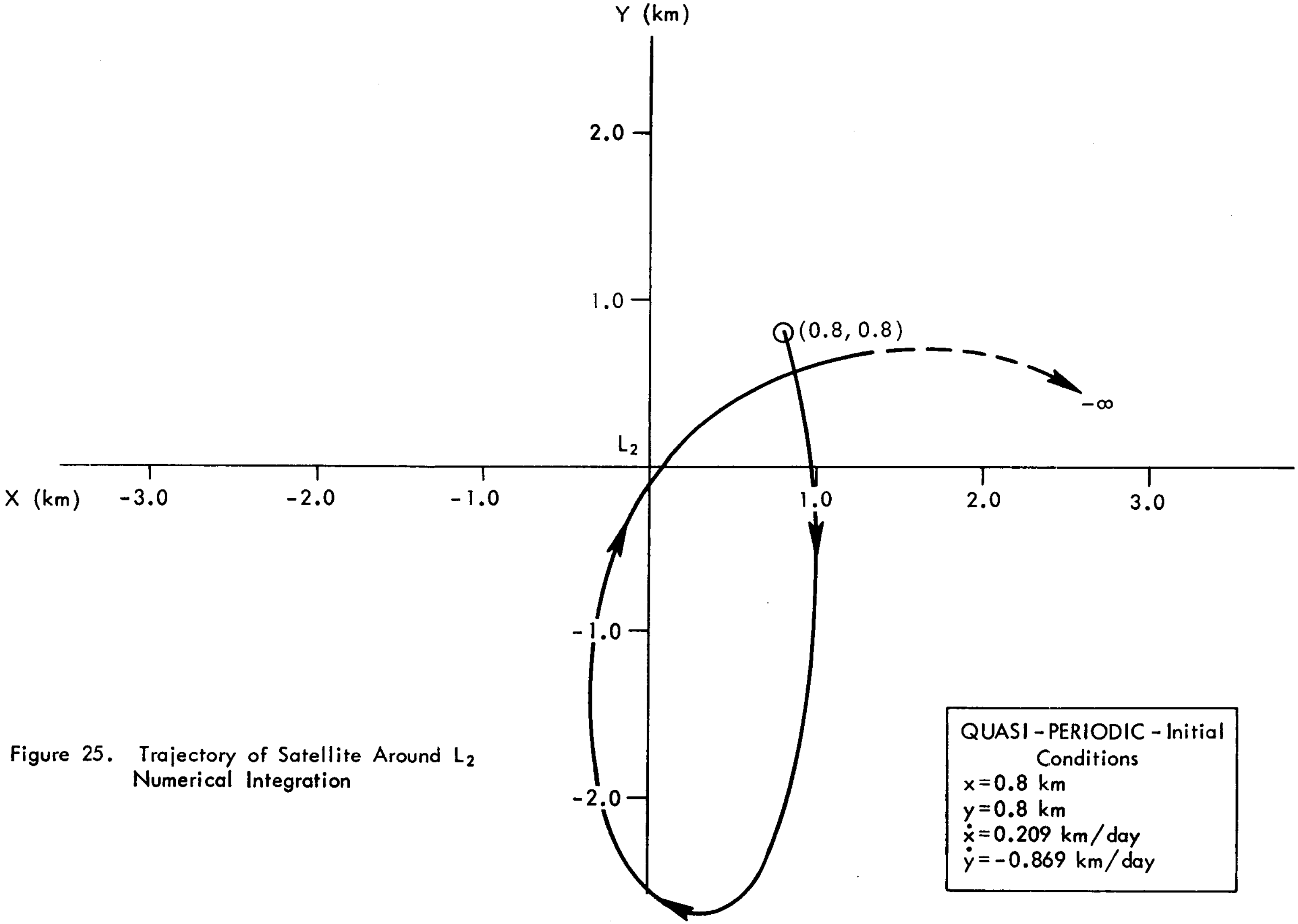

Figure 25. Trajectory of Satellite Around L_2 Numerical Integration

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