FORTRAN SUBPROGRAMS FOR COMPLETE ELLIPTIC INTEGRALS

by

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ABSTRACT

Fortran II subprograms have been developed for evaluating the complete elliptic integrals of the first and second kinds on the RCA 601 computer. These subprograms exhibit an error no greater than $2\ (10)^{-8}$ over the entire range of definition.

DEFINITIONS

The complete elliptic integral of the first kind is defined as

$$K(m) = \int_0^1 \left[\left(1 - t^2\right)\left(1 - mt^2\right)\right]^{-1/2} dt$$

$$= \int_0^{\pi/2} \left(1 - m \sin^2 \theta\right)^{-1/2} d\theta, \tag{1}$$

and the complete elliptic integral of the second kind is defined as

$$E(m) = \int_0^1 \left(1 - t^2\right)^{-1/2} \left(1 - mt^2\right)^{1/2} dt$$

$$= \int_0^{\pi/2} \left(1 - m \sin^2 \theta\right)^{1/2} d\theta. \tag{2}$$

In the above expressions m is the <u>parameter</u> of the integrals. This quantity is related to the <u>modulus</u> k and the <u>modular angle</u> α by the relations

$$m = k^2 = \sin^2 \alpha. \tag{3}$$

One may also define the <u>complementary parameter</u> m_1 and <u>complementary modulus</u> k' through the relations

$$m_1 = 1 - m$$

$$= (k')^2 = \cos^2 \alpha \tag{4}$$

The user is cautioned that many texts define elliptic integrals in terms of the modulus k.

PROGRAMMING METHOD

The complete elliptic integrals are evaluated using the polynomial approximations given by Equations 17.3.34 and 17.3.36 of the <u>Handbook of Mathematical Functions</u>:(1)

<u>17.3.34</u>

$$K(m) = \left[a_o+a_1m_1+\ldots+a_4m_1^4\right] + \left[b_o+b_1m_1+\ldots+b_4m_1^4\right] \ell n\ (1/m_1) + \epsilon(m)$$

$$|\epsilon(m)| \leq 2 \times 10^{-8}$$

a_o = 1.38629 436112		b_o = .5	
a_1 = .09666 344259		b_1 = .12498 593597	
a_2 = .03590 092383		b_2 = .06880 248576	
a_3 = .03742 563713		b_3 = .03328 355346	
a_4 = .01451 196212		b_4 = .00441 787012	

<u>17.3.36</u>

$$E(m) = \left[1+a_1m_1+\ldots+a_4m_1^4\right] + \left[b_1m_1+\ldots+b_4m_1^4\right] \ell n\ (1/m_1) + \epsilon(m)$$

$$|\epsilon(m)| < 2 \times 10^{-8}$$

a_1 = .44325 141463	b_1 = .24998 368310
a_2 = .06260 601220	b_2 = .09200 180037
a_3 = .04757 383546	b_3 = .04069 697526
a_4 = .01736 506451	b_4 = .00526 449639

Listings of the subprograms are given in Figures 1 and 2. In these subprograms the Fortran floating point variable B plays the role of the complementary parameter m_1. The natural logarithm subroutine LOG(B) in the RCA 601 Fortran II package normally causes a loss of significant figure accuracy halt if $\left|1-B\right| < 10^{-2}$. In the present case, however, the logarithm will maintain sufficient accuracy when used in an expression of the form

$$\sum_i a_i\, m_1^{\,i} - \ell n\,(m_1) \sum_i b_i\, m_1^{\,i}$$

with

$$\Sigma\, a_i\, m_1^{\,i} \approx \Sigma\, b_i\, m_1^{\,i}.$$

so that the result will be accurate to full significance. Thus the loss of accuracy halt was inhibited in these subprograms by using the special call BYPASS(LOG,B).

CALLING PROCEDURE

The complete elliptic integrals of the first and second kind are evaluated by placing the terms ELK(B) and ELE(B), respectively, in any floating-point Fortran arithmetic expression. Note that the calls are in terms of the complementary parameter $B = m_1$. This was done, on the suggestion of R. W. Klopfenstein, to avoid the loss of accuracy in the machine computation of

$$B = 1 - (1 - B) \tag{5}$$

when B is known to full accuracy. (Note in Figures 1 and 2 that the integrals are evaluated in terms of B.)

EXAMPLE: To evaluate the expression

$$C = K\ (m = A) + E\ (m_1 = B) + 2\ K\ (\alpha = D^{o}) \tag{6a}$$

with A, B, and D known, one could write the Fortran statement

$$C = ELK\ (1.0 - A) + ELE\ (B) + 2.0 * ELK\ (COS\ DF\ (D)\ ** \ 2) \qquad (6b)$$

ERROR STOPS AND SPECIAL CONDITION

B must satisfy $0 < B \leq 1$ for ELK (B) and $0 \leq B \leq 1$ for ELE (B). For B outside these limits an error message is printed giving the value of B. The job is then terminated with a dump.

For B = 0 in ELE (B) the polynomial approximation is bypassed, ELE is set equal to 1.0, and control is returned to the calling program.

ACCURACY

Provided B is given to full (i.e., 9 significant figures) accuracy, ELE and ELK will exhibit an error of no more than $2\ (10)^{-8}$, that is, 2 units in the ninth significant figure.

In testing the ELK program, two situations were encountered where poorer accuracy was obtained. First, as would be expected, calls of the form ELK $\left(1.0 - (1.0 - B)\right)$ for small B yielded results which were accurate to the same number of significant figures as $\left(1.0 - (1.0 - B)\right)$. Thus, the above form of the call should be avoided when the complementary parameter B is available. Second, calls of the form ELK (COS DF (D) ** 2) lost several significant figures of accuracy when D was very close to 90°. This is due to a loss of significance in COS DF at these values, as is illustrated below.

ELE is not subject to the same loss of significance since in this program there is no constant term in polynomial which multiplies the logarithm. [See 17.3.36 of Reference (1).]

TESTING

Both ELK and ELE were evaluated for the following values of m and α

$$m = 10^{-5},\ 10^{-4},\ 10^{-3},\ 10^{-2},\ 0.1(0.05)0.9(0.01)0.99,\ 0.999,\ 0.9999,\ 0.99999$$

$$\alpha = 0\ (10)\ 80\ (1)\ 89\ (0.1)\ 89.9\ \text{degrees}$$

The results are shown in Figures 3 through 8.

The computed values were compared with values taken from Tables 17.1 and 17.2 of Reference (1) or values computed by a special double precision program described below. The Δ columns following the computed values of ELK = FIRST and ELE = SECOND in Figures 3 through 8 give

$$\Delta = \left(E_{comp.} - E_{exact}\right)(10)^8 \qquad (7)$$

where E is the value of the elliptic integral.

It can be seen that both subprograms maintain the specified accuracy over the entire range of m. However, the elliptic integral of the first kind looses some significance for α close to 90°. That this is due to a loss of significance in the cosine evaluation can be seen in Figures 6 through 8 where the column headed Δ_c gives the error in the ninth significant figure of the computed value of the cosine. Note in particular the loss of significant figures in Figure 8.

Figure 9 shows the results of the evaluation of ELE ($\alpha = 90^\circ$) and also shows the error message printed out when ELK (90°) was called.

SPECIAL TEST PROGRAM

In order to obtain accurate values of the elliptic integrals outside the range covered in the tables, a special test program was written in double precision for the 70/45 Phase I Basic Time Sharing System. This program is shown as Figure 10.

This program uses Equation (8) to obtain three stages of reduction of the parameter m:

$$m_{i+1} = \left(\frac{1 - \sqrt{1 - m_i}}{1 + \sqrt{1 + m_i}} \right)^2, \quad i = 0, 1, 2 \tag{8}$$

where

$$m_o = m \tag{9}$$

and

$$m_{i+1} < m_i \tag{10}$$

Next, $K(m_3)$ and $E(m_3)$ are evaluated using the series expansions 773.2 and 774.2 of Dwight.[2] (The reduction on m assures that these expansions are rapidly convergent, even for m very close to 1.) Finally, Equations 17.3.29 and 17.3.30 of Reference (1) are applied three times to obtain K(m) and E(m), respectively.

Figure 11 shows the results obtained from this program for selected values of m. Comparison with tabulated values show the results are accurate to 13 significant figures.

To obtain Figure 12, the statement

```
35      M = SIN(1.5707963267948966*M/90)**2
```

was inserted between statements 30 and 40 in the test program. Thus, in this table M represents the modular angle (in degrees). Better than 9 significant figure accuracy was obtained over the entire range.

The coding procedure used in the test program was not used in the subprograms since it requires more code and takes longer to execute.

ACKNOWLEDGMENT

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REFERENCES

1. M. Abramowitz and I. A. Steguh (Ed.), Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series 55, GPO, Washington, D. C., 1964.

2. H. B. Dwight, Tables of Integrals and Other Mathematical Data, 4th Ed., Macmillan, New York, 1961.

```
*COMPILE
        C      COMPLETE ELLIPTIC INTEGRAL OF FIRST KIND
        C
        C
        C      ERROR LESS THAN 0.00000002
        C
000001         FUNCTION ELK(B)
000002  F      LOG
000003         IF(B) 130,130,100
000004     100 IF(1.0-B ) 130,110,110
000005     110 ELK=((((0.0145119621*B+0.0374256371)*B+0.0359009238)*B
             1     +0.0966634426)*B+1.38629436)-((((0.00441787012*B
             2     +0.0332835535)*B+0.0688024858)*B+0.124985936)*B+0.5)
             3     *BYPASS(LOG,B)
000006         RETURN
000007     130 PRINT 900,B
000008     900 FORMAT(27H0COMPLEMENTARY PARAMETER B=,
             X                                E18.9,38H IS OUT OF RANGE IN ELK-JOB TERMINA
             1TED)
000009         CALL PMDUMP
000010         END
```

Figure 1.

```
*COMPILE
       C      COMPLETE ELLIPTIC INTEGRAL OF SECOND KIND
       C
       C
       C      ERROR LESS THAN 0.00000002
       C
              FUNCTION ELE(B)
       F      LOG
              IF(B) 130,140,100
          100 IF(1.0-B ) 130,110,110
          110 ELE=((((0.0173650645*B+0.0475738355)*B+0.0626060122)*B
             1      +0.443251415)*B+1.0)-((((0.00526449639*B+0.040696975)*B
             2      +0.0920018004)*B+0.24998368)*B)*BYPASS(LOG,B)
              RETURN
          130 PRINT 900,B
          900 FORMAT(27H0COMPLEMENTARY PARAMETER B=,
             X                          E18.9,38H IS OUT OF RANGE IN ELK-JOB TERMINA
             1TED)
              CALL PMDUMP
          140 ELE=1.0
              RETURN
              END
```

Figure 2.

M	FIRST	Δ	SECOND	Δ
0.00000	1,57079633	0	1,57079633	0
0.00001	1.57060026	+1	1,57079240	0
0.00010	1.57083560	0	1,57075706	0
0.00100	1.57118926	+1	1,57040355	0
0.01000	1.57474557	+1	1,56686195	+1
0.10000	1.61244136	+1	1,53075764	0

Figure 3.

M	FIRST	Δ	SECOND	Δ
0.15000	1.63525672	-1	1.51012182	-1
0.20000	1.65962358	-2	1.48903504	-2
0.25000	1.68575035	0	1.46746220	-1
0.30000	1.71388945	0	1.44536306	0
0.35000	1.74435060	0	1.42259115	+2
0.40000	1.77751939	+2	1.39939215	+1
0.45000	1.81388395	+1	1.37540199	+2
0.50000	1.85407469	+1	1.35064389	+1
0.55000	1.89892491	0	1.32502450	0
0.60000	1.94956773	-2	1.29842802	-1
0.65000	2.00759838	-2	1.27070746	-2
0.70000	2.07536314	0	1.24157057	0
0.75000	2.15651565	0	1.21105603	0
0.80000	2.25720534	+1	1.17848994	+2
0.85000	2.38901650	+1	1.14339580	+1
0.90000	2.57609210	-1	1.10477472	-1

Figure 4.

M	FIRST	Δ	SECOND	Δ
0.91000	2.62777331	-2	1.09647751	-1
0.92000	2.68355139	-2	1.08793749	-1
0.93000	2.74707299	-1	1.07912139	-2
0.94000	2.82075248	-2	1.06998612	-1
0.95000	2.90833724	-1	1.06047372	-1
0.96000	3.01611249	0	1.05050223	0
0.97000	3.15587496	+1	1.03994687	+1
0.98000	3.35414146	+1	1.02859453	+1
0.99000	3.69563736	0	1.01599355	0
0.99900	4.84113254	-2	1.00217077	-2
0.99990	5.99158933	-1	1.00027458	0
0.99999	7.14277247	-2	1.00003321	0

Figure 5.

ANGLE	FIRST	Δ	SECOND	Δ	COS	Δ_c
0.0	1.57079633	0	1.57079633	0	1.00000000000	0
10.0	1.58284282	+2	1.55888721	+1	0.98480775400	+1
20.0	1.62302590	0	1.52379920	-1	0.93969262200	+1
30.0	1.68575034	-1	1.46746220	-1	0.86602540500	+1
40.0	1.78676915	+2	1.39314027	+2	0.76604444200	-1
50.0	1.93558109	-1	1.30553908	-1	0.64278760900	-1
60.0	2.15651566	+1	1.21105603	0	0.49999999900	-1
70.0	2.50455006	-2	1.11837774	0	0.34202014600	+3
80.0	3.15338525	0	1.04011441	+1	0.17364817900	+1

Figure 6.

ANGLE	FIRST	Δ	SECOND	Δ	COS	Δ_c
81.0	3.25530295	+1	1.03378948	+2	0.15643446500	0
82.0	3.36986804	+1	1.02784364	+2	0.13917310000	0
83.0	3.50042252	+2	1.02231260	+1	0.12186934100	-2
84.0	3.65185596	-1	1.01723693	+1	0.10452846500	+2
85.0	3.83174199	-1	1.01266351	0	0.08715574354	+8
86.0	4.05275817	0	1.00864795	-1	0.06975647314	-6
87.0	4.33865401	+3	1.00525857	-2	0.05233595424	-20
88.0	4.74271717	-9	1.00258407	-2	0.03489949954	+28
89.0	5.43490974	-9	1.00075156	-2	0.01745240764	+14

Figure 7.

ANGLE	FIRST	Δ	SECOND	Δ	COS	Δ_c
89.1	5.54020302	-1	1.00062176	-2	0.01570731734	0
89.2	5.65792447	+8	1.00050275	-1	0.01396217904	-13
89.3	5.79140015	+21	1.00039489	0	0.01221699804	-28
89.4	5.94550061	-20	1.00029657	-1	0.01047178624	+21
89.5	6.12777873	-9	1.00021429	0	0.00872653620	+70
89.6	6.35088547	+10	1.00014257	-1	0.00698125960	-70
89.7	6.63853776	+42	1.00008414	-1	0.00523596174	-209
89.8	7.04397887	-81	1.00003987	0	0.00349065421	-279
89.9	7.73711124	-81	1.00001102	0	0.00174532976	+139

Figure 8.

ANGLE	FIRST	SECOND	COS
90.0	**********	1.00000000	
90.0			

COMPLEMENTARY PARAMETER B= 0.000000000E 00 IS OUT OF RANGE IN ELK-JOB TERMINATE

Figure 9.

```
/RESEQ ELLP3
      D.P. M,E,K
1     FORMAT(E20.6)
2     READ 1,M
      CALL ELLP(M,K,E)
3     FORMAT(3H K=,E20.6,3H E=,E20.6)
      PRINT 3,K,E
      GO TO 2
      END
      SUBROUTINE ELK(M,K)
      D.P. K,M,B,C,D,F,A
      F=SQRT(1.0-M)
      B=1.0
      F=((1.0-F)/(1.0+F))
      K=1.0
      A=F*F
      D=1.0
1     C=(B/(B+1))**2
      D=C*D*A
      K=K+D
      B=B+2.0
      IF(D.GT.1E-20) GO TO 1
      K=K*1.5707963267948966*(1.0+F)
      RETURN
      END
      SUBROUTINE ELE(M,K)
      D.P. K,M,B,C,D,F,A
      B=1.0
      F=SQRT(1.0-M)
      F=((1.0-F)/(1.0+F))
      A=F*F
      D=A/4.0
      K=1.0+D
1     C=(B/(B+3.0))**2
      D=C*D*A
      K=K+D
      B=B+2.0
      IF(D.GT.1E-20) GO TO 1
      K=K*1.5707963267948966/(1.0+F)
      RETURN
      END
      SUBROUTINE ELLP(M,K,E)
      D.P. F(3),G(3),H(3),M,K,E,A,B
      B=M
      DO 1 I=1,3
      A=SQRT(1-B)
      F(I)=1+A
      H(I)=2/(1+A)
      G(I)=H(I)*A
1     B=((1-A)/(1+A))**2
      CALL ELK(B,K)
      CALL ELE(B,E)
      E=F(1)*F(2)*F(3)*E-(F(1)*F(2)*G(3)+F(1)*G(2)*H(3)+G(1)*H(2)
     *H(3))*K
      K=H(1)*H(2)*H(3)*K
      RETURN
      END
```

Figure 10.

```
ÆCODE ELLP3
   560
M = .00000001
K=  .15707963307218795E 01 E=  .15707963228679061E 01
M = .0000001
K=  .15707963660647964E 01 E=  .15707962875249847E 01
M = .000001
K=  .15707967194941894E 01 E=  .15707959340957412E 01
M = .00001
K=  .15708002538077987E 01 E=  .15707923997967117E 01
M = .0001
K=  .15708355989121459E 01 E=  .15707570561503829E 01
M = .001
K=  .15711892469233383E 01 E=  .15704035546514160E 01
M = .01
K=  .15747455615173461E 01 E=  .15668619420216629E 01
M = .1
K=  .16124413487202102E 01 E=  .15307576368977569E 01
M = .2
K=  .16596235986105117E 01 E=  .14890350580958648E 01
M = .3
K=  .17138894481787786E 01 E=  .14453630644126652E 01
M = .4
K=  .17775193714912399E 01 E=  .13993921388974456E 01
M = .5
K=  .18540746773013980E 01 E=  .13506438810476806E 01
M = .6
K=  .19495677498060134E 01 E=  .12984280350469100E 01
M = .7
K=  .20753631352924591E 01 E=  .12416705679458343E 01
M = .8
K=  .22572053268208404E 01 E=  .11784899243278442E 01
M = .9
K=  .25780921133481506E 01 E=  .11047747327040893E 01
M = .95
K=  .29083372484445388E 01 E=  .10604737277662719E 01
M = .96
K=  .30161124924776275E 01 E=  .10505022269844482E 01
M = .97
K=  .31558749478918197E 01 E=  .10399468608930877E 01
M = .98
K=  .33541414456991409E 01 E=  .10285945190307708E 01
M = .99
K=  .36956373629898513E 01 E=  .10159935450252166E 01
M = .999
K=  .48411325605502166E 01 E=  .10021767908344512E 01
M = .9999
K=  .59915893405099173E 01 E=  .10002745824306549E 01
M = .99999
K=  .71427724505784512E 01 E=  .10000332138990753E 01
M = .999999
K=  .82940514635455680E 01 E=  .10000038970261672E 01
M = .9999999
K=  .94453423974408320E 01 E=  .10000004472671157E 01
M = .99999999
K=  .10596634749024688E 02 E=  .10000000504831680E 01
M =
```

Figure 11.

```
K=   .15707963267948944E 01 E=   .15707963267948944E 01
M = 10
K=   .15828428043383439E 01 E=   .15588871966615914E 01
M = 20
K=   .16200258991241928E 01 E=   .15237992052597763E 01
M = 30
K=   .16857503548125823E 01 E=   .14674622093394332E 01
M = 40
K=   .17867691348850021E 01 E=   .13931402485238265E 01
M = 50
K=   .19355810960047058E 01 E=   .13055390942978027E 01
M = 60
K=   .21565156474996231E 01 E=   .12110560275684661E 01
M = 70
K=   .25045500790016128E 01 E=   .11183777379698745E 01
M = 80
K=   .31533852518878112E 01 E=   .10401143957065043E 01
M = 81
K=   .32553029421435226E 01 E=   .10337894623907501E 01
M = 82
K=   .33698680266584051E 01 E=   .10278436197408340E 01
M = 83
K=   .35004224991717971E 01 E=   .10223125881675827E 01
M = 84
K=   .36518559694787094E 01 E=   .10172369183410156E 01
M = 85
K=   .38317419997840988E 01 E=   .10126635062343963E 01
M = 86
K=   .40527581695493884E 01 E=   .10086479569078999E 01
M = 97
K=   .35004224991718246E 01 E=   .10223125881675827E 01
M = 87
K=   .43386539759996475E 01 E=   .10052585872091476E 01
M = 88
K=   .47427172652787094E 01 E=   .10025840855275512E 01
M = 88.1
K=   .47938936793338353E 01 E=   .10023603331424942E 01
M = 88.2
K=   .48478485363025955E 01 E=   .10021450964784072E 01
M = 88.3
K=   .49048991835163269E 01 E=   .10019385221423065E 01
M = 88.4
K=   .49654207227696609E 01 E=   .10017408021770492E 01
M = 89
K=   .54349098296252899E 01 E=   .10007515777018264E 01
M = 89.1
K=   .55402030251903152E 01 E=   .10006217753080154E 01
M = 89.2
K=   .56579243899833175E 01 E=   .10005027597561798E 01
M = 89.3
K=   .57913999403597030E 01 E=   .10003948925620452E 01
M = 89.4
K=   .59455808101587615E 01 E=   .10002985756079849E 01
M = 89.5
K=   .61277788245260818E 01 E=   .10002142962614456E 01
M = 89.6
K=   .63508853799722747E 01 E=   .10001425809695159E 01
M = 89.7
K=   .66385373456341583E 01 E=   .10000841452274887E 01
M = 89.8
K=   .70439796803819920E 01 E=   .10000398580959457E 01
M = 89.9
K=   .77371120574368960E 01 E=   .10000110227445724E 01
M = 90
K=   .19408121051954432E 02 E=   .99999999999999060E 00
```

Figure 12.