

A MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL
PROBLEMS WITH FUNCTIONAL DIFFERENTIAL SYSTEMS

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In this note we present a maximum principle in integral form for optimal control problems with delay-differential system equations which also contain delays in the control. Recent related results for particular cases of the systems discussed below may be found in [1], [5], and [6]. Vector matrix notation will be used and we shall not distinguish between a vector and its transpose.

Let α_0 and t_0 be fixed in R^1 with $-\infty < \alpha_0 < t_0$, $I = [\alpha_0, a)$ be a bounded interval containing $[\alpha_0, t_0]$, and put $I' = (t_0, a)$. For x continuous on I and t in I' , the notation $F(x(\cdot), t)$ will mean that F is a functional in x , depending on any or all of the values $x(\tau)$, $\alpha_0 \leq \tau \leq t$. $\bar{\Phi}$ will denote the class of absolutely continuous $n-1$ vector functions on $[\alpha_0, t_0]$. Let Ω be a given convex subset of the class of all bounded Borel measurable functions u defined on I into R^r , and \mathcal{T} be a given C^1 manifold in R^{2n-1} . The problem considered is that of minimizing

$$J[\bar{\Phi}, u, \bar{x}, t_1] = \int_{t_0}^{t_1} f^0(\bar{x}(\cdot), u(\cdot), t) dt$$

over $\bar{\Phi} \times \Omega \times C(I, R^{n-1}) \times I'$ subject to

$$(i) \quad \begin{aligned} \dot{\bar{x}}(t) &= \bar{f}(\bar{x}(\cdot), u(\cdot), t) && \text{a.e. on } [t_0, t_1] \\ \bar{x}(t) &= \bar{\varphi}(t) && \text{on } [\alpha_0, t_0] \end{aligned}$$

$$(ii) \quad (\bar{x}(t_0), \bar{x}(t_1), t_1) \in \mathcal{T}.$$

We assume that $f = (f^0, \bar{f}) = (f^0, f^1, \dots, f^{n-1})$ is an n -vector functional of the form

$$(1) \quad f^i(\bar{x}(\cdot), u(\cdot), t) = h^i(\bar{x}(\cdot), t) + \int_{\alpha_0}^t u(s) d_s \eta(t, s) g^i(\bar{x}(s), t)$$

$$\text{for } i = 0, 1, \dots, n-1,$$

where the integral is a Lebesgue-Stieltjes integral. Each $h^i(\bar{x}(\cdot), t)$ is assumed C^1 in \bar{x} and measurable in t , and each $g^i(\bar{y}, t)$ is C^1 in (\bar{y}, t) on R^n . The $r \times 1$ vector function $\eta(t, s)$ is measurable in t, s , and of bounded variation in s on $[\alpha_0, t]$. It is also assumed that the variation of η is dominated by an $L_1(I')$ function m . That is, $\bigvee_{s=\alpha_0}^t \eta(t, s) \leq m(t)$ for $t \in I'$. Finally, suppose that given \bar{X} compact, $\bar{X} \subset R^{n-1}$, there exists an \tilde{m} in $L_1(I')$ such that $h = (h^0, h^1, \dots, h^{n-1})$ satisfies

$$|h(\bar{x}(\cdot), t)| \leq \tilde{m}(t)$$

and

$$|dh[\bar{x}(\cdot), t; \bar{\psi}]| \leq \tilde{m}(t) \|\bar{\psi}\|_t$$

for any $\bar{\psi} \in C(I, R^{n-1})$ and $\bar{x} \in C(I, \bar{X})$, where $\|\bar{\psi}\|_t = \sup \{ \|\bar{\psi}(s)\| : s \in [\alpha_0, t] \}$ and dh is the Fréchet derivative of h with respect to \bar{x} . ($|A|$ denotes the Euclidean norm of A .)

If $(\bar{\varphi}^*, u^*, \bar{x}^*, t_1^*)$ is a solution of the above problem, we define the $n \times n - 1$ matrix function $\bar{\eta}^*$ for $t \in I'$, $s \in [\alpha_0, t]$ by

$$(2) \quad \bar{\eta}^*(t, s) = \bar{\eta}_1^*(t, s) + \bar{\eta}_2^*(t, s),$$

where $\bar{\eta}_1^*$ is such that

$$dh[\bar{x}^*(\cdot), t; \bar{\psi}] = \int_{\alpha_0}^t d_s \bar{\eta}_1^*(t, s) \bar{\psi}(s)$$

for $t \in I'$ and all $\bar{\psi} \in C([\alpha_0, t], R^{n-1})$, and

$$\bar{\eta}_2^*(t, s) \equiv \begin{cases} -\int_s^t u^*(\beta) d_\beta \eta(t, \beta) g_{\bar{y}}(\bar{x}^*(\beta), t) & s < t \\ 0 & s \geq t. \end{cases}$$

(The existence of $\bar{\eta}_1^*$ is guaranteed by the Riesz theorem.) Then we have the following necessary conditions:

Theorem: Let $(\bar{\varphi}^*, u^*, \bar{x}^*, t_1^*)$ be a solution to the problem under the assumptions above. In addition, suppose that t_1^* is a Lebesgue point of $f(\bar{x}^*(\cdot), u^*(\cdot), t)$. Then there exists a non-trivial n -vector function $\lambda(t) = (\lambda^0(t), \bar{\lambda}(t))$ of bounded variation on $[t_0, t_1^*]$, continuous at t_1^* , satisfying:

$$(a) \quad \lambda^0(t) = \text{constant} \leq 0, \quad \lambda(t_1^*) \neq 0$$

$$\bar{\lambda}(t) + \int_t^{t_1^*} \lambda(\beta) \bar{\eta}^*(\beta, t) d\beta = \bar{\lambda}(t_1^*) \quad \text{for } t \in [t_0, t_1^*]$$

where $\bar{\eta}^*$ is defined by (2).

$$(b) \quad \int_{t_0}^{t_1^*} \lambda(t) f(\bar{x}^*(\cdot), u^*(\cdot), t) dt \geq \int_{t_0}^{t_1^*} \lambda(t) f(\bar{x}^*(\cdot), u(\cdot), t) dt$$

for all $u \in \Omega$.

(c) The $2n-1$ vector

$$(-\bar{\lambda}(t_0) + \int_{t_0}^{t_1^*} \lambda(\beta) \{\bar{\eta}^*(\beta, \alpha_0) - \bar{\eta}^*(\beta, t_0)\} d\beta, \bar{\lambda}(t_1^*), -\lambda(t_1^*) \cdot f^*(t_1^*))$$

is orthogonal to \mathcal{F} at $(\bar{x}^*(t_0), \bar{x}^*(t_1^*), t_1^*)$, where $f^*(t_1^*) \equiv f(\bar{x}^*(\cdot), u^*(\cdot), t_1^*)$.

The proof of this theorem involves showing that the class of functions $\mathcal{F} = \{F(\bar{x}(\cdot), t) : F(\bar{x}(\cdot), t) = f(\bar{x}(\cdot), u(\cdot), t), u \in \Omega\}$

is absolutely quasiconvex [2] and then using necessary conditions for extremals given in [2]. Absolute quasiconvexity is a generalization of ideas due to Gamkrelidze [4], who first obtained an integral maximum principle for control problems with ordinary differential system equations. The inequality in (b) is a maximum principle in integral form for the above described optimal control problem.

In many particular cases of the systems defined by (1), one can show that the multipliers $\bar{\lambda}$ are actually absolutely continuous and satisfy (a) in differentiated form. This differentiated form becomes the usual known multiplier equation for systems with simple time lags in the state variables (see [1]). The transversality conditions given in (c) also can be reduced to a simpler form for many special cases of (1).

Included in (1) are many integro-differential systems and time lag systems which appear in physical problems. For example, if one modifies slightly the biological population model formulated by Cooke in [3], one obtains the system equation

$$\dot{x}(t) = u(t-\tau)x(t-\tau) + \beta(t)u(t-\tau-\theta(t))x(t-\tau-\theta(t))$$

where $x(t)$ is the number in the population at time t , $u(t)$ is the birth rate at time t , and τ is the gestation period. Systems with

$$h(\bar{x}(\cdot), t) = \int_{\alpha_0}^t A(t, s) q(\bar{x}(s), t) ds,$$

which arise in the study of reactor dynamics [7], and with

$$h(\bar{x}(\cdot), t) = G(\bar{x}_t, t),$$

where $\bar{x}_t = \bar{x}(t+\theta)$, $\theta \in [-T, 0]$, are also special cases of (1).

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