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STUDY OF THE POSSIBILITY OF AMPLIFICATION OF A VERY RELATIVISTIC SYNCHROTRON OR GYROMAGNETIC RADIATION BY A NONCOLLECTIVE PLASMA

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SUMMARY

An analysis is made of the possibilities of amplification of synchrotron radiation taking into account the detailed form of the radiation pattern; collective effects are neglected. While amplification is never possible in one polarization, it may occur in another in the presence of particle beams with angular spread of the order of the radiation pattern's aperture angle.

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1. Study of the possibilities of amplification of synchrotron radiation, as far as we know, was made only by utilizing two essential simplifications.

a) One assumes that for a particle with a given dynamic state, emissivity takes place exclusively in the velocity direction:

$$Q_\lambda (E, \theta, \Phi) = P_\lambda (E)\, \delta (\theta - \dot{\Phi}) \qquad (1)$$

i.e., one ignores the detailed structure of the radiation pattern of the particle.

b) One assumes the distribution function as being isotropic, i.e., in the absorption operator $\mathcal{O}_\lambda$ (*cf.* (1) (2) (3))

$$\mathcal{O}_\lambda = \frac{\partial}{\partial E} + \frac{\Delta\, \Phi}{\hbar \omega} \frac{\partial}{\partial \Phi} .$$

Only the part $\frac{\partial}{\partial E}$ is retained.

Certain authors (Twiss [1], Wild, Weiss, Smerd [2]) do not advance this hypothesis, but show that the anisotropy term is generally negligible, for the impulsion transfer proceeds in the direction of particle velocity. Here we have an approximation that stems also from the simplification (1). The form obtained for ϖ_λ in a preceding work [3] justifies this approximation in the variables (E, Φ).

The greater the energy of the considered particles, the more correct are the simplifications. However, we felt that it would be more appropriate not to make use of them so as to study the aspect of the phenomena for nonultra-relativistic particles. The results obtained blend with those of Wild, Weiss and Smerd [2] at very great energy threshold, but one finds a possibility of amplification of one of the polarizations, when particles form a sufficiently directive beam.

In order to conduct this study we start from the expressions obtained in the work [3], which takes account of anisotropy effects of the distribution function assumed nevertheless to be "gyrotropic", that is, of revolution about the uniform and constant magnetic field $\vec{B}_0$ into which the plasma is immersed. The system is assumed to be homogenous. We neglected the collisions and the collective effects: the only phenomena retained are the interactions between the transverse polarization oscillators and plasma particles. The absorption coefficient will then be written:

$$b_\lambda = \sum_{S=1}^{\infty} \sum_{j} \frac{4\pi^2 e_j^2}{V \nu_\lambda} \int_0^{\infty} \int_{-\infty}^{+\infty} 2\pi p_{j\perp}\, dp_{j\perp}\, dp_{j\parallel}$$
$$\times \left| P_{\lambda j}^{s} \right|^2 \delta\left(s \left| \Omega^*_j \right| - (\nu_\lambda - k_{\lambda\parallel} v_{j\parallel})\right) \varpi_\lambda g \tag{2}$$

$2\pi p_\perp g$ is the distribution function of one particle pulses, $|P_{\lambda j}^s|^2$ is the emissivity on the oscillator λ by a particle in a given dynamic state $(p_\perp, p_\parallel)$.

In the polarization ϕ(azimuthal polarization, perpendicular to the magnetic field), we have

$$|P_{\lambda j}^{s}|^2 = v^2_{j\perp}\; J'^2_s \left(\frac{v_{j\perp}}{c} \frac{\nu_\lambda}{\Omega^*_j} \sin\theta_\lambda \right) \tag{3}$$

and in the polarization θ, perpendicular to the former, we have

$$|P_{\lambda j}^{s}|^2 = c^2 \left(-\frac{s\,\Omega^*_j}{\nu_\lambda} \operatorname{cotg}\theta + \frac{v_{j\parallel}}{c} \sin\theta \right)^2 J_s^2\,(t_{\lambda j}) \tag{4}$$

In expression (2) we may pass to dimensionless variables

$$\varepsilon_j = \left(1 + \frac{p_{j\perp}^2 + p_{j\parallel}^2}{m^2 c^2}\right)^{1/2}$$

$$x_{j\parallel} = \frac{p_{j\parallel}}{m_j c}$$

$$u = \frac{\nu_\lambda}{|\Omega_j|}$$

In these variables operator $\mathfrak{D}_\lambda$ is written :

$$\mathcal{O}_\lambda g = -\nu_\lambda \left(\frac{\partial g}{\partial \varepsilon} + \cos\theta \frac{\partial g}{\partial x_\parallel}\right).$$

Taking into account that

$$\delta\left(s\,|\Omega^*_j| - (\nu_\lambda - k_{\lambda\parallel} v_{j\parallel})\right) = \frac{\varepsilon}{\nu_\lambda}\,\delta\left(\frac{s}{u} - \varepsilon + \cos\theta x_\parallel\right),$$

we obtain

$$b_\lambda = -\sum_{S=1}^{\infty}\sum_j \frac{4\pi^2 e_j^2 m_j^2 c^3}{\nu_\lambda}\int_{-\infty}^{+\infty}\int_0^{\infty} 2\pi\,\varepsilon^2\,d\varepsilon\,dx_\parallel$$
$$\times\,\delta\left(\frac{s}{u} - \varepsilon + \cos\theta x_\parallel\right)|P^s_{\lambda j}|^2\left(\frac{\partial g}{\partial \varepsilon} + \cos\theta\frac{\partial g}{\partial x_\parallel}\right) \tag{5}$$

The study of this expression's sign assumes the study of the kernel $N(\varepsilon, x_\parallel) = \varepsilon^2\,|P^s_{\lambda j}|^2$ and that of the sign of the expression

$$\frac{\partial g}{\partial \varepsilon} + \cos\theta\,\frac{\partial g}{\partial x_\parallel}$$

2. STUDY OF THE KERNEL ON THE "INTERACTION STRAIGHT LINE"

Taking into account the Doppler effect, oscillator λ interacts "as far as" a harmonic of number s with particles whose dynamic states are situated on a straight line $D_{\lambda,s}$ of the plan $(\varepsilon, x_\parallel)$:

$$\varepsilon = \cos\theta\, x_\parallel + \frac{s}{u} \tag{6}$$

When s varies, these lines have a regular spacing between them. The greater the pulsation ν_λ of the oscillator λ the smaller the margin. One may notice that the region of the useful plane $(\varepsilon, x_\parallel)$ is defined by $\varepsilon^2 \geqslant 1 + x_\parallel^2$ and represents the interior of an hyperbola. The interaction lines intersect it always at two points of the upper branch (eventually at infinity if cosθ = 1).

Taking into account the interaction condition (6), the kernel assumes on the line $D_{\lambda,s}$ the form

$$N(\varepsilon, x_\parallel) = c^4\left(-\frac{s}{u}(\cot\theta + \operatorname{tg}\theta) + \varepsilon\operatorname{tg}\theta\right)^2 J_s^2(t_{\lambda j}) \tag{7}$$

in the polarization θ and

$$N(\varepsilon, x_\parallel) = \frac{c^2}{u^2\sin^2\theta}\,x_\perp^2\;u^2\sin^2\theta\,J_s'^2(u\sin\theta\, x_\perp) \tag{8}$$

in the polarization ϕ. We postulated $t_{\lambda j} = u\sin\theta\, x_\perp$.

We have $\varepsilon^2 = 1 + x_\perp^2 + x_\parallel^2$, which defines $x_\perp$ as a function of two variables ε and $x_\parallel$.

It is easy to verify that $u \sin\theta\, x_\perp$ does not exceed on $D\lambda,s$ the value of $\underline{s}$ by making intervene the angle Ψ of particle velocity with $\vec{B}_0$.

On the other hand, the curves of Eq. $x_\perp$ = const are branches of hyperbolas with asymptote $x_\parallel = \pm\varepsilon$, just as the limit hyperbola which corresponds anyhow to $x_\perp = 0$ They are interior to each other in the order of $x_\perp$ which acts as their parameter forming a linear beam, bi-tangent to the right at infinity. These remarks allow us to see that the point on $D\lambda,s$ realizing the maximum of $x_\perp$, is at the contact of $D\lambda,s$ and the hyperbola of the family tangent to it. The region of these points is the conjugate diameter of the direction of $D\lambda,s$ relative to all the hyperbolas of the beam. The property of conjugation shows that two points of same $x_\perp$ on $D\lambda,s$ are symmetrical relative to the point realizing $x_\perp$ maximum. One may easily see that this median point has for coordinates:

$$\varepsilon_m = \frac{s}{u \sin^2 \theta}$$
$$x_{\parallel m} = \frac{s}{u \sin \theta} \operatorname{cotg} \theta \tag{9}$$

and realizes:

$$u \sin \theta\, x_{\perp\max} = s\left(1 - \frac{u^2 \sin^2 \theta}{s^2}\right)^{1/2} \tag{10}$$

Thus we make appear a parity property of expression (8), which suggest to use for axes the direction of $D\lambda,s$ and the conjugate one ($\cos\theta \neq 1$).

We shall postulate

$$\left\{\begin{aligned} p_{\lambda j} &= \varepsilon_j - \frac{1}{\cos\theta_\lambda} x_{j\parallel} \\ q_{\lambda j} &= \varepsilon_j - \cos\theta_\lambda\, x_{j\parallel} \end{aligned}\right. \quad \text{say} \quad \left\{\begin{aligned} \varepsilon_j &= \frac{1}{\sin^2\theta_\lambda}(q_{\lambda j} - p_{\lambda j}\cos^2\theta_\lambda) \\ x_{j\parallel} &= \frac{1}{\sin^2\theta_\lambda}\cos\theta_\lambda\,(q_{\lambda j} - p_{\lambda j}) \end{aligned}\right. \tag{11}$$

In the following we shall omit indices λj, but $p_{\lambda j}$ must not be confused with the particle's momentum.

In these variables we obtain

$$\omega_\lambda = \sin^2\theta \frac{\partial}{\partial q_\lambda}$$
$$u \sin\theta\, x_\perp = uq\left(1 - \frac{\sin^2\theta + p^2\cos^2\theta}{q^2}\right)^{1/2} \tag{12}$$

in particular on an interaction line $D\lambda,s$ of equation

$$q = \frac{s}{u} \tag{14}$$

we obtain an expression quite similar to (10)

$$u \sin\theta\, x_\perp = s\,x = s\left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2} \tag{15}$$

by postulating

$$\begin{aligned} \alpha_p^2 &= u^2 \sin^2\theta + u^2 \cos^2\theta\, p^2 \\ \alpha_0^2 &= u^2 \sin^2\theta \end{aligned} \tag{16}$$

b_λ then has for expression:

$$b_\lambda = -\sum_j \sum_{s=1}^{\infty} \frac{4\pi^2 e_j^2}{V\nu_\lambda} m_j^2 c^2 \int_{-\infty}^{+\infty} dp \int_{\sin\theta}^{\infty} dq$$
$$\frac{\sin 2\theta \cos\theta}{\sin^2\theta}\, \delta\left(\frac{s}{u} - q\right) 2\pi N(p,q) \frac{\partial g}{\partial q}$$

In these variables coefficient b_λ has a simple form. We now must study the function N(p, q) in each polarization.

In the polarization φ:

$$N(p,q) = N_\varphi(p,q) = \frac{c^2}{\alpha_0^2} s^2 \left(1 - \frac{\alpha_p^2}{s^2}\right) J'^2_s \left(s\left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2}\right) \tag{18}$$

In the polarization θ:

$$N(p,q) = N_\theta(p,q) = \frac{c^2}{\alpha_0^2} u^2 p^2 \cos^2\theta\, J_s^2 \left(s\left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2}\right) \tag{19}$$

As one could have expected these functions are even with respect to p. We should note that (19) is zero at p = 0. This is explained by the easily verifiable fact that in this polarization the radiation pattern is divided into two lobes. In the direction of particle velocity the emitted power is zero (Fig.1).

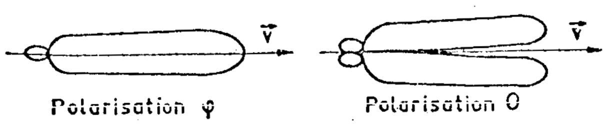

Fig.1

These two properties are not linked in a perfectly trivial fashion since the radiation pattern yields for a given dynamic state the emitted power on different oscillators, whereas kernel N(p, q) yields the power emitted on a single, quite precise an oscillator, as a function of the dynamic state of the emitted particle.

It is easy to see that for a fixed s, kernels as a function of p, have the following course (Fig.2, next page).

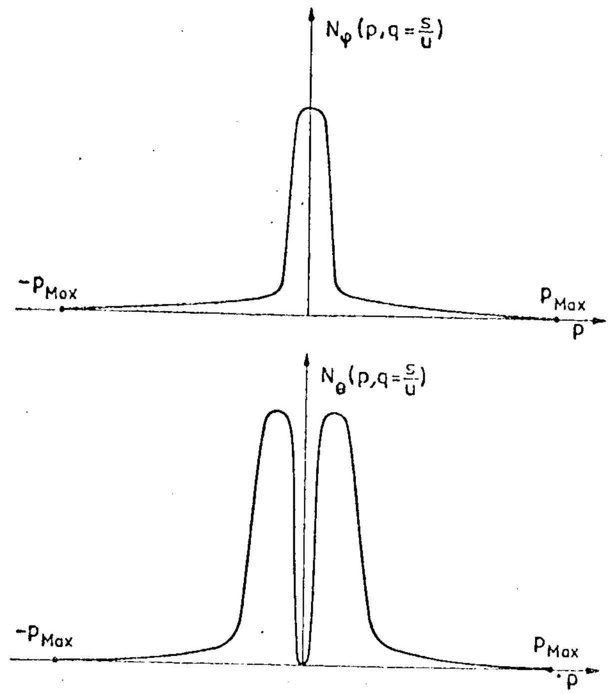

Fig.2

3. STUDY OF THE KERNEL WHEN s VARIES

When s increases, the relative width of the peaks decreases. Let us now see the aspect of a N(p, q) cutaway for p = constant. We must study

$$\frac{c^2}{\alpha_0^2}\, s^2 \left(1 - \frac{\alpha_p^2}{s^2}\right) J'^2_s \left(s\left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2}\right) \tag{20}$$

as a function of p in the polarization ϕ and

$$\frac{c^2}{\alpha_0^2}\, u^2 p^2 \cos^2\theta\; J_s^2 \left(s\left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2}\right) \tag{21}$$

in the polarization θ.

a) Case of Polarization θ

Case (21) is particularly clear. Indeed (cf.[5], p.260)

$$J_s(sx) \leqslant J_s(s) \text{ for } x \leqslant 1$$
$$J_s(s) < \frac{\Gamma(1/3)}{\pi\, 2^{2/3}\, 3^{1/6}}\, s^{-1/3}$$

Thus, $N_\theta(p, q)$ approaches zero as $s \to \infty$.

On the other hand, for $s = \alpha_p$, $N_\theta(p, \alpha_p) = 0$; ($x_1 = 0$). Thus there is at least one maximum that has to be situated. To do so one may employ approximation formulas for Bessel function of great index. The argument being close to the index, one must be very cautious in the choice of approximations.

We may use the Cauchy-Meissel approximation (cf. [5], p.245) in the zero order. One must then be assured that

$$s - s\left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2} \gg s^{1/3}$$

say:

$$s \gg \alpha_p^{3/2}$$

We then have :

$$J_s\left(s\left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2}\right) \simeq \frac{\Gamma(1/3)}{\pi\, 2^{2/3}\, 3^{1/6}}\, s^{-1/3}.$$

In principle, the Carlini approximation is valid for a fixed x smaller than the unity. The closer x to the unity, the greater the minimum rank of S starting with which it constitutes a good approximation. We have not found any estimate of S as a function of x. However, we shall make use of this approximation when

$$s \ll \alpha_p^{3/2}.$$

Then we have

$$J_s\left(s\left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2}\right) \simeq \frac{\alpha_p^{-1/2}}{\sqrt{2\,\pi}}\left(\exp \alpha_p\right)\left(\frac{1 - \frac{\alpha_p}{s}}{1 + \frac{\alpha_p}{s}}\right)^{s/2}.$$

The Cauchy approximation yields a decreasing behavior for $s \gg \alpha_p^{3/2}$ and that of Carlini yields an increasing one for $s \ll \alpha_p^{3/2}$. Therefore, the maximum must be located in the neighborhood of $s = \alpha_p^{3/2}$, but function J_S (sx) is represented by neither of the preceding approximations in this domain (transition zone). The Nicholson approximation or its improved version by Watson are not algebraic. Thus we shall renounce here any detailed study of its behavior in this region. We shall simply consider that the maximum will be located at $s = \alpha_p^{3/2}$ and we shall evaluate it with the aid of the Nicholson formula:

$$J_s(s\,x) \simeq \frac{1}{\pi}\left(\frac{2}{3}\right)^{1/2}\left(\frac{1-x}{x}\right)^{1/2} \times K_{1/3}\left(\frac{2^{3/2}\, s^{3/2}\,(1-x)^{3/2}}{3\, s^{1/2}\, x^{1/2}}\right)$$

for

$$x = \left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2} \text{ et } s = \alpha_p^{3/2},$$

the argument of $K_{1/3}$ becomes a constant independent of p:

$$J_s(s\,x) \sim (\pi\, 2^{1/2}\, 3^{1/2})^{-1}\, K_{1/3}(2^{-3/2}) \times \alpha_p^{-1/2}$$

For a fixed p, $N_\theta(p,q)$ maximum thus has the value:

$$\frac{c^2}{\alpha_0^2}\, u^2 p^2 \cos^2\theta \times \frac{c^{te}}{\alpha_p} \sim \frac{p^2}{(p^2 u^2 \cos^2\theta + u^2 \sin^2\theta)^{1/2}}$$

This function increases from zero to infinity. Surface $N_\phi(p,q)$ has the shape of a basin of which the edges ascend to infinity. Figure 3 gives a perspective view of the surface N_θ.

b) Case of Polarization ϕ

Let us now examine the second polarization. The kernel is written with s fixed:

$$N(p,s) = \frac{c^2}{\alpha_0^2} s^2 \left(1 - \frac{\alpha_p^2}{s^2}\right) J'^2_s \left(s\left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2}\right),$$

N_ϕ is maximum at p = 0; the greater s, the sharper N_ϕ.

When p is fixed, we may utilize the Cauchy approximation for $J_s(sx)$ if $s \gg \alpha_p^{3/2}$

$$J'_s(sx) \simeq J'_s(s) \simeq \frac{3^{1/6}}{\pi\, 2^{2/3}} \Gamma\left(\frac{2}{3}\right) s^{-2/3} \text{ and } s^2\left(1 - \frac{\alpha_p^2}{s^2}\right) J'^2_s(sx) \sim K\, s^{+2/3}.$$

$N_\phi(p,s)$ grows asymptotically.

For $s \ll \alpha_p{}^{3/2}$, we utilize the Carlini formula, thus obtaining

$$s^2\left(1 - \frac{\alpha_p^2}{s^2}\right) J'^2_s\left(s\left(1 - \frac{\alpha_p^2}{s^2}\right)^{1/2}\right) \simeq \frac{1}{2}\, \frac{\alpha_p}{\pi}\, e^{2\alpha_p} \left(\frac{1 - \frac{\alpha_p}{s}}{1 + \frac{\alpha_p}{s}}\right)^s.$$

This function increases with s.

Thus, one may think without risking the error that function $N_\phi(p,s)$ is continually rising with respect to s, so long as p is fixed.

The shape of function N_ϕ is represented in Figure 4.

4. "THE CONTINUOUS SPECTRUM" APPROXIMATION

The higher the frequency, the closer together the interaction lines so that one may often do away with the summation on s and the function δ in the expression (17) for b_λ. This is tantamount to taking the "continuous spectrum" limit". We then have:

$$b_\lambda = -\sum_{j=1}^{N} \frac{4\pi^2 e_j^2}{V \nu_\lambda} m_j^2 c^3 \int_{-\infty}^{+\infty} dp \int_{\sin\theta}^{\infty} dq \times 2\pi \frac{\sin 2\theta \cos\theta}{\sin^3\theta} N(p,q) \frac{\partial g}{\partial q} \tag{22}$$

This approximation is valid if there are many interaction lines in the energy interval $\Delta\varepsilon$ of the variation scale of the distribution function, say:

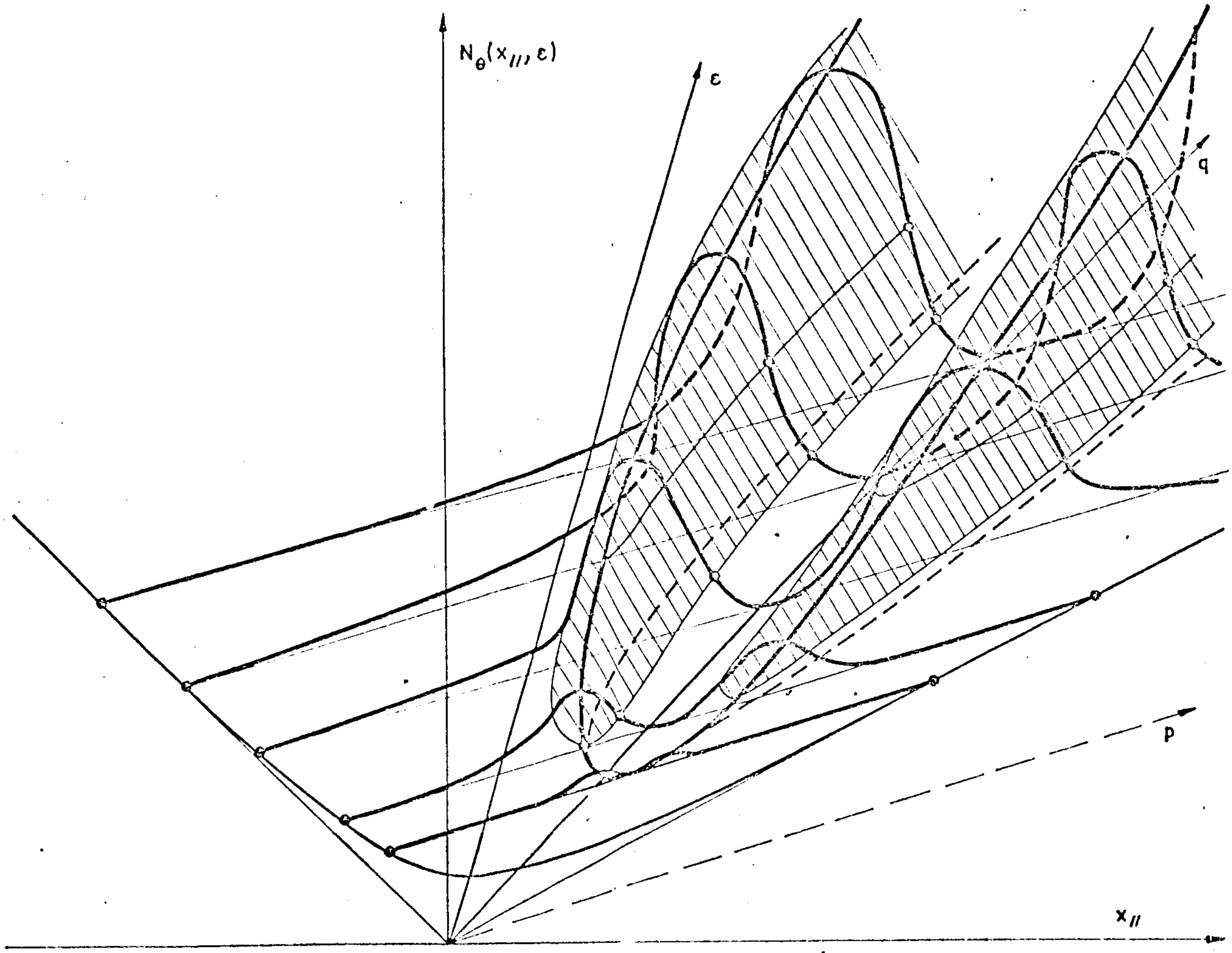

Fig.3

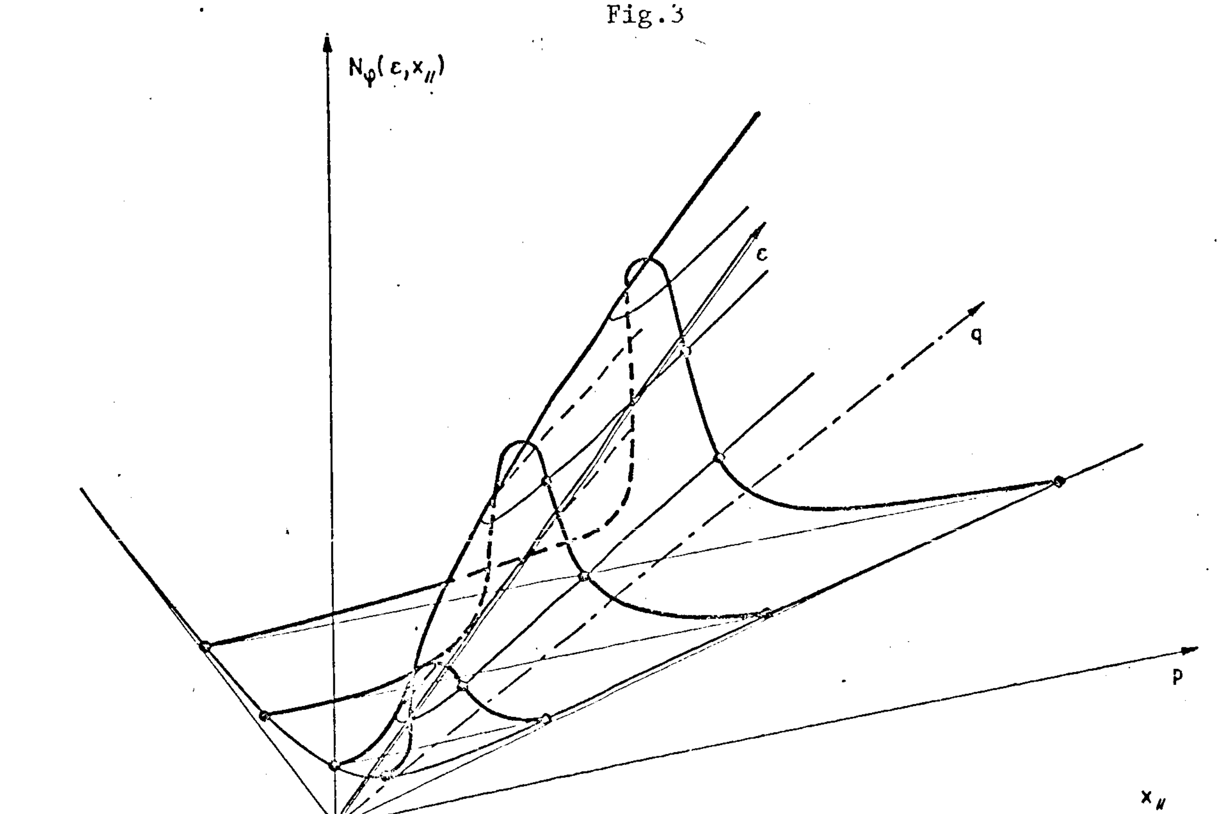

Fig.4

$$\frac{\Delta\,\varepsilon}{u^{-1}} \sim \frac{u}{\dfrac{\partial g}{\partial \varepsilon}} \gg 1$$

For rather regular functions $\dfrac{\partial g}{\partial \varepsilon} \sim \dfrac{1}{\varepsilon}$, and we then have the criterion

$$\varepsilon u \gg 1$$

In general we shall be interested in frequency amplifications in which the considered particles are best emitters, i.e., in frequencies close to the critical frequency for energies typical of plasma particles (cf. [4]). The preceding criterion then states that the number of the critical harmonic must be great ahead of the unity.

This criterion is not always applicable, for the estimate made by us there concerning $\partial g/\partial \varepsilon$ may result faulty. This question has to be reconsidered every time.

β	0,5	0,6	0,7	0,8	0,9	0,95	0,96	0,97	0,98	0,99	0,999	0,9999
$1-\beta^2$	0,75	0,64	0,53	0,36	0,19	9,6 10^{-2}	7,8 10^{-2}	5,72 10^{-2}	4 10^{-2}	2,25 10^{-2}	25 10^{-4}	2 10^{-4}
$(1-\beta^2)^{1/2}$	0,866	0,8	0,73	0,6	0,435	0,31	0,28	0,24	0,2	0,15	0,05	0,014
$\Delta\theta^{\circ}$ $\left\{ \Delta\theta_{rad} \simeq \sqrt{(1-\beta^2)} \right\}$	Emissivity direction almost inexistant					17°	16°	13°	11°	8°30'	3°	0°42'
$n_c = \sin^3\theta(1-\beta^2)^{-3/2} \sim \frac{1}{10}(1-\beta^2)^{-3/2}$	1	1	1	1	1,2	3,3	4,5	7,3	12,5	29	800	36 000
$u_c - \frac{\nu c}{\Omega j} = n_c\sqrt{1-\beta^2}$	0,86	0,8	0,73	0,6	0 535	1,04	1,26	1,75	2,5	4,35	40	500
Remarks	Low cyclotron and gyromagnetic energy					Great gyromagnetic energy				Synchrotron		
	Only a limited number of harmonics participate in the interaction					The "continuous spectrum" approximation is the most often valid				Application of "contin. spectrum" nearly always correct		

TABLE 1

Table 1 gives the order of magnitude of the quantities of interest: the width $\Delta\theta$ of the associated synchrotron cone is computed as a function of $\beta = v/c$, typical of the population envisaged.

We may see that for $\beta < 0.95$, radiation is not seriously concentrated within an angle surrounding the velocity. Associated to these typical energies are critical frequencies (cf.[4]), giving an order of magnitude of the "most interacting" frequencies with particles.

We computed for each value of β the order of magnitude of parameter $\underline{u}$ for the critical frequency and the number n_c of the critical harmonic (this makes sense only if $n_c > 1$). It is thus possible to make more precise the extent to which we are faced with synchrotron radiation and to what measure the "continuous spectrum" approximation is correct.

We may see that the directivity condition required to bring about the amplification is not too severe within a fairly broad region. This region covers even an energy range where radiation is of synchrotron type, i. e. toward $(1 - \beta) \sim 10^{-3}$.

6. ULTRARELATIVISTIC CASE

We must now show that at the limit of very high energies amplification is not possible.

We may indeed consider that in this case neither $\underline{g}$ nor $\frac{\partial g}{\partial q}$ vary for a $\underline{q}$ on an angle of the order $\Delta\theta$ synchrotron. The quantity $\partial g/\partial q$ may be taken out of the integral over $\underline{p}$ in (22). It is easy to be convinced that

$$\int N_0(p, q)\, dp = \mathfrak{N}(q).$$

is a rising function of $\underline{q}$. We obtain

$$b_\lambda \sim - \int dq \frac{\partial g}{\partial q} \mathfrak{N}(q) \tag{23}$$

For reasons already indicated and in agreement with [2], we find a positive result. We therefore may state that within this limit the details of the radiation pattern become too fine relative to those of the distribution function.

7. EXAMPLES

We shall study in Appendix I the case of a thermal beam launched in the direction of the field with a relativistic velocity, and show that such a distribution may in no case lead to the amplification, no matter what the temperature of the beam.

In Appendix II we shall also consider the case of distribution, where an orthogonal particle impulse is concentrated around a mean nonzero value, by taking interest in an oscillator perpendicular to the magnetic field. The various situations are classified with the aid of three parameters:

- the mean thermal energy of particles

$$t = \frac{kT}{mc^2}$$

- the directed energy of the beam

$$d = \left(1 + \frac{P_\perp^2}{m^2c^2}\right)^{1/2}$$

- the energy, starting with which particles interact in a notable fashion with the oscillator λ (ε_3 in the Appendix II)

$$i = \varepsilon_3$$

We reach the following criteria:

- if i > d, there can be no amplification (particles insufficiently energetic;

- if i < d and $(d/i)^{4/3} - 1 \gg 2t$, there is no amplification (Beam too cold);

- if i < d and $(d/i)^{4/3} - 1 < 2t$, no amplification, the beam being too hot;

- if i < d and $(d/i)^{4/3} - 1 \gtrsim 2t$, there is amplification;

- if i < d and $(d/i)^{4/3} - 1 \ll 2t$, we find ourselves in a case similar to that of very high energies. There is no amplification.

8. CONCLUSION

We note the possibility of amplification of gyromagnetic or nearly synchrotron radiation by jets of energetic particles rather directive only on one of the polarizations. However, we note that favorable conditions are rather difficult to find. Notwithstanding, this mechanism is of interest for lending itself fairly well to the creation of an anomalously polarized radiation.

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See Appendices I and II .....

APPENDIX I

CASE OF A BEAM PARALLEL TO THE MAGNETIC FIELD

It is necessary to carefully identify function g intervening in (5).

R. G. Synge [6] gives the following expression (Maxwell-Juttner distribution) :

$$\begin{aligned} & f\, dr_1\, dr_2\, dr_3\, dM_1\, dM_2\, dM_3 \\ & = K\ exp\ \xi\ M_\alpha\lambda_\alpha\ dr_1\, dr_2\, dr_3\, dM_1\, dM_2\, dM_3 \\ & \xi = \frac{c^2}{kT} \end{aligned}$$

For a = 1, 2, 3 $\quad M_\alpha = \dfrac{m^* u_\alpha}{c} = m x_\alpha$

For a = 4 $\quad M_4 = im^* = im\varepsilon = mx_4$

λ_α is the mean velocity four :

$$\begin{aligned} & \lambda_\alpha \lambda_\alpha = -1 \\ & \lambda_1 = \lambda_2 = 0;\ \ \lambda_3 = \frac{V_3\, m^*}{mc} = X_3 = X_{\parallel} \\ & \lambda^2_4 = -1 - X_3^2 \quad \Rightarrow \quad \lambda_4 = i\,(1 + X_3^2)^{1/2} \end{aligned}$$

In order to recognize function g, we must pass in the variables

$$\begin{cases} x_{\parallel} = x_3 \\ x_{\perp} = (x^2_1 + x^2_2)^{1/2} \\ 0 \ = \text{Arc tg}\, \dfrac{x_1}{x_2} \end{cases}$$

Jacobian calculation gives:

$$f\, dr_1\, dr_2\, dr_3\, dx_1\, dx_2\, dx_3\, m^3 = K \exp m_\alpha\ \xi\ (x_\alpha\, \lambda_\alpha)\ x_\perp\ \ dx_\perp\ dx_\parallel\ d0\ dr_1\, dr_2\, dr_3.$$

Whence

$$\begin{aligned} g &= K \exp m\, \xi\, x_\alpha \lambda_\alpha \\ &= K \exp \frac{mc^2}{kT} \left(x_\parallel X_\parallel - (1 + X^2_\parallel)^{1/2}_{\varepsilon} \right) \end{aligned}$$

The contour lines of g are the straight lines:

$$\varepsilon = x_\parallel \frac{X_\parallel}{(1 + X^2_\parallel)^{1/2}} + H$$

Their inclination does not exceed 1 and approaches the unity when $X_\parallel \to \infty\ (V_\parallel \to c)$, the smaller H, the more important are the values assumed by g .

Among these lines the one that crosses the useful region ($\varepsilon^2 \geqslant 1 + x_{\parallel}{}^2$) and possesses the smallest ordinate at origin, is tangent to the limit hyperbola.

It is easy to see that the variation of g assumes a decreasing trend pursuant to any line of direction interior to the asymptote angle, so that in this case the amplification is not possible, no matter what the temperature. This result stems from the fact that the direction of the beam was taken according to B_0, thus rejecting the most probable state at the limit of the useful region.

APPENDIX II

CASE OF A DISTRIBUTION WITH AVERAGE OR ZERO ORTHOGONAL IMPULSION

Let us examine the case of an f.d.d. (?) constituting an overpopulation in the vicinity of the state $(X_\perp, X_\parallel)$.

The Maxwell-Jüttner distribution function provides an example of such a distribution which, unfortunately, is not gyrotropic as required by theory that yielded the expression for b_λ. This quality can be obtained by performing integration of the orthogonal beam velocity on the polar angle. This amounts to taking the sum of a large number of distributions differing by this angle. We thus obtain the distribution

$$K \frac{1}{2\pi} \int d\alpha \exp \frac{mc^2}{kT} \left(x_\parallel X_\parallel - x_\perp X_\perp \sin(\alpha - \varphi) - \left(1 + X_\perp^2 + X_\parallel^2\right)^{1/2} \varepsilon\right)$$

ϕ being the polar angle of particle orthogonal velocity. We shall have:

$$g(\varepsilon, x_\parallel) = K \exp \frac{mc^2}{kT}\left(x_\parallel X_\parallel - \varepsilon\left(1 + X_\perp^2 + X_\parallel^2\right)^{1/2}\right) I_0 \left(\frac{mc^2}{kT} x_\perp X_\perp\right)$$

I_0 is a Bessel function, modified, of first kind and of the zero order.

It is clear that the contour lines of such a function are not simple. In the majority of cases and particularly for those of interest to us, we have $mc^2 \gg kT$, so that it will be possible to take an asymptotic approximation of the Bessel function. Taking the lowest order according to

$$I_0(x) \sim \frac{1}{\sqrt{2\pi x}} e^x$$

It is clear that this approximation will yield poor results in the neighborhodd of the limit hyperbola where $x_\perp \sim 0$. We then have

$$I_0(x) \sim I.$$

In the neighborhood of the limit hyperbola the contour lines just about overlap the straight lines

$$x_\parallel X_\parallel = \varepsilon\left(1 + X_\parallel^2 + X_\perp^2\right)^{1/2} + \text{const.}$$

In other cases we shall have to study the countour lines of

$$\exp \frac{mc^2}{kT}\left(x_{\parallel} X_{\parallel} + x_{\perp} X_{\perp} - \varepsilon\left(1 + X_{\parallel}^2 + X_{\perp}^2\right)^{1/2}\right) \times \sqrt{\frac{1}{2\pi \frac{mc^2}{kT} x_{\perp} X_{\perp}}}$$

Since we search for quantitative results, and in view of slowness of $x_{\perp}^{-1/2}$ variations relative to the exponential, the course of contour lines will be sensibly given by

$$x_{\parallel} X_{\parallel} + x_{\perp} X_{\perp} - \varepsilon\left(1 + X_{\parallel}^2 + X_{\perp}^2\right)^{1/2} = H$$

To greatest values of H correspond the greatest values of the distribution function.

These considerations tend only to provide a physical justification for the study of a distribution function of the form

$$\exp H \frac{mc^2}{kT}.$$

For simplicity let us further assume that $X_{\parallel} = 0$ and that from now on we shall postulate $X_{\perp} = a$ so as to alleviate the writings.

The contour lines constitute ellipses of equation

$$(\varepsilon + H(1 + a^2)^{1/2})^2 + a^2 x_{\parallel}^2 = a^2 (H^2 - 1)$$

They are real if $|H| > |$ and the only ones to intersect the useful region $(\varepsilon^2 - x_{\parallel}^2 - 1 \geqslant 0)$ corresponding to a negative H.

It is of interest to study their intersection with the limit hyperbola. The equation at ε of the points of intersection is written:

$$(\varepsilon(1 + a^2)^{1/2} + H)^2 = 0$$

This equation always has a double root for H < 0, which corresponds effectively to tangency with the hyperbola, provided $\varepsilon > 1$.

For $-(1 + a^2)^{1/2} < H \leqslant -1$ the ellipses are entirely within the region.

For $+ H \leqslant -(1 + a^2)^{1/2}$ they are bi-tangent to the hyperbola. In reality only part of these ellipses constitutes a contour line; it is very perceptible in a representation in the plane $(x_{\perp}, x_{\parallel})$: for $H \leqslant -(1 + a^2)^{1/2}$ the curves pass in the region $x_{\perp} < 0$; the corresponding points have no physical significance of any kind. In the last resort we must preserve the part of the chord of the points of contact.

Let us take interest to the amplification by an oscillator perpendicular to $\vec{B}_0$ for such a distribution.

To that effect it is convenient to refine the sign of the gradient of g in the direction of axis ε.

The curve for sign change is the location of the points of contact with the contour lines of lines, parallel to that direction.

We find a hyperbola of equation

$$\frac{\varepsilon^2}{1+a^2} - x^2_{\parallel} = 1$$

Fig.II, 1 shows these different curves. Dashes indicate the assumed course of the function

$$e^{-\frac{mc^2}{kT}\varepsilon(1+a^2)^{1/2}} I_0\left(\frac{mc^2}{kT} x_\perp, X_\perp\right)$$

The zone corresponding to important values of g and $\mathcal{D}_\lambda g$ is shaded. This zone is concentrated around the contour line at 1/e from the maximum. The value of H corresponding to it is given by

$$\frac{mc^2}{kT} H = -\frac{mc^2}{kT} - 1$$

$$H = -1 - \frac{kT}{mc^2}$$

The corresponding point on the curve of gradient nullification has for ordinate

$$\varepsilon_1 = \sqrt{1+a^2}\left(1 + \frac{kT}{mc^2}\right)$$

The semi-axis about $x_{\parallel}$ is worth

$$(H^2 - 1)^{1/2} \sim \left(\frac{2kT}{mc^2}\right)^{1/2}$$

Given this, we shall be in a position to decide whether or not there is amplification by placing upon this diagram the curve $q = \alpha_p^{3/2}$, say in this case

$$q = \varepsilon$$
$$p \cos 0 = x_{\parallel}$$
$$\alpha_p = u(1 + x^2_{\parallel})^{1/2}$$

whence

$$\varepsilon = \left(\frac{\nu_\lambda}{\Omega}\right)^{3/2}(1 + x^2_{\parallel})^{3/4}$$

This curve crosses the hyperbola

$$\varepsilon^2 = (1+a^2)(1+x^2_{\parallel})$$

at a point of ordinate

$$\varepsilon_2 = (1+a^2)^{1/2}\frac{1+a^2}{u^3}$$

A real intersection point will be obtained if $\frac{1-a^2}{u^3} > 1$

say $$u < (1 + a^2)^{1/2}$$

or still $$u^{3/2} < (1 + a^2)^{3/4}$$

The ordinate at the origin of the curve $q = \alpha_p^{3/2}$ is . $\varepsilon_3 = u^{3/2}$ and the preceding inequality is interpreted in a very simple manner: it suggests that this curve places itself above the summit of g distribution.

The relative position of the curve $\varepsilon = \alpha_p^{3/2}$ and of the region where $\omega_\lambda g$ takes important values are indicated on the following figures (II, 2,3,4,5). Represented in them is the contour line of g limiting the region where $\omega_\lambda g$ can assume important values, and also the curve of $\omega_\lambda g$ sign change in the same region.

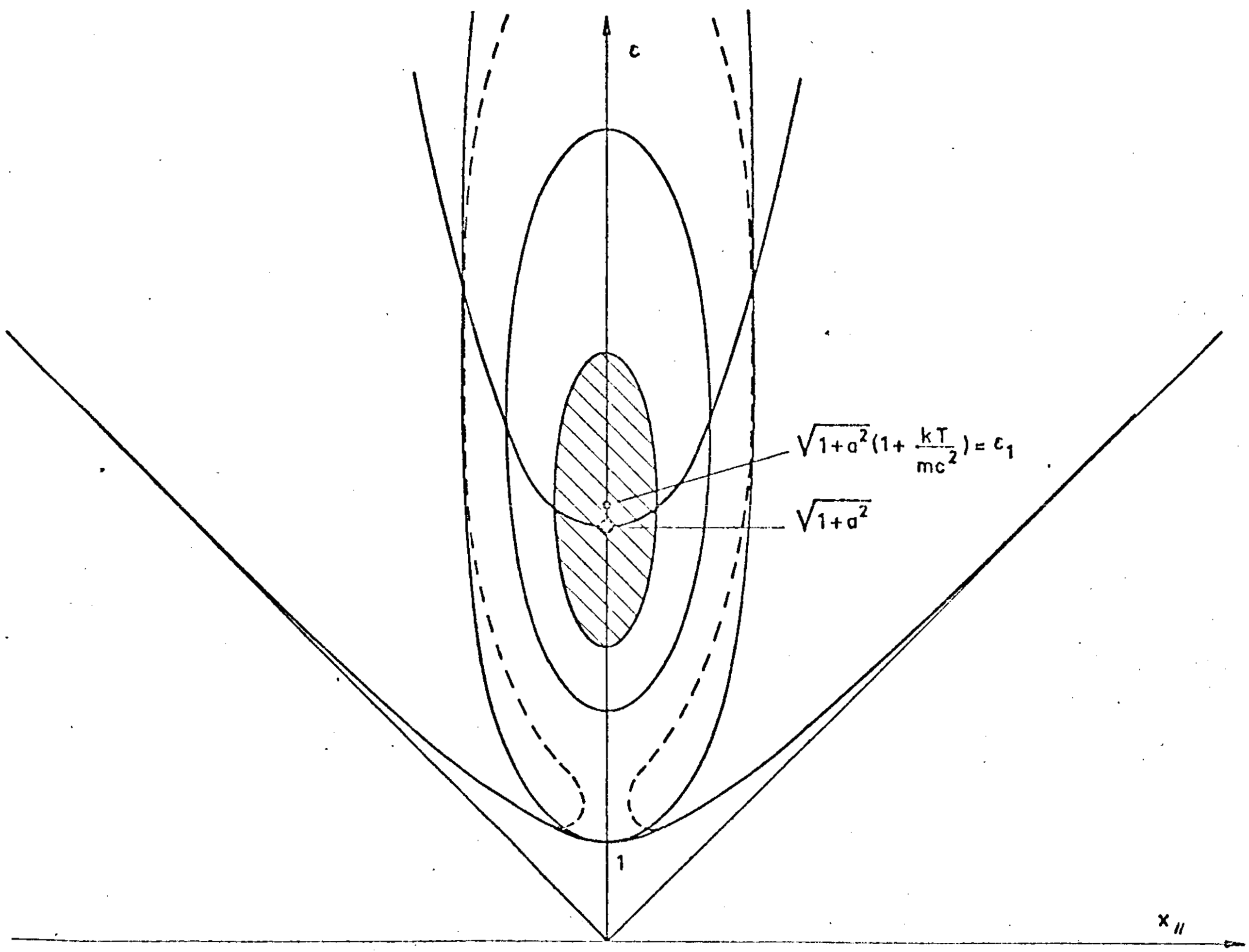

Fig. II, 1

Around the curve $\mathcal{E}$ ($\varepsilon = \alpha_p^{3/2}$ we represented the region where kernel $N_\theta(p,q)$ assumes important values.

1) $\varepsilon_3 > \sqrt{1 + a^2}$ (fig. II, 2)

The curve $\mathcal{E}$ is located entirely within a zone where $\omega_{\lambda g}$ is positive. There is no amplification.

Explicit Criterion :

$$\left(1 + \frac{P_\perp^2}{m^2c^2}\right) < \frac{\nu_\lambda^3}{\Omega_i^3}$$

say, $d < i$

2) $\varepsilon_3 < \sqrt{1 + a^2}$ (fig. II, 3)

The width of the ellipse is notably less than the width at the lowest point of the limit ellipse of $\mathcal{E}$. We obtain the criterion

$$\frac{(1 + a^2)^{3/2}}{u^2} - 1 \gg \left(\frac{2kT}{mc^2}\right)$$

say $\left(\frac{d}{i}\right)^{4/3} - 1 \gg 2t$

Then there is practically no interaction with this polarization the beam being too much directed so that the particles, and nearly all of them, have the direction of the oscillator and interact little, on account of breaking in

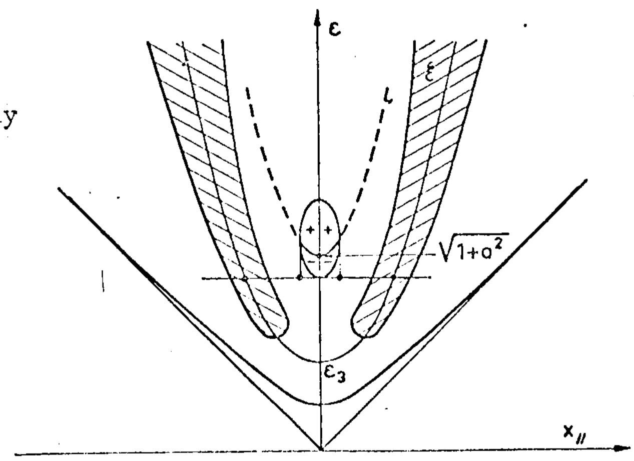

Fig. II, 3.

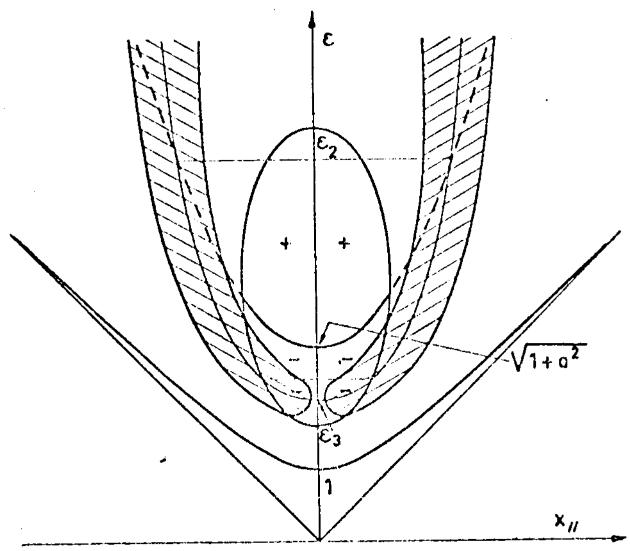

Fig. II, 4.

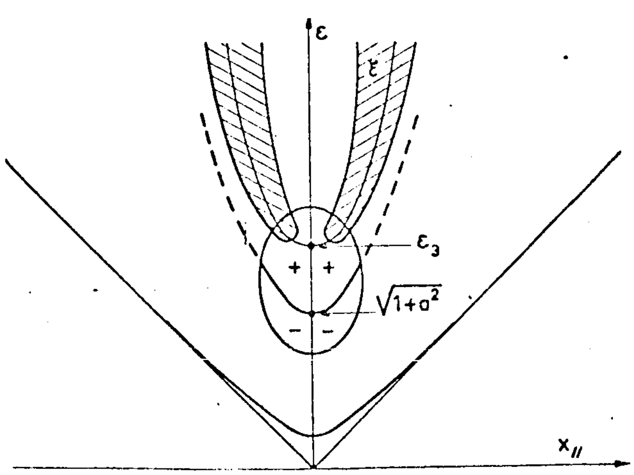

Fig. II, 2.

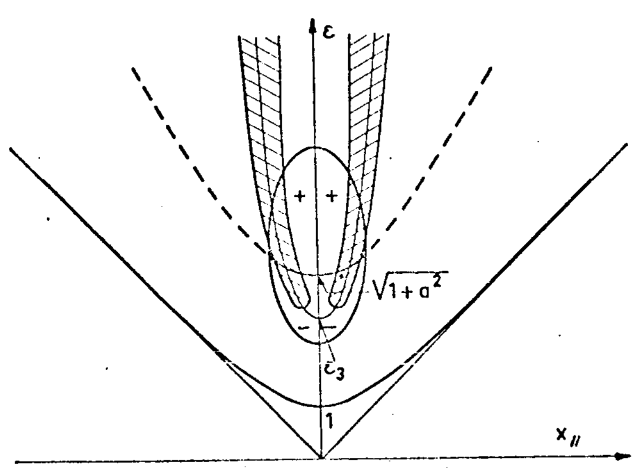

Fig. II, 5.

two parts of the radiation pattern.

3) $\mathcal{E}$ emerges from the zone where $\varpi_\lambda g$ assumes notable values prior to passing into that where $\varpi_\lambda g$ is positive; or more simply, the width of the limit ellipse at $\varepsilon = \sqrt{1+a^2}$ is smaller than that of the courbe for $\mathcal{E}$ (Fig.II,4).

The half-width of the ellipse is of the order of $\left(\frac{2kT}{mc^2}\right)^{1/2}$ and the width of $\mathcal{E}$ at this point is easy to compute.

We obtain the criterion:

$$\frac{(1+a^2)^{1/2}}{u^2} - 1 > \left(\frac{2kT}{mc^2}\right)$$

$$\left(\frac{d}{i}\right)^{4/3} - 1 > 2t$$

4) (Fig.II,5)

$$\varepsilon_3 < \sqrt{1+a^2}$$

$$\frac{(1+a^2)^{1/2}}{u^2} - 1 < \left(\frac{2kT}{mc^2}\right)$$

say

$$\left(\frac{d}{i}\right)^{4/3} - 1 < 2t$$

The zone where the kernel assumes notable values crosses the regions where $\varpi_\lambda g$ assumes either sign.

The kernel growth along $\mathcal{E}$ being given, the regions where $\varpi_\lambda g$ is positive contribute to b_λ with a weight $\varepsilon^2 |P_{\lambda j}^{i}|^2$ which is superior, though $\varpi_\lambda g$ is of same order. One may thus feel that, generally, such a situation would not lead to the amplification.

5) When the inequality of 4° is made very strong, we find ourselves confronted with the classical case where the detail of the radiation pattern does not intervene. In this we find again the case whereby amplification is impossible.

*** END OF APPENDIX II ***

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