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PRACTICAL RELIABILITY

Volume IV - Prediction

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FOREWORD

The typical few-of-a-kind nature of NASA systems has made reliability a premium en on the initial items delivered in a program. Reliability defined and treated on basis of percentage of items operating successfully has much less meaning than ıen larger sample sizes are available as in military and commercial products. Relia- ility thus becomes based more on engineering confidence that the item will work as intended. The key to reliability is thus good engineering--designing reliability into the system and engineering to prevent degradation of the designed-in reliability from fabrication, testing and operation.

The PRACTICAL RELIABILITY series of reports is addressed to the typical engineer to aid his comprehension of practical problems in engineering for reliability. In these reports the intent is to present fundamental concepts on a particular subject in an interesting, mainly narrative form and make the reader aware of practical problems in applying them. There is little emphasis on describing procedures and how to implement them. Thus there is liberal use of references for both background theory and cookbook procedures. The present coverage is limited to five subject areas:

Vol. I. - Parameter Variation Analysis describes the techniques for treating the effect of system parameters on performance, reliability, and other figures-of-merit.

Vol. II. - Computation considers the digital computer and where and how it can be used to aid various reliability tasks.

Vol. III. - Testing describes the basic approaches to testing and emphasizes the practical considerations and the applications to reliability.

Vol. IV. - Prediction presents mathematical methods and analysis approaches for reliability prediction and includes some methods not generally covered in texts and handbooks.

Vol. V. - Parts reviews the processes and procedures required to obtain and apply parts which will perform their functions adequately.

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This report is Vol. IV - Prediction. This subject has been of interest in reliability work since the earliest efforts of organized reliability activity. In these ensuing years much has been written on reliability prediction, but often the item concentrates on limited facets of the subject. This report synthesizes reliability prediction, with emphasis on the basics and the scope. C. A. Krohn selected and organized the contents, and together with A. C. Nelson, Jr. prepared the material. W. S. Thompson provided helpful comments.

ABSTRACT

The features and techniques of reliability prediction are identified and brought together in this report. The approach is to:

(a) Bring together scattered material,

(b) Present some material not in books or handbooks,

(c) Identify several points which have a tendency to be missed,

(d) Present some ideas which may be helpful to others involved in development of reliability prediction techniques, and

(e) Express some opinions related to the role of reliability prediction.

Material presented in this report is grouped into four major categories. Part I is largely qualitative discussion concerned with introduction and perspective. Contents include discussion and opinions on the role of reliability prediction, on perspective features, e.g. program phase and hardware level, on the relation to other analyses, and on the problems. Part II is concerned with reliability measures or definitions concerning single items, including data sources. Part III is devoted to the reliability prediction techniques which are suitable for general use and to classical reliability models. This material is scattered throughout the references; the treatment here mainly identifies approaches and relates them, with reliance on the references. Included for multi-item models are logic, lifetime, environment and bound-crossing topics. The remaining Part IV is concerned with concepts related to the detailed treatment of failure modes without independence assumptions. This is food-for-thought material from the results of research on reliability prediction techniques. This material in Part IV, in general, is not suited for widespread application. The Appendix presents a ready-reference on some basic probability laws and on various probability distributions.

REPORT PREFACE

In this report the subject of reliability prediction is synthesized. It is an attempt to "see the forest", but done while keeping both feet on the ground. Fundamentals are stressed in order to help develop a better understanding of what is involved. Expanded treatment is given to basic reliability measures, to some points which have a tendency to be somewhat misunderstood in the literature, and to several topics which are not covered in existing books and handbooks but where RTI has delved into them. Other topics are identified and related to one-another.

Reliability prediction as an organized discipline is approximately 15 years old. There are approximately a dozen books on the subject and approximately the same number of handbooks. There are many hundreds of reports and papers. Some which treat the fundamentals will be relied on heavily.

The qualitative discussion of Part I on the scope of reliability prediction is suitable for any reader - design engineer, manager, reliability generalist, or reliability analyst. Parts II and III cover mainly conventional and classical approaches to reliability prediction and Part IV reports on some research on structuring certain detail into a prediction. Parts II, III and IV will not be easy reading for persons who are not knowledgeable in the mathematics of probability. Of course perusing these parts will give any reader a flavor of the subject. If a reader wants to understand the subject he will have to study the material as introduced here and as elaborated in the references. If he does not know the mathematics of probability then he will first need to learn its fundamentals. The practicing reliability analyst should be familiar with most of the contents. For him, perhaps the manner in which the material is organized, the identification of references, and the results of research on reliability prediction techniques in Part IV will be of interest.

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Part I: Perspective

Reliability is the moniker which has been attached to those questions concerning whether or not an item will perform its intended function when it is ultimately used. Different ways of expressing a reliability index will be described later; some of the more common are mean time between failure, reliability, or failure rate. Reliability is different from traditional performance concerning explicit quantitative requirements; a reliability requirement can be avoided by just not introducing it. As long as the resultant item has subjectively acceptable reliability, then there is no concern. On the other hand, if the item turns out to be excessively failure prone, "What went wrong?" There was no requirement...there was no analysis...there was no measurement, even if this last task was plausible.

The current tendency is to treat this question quantitatively to the maximum extent which is "sensible;" otherwise the risk of unacceptable reliability is higher than is necessary. When either the initial requirements for an item are being prepared or are being responded to, they will typically contain a reliability index. Treating reliability quantitatively brings the subject into open consideration. By explicitly treating reliability the designers will think about what is needed to fulfill the requirement. This considerably enhances the odds of getting something useful and on schedule.

Even if an individual or a designer is not convinced of the need for quantitative treatment of reliability, he still cannot avoid the subject nor the need for some knowledge of it. When the designer is associated with the organization which will actually use the item, often the management pressure for a minimum total ownership cost compels quantitative reliability analysis by procedural requirements. When the designer is associated with an organization which is providing items to customer requirements, then there will usually be a quantitative reliability requirement and less often a specified procedure for measuring reliability. The pressure of this contractual requirement may be further increased if the contract is fixed price or has a fee incentive for measured reliability.

In the early days of an engineering project the situation is such that treatment of chance is with probabilistic modeling. As the new designs evolve into physical items and measurements are made from testing, the situation changes into one where the treatment of chance is also with statistical inference. The material presented in this report is probabilistic. Measurement of reliability and the use of statistical inference is given some coverage in Vol. III - Testing of this Report Series.

1. Treating Reliability Quantitatively

This section contains some definitions and discussion of the role, uses and accuracy of reliability prediction.

1.1 NASA Definition of Reliability

Reliability is defined in NASA Reliability Publication NPC 250-1 as: the probability that a system, subsystem, component or part will perform its intended functions under defined conditions at a designated time for a specified operating period [Ref. 1]. This definition will be used in this report. In the discussions the system, subsystem, component, or part will simply be referred to as a system or an item. When item is used, the material under discussion is potentially pertinent to any hardware level of aggregation. Multi-item or system will be used for bringing items together.

1.2 Probabilistic Approach

Reliability, in the quantitative sense as used here, is defined above as a probability. Perhaps another quantitative definition of reliability will evolve in the future which is not based on probabilistic concepts. For the present, however, it seems that quantitative treatment of reliability will involve probability and statistical inference. In one sense, this is unfortunate, as many engineers and managers have not had meaningful academic or other exposure to this subject. The subject is no more difficult than other ones of mathematics, but as with the other ones, it does take continued exposure to it over a period of time in order to be comfortable and confident with it.

The material in this report relies on the basic probability concepts and laws which are briefly reviewed in the Appendix. The reader is encouraged to review them and, if this is new material to study the references of the Appendix or other modern books on probability. In particular, the plea is made to avoid what seems to be a tendency to pick-up a few formulas such as some from Parts II and III and to overgeneralize their applicability. Rather, rely more on the fundamentals of the Appendix. To the engineers, do not be hesitant about seeking consultation from a probabilistic mathematician or a statistician.

The terms probability and statistical inference were used in the preceding paragraph. Probability is used in reference to an a priori situation, where assumptions are made concerning the probability descriptions of input information. Probability predicts the outcome from a set of assumptions. Statistical inference is used in reference to an a posteriori situation, where data is used to make inferences about the form of the distribution and to make estimates about the parameters of the distribution. Thus, probability is deductive and statistics is inductive.

1.3 Role

Reliability predictions may be performed for any of the following reasons:

(1) Potential technical contribution,

(2) Financial implications, and

(3) Compulsory.

Each of these could apply to the user (or buyer) of a system as well as to the supplier. The potential technical contribution is the most satisfying reason to the engineer. For example, he may decide to search for areas needing reliability improvement. However, the other reasons do occur. Financial implications arise in a fixed price or incentive contract which also has an associated reliability requirement and method of measurement. The compulsory reason may typically apply to a government agency because of policy and to a supplier because of contract requirements. There is nothing derogatory about any of these reasons; each has a role in the mature blending of technology, competition, and checks and balances.

1.4 Uses

Major uses of reliability prediction are:

(1) To obtain a numerical value of a reliability index,

(2) To obtain a numerical measure of uncertainty of the reliability prediction value,

(3) To search for needed improvements in the design or the operational procedure,

(4) To allocate total system reliability optimally to the sub-items.

The numerical reliability prediction number and its attendant measure of uncertainty are usually necessary in order to respond to any of the reasons for performing a prediction which are noted above. That is, response to such questions as "Can the mission be achieved?" or "What are the possibilities of making a profit?" or simply here is what the customer asked for. Searching for reliability improvements and probing around for weaknesses in the design and the operational procedure is the most technically appealing use. It is this use that often results in a reliability prediction going into more detail than it otherwise might. That is, comparative detailed values are sought rather than absolute gross values. Hopefully new alternatives will be opened up and the really bad choices can be eliminated. Literal optimization techniques, such as dynamic programming algorithms, offer the potential of improved allocation of overall reliability among the items comprising the system. Of these uses, obtaining the prediction number and searching for improvements have seen more application than the other two.

1.5 Accuracy

With the extensive experience accumulated with reliability prediction, it is now possible to make some intelligent judgments on accuracy even if only qualitative. When there is a fair amount of historical data and the equipment is not excessively complex or new, a crude rule-of-thumb for electronic equipment would be to expect the actual mean time between failure (MTBF) to be within the range of 50 to 200 percent of the predicted MTBF. This accuracy would apply to the case of an experienced analyst making his best effort, i.e., one which is not unduly optimistic or pessimistic. At the equipment level and the parts level, it is often possible to give the most accurate prediction possible with only a small amount of effort. That is, the point of diminishing returns is quickly reached in reliability predictions as far as the accuracy of the prediction number is concerned. It must be noted that the prediction analysis will usually go into more detail in searching for reliability improvements. If the inputs, the tools, and the assumptions of the reliability predictions are reasonably accurate and understood, then there is no reason why the results should not be able to be appraised so that the prediction can be intelligently used.

The competitive nature of the buyer-seller environment quite understandably has an influence on the accuracy of reliability prediction. There is probably a tendency to get more accurate predictions, at least more conservative ones, if there exists a firm reliability requirement, a method of reliability measurement, and firm dollar implications. Those who use reliability predictions of others, e.g. those at higher levels of system aggregation, must realize that those at the lower levels will tend to present predictions which will make the supplier look best at the time the prediction is made. That is, the equipment supplier will often not account for rough handling, for unverified failures on the part of the operators, for unforeseen environments, or possibly for burn-in. A final remark on the accuracy of reliability prediction is the realization that other system characteristics such as cost, schedule, repair time, or even performance tend to have inaccurate predictions at the early stages in the life cycle. As the program progresses through the life cycle there is an opportunity to measure some of these characteristics, whereas reliability may never really be able to be accurately measured.

2. Prediction and Allied Approaches

In this section the classical reliability prediction techniques and those which are suitable for active program usage are briefly identified by key words. Also, selection of a particular technique and other analyses related to reliability prediction are briefly discussed. Parts II and III of this report will give further introduction to the reliability prediction techniques which are only cited here. The purpose is to identify reliability predictions approaches and to fit them into related analyses.

2.1 Prediction Techniques

Figure 2-1 shows key words associated with reliability measures of single items and Fig. 2-2 does the same for the conventional and classical reliability prediction modeling approaches for multi-item systems. The sections of this report where the topic is covered are cited on the figures. Broadly speaking, the single item measures of Fig. 2-1 can apply to various levels of system aggregation, e.g., parts, equipment, and system levels as well as to human events. That is, they offer indices by which to describe some of the inputs to a multi-item reliability prediction and by which to express some of the outputs.

Reliability predictions implemented with the approaches of Figs. 2-1 and 2-2 have typical assumptions and characteristics. Quite often these are unstated; they are just implied. Some typical assumptions and characteristics are:

(1) A "fuzzy" definition of the failure of items and system, is it: Out of specification? Simply inoperative? Complete catastrophic failure?

(2) The prediction usually considers each item involved to have two states, either good or bad. In most predictions this is reasonable; however, there are certain situations where this can lead to grossly incorrect results. A familiar example is ignoring the two failed states of open or short of diodes in redundant arrangements.

(3) Independence among items is liberally assumed. Included here is the impact of not considering uncertainty in the natural or induced environments.

(4) Prediction is for a mature product. A prediction is quite often mute on the assumption that most design and manufacturing "goofs" have been removed and that the necessary burn-in period has been passed. This has serious implications for items such as those intended for space which are produced in small quantity.

(5) Omission of the human element during operation.

(6) Uncertainty in the parameters of single items are often not considered explicitly. Techniques of sensitivity analysis and of probabilistic treatment of uncertainty are potentially applicable.

Reliability Measures	
• Item states as attributes	(4.1)*
• Lifetime distributions	(4.2,A.1,A.2)
• Bathtub curve for a part	(4.3)
• Environment consideration	(4.4)
• Poisson process	(4.5)
• Repairable equipment	(4.6)
• Replaced items	(4.7)
• Bathtub curve for an equipment	(4.8)
• Data	(6.1,6.2)

Bound-Crossing	
• Fixed bounds	(5.2)
• Stress-strength	(5.3)
• Time-dependency	(5.4)
Slowly varying	
Random process	
• Data	(6.1,6.3)

* Sections where the topic is presented are shown in parentheses.

Figure 2-1 Reliability Definitions for Single Items Cited in Part II

Logic Models

- Reliability block diagram (7)*
- Readily apparent configurations (7.2)
- Complex configurations
  - Conditional probability (7.3)
  - Cuts and paths (7.4)
- Multi-function and/or phase
  - Conditional probability (7.4, 7.5)
  - Cuts and paths (7.4, 7.5)
- N-State logic (7.6)

Time-Dependency

- Substituting into logic models (8.1)
- System state dependence
  - Standby redundancy (8.2.1)
  - "Rope" (8.2.3)
- Additional approaches
  - Continuous Markov process (8.3.1)
  - Extreme value (8.3.2)
  - Flowgraphs (8.3.3)
- General redundancy model (8.4)

Environment and Bound-Crossings

- Environment uncertainty (9.1)
- Stress-strength (9.2)
  - Expected value
  - Extreme value
- Functional relationships (9.3)
  - Method of moments (9.3.1)

* Sections where the topic is presented are shown in parentheses.

Figure 2-2 Multi-Item Reliability Prediction Approaches in Part III

Of course, exceptions in particular predictions will exist for these typical assumptions and characteristics. These apply to the majority, not the exceptional case.

2.2 Prediction Technique Choice

The approach used for a particular prediction is influenced by a myriad of factors. Implications of some factors such as the commodity and its intended operational profile tend to be somewhat apparent. Here the need is mainly that of knowledge about the various reliability prediction indices and equations of the sort noted in Figs. 2-1 and 2-2. Other factors such as schedule and the extent of the intended reliability program are more subjective in their implications. Here the influence is more on the choice of parameter values rather than the choice of equations.

There is little which can be said on applying this technique to this situation and that technique to that situation which is not apparent to someone who understands what you are talking about. About all that seems appropriate is a plea to use common sense, e.g., use the simplest approach commensurate with the purpose of the analysis and the accuracy of the parameter data.

Life Cycle. Reliability prediction plays its major role during the planning and early design phases of a program's life cycle. In these earlier phases of the program a priori techniques of the sort described in this report are used. It can be expected that there will be many iterations of a prediction as the program progresses, and that the prediction model will become more complex. When the program progresses to the point that test and operational data start to become available, then the a priori prediction techniques start to give way to the a posteriori techniques of statistical inference. The reliability prediction model still has a role. It provides a means for combining statistical inference estimates on items into a composite measure for multi-item levels.

Commodity Considerations. Reliability prediction at the nonrepairable part level is largely a matter of selection of the appropriate form from Part II by which to express the reliability measure. This includes the designation of the stresses which are appropriate. At the equipment level, say in the order of hundreds of piece parts, the main consideration as to what technique is used depends largely on whether the equipment is electrical or non-electrical. Electronic equipment typically uses a very straightforward approach. The parts are assumed to have a constant failure rate and failure rates are added to obtain the failure rate (or its reciprocal mean-time-between failure) for the equipment (See Sec. 8.1.1). This approach is also sometimes used for nonelectrical commodities, but often more involved prediction techniques will be applied to mechanical or structural commodities. The stress-strength approaches of Secs. 5.3 and 9.2 are structurally oriented, but they are really specialized applications of broadly applicable approaches concerning environment effects which are noted in Secs. 4.4 and 9.1. At the system level the techniques which are used tend to be quite varied and the entire scope of Part III is applicable.

Mission Implications. The operational time periods and the environments of the operational profile, of course, affects the choice of the reliability prediction technique. Space systems have been thus far of a one-shot nature with the main periods of launch, orbit, and recovery. Launch and recovery environments tend to be somewhat severe, whereas orbital environments tend to be moderate. A major constraint handicapping reliability measurement prior to use has been the combined effects of the lack of experience, the cost, and the small quantity of some commodity types. The decade of space experience has alleviated these conditions to some extent. However, presently the nature of space missions is being extended to deep space missions and eventually to commodities reusable after recovery. Thus an increasing number of one-shot, non-reusable commodities are on the verge of giving way to multiple use, repairable commodities. Space reliability predictions will thus start to take on more of a similarity to predictions for systems intended for airborne and surface missions. These latter types are as much concerned with the implications of repair, that is maintainability, as they are with reliability. The typical airborne system which is not operating continuously is desired to have a very high overall availability followed by a very high reliability for relatively short missions. The overall availability here is of a continuous nature and the missions are of a cyclic nature. Systems for surface missions often are continuously operational, but they can quite often be removed from operation for repair or maintenance. However, they usually will have short periods of intense operation where no repair is possible. These short periods may be somewhat predictable, as for space-oriented services, or they may be nonpredictable as for military uses. Presentations on maintainability and availability prediction techniques are available in Refs. 2 and 3.

Subjective Factors. The ultimate accuracy is primarily affected by the subjective judgment of the person performing the reliability prediction. Main considerations are the kind of reliability program with attendant influences of budget, schedule, and the operational environment. Historical experience in reliability prediction, particularly where it has been followed up with reliability measurements, have helped considerably in this area. Detailed listings have been made of the many variables which are pertinent [Ref. 4] but in the final analysis this is largely a matter of mental assimilation on the part of the person performing the prediction.

2.3 Related Analyses

Other types of analyses overlap and interface with reliability predictions.

Brief comments are given below on these allied studies. The comments are aimed primarily toward the equipment and system level of commodity complexity, and particularly toward the latter. System effectiveness is currently a popular phrase which is used to cover the scope of the considerations cited below. Some system effectiveness models have been proposed which attempt to pull together the appropriate ingredients [Refs. 5 and 6]. This is possible to some extent with the gross models and their attendant assumptions. A system effectiveness analysis will typically reflect the effort of various individuals as it is unlikely that any one individual can master or have the time to perform all of the analysis areas intended in any one program.

Other Reliability Analyses. During the planning and early design program phases the other reliability analyses, in addition to prediction, can be classified into failure mode and effects, performance variation, and stress as suggested in Ref. 7. Failure mode and effects analyses are often probing to a level of detail which is not reflected in the reliability prediction model. It is often of a semi-qualitative nature. It is conceptually possible to reflect extremely detailed failure modes into reliability prediction.* However, there are usually the practical reasons of the unavailability of data and the complexity of such models which prevent a literal one-to-one correspondence between the failure mode and effects analysis and a reliability prediction.

The performance variation analysis is concerned with the area of reliability prediction which in this report is referred to as bound-crossing. The reliability discipline has promoted approaches for drift failure analysis of electronic circuits which are commonly referred to as worst-case or as tolerance analysis techniques. These have proved to be of value for purposes of reliability improvement. However, they almost invariably are not extrapolated over into the reliability prediction analysis. Again this is conceptually possible but usually not done for sound reasons.* It should be noted that with mechanical and structural commodities there has been greater use made of bound-crossing techniques for reliability prediction purposes. A prominent example here would be the classical stress-strength model. Those which are conventional or classical are noted in Parts II and III of this report.

Stress analysis typically has the most explicit relationship with a reliability prediction.* This is because many of the reliability prediction manuals include the applicable stress derating and failure rate adjustment tables and curves. Examples are the effect of temperature, current or wattage levels on the failure rate. In the nonelectronic commodity the stress-strength model would be an example of a technique which is common to stress and prediction analysis.

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* Part IV of this report presents some thoughts on structuring a detailed reliability prediction model which explicitly incorporates this detail.

Conventional Design Analyses. These traditionally involve both performance and stress calculations and are what the design engineer would do to some extent regardless of whether he is explicitly concerned with reliability analysis. The performance analysis is mainly related to the bound-crossing type of reliability measure and to the performance variation analysis. The traditional deterministic, engineering equations, relating performance attributes to part characteristics and other variables, become part of the performance variation analysis. Similarly, the traditional deterministic stress equations are developed and used by the designers. Calculation of safety margins to such factors as voltage, power and temperature is a familiar form of this type of stress investigation.

Safety. Systems analyses for manned space missions have always been directed toward both safety and reliability. In terms of the impact on the reliability prediction model, it usually turns out that the same, or slightly modified, prediction approach will serve the safety prediction needs as well as those of reliability. For safety there will typically be a different criterion of failure and a different operational profile than for reliability. Also note that in nonspace types of systems, safety analyses are also being performed [Ref. 8].

Availability and Maintainability. When repair is possible during application, then availability and maintainability cannot be avoided in the prediction. This adds a measure of complexity to the prediction technique, as the reliability prediction literally becomes absorbed by the availability prediction. Some comments have previously been made on these analyses in Sec. 2.2 under the heading of Mission Implications.

Spares. Reliability prediction techniques have been experiencing increased applications in spares planning and optimization. These may seem to be inseparable; nevertheless, reliability analysis and spares analysis have been traditionally performed by separate groups. Furthermore there are reasons which from the spares viewpoint cause items to have higher failure rates than from the operational viewpoint. Examples here would be the effects of secondary failures and the replacement of incorrectly diagnosed failures. It is also noted that optimum spares allocation and optimum redundancy allocations can use identical approaches.

Cost Trade-off. If cost-reliability relationships are available for single items, then for some forms of multi-item configurations the literal optimization techniques can be applied in order to obtain optimum reliability values for items. Also, to some limited extent this can be expanded to include simultaneous optimization of reliability and spares or reliability and maintainability. The main limitations here are the accuracy of the cost-reliability or cost-maintainability relationships and those of optimization techniques. Note also that the optimum allocation techniques find application for other penalties than cost, e.g., volume, weight, power, or perhaps simultaneous treatment for several of these.

The basic reliability allocation problem which is amenable to analytical solution is that of selecting an optimum system configuration from allowable alternate design approaches so that reliability is maximized subject to a penalty constraint, or vice-versa. It is necessary to have a reliability prediction equation which covers the range of allowable alternate design approaches and similarly a penalty prediction equation. Thus one use of the suitable reliability prediction equations of this report is to provide an input for a reliability allocation. An approach to this problem can be developed based on the dynamic programming principle. As would be expected exact solutions can be obtained from problems which are usually too simple to be of practical value. For example, Ref. 9 gives a dynamic programming procedure for selecting exactly the order of the active redundancy in the case of one constraint and of the active form of redundancy. Procedures for more realistic problems can be developed but they usually yield an approximate solution. However, the incompleteness usually will not result in differences of practical importance.

Refs. 10 and 11 describe computerized approaches which are suitable for realistic problems. The approach in Ref. 10 is for identifying an optimum redundancy configuration where each item in the system can be active, standby with switch, or spare redundancy. It is assumed that only one item must work, that the items have an exponential failure distribution, and that the failure (or success) events for the items are mutually independent. Ref. 11 treats essentially the same problem ignoring the switch but introducing the non-serial, e.g., a "bridge," configuration. The former paper is patterned after the results in Ref. 10 but allows for more practical redundancy alternatives.

It was decided that the result given in Ref. 10 could be generalized to include the case in which at least n_o items must work out of n items ($n_o < n$). In order to do this it was necessary to derive a general reliability prediction formula for parallel arrangements, as shown in Sec. 8.4. This formula has been computerized and the program is discussed in Volume II - Computation. This program is actually a subroutine in the general Reliability Cost Trade-off Program (RECTA). The subroutine enables one to consider majority voting redundancy as well as the three types of redundant items as noted above. Practical procedures for obtaining an optimum selection of the reliability of items in series can also be based on a dynamic programming procedure. This is where reliability improvement of an item is improved by such means as design and manufacturing emphasis on reliability and redundancy is not allowed. The largest difficulty here is obtaining an accurate relationship between item reliability and cost. The general reliability cost trade-off program (RECTA) cited above simultaneously treats configurations containing series and the various redundancy approaches. Here the allowable alternative for an item includes increasing the reliability of the non-redundant item and/or making the item redundant. Any of these alternatives can be disallowed, thus a generalized series-parallel reliability allocation procedure.

RECTA as cited above was developed as part of an evaluation of computer programs for system effectiveness [Ref. 12]. This reference and other sources will call attention to the possible use of allocation procedures based on linear and quadratic programming and on Lagrange multipliers. These approaches have usually not proven suitable for realistic reliability-cost allocation problems.

3. Needs and Problems

The largest need is that reliability prediction be included or be considered as an essential element of the actual decision-making process. This is not just a matter of design engineers and managers tolerating the reliability prediction, but rather one where the desired situation is that these persons need and want the results of the prediction. The reliability prediction should be influencing the design and operating plans, rather than a separate exercise whose outputs are ignored or forced to justify a preconceived design and operating plan. The problems here are grouped into those concerning people, data, and techniques. These remarks are not in the sense of criticizing anyone or any discipline; rather, they are intended as unemotional commentary.

3.1 People

Reliability prediction utilizes heavily the mathematics of probability. Reliability measurement and testing utilize the mathematics of statistical inference. These are both complex subjects that are simply difficult to really learn. In addition to the practical knowledge required for applying them, the theory is also important. The majority of technical persons, including designers, management, as well as reliability engineers, typically have not had the opportunity to become well-versed in the mathematics of probability and statistical inference prior to their initial attempts at using them.

The solution here is not at all readily apparent. A probability or statistics course or two in the college curriculum or a concentrated short course after college really only helps the person communicate better with someone who is well-versed in these subjects. Persons specifically trained in the mathematics of probability or statistics, on the other hand, have their difficulties in understanding the engineering applications. Such a person in a product-oriented organization will typically have difficulty adjusting to the approximate nature of engineering mathematical models, to the myriad of pertinent variables which cannot be reflected simultaneously in equations, and to the situation that testing to satisfy statistical confidence often requires unrealistically large sample sizes due to cost considerations.

There is some sentiment for having the design engineer also pick up the task of reliability prediction and the other reliability analyses. There is much to be said for this; after all, these persons, particularly at the equipment level, are usually called upon to provide cost, weight, and other predictions in addition to strict performance. It is generally accepted that the designer is "responsible" for "designing reliability" into the equipment; it follows that he should have some degree of responsibility for the reliability analysis of his design. For electronic equipment this may be a reasonable approach. Some few suppliers are doing this.

They do not have reliability specialists, or if they do he performs the role of a consultant. For structural and mechanical commodities and for systems, the reliability prediction is more complex than for electronic equipment. The approach of having the designer also perform the reliability prediction is more difficult here. Even if management decides that the approach of having the designer perform the prediction is desirable, it is still difficult to implement. These design people are already generally overloaded in work schedules. Also, they may not be interested.

A nagging consideration to many persons is that the mathematics of probability and statistics have enjoyed successful application in many areas, for example, communications, economics, biology, agriculture, and information theory. It seems that it is the reliabillity area which perhaps uniquely has a somewhat unsuccessful history of application of probability and statistics. At least the road here has been a lot rougher than in other areas. One cannot help but feel that a major reason is that many of the people who have been involved in reliability prediction - the people doing them as well as other persons who are expected to use the results - have just been weak in the theory and practical applications of probability and statistics.

3.2 Data

Data refers to the actual numerical value of reliability indices for various items. Thus data, one way or another, revert back to some type of reliability measurement. Even once the need is recognized, there is the problem of how to go about making reliability measurements. What is the best index? The greater the reliability of any item, the more difficult it is to measure. Who is to pay for it? A part or equipment supplier often will deliver his product and will never hear anything further regarding reliability, particularly if it is satisfactory. Experience with the reliability measurements of operational items have indicated that it is near impossible to rely on operational and maintenance personnel to supply this data; special persons have to go along just to record the reliability information. In addition, there is recognition of the situation that it is more glamorous to work with models and equations than to try to record and interpret data.

Efforts, of course, have been made at gathering and disseminating data, and these continue (Sec. 6 of this report contains some data references). These are to be commended. Contractors are more and more developing their own data and making data banks, but they are handicapped. More efforts are needed in the data area and of necessity will require funding by various government agencies. Individual contractors do not have wide access to operational sites, nor do they have much funding for this. With regard to individual programs, there is a great opportunity for using sampling techniques rather than to record everything, particularly with actual operating equipment and systems, in order to gather needed information.

3.3 Techniques

A few comments are given here on the area of reliability prediction technique needs; however, this is not as large a problem area as those of people and data. Conflicting positions can be easily taken. On one hand, it can be said that more complex techniques are not generally going to be applied because invariably better data is needed. It is unlikely that such data will become generally available. On the other hand, it can be argued that complex situations require complex mathematical models. In any case, efforts will continue for technique development. It is something that can be done individually and without major funding. It is the sort of thing that people who are inclined in this direction will continue to do whether they have a great deal of support or not.

Computers seems to continue to get faster with larger storage. This opens the door to more involved and more complex analyses. Rationale-wise, there is a need to cycle more practical experience back into the development of prediction techniques. This is now becoming possible more than previously because of the increased experience with reliability prediction.

At the system reliability level, opportunity areas are more explicitly bringing in the human impact and the environment, that is, treating the reliability of man as well as the machine and treating other unknowns such as possibly the environment as a probabilistic variable. At the equipment level a need is how to formulate probabilistic models for treating distinct failure modes simultaneously with environment (Part IV of this report presents some thoughts on this). At the systems level again, there is a need for improved methods to tie together maintainability, spares, performance, and cost with reliability. This has been labeled systems effectiveness, and there are efforts under way here as noted in Sec. 2.2.

Also to be given due consideration is the opportunity for less complex methods, that is, striving for balance between complexity of the prediction technique and accuracy of the result. There are places for simple rules of thumb and for simple estimating relationships.

Part II. Single Item Reliability

In this part the concept of reliability measures for a single item are discussed from a broad viewpoint. The reliability measures consider two basic categories of problems: (1) those in which an item is in either a success or in a failed state (considered in Sec. 4) and (2) those in which certain characteristics of an item may be of an unacceptable value, the "bound-crossing" problem (considered in Sec. 5). Guidance on obtaining numerical index values for a single item of both categories is given in Sec. 6.

These reliability measures are potentially applicable to any item or event to be considered in a prediction. Thus inputs for multi-item prediction equations would be of one of the forms covered, as would the output of the prediction. Or, if the reliability measures for an item is obtained from testing, then inferences would be made concerning these measures.

These definitions will, in the main, be well-known to reliability workers. Some features are covered, however, which are not emphasized in existing handbooks and books. These are the following: Possible confusion concerning mathematical descriptions of the widely cited bathtub curve when it is used for non-repairable items such as parts versus for repairable items such as equipment is discussed in Secs. 4.3 and 4.8. Uncertainty in the environment is discussed in Sec. 4.4. Explicitly bringing time into consideration for bound-crossing problems is introduced in Sec. 5.4, where possible failure criteria for non-monotonic drift requires careful treatment.

4. Reliability Measures

Various indices used for reliability measures are described in this section, and there is a probing beyond conventional assumptions. The material gets progressively more involved, starting with simpler notions and models. The later part of this section goes into considerations of reliability measures for repaired items.

4.1 Definitions of States and Reliability

The simplest way of classifying the state of an item is as two states, success (S) and failure (F). Let P(S) be the probability of success and P(F) be the probability of failure of the item subject to given conditions under which the probability measures are to be defined. Then

$$R = P(S) = \text{probability of success,}$$

$$1 - R = P(F) = \text{probability of failure, and clearly}$$

$$P(S) + P(F) = 1.$$

This simple classification and the associated indices of reliability and unreliability are based on several assumptions such as the following:

(1) a definition of failure exists,

(2) the probabilities of success (or failure) are conditional on a known (deterministic environment, or on known characteristics of environment described by probabilistic measures, and

(3) the classification is for a certain future time instant or time interval.

Much of the subsequent material in this section involves expanded treatment of these assumptions. The assumptions should be kept in mind, but more important, they should also be kept in perspective. Most definitions and mathematical models are based on assumptions which are not fully met when associated with real world situations. The delicate question is always one of the effects of violation or relaxation of the assumptions for the problem which is at hand. Sometimes extremely simple equations will do the job; at other times extremely complex equations are needed. In some situations the state of an item should be subdivided into three states S, F_1, and F_2 for an adequate approximation to real world application. F_1 and F_2 are two different failure modes and the probability identity can be written as

$$P(S) + P(F_1) + P(F_2) = 1.$$

Some examples for consideration of two failure modes are digital circuits, relays, switches, and diodes. In general, any reasonable number of states may be associated with the various modes which an item might assume. Some additional comments concerning mathematical descriptions of multiple failure modes appear in Sec. 4.3.

It is desired to broaden one's concept of failure to include the many possible types which may occur. Some examples of failure modes are given below.

(1) The performance of an item deviates from its nominal value by more than 10 percent.

(2) A diode opens or shorts.

(3) An amplifier is "noisy".

(4) An accumulation of the effects of a somewhat periodic variation of the performance of an item outside given bounds.

(5) Corrosion of a boiler tube.

(6) Fracture of a pressure vessel.

These various types of failure are introduced to motivate one to pay attention to possible ways in which items can fail and hence not overlook any important details.

4.2 Reliability as Function of Time

The probability density function of time to failure of an item will be used as the starting point, as this can be visualized easily from a histogram of time to failure data. In Fig. 4-1 a histogram is shown as dashed and the associated probability density is the continuous function.

(1) The probability density of failure as a function of time t is

$$p(t), \quad t \geq 0. \tag{4-1}$$

(2) The probability of failure of the item by time t is the cumulative probability

$$F(t) = \int_0^t p(t)dt. \tag{4-2}$$

(3) Reliability is the probability of no failure by time t

$$R(t) = 1 - F(t) = \int_t^\infty p(t)dt. \tag{4-3}$$

(4) The hazard rate is the conditional probability of failure given that the item has not failed by time t. Other terms widely used for hazard rate are failure rate (when exponential failure density function applies), instantaneous failure rate, or force of mortality.

The probability relationship concerning two dependent events can be used to develop the hazard rate. Recall that*

$$P(A|B) = \frac{P(AB)}{P(B)}$$

---

* Basic probability definitions and relationships are presented in Appendix A.4.

If: $P(A|B) = h(t)dt =$ probability that an item fails between t and t+dt, given that it has not failed by t,

$P(AB) = p(t)dt =$ probability that an item has not failed by t and that it fails between t and t+dt, and

$P(B) = R(t) =$ probability that an item has not failed by t.

Hence

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{p(t)dt}{R(t)} ,$$

$$h(t)dt = \frac{p(t)dt}{R(t)} , \text{ or } h(t) = \frac{p(t)}{R(t)} . \tag{4-4}$$

The hazard rate function h(t) can also be obtained using the fact that it is an instantaneous failure rate.

$$h(t) = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t} \frac{1}{R(t)} = \frac{p(t)}{R(t)} .$$

It can also be expressed as follows:

$$h(t) = \frac{-R'(t)}{R(t)} = -\frac{d\ln R(t)}{dt} ,$$

where R'(t) is dR/dt. Reliability can now be expressed as

$$R(t) = \exp\{-\int_0^t h(t)dt\}.$$

(5) The mean time to failure, MTTF, is the expected time to failure. The expected value of a random continuous variable x is

$$E(x) = \int_{-\infty}^{\infty} x\, p(x)dx$$

or in the above notation

$$E(t) = MTTF = \int_0^{\infty} t\, p(t)dt = \int_0^{\infty} R(t)dt . \tag{4-5}$$

The last result can be seen by integrating by parts the following

$$\int_0^{\infty} t\, R'(t)dt, \; R'(t) = \frac{dR}{dt} .$$

The definitions in Eqs. 4-1 through 4-5 were developed for time as a continuous variable. In some situations it is appropriate to measure time as a discrete variable, where the number of cycles or operations to failure is a discrete variable. The definitions in Eqs. 4-1 through 4-5 have direct counterparts for handling discrete variables. These counterparts for the discrete variable case are shown below, where n is the number of cycles to failure [Ref. 13].

Probability density: $p(n), \; n = 1, 2, 3, \ldots$ (4-6)

Probability of failure: $F(n) = \sum_{1}^{n} p(n)$ (4-7)

Reliability: $R(n+1) = 1 - F(n)$ (4-8)

Hazard rate: $h(n) = \dfrac{p(n)}{R(n-1)}$ (4-9)

Mean cycles to failure: $\mathrm{MCTF} = \sum_{1}^{\infty} n \, p(n).$ (4-10)

A large number of possible probability density functions (discrete and continuous forms) have been proposed. Several are shown in Appendices A.1 and A.2. Although these density functions are presented with reference to lifetimes, there are also other possible applications of these same density functions in reliability analysis. Some of these density functions will again appear in subsequent sections of this volume. See Ref. 14 for some good examples of application of various density functions for reliability purposes.

The exponential density function is widely used in reliability prediction and its key feature, a constant hazard rate, is illustrated below in Ex. 4-1. One of the most common misconceptions appearing in the reliability literature is the implication that a random failure law and the exponential failure law are one and the same. Assuming a random failure law simply implies that failure times occur randomly over time according to the stated probability distribution, there can be any number of distributions or laws depending upon whether the log-normal, the Weibull, the gamma or some other distribution is assumed to best describe the distribution of failure times.

<u>Example 4-1</u>

A high-power magnetron has an exponential distribution of time to failure with a failure rate of 2.5×10^{-3} failures per hour. The probability density function of Fig. 4-1 is of this item. (a) What is the reliability for a new magnetron for the first 40 hours of operation? (b) What is the reliability for the following 40 hours of operation if the magnetron has not failed during the first 40 hours?

Solution:

(a) The probability density function (pdf) of the exponential distribution is

$$p(t) = \lambda e^{-\lambda t}.$$

Using Eq. 4-3, the reliability equation for the exponential distribution is

$$R(t) = \int_t^\infty \lambda e^{-\lambda t} dt = e^{-\lambda t}.$$

For $\lambda = 2.5 \times 10^{-3}$, $t = 40$ hours the reliability is

$$R = e^{-2.5\times 10^{-3}\times 40} = 0.905.$$

(b) Rephrasing the second question, what is the probability that failure will not occur in an interval $\Delta t = t'' - t'$, given that it has not failed up to time t'? Using the probability that an item failed between t and t+dt if it has not failed by t shown in development of the Eq. 4-3:

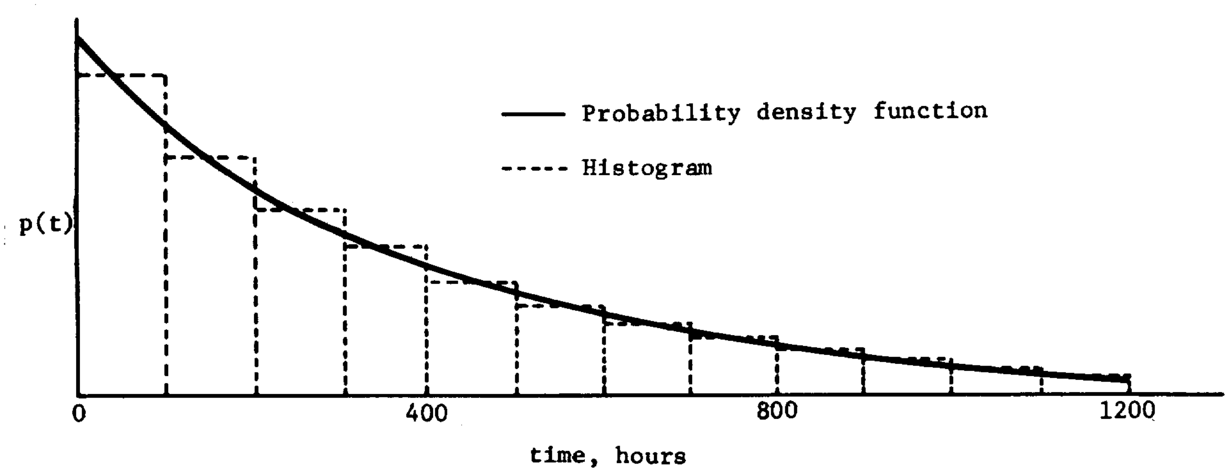


Figure 4-1 Exponential Probability Density With $\lambda = 2.5 \times 10^{-3}$ and a 400 Hour MTTF

$$R = 1 - \frac{\int_{t'}^{t''} p(t)dt}{R(t')}$$

$$= 1 - \frac{R(t') - R(t'')}{R(t')} = \frac{R(t'')}{R(t')} .$$

In the example problem for the exponential distribution

$$R = \frac{e^{-\lambda t''}}{e^{-\lambda t'}} = \frac{e^{-\lambda(t'+\Delta t)}}{e^{-\lambda t'}} = e^{-\lambda\Delta t}$$

$$= e^{-2.5\lambda 10^{-3}\times 40} = 0.905.$$

Thus the same solution applies to questions (a) and (b). Let us now apply Eq. 4-4 for the hazard rate to the exponential distribution to assist in understanding this result.

$$h(t) = \frac{p(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda.$$

The hazard rate for the exponential distribution is constant. For the exponential distribution the same reliability equation applies regardless of how much operating time has been accumulated. Only the exponential distribution is like this, which is one reason why it is widely used in reliability analysis.

4.3 Bathtub Curve

A form of the hazard rate which is widely cited in reliability literature is the bathtub curve as shown in Fig. 4-2(a). A popular reasoning on how such a curve would come about is as follows. The early decreasing hazard rate is thought of as resulting from manufacturing defects, and early operation will remove these items from a population of like items. The remaining items have a constant hazard rate for some extended period of time where the failure cause is not readily apparent and finally those items remaining reach a wear-out stage. There is a strong parallel between the above curve and the instant mortality curve for human beings.

None of the commonly used reliability distributions such as those cited in Sec. 4.2 and expanded on in Appendix A.1, e.g. log-normal or Weibull, individually has a form which has this bathtub shaped hazard function. Thus if a mathematical description of the bathtub curve is desired then it must be developed. One approach would be to first select an appropriate probability density for each of the three periods of decreasing, constant, and increasing hazard rates as shown in Fig. 4-2(b). These will respectively be $p_d(t)$, and $p_c(t)$, and $p_i(t)$. These could each be for the Weibull or gamma distribution with different shape and location parameters for each of the three periods. The $p_c(t)$ for the constant hazard-rate will be the exponential distribution, which is one case of both the Weibull and gamma. Further, there is a probability that only one of the failure causes will occur for an item, where $P(d)$, $P(c)$, and $P(i)$ are respectively these probabilities for each of the three causes and $P(d) + P(c) + P(i) = 1$. These probabilities for a single item will be the same as the percentages for a large population of these failed items which would fail from each of the causes. A probability density for an item such as that shown in Fig. 4-2(b) could be developed from

$$p(t) = P(d)\, p_d(t) + P(c)\, p_c(t) + P(i)\, p_i(t). \tag{4-11}$$

where the terms are discussed above. The reliability function and hazard rate can then be developed using Eq. 4-3 and 4-4.

Another approach to the development of a reliability function for the bathtub shaped hazard curve is to treat the reliability of each of the causes as conditional evenets. Here the probability that an item will not fail as a function of time is

$$R(t) = R(\bar{d};t)\, R(\bar{c};t|\bar{d})\, R(\bar{i};t|\bar{d},\bar{c}). \tag{4-12}$$

where $\bar{d}$ is the event of no failure from the cause described with a decreasing hazard rate and similarly for $\bar{c}$ and $\bar{i}$. Development of this function will lead to the same results as development of Eq. 4-11.

There are two reasons for this discussion. One is that the development of Eqs. 4-11 and 4-12 illustrates how the same item would be mathematically described where time and multiple failure modes (or failure states of Sec. 4.1) are both explicitly considered. This approach will be used later in Part IV where detailed failure-modes are again treated. Another reason for development of mathematical models which would have a bathtub shaped hazard function is to assist in the understanding of the implications of these curves.

The above discussion is for a bathtub shaped curve for an item or for a population of identical items where a failed item is not repaired or replaced. The bathtub shaped curve is also used in association with the situation where an item is repaired or replaced, in particular for repairable equipment. When failed items are repaired or replaced, then the mathematical development of the bathtub curve is different than noted above. This use is discussed in Sec. 4.8 following the presentation of some groundwork in Secs. 4.5 through 4.7 concerning reliability measures of repaired and replaced items.

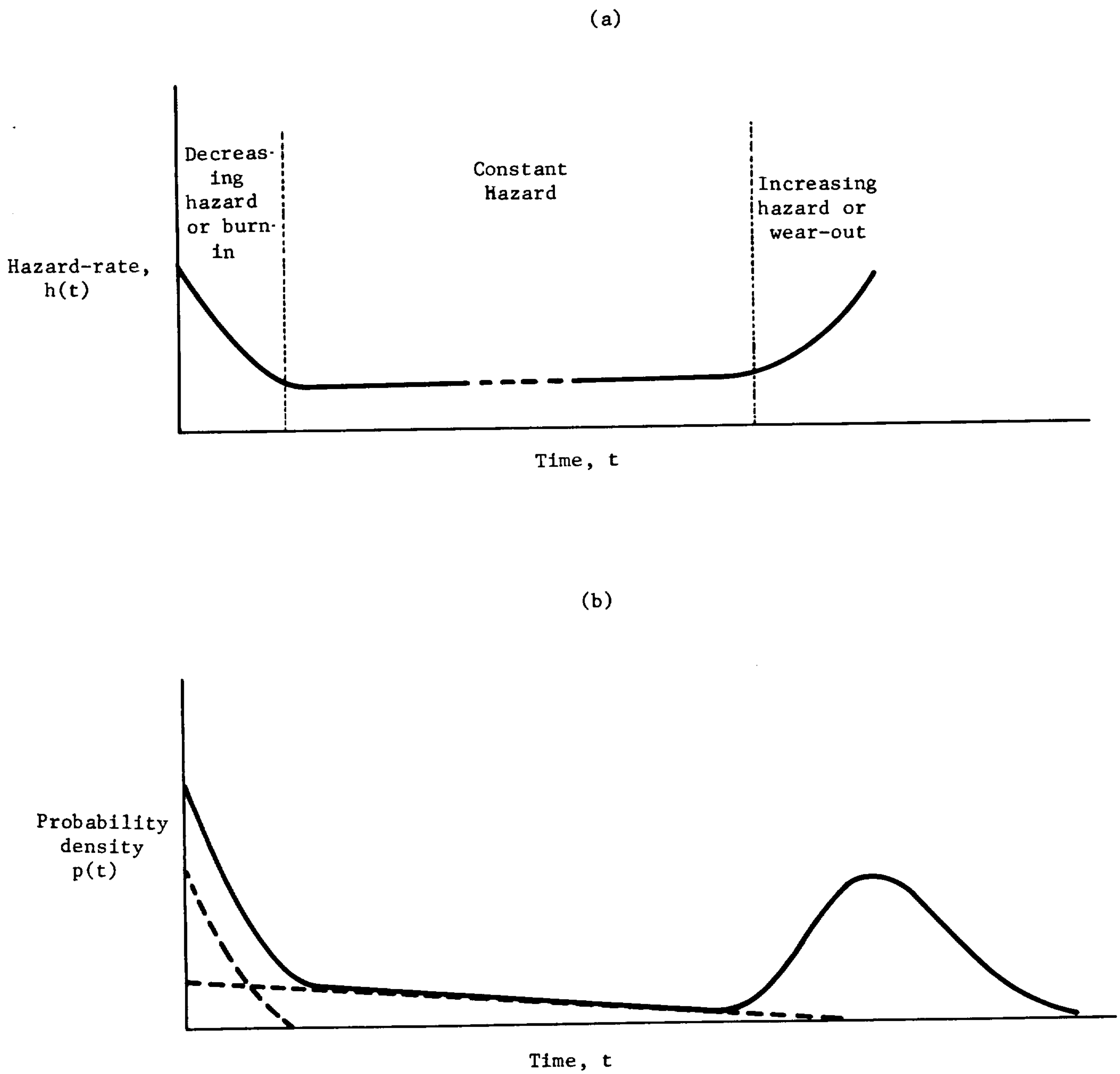


Figure 4-2 The Bathtub Shaped Hazard-Rate Curve
and Its Probability Density

4.4 Consideration of the Environment

The reliability of an item is defined as the probability that an item performs its intended function under defined conditions at a designated time for a specified operating period. Thus the reliability is conditional on a specific environment or environmental profile whether it is estimated by a simulated test or from results of items used in previous missions. The environment might be characterized by fixed conditions, such as temperature equal to 30°C, or it may be described by a deterministic profile, such as that shown in Fig. 4-3.

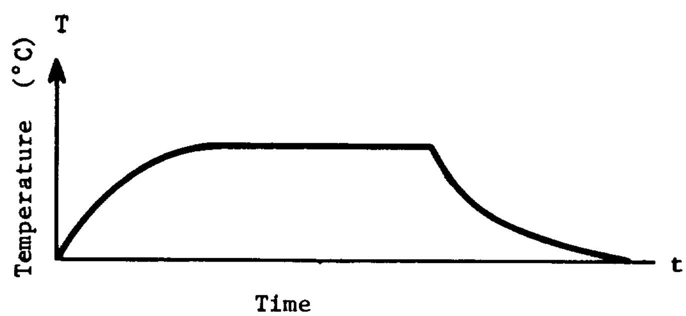


Figure 4-3 Example of Deterministic Environmental Profile

The environment might also be characterized by a random variable or a random process where time is explicitly considered. Some approaches to considering the effect of random environments on reliability measures are discussed below.

If the environmental stress is described by its density function p(E), then the probability of successful operation is given by the following procedure. Let the conditional probability of success given E be denoted by

$$P(S|E),$$

then the unconditional probability of success for continuous density function, p(E), is given by

$$P(S) = \int_E P(S|E)\, p(E)dE \tag{4-14}$$

and for discrete density function, $P(E_i)$, $i = 1, 2, \ldots$, by

$$P(S) = \sum_i [P(S|E_i)\ P(E_i)]. \tag{4-15}$$

Example 4-2

Consider the simple example in which the probability density for the environment is discrete as given below.

$$P(E_i) = \begin{cases} \frac{1}{4} \text{ for } E_1 \\ \frac{1}{2} \text{ for } E_2, \\ \frac{1}{4} \text{ for } E_3 \end{cases}$$

and let the probability of success conditional on these environments be

$$P(S|E_1) = \frac{4}{8}$$

$$P(S|E_2) = \frac{3}{8}$$

$$P(S|E_3) = \frac{1}{8}\ .$$

Then the unconditional probability of failure is

$$\sum_i [P(S|E_i)\ P(E_i)] = \frac{1}{4} \times \frac{4}{8} + \frac{1}{2} \times \frac{3}{8} + \frac{1}{4} \times \frac{1}{8} = \frac{11}{32}\ .$$

The above concept also can be used when an event requires an elapsed time period (such as S = no failure to time t) and also when the environments are time dependent.

In some situations it is necessary to explicitly consider the environment as a random process with known characteristics. Consider the problem where an item will sometimes fail when an environment which is a random process reaches a certain level. If the environment is a random process with peaks the distance between which is given by the negative exponential distribution (assuming they occur with rate λ per unit time period) and if the conditional probability of failure is p given that a peak has occurred, the probability that the item does not fail in the interval (0, t) is given by

$$
\begin{aligned}
P(S) &= P(\text{no peaks in } (0,\ t)) + P(\text{1 peak in } (0,\ t))q \\
&\quad + P(\text{2 peaks in } (0,\ t))\ q^2 + \ldots \\
&= e^{-\lambda t} + \frac{(\lambda t)\ e^{-\lambda t}}{1!}\ q + \frac{(\lambda t)\ \ e^{-\lambda t}}{2!}\ q^2 + \ldots \\
&= e^{-\lambda p t}.
\end{aligned}
$$

Thus the failure time distribution is exponential under this environment. See Refs. 15 and 16 for further discussion on this and related descriptions of a random environment. Further, if an item will fail only after k peaks or shocks have occurred, the gamma density function is appropriate. That is

$$
\begin{aligned}
p_k(t) &= \frac{\lambda^k\ t^{k-1}\ e^{-\lambda t}}{\Gamma(k)}, \qquad t \geq 0 \\
p_k(t) &= 0, \qquad\qquad \text{elsewhere,}
\end{aligned}
\tag{4-16}
$$

where

t is time,

λ is the rate at which the shocks occur,

k is the number of shocks for failure, and

$\Gamma(k) = (k-1)! = (k-1)(k-2) \cdots 1.$

In summary, the nature of the environment must be considered carefully to hypothesize models for behavior of the reliability function.

4.5 Poisson Processes

The Poisson process is widely assumed in reliability prediction, particularly for repairable items such as the typical electronic equipment.

Let $P_n(t)$ = probability that exactly n occurrences are recorded during a time interval of length t.

Thus $P_0(t)$ = probability of no occurrences, and

$1 - P_0(t)$ = the probability of one or more occurrences.

It is assumed that

$$\lim_{t \to 0} \frac{1-P_0(t)}{t} \to \lambda,$$ that is the probability of one

or more occurrences is proportional to the length of the interval, λ is a positive constant, the failure rate of an item. See Ref. 17 for a detailed development of this process and related birth and death processes.

Postulates: Whatever the number of occurrences in the interval (0, t), the following conditional probabilities hold

$$P(\text{an occurrence in the interval } (t, t+h)) = \lambda h + o(h),^{*}$$

$$P(\text{more than one occurrence in } (t, t+h)) = o(h).$$

The above postulates yield the following difference equations.

$$P_n(t+h) = P_n(t)(1-\lambda h) + P_{n-1}(t)\lambda h + 0(h), \ n \geq 1 \tag{4-17}$$

i.e., the probability that there are n occurrences in the interval (0,t+h) is the probability of n occurrences in the interval (0,t) multiplied by the probability of no occurrences in the interval (t,t+h), plus the probability of n-1 occurrences in the interval (0,t) and one occurrence in the interval (t,t+h), plus the probability of n-x ($x \geq 2$) occurrences in (0,t) and x(≥ 2) in the interval (t,t+h), (the latter is of order o(h)). For n = 0

$$P_0(t+h) = P_0(t)(1-\lambda h)$$

or

$$\frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t),$$

and as $h \to 0$ one obtains

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) \text{ or } P_0'(t) = -\lambda P_0(t).$$

Using $P_0(0) = 1$ we get $P_0(t) = e^{-\lambda t}$. Equation 4-17 similarly can be reduced to the differential equation

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \ n \geq 1. \tag{4-18}$$

Substituting into Eq. 4-18, we obtain

$$P_1(t) = \lambda t e^{-\lambda t}.$$

We derive successively all the terms to obtain the general terms

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \ n = 0, 1, 2, \ldots, \infty. \tag{4-19}$$

---

* o(h) is a function of h such that $\lim_{h \to 0} \left(\frac{o(h)}{h}\right) = 0$.

This formula gives the probability that n occurrences will be observed in a time interval (0,t) with a constant rate of occurrence per unit time equal to λ. The quantity λt is the expected number of occurrences in the time interval of length t and one frequently sees the form

$$P_n(t) = \frac{e^{-\mu} \mu^n}{n!} , \qquad (4\text{-}20)$$

where μ is the expected number of occurrences in the time interval of length t.

Example 4-3

Suppose that an item has a failure rate, λ, of 0.001/hour. What is the probability that no failures occur in 100 hours?

Solution:

$$\lambda t = 100 \ (.001) = 0.1.$$

Thus

$$P(0 \text{ failures}) = e^{-\lambda t} = e^{-0.1}.$$

Note that this is equivalent to the reliability of the item. Hence one can better understand the tie-in between the Poisson distribution and the exponential distribution.

Example 4-4

Suppose that a certain item is tested as follows. One item is placed on test until failure and it is then replaced by another identical item, etc. Suppose further that the failure rate of the item is 0.001/hour and that the test is for 10,000 hours. What is the probability of at least 15 failures?

Solution:

λt is equal to 0.001 (10^4) = 10, the expected number of failures in 10^4 hours. Thus the probability of at least 15 failures is given by

$$P(n \geq 15) = \sum_{n=15}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} = \sum_{n=15}^{\infty} \frac{e^{-10} \ 10^n}{n!} = 0.083$$

using Molina's Tables [Ref. 18] for the Poisson distribution. The same solution would apply to the above problem if a single item was repaired and returned to operation. Here the operating times between failure would be exponentially distributed, and the equipment reliability index, Mean Time Between Failure (MTBF), is the mean of this distribution. Only the operating time would be considered. Further, the same solution would apply to any number of identical items operated for a total time of 10,000 hours, regardless of how much time was accumulated on any item.

A reason why the Poisson process is widely assumed in reliability prediction is that in this mathematical model past operation has no influence on future reliability. This simplifies a prediction analysis; for some complex systems it makes the prediction practical.

4.6 Reliability Measures for Repaired Items

Reliability descriptions for repairable items are discussed here for a general situation where an example of such an item would be a motor or typical electronic equipment. With repair, there are time (of operation) to first failure, time between first and second failure, time between second and third failure, and so on. Each of these failure times when considered for a large number of identical items will have a distribution associated with it; these distributions may or may not be identical.

The data from motor failures [Ref. 19] have indicated time between failure patterns as in Fig. 4-4. Density functions of the time to first failure, time between first and second failure, time between second and third failure, and so on are shown in Fig. 4-4, and these could be fitted with Weibull distributions with different shape and scale parameters. The origin of each density function is when operation is resumed after the motor is repaired. When the density functions are plotted on an elapsed operating time scale, starting with the earliest initial operation, then only the time to first failure density function is as shown in Fig. 4-4 and the others have a different shape. This is illustrated in Fig. 4-5. The density function of the time to second failure on the scale in Fig. 4-5 is the sum of the time to first and time between first and second failure; the time to third failure is the sum of the first, second and third, and so on. There is considerable overlap on the time scale of Fig. 4-5; as an example the early overlap of the first and second times comes about because some of the first failures occur late, which are repaired, and the second failure occurs shortly. The overlap reflects the many possible combinations by which the first and second failures can occur. As the third, fourth, and additional failures are brought into consideration, they enter into the overlap on the elapsed operating time scale in a similar manner. The summing operation is referred to as convolution. Fig. 4-5 also illustrates the renewal rate, which represents the total number of items failing per unit of time, divided by the original population. It can be seen to be the sums of the ordinates of all the density functions of the time to failure as a continuous function of time. Note that this is a conventional deterministic, algebraic summing, and is thus different from the probabilistic convolution type summing noted above. The renewal rate where an item is repaired is similar in one sense to the density function of the non-repairable item, as their shapes are what the smoothed curves for histograms

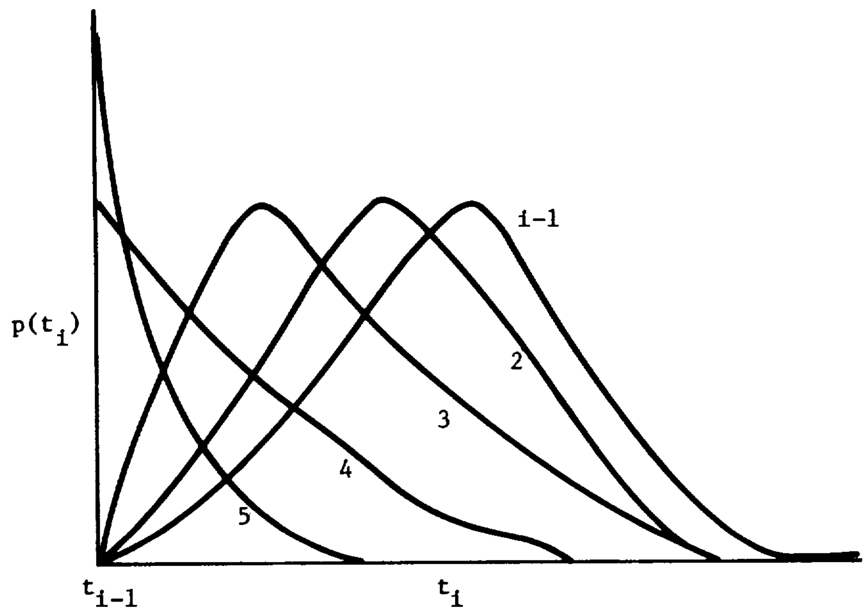


Figure 4-4 Time Between Failures With Different Density Functions

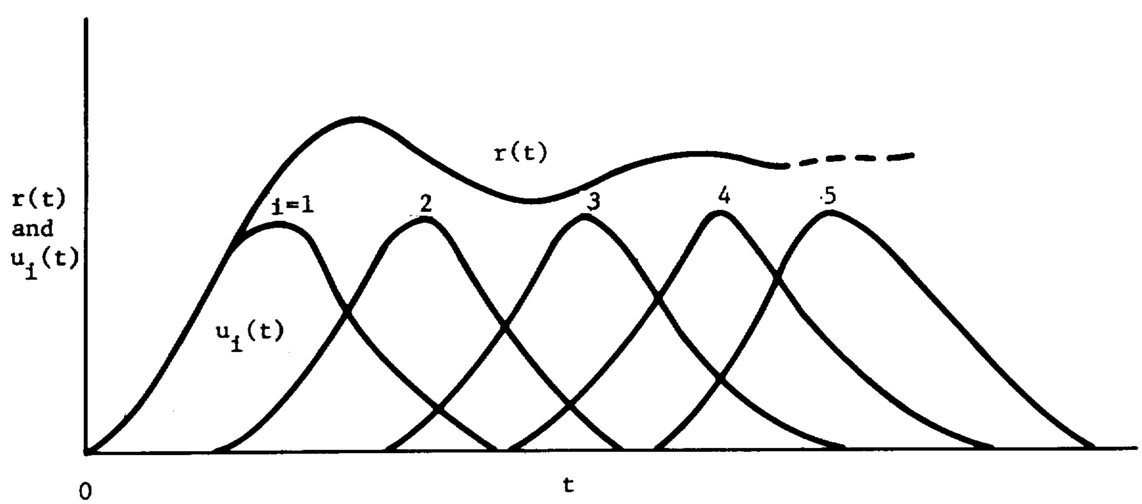


Figure 4-5 Renewal Rate Associated With Fig. 4-4

of populations of items would look like. However, they are different in the sense of predicting the reliability for a single item. If for a repairable item the accumulated operating time since the last failure, and the shape of the density function of the pending failure is known, then a reliability prediction equation would be based on Eq. 4-4. If this information is not known then how a reliability prediction would be made would depend on just how much is known concerning accumulated operating time, accumulated failures, and the time of last failure.

The mathematical description of the renewal rate is sketched below. Extensive treatment of it can be found in Ref. 20.

$$
\begin{aligned}
u_1(t) &= p_1(t) \\
u_2(t) &= \int_0^t u_1(t_1)\, p_2(t-t_1)dt_1 \\
&\cdots \\
u_i(t) &= \int_0^t u_{i-1}(t_{i-1})\, p_i(t-t_{i-1})dt_{i-1} \\
&\cdots \\
u_n(t) &= \int_0^t u_{n-1}(t_{n-1})\, p_n(t-t_{n-1})dt_{n-1} \,. \qquad (4\text{-}21)
\end{aligned}
$$

The renewal rate is their sum:

$$r(t) = \sum_{i=1}^{n} u_i(t_i). \qquad (4\text{-}22)$$

Here

$p_i(t)$ = the density function of the time between the (i-1)th and ith failure where elapsed time only includes that of the ith failure.

$u_i(t)$ = the density function of the time to the ith failure, where elapsed time includes that of previous failures.

$r(t)$ = the renewal rate where elapsed time is continuous.

The renewal rate has not received much explicit application to conventional reliability predictions, as conventional predictions typically assume that time between all failures during the period of interest have an exponential distribution and thus are a Poisson process as discussed in Sec. 4.5. The Poisson process was developed in Sec. 4.5 using difference equations, but it could also be developed using the renewal rate as the basis. However, for various mixtures of non-exponential distributions where the difference equation approach is not applicable, the renewal rate offers an approach for developing appropriate mathematical models. The discussion of renewal rates is included here to give those interested in using non-exponential distributions for repairable items an indication of how to get started and also to support the later discussion in Sec. 4.8 concerned with the use of bathtub shaped curves for repairable items.

4.7 Reliability Measures for Replaced Items

A somewhat similar situation to the repairable item exists where identical, non-repairable items are used in large quantities and are replaced with new items when a failure occurs. Examples of this are light bulbs of fluorescent tubes in large buildings. Here the mathematical description of the density functions of the time to first failure, time between second and third failure, and so on are identical.

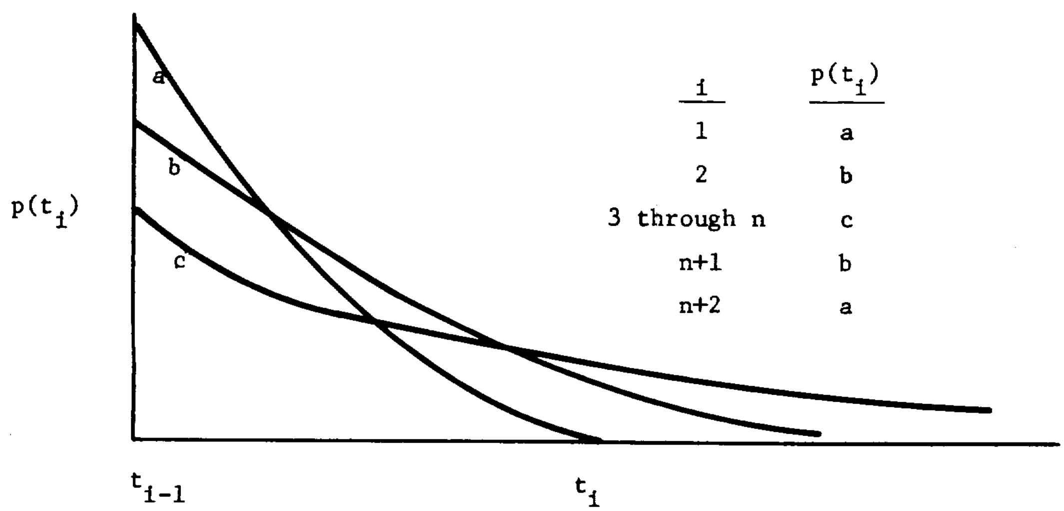
The renewal rate of Sec. 4.6 becomes the replacement rate, where the later has the feature that all densities of time between failure are identical. Where this feature exists, then $r(t)$ of Eq. 4-22 and Fig. 4-5 becomes constant and equal to the reciprocal of the mean lifetime after several generations [Ref. 20]. This is a classical problem in renewal theory, but has limited applicability for real world reliability analysis problems.

4.8 Bathtub Curve for Repaired Items

The familiar bathtub shaped curve, which has previously been discussed in Sec. 4.3 for a single lifetime hazard rate where there was no repair or replacement, is also used on occasion for repairable equipment. Typically there is no discussion of the appropriate mathematical development [Ref. 21 , p. 24]. Such a bathtub shaped curve for the repaired item, however, implies a different mathematical model than for the non-repairable item. (The repairable equipment is confused with the non-repairable part on page 19 of Ref. 22 .) A mathematical model which would lead to a bathtub shaped curve for repaired items could result from application of the renewal rate of Sec. 4.6, which is quite different from that of Sec. 4.3 based on the hazard rate.

In Fig. 4-6 some time between failure density functions are shown for the time to the first, time between first and second, and so on . Figure 4-7 shows the renewal rate as well as the elapsed operating times to the first, first plus second, first plus second plus third, and so on . Figures 4-6 and 4-7 do not come directly from data, but are a judgement assumption which is believed to be somewhat similar to that which would be found for some electronic equipment.

Each of the time between failure density functions of Fig. 4-6 is assumed to be exponential in shape, but with some differences in the mean time to failure parameters. The first two density functions have successively increasing means, the third density function on through a very large number, n, have the same mean, and the n+1st and successive density functions have decreasing means. Combining these time between failure density functions into a renewal rate is illustrated in Fig. 4-6, which has the familiar bathtub shape.



i	$p(t_i)$
1	a
2	b
3 through n	c
n+1	b
n+2	a

Figure 4-6 Exponential Time Between Failure Densities with Different Means

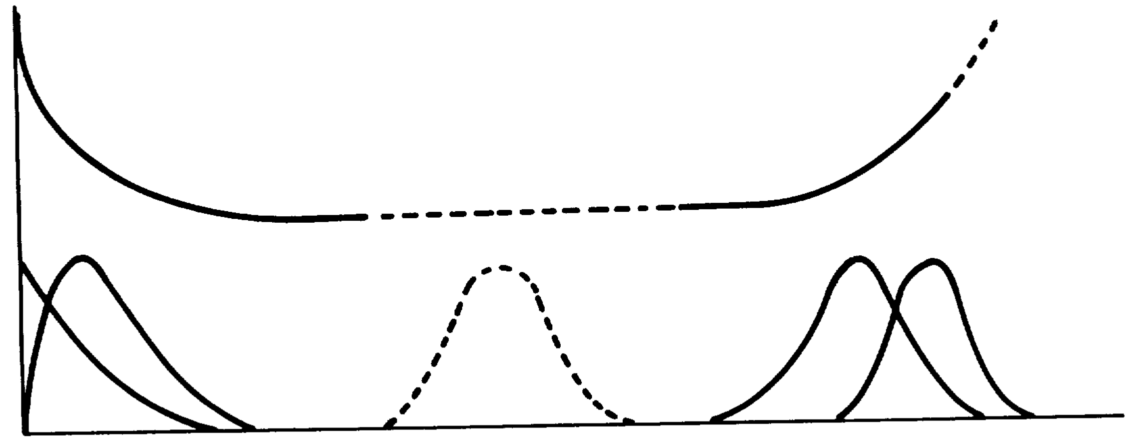


Figure 4-7 The Bathtub-shaped Renewal-rate Curve
for Repaired Items of Fig. 4-6

The flat portion of the renewal rate of Fig. 4-6 is the situation often assumed in reliability prediction. Here the accumulated operating time does not affect the reliability of an equipment, and the reliability model of

$$R(t) = e^{-\lambda t}$$

applies regardless of age. This period is also described by the Poisson process discussed in Sec. 4.5. Exponential distributions with identical λ will always result in a constant renewal rate. On the other hand, a constant renewal rate does not mean that the times between successive failures have an exponential distribution and that a Poisson process exists. Recall that Sec. 4.7 noted that a constant renewal rate will ultimately result from any stable distribution of time between failures.

One reason for going into this discussion of the widely cited bathtub curve is to point out that a bathtub curve could arise from other than identical exponential distributions. As reliability analysis matures and is extended to a wider diversity of commodities it will be increasingly necessary to be aware of the possibility that non-exponential distributions might exist. For instance, data from a population of repaired equipment when plotted in histogram fashion might resemble the bathtub curve, but the distribution of time between failures need not be exponential. Correct choice of underlying distributions can have high implications for the accuracy of reliability predictions, for the validity of statistical tests, and for the optimization of preventive maintenance actions based on assumed distributions.

5. Bound-Crossing

The type of reliability measures treated in this section are those sometimes labeled tolerance, drift or degradation, better described as a "bound-crossing" type. Items are designed to meet given requirements such as the output voltage of an electronic power supply shall be 115 $\pm$ 1 volt ac and it is designated as failed if the voltage crosses one of the bounds 114 and 116 volts ac. In a mechanical system it may be desired to estimate the probability that the strength of an item will exceed the stress to which it is subjected. In some environments the strength of an item will be a function of time as a result of fatigue due to thermal cycling or stress cycling. In this case we will be interested in the probability that at the mission end the item will have sufficient strength to meet the applied stress. The bound in this case is not necessarily a fixed level but may be a distribution of stress levels.

5.1 Fundamentals

5.1.1 Notation

The notation to be used in this section will be y for a performance characteristic, s for stress level or environment level, and t for time. The bound will be denoted by ℓ for lower and u for upper.

5.1.2 Bound-Crossing Reliability

The probability that a performance characteristic y does not exceed the upper bound u is denoted by

$$P_u = P(y \leq u),$$

and the probability that u is exceeded by y is

$$1 - P_u = P(y > u).$$

Similarly, P_ℓ is the probability that y is less than the lower bound ℓ, i.e. $P_\ell = P(y \leq \ell)$. If the bound has a distribution of values such as the probability density p(s) for stresses, then the probability that y-s exceeds 0,

$$R = P(y-s > 0)$$

is a measure or index of the performance of the item. To consider more than one performance characteristic and stress, vector notation can be replaced for the single values y and s respectively, with due consideration for probabilistic dependence. To consider time the appropriate distributions become time varying, and additional criteria of failure become possible; this will be discussed in Sec. 5.4.

5.1.3 Distribution Types

In order to estimate the probabilities P_u and P_ℓ it is necessary to know the distributions of the performance characteristics and stress levels. These distributions may be any one of several common distribution forms given in Appendix A.1, e.g. normal, log-normal, uniform, gamma, etc. The selection of the distribution form can sometimes be made on the basis of technical considerations such as positive and negative deviations of the same magnitude are equally likely (normal), or that the incremental changes are proportional to the measurement value (log-normal). Refer to Ref. 15 for basic assumptions underlying some distribution forms. Often the distributions are selected on a subjective basis to describe one's feeling about the variation of the characteristics and perhaps more often they are chosen for convenience of the analytical methods. The latter is often not necessary due to the capabilities of modern electronic computers and the availability of "canned" computer programs to perform the necessary analyses as described in Volume II - Computation. Time varying distributions for bound-crossing problems introduce additional considerations which will be covered in Sec. 5.4.

5.2 Fixed Bounds

In this situation it is assumed that a distribution form can be selected which describes the variation in the performance attribute at some specified time in its life when used under certain environmental conditions. The distribution can sometimes be selected by basic considerations of the physical process, by fitting a few distributions by graphical techniques, or by using more sophisticated statistical techniques for estimating the distribution parameters. In some cases the form of the distribution may not be specified and a distribution free or non-parametric method used. The bounds are assumed to be known from technical considerations of the application of the item in the system.

Example 5-1

A performance attribute of interest at the end of 10,000 hours under specified environmental conditions is normally distributed with mean 95 and standard deviation 10 units. For example, this might apply to h_{fe}, an equivalent circuit h-parameter for a transistor. If it is desired that h_{fe} exceed 70, then what is the probability that a transistor selected at random from a population of similar items and subjected to 10,000 hours of operation under the same conditions will have an h_{fe} greater than 70?

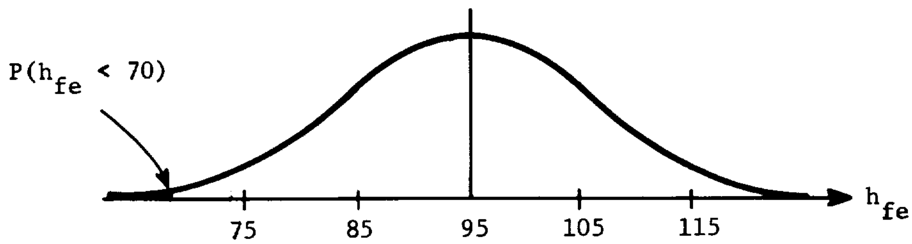


Figure 5-1 Probability Density Function for h_{fe}

The probability that h_{fe} exceeds 70 is

$$P(h_{fe} > 70) = 1 - P(h_{fe} \leq 70)$$

$$= 1 - \Phi\left(\frac{70 - \mu}{\sigma} \right)$$

$$= 1 - \Phi\left(\frac{70 - 95}{10} \right)$$

where μ = mean of h_{fe} = 95,

σ = standard deviation of h_{fe} = 10,

and $\Phi(X)$ is the area udder the standard normal distribution curve to the left of X. In this example X is -2.5 and the area to the left of -2.5 can be obtained from a table of areas under a normal curve such as given in standard probability and statistics books,

$$\Phi(-2.5) = 0.0062.$$

Thus the probability that h_{fe} exceeds 70 is

$$P(h_{fe} > 70) = 1 - 0.0062 = 0.9938.$$

Figure 5-1 illustrates this example.

The assumptions under which the above result was obtained are given below and should be carefully noted when using these techniques:

(1) Normal distribution of values of h_{fe},

(2) Known mean and standard deviation, and

(3) Conditions of manufacture and operation of items are same as those to which the probability estimate is to apply.

Checking the first and second assumption would depend on the source of the information for the h_{fe} distribution. Often this will come from special tests for this purpose. If so, the first assumption above can be checked graphically by plotting the sample distribution function on normal probability paper. The extent to which one can check the adequacy of the normality in the region of the tails depends upon the amount of data. The second assumption is really never satisfied but for very large sample sizes the results would be practically unaltered by using procedures which depend on the sample mean $\bar{x}$ and standard deviation s. The third assumption is of special importance to the design engineer in that he will specify the test conditions to correspond as nearly as possible to those conditions under which he wishes to infer the quality concerning the items tested.

Similar results can be obtained using other forms of distributions such as log-normal, Weibull, extreme-value, etc. In each case the "goodness" of the distribution can be checked by a probability graph of appropriate form and/or by analytical techniques such as given in statistical texts, e.g. see Ref. 23.

5.3 Stress-Strength Model (Bound Distribution)

In this case the performance of an item is considered to be satisfactory if the strength of the item exceeds the stress to which it is to be exposed in application. Thus the bounds may not be fixed in that an item selected at random and used in a specific system may be subjected to one of a distribution of stresses rather than a known fixed stress. In actual practice the stress may vary over the life of the item but consider for the moment that an item is subjected to a constant stress over its life and that the stress level may vary from item to item.

The approach to this problem is to specify the stress and the strength density functions by one of the methods of Appendix A.1. Then the parameters of the distributions are estimated and one then computes the estimate of the desired probability. Thus in the notation suggested earlier, the strength density function is p(y) and the stress by p(s). Then it is desired to determine the probability that y is larger than s.

$$P(y > s)$$

or the equivalent

$$P(y - s > 0).$$

This is found by

$$P(y > s) = \int_0^\infty p(s) \left[\int_s^\infty p(y)dy\right]ds \tag{5-1}$$

where the range of s and of y does not contain negative values and the distributions are independent. An example is given below in which both distributions are assumed to be independent and normal. In this case the difference y-s is also normally distributed and the parameters for this distribution of y-s are given in terms of those for the individual distributions of y and s respectively.

<u>Example 5-2</u>

Consider a simple stress-strength analysis of a part with strength density function assumed to be normal with mean (μ) 25K psi and standard deviation (σ) 3K psi and stress distribution with mean 15K psi and standard deviation 2K psi. The two density functions are illustrated in Fig. 5-2.

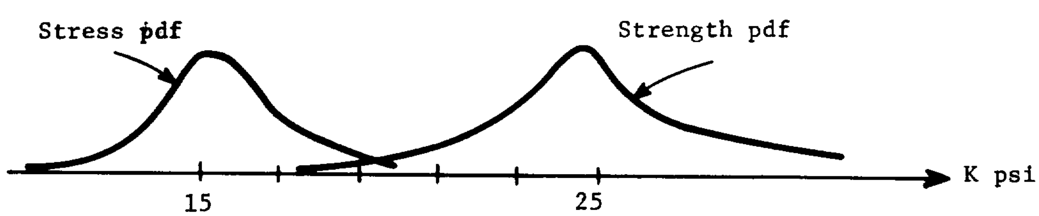


Figure 5-2 Probability Density Functions for Stress and Strength

The probability that the strength exceeds the stress is given by the probability,

$$P(y - s > 0),$$

where y is the strength and s is the stress. Now y-s is also normally distributed with mean 10K and standard deviation

$$\sigma(y-s) = \sqrt{\sigma^2(y) + \sigma^2(s)}$$

$$= K\sqrt{9+4} \approx 3.6K \text{ psi.}$$

and hence

$$P(y - s > 0) = P\left(\frac{y-s - (10K)}{3.6K} > \frac{-10K}{3.6K} \right) = P\left(u > \frac{-10K}{3.6K}\right)$$

where u is a standard normal variable and thus

$$P(y - s > 0) = 1 - \Phi(-2.78) = 0.9973.$$

One of the major difficulties in stress-strength problems is obtaining sufficiently good estimates of the stress and strength distributions and hence the difference y-s. Given these estimates the problem of estimating the probability is a difficult one even if one assumes a normal distribution. Often the difficulty is aided by using the estimated safety margin as a measure or index of adequate strength-stress margin. The safety margin is

$$\text{Safety Margin} = \frac{\mu_y - \mu_s}{\sqrt{\sigma^2(y) + \sigma^2(s)}} . \qquad (5\text{-}2)$$

5.4 Time Dependency*

The random behavior over time of a performance attribute can be visualized as a time-varying probability density function as illustrated in Fig. 5-3. Such sketches are sometimes used for data for a part characteristic obtained from life testing. Where the criterion of failure is that the performance attribute y(t) go outside some fixed bound, then the reliability measure is a straightforward extension of the approach in Sec. 5.2 if the performance attribute drift is always either increasing or decreasing (monotonic) such as shown in Fig. 5.4.

Here the reliability is

$$R(t) = \text{Prob}[y_\ell < y(t) < y_u]$$

*Lower case letters are used in this report for random processes and variables.

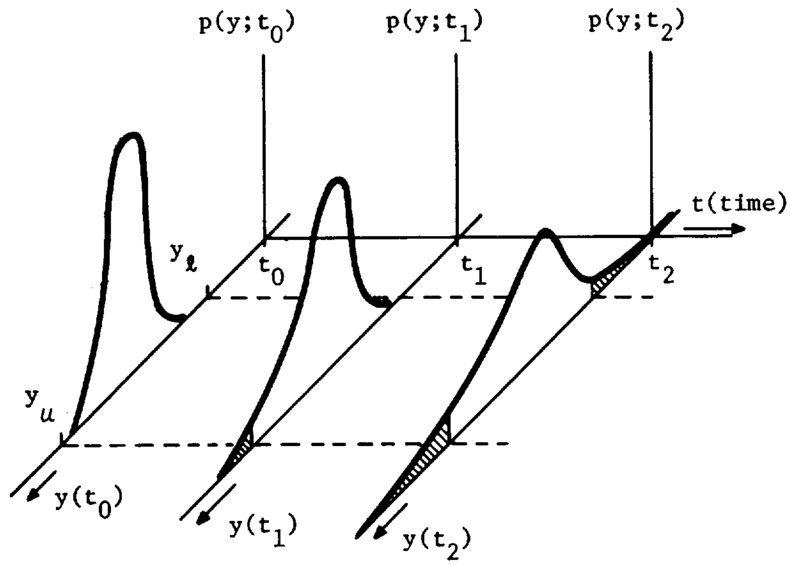


Figure 5-3 Drift of y(t) Illustrated as a Time-varying Density Function

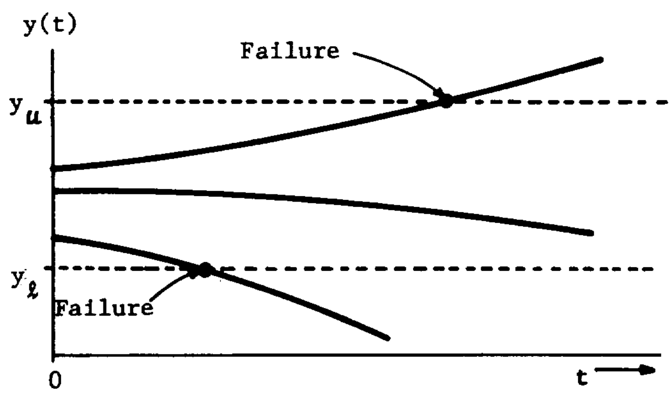


Figure 5-4 Examples of Monotonic Drift Behavior

which is also

$$R(t) = \int_{y_\ell}^{y_u} p(y;\ t)dy$$

where the integration is over y at time t and R(t) is a monotonically decreasing function. An approximation to R(t) by performing this integration on the p(y; t) at selected times t will often suffice. It is stressed that this approach is for a monotonic drift. Non-monotonic drift such as shown in Fig. 5-5 introduces additional considerations.

For non-monotonic drift first consider the failure criterion treated above where failure is defined as the performance attribute going outside some fixed bound. If all that is known is some p(y; t), then the drift reliability cannot be obtained. For instance, the p(y; t) at time t_1 and at some later time t_2 might be identical, but this does not mean that no additional failures have occurred because in a population of items some which were out of bound may have drifted back in and others may have drifted out. Therefore it is necessary to describe the time-varying distributions of the performance attributes with a functional form. Here the performance attribute is expressed as a deterministic function $y(t) = y(\underline{\alpha};\ t)$ where the $\underline{\alpha}$ are probabilistic with known probability density $p(\underline{\alpha})$. This method can be used where the drift failure criterion is a first crossing of a bound for either monotonic or non-monotonic drift such as in the above discussions, and it also can be used for other criteria for non-monotonic drift such as the following:

(1) the cumulated area outside a bound(s),

(2) the number of crossings of a bound(s),

(3) the cumulated time outside a bound(s).

The approach for non-monotonic drift is to reduce the failure criterion to a first crossing problem. A new function w(t) is defined such that reliability becomes

$$R(t) = P(y_\ell < w(t) < y_u).$$

As an example, Fig. 5-5 illustrates the last failure criterion of the amount of time that y(t) is outside the bounds. Here a corresponding function w(t) is defined, and the failure criterion becomes w(t) first crossing a specified level w_u. Other w(t) functions could be established for other failure criteria.

An example of a possible mathematical form for describing the performance attribute is the polynomial expression $y(t) = b_0 + b_1\ t + \ldots + b_n t^n$ where the b's are random variables of the same sign for monotonic drift and of mixed signs for non-monotonic drift. Trigonometric series offer forms for periodically varying attributes.

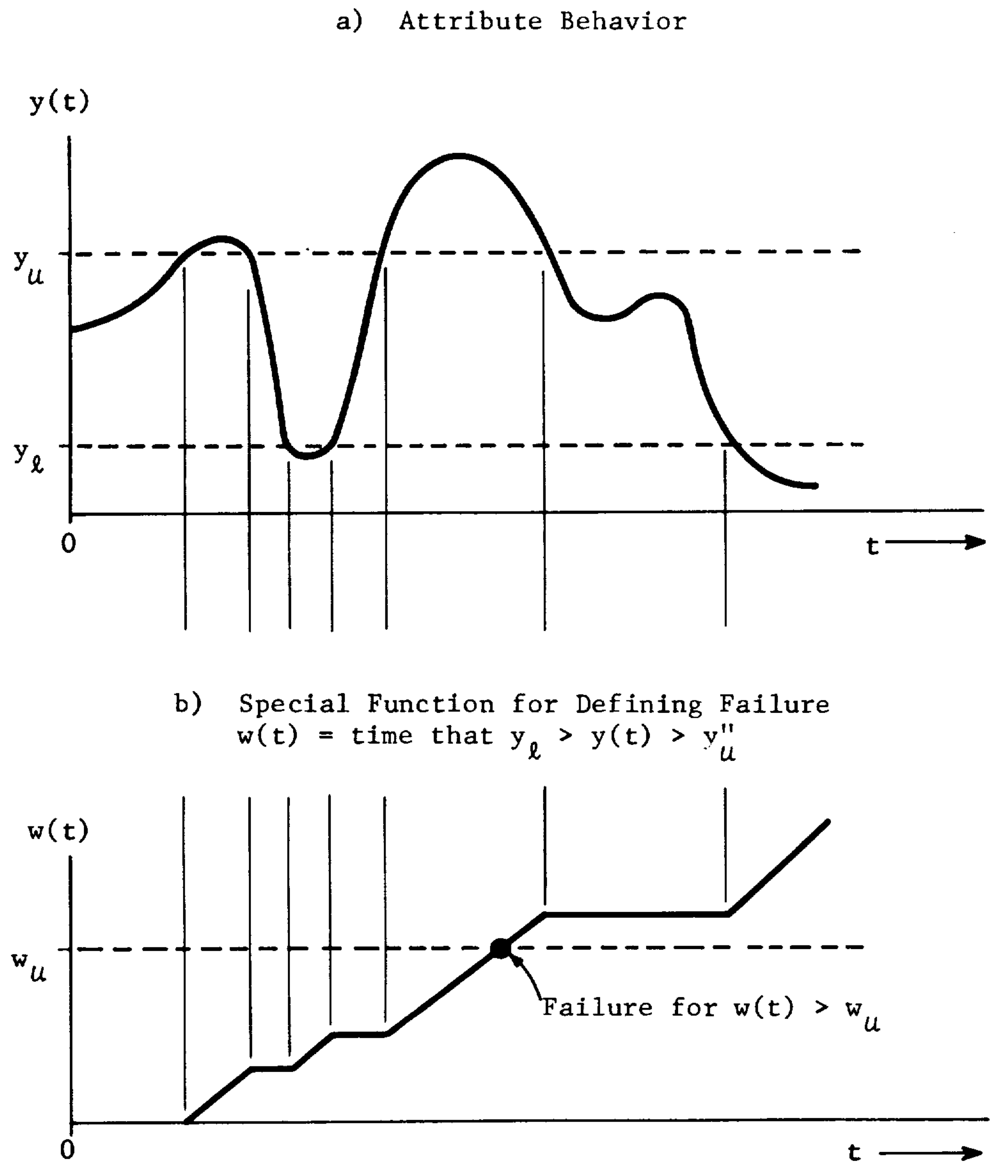


Figure 5-5 Example of Non-monotonic Drift Behavior and One Possible Method for Defining Failure

In some situations where the drift of the performance attribute is non-monotonic it may be represented by a stochastic process. Such a situation could be the error in a system output. For example, a stationary Markov Gaussian noise process may be completely described by its auto-correlation function or its power spectral density. An experimental application has been made using this general approach for the error in a tilt stabilization assembly for an airborne radar antenna [Ref. 24]. A recent theoretical book [Ref. 25] discusses various reliability indices for stochastic processes.

The discussion in this section is about fixed-bounds. The reader who is interested in time-dependent problems where the bound is a distribution (such as in Sec. 5.3 for the stress-strength problem) would find some guidance in the discussion of Part IV. The basic idea would be to treat both the performance attribute and the bound as independent variables in a deterministic function. The dependent variable then becomes a single performance attribute which has a fixed bound for the failure criterion. For example, let $w(t) = y(t) - s(t)$ where y is the strength, s is the stress, and w is the new performance attribute which has the bound $w(t) > 0$.

Two examples follow where the performance attribute is a time-varying normal distribution. In these examples the performance attributes are of the form $y(t) = y(\underline{\alpha};\, t)$ where the $\underline{\alpha}$ are normally distributed.

Example 5-3

Suppose that the resistance of a particular electrical resistor changes over the interval $0 < t \leq T$ according to

$$r(t) = \alpha_1 + \alpha_2 t \quad \text{ohms}$$

where

α_1 is normal with mean 330Ω and standard deviation 33Ω,

α_2 is normal with mean -0.003Ω/hour and standard deviation 0.001Ω/hour.

Let r(t), the resistance at time t be the performance measure of interest and hence r(t) is also normally distributed with mean and standard deviation,

$$\mu\{r(t)\} = \{330 - 0.003t\}\Omega$$

$$\sigma\{r(t)\} = \{(33)^2 + t^2(.001)^2\}^{1/2}\ \Omega.$$

$$= \{1089 + 1 \times 10^{-6}t^2\}^{1/2}\Omega.$$

For t = 1000 hrs.,

$$\mu\{r(1000)\} = 327\Omega$$

$$\sigma\{r(1000)\} = (1090)^{1/2} \simeq 33\Omega.$$

and the density function of resistances at 1000 hrs. is shown in Fig. 5-6.

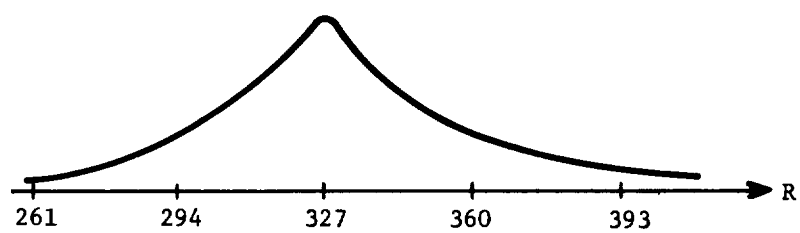


Figure 5-6 Probability Density Function of Resistance at t = 1000 Hours.

If the reliability were defined as the probability that the resistance lies between 270 and 390, then it would be given by the following, at t = 1000 hours,

$$R = \Phi\left(\frac{390-327}{33}\right) - \Phi\left(\frac{270-327}{33}\right)$$

$$= \Phi(1.91) - \Phi(-1.72) = 0.97 - 0.04 \simeq 0.93,$$

where $\Phi(x)$ is the cumulative standard normal distribution function obtained from standard texts.

Example 5-4

Suppose that a mechanical part under consideration has a strength which decreases with time in accordance with some function of time under stress. Let the strength be described by

$$y(t) = \alpha_1 e^{-kt} + \alpha_2(1-e^{-kt})$$

where α_1 is the initial strength, α_2 the strength as $t\to\infty$, k is a constant determined by the particular part. Let also

α_1 be normally distributed with mean 50K psi and standard deviation 4K psi,

α_2 be normally distributed with mean 20K psi, standard deviation 2K psi and assume that it is correlated with y(0), i.e. $\rho = 0.90$, and

$k = 0.001$.

Thus y(t) is also normally distributed with mean

$$\mu\{y(t)\} = [50K\ e^{-0.001t} + 20K(1-e^{-.001t})]\text{psi},$$

and standard deviation

$$\sigma\{y(t)\} = \{4^2K^2\ e^{-0.002t} + (1-e^{-.001t})^2\ 2^2K^2$$

$$+ 2(0.90)\ e^{-0.001t}\ 4K \cdot 2K(1-e^{-0.001t})\}^{1/2}\text{psi}$$

$$= K\{5.6e^{-0.002t} + 6.4^{-.001t} + 4\}^{1/2}\text{psi}.$$

For t = 1000 cycles, $\mu\{y(t)\} = 31{,}000$ and $\sigma\{y(t)\} = 2670$. If the prescribed lower limit were $y_\ell = 30K$ psi, what is the maximum number of cycles to which the part should be exposed in order that the probability of its strength exceeding 30,000 psi will be 0.95?
To solve this problem we must find t such that

$$\mu\{y(t)\} - 1.645 \cdot \sigma\{y(t)\} = 30{,}000$$

$$50K\ e^{-0.001t} + 20K(1-e^{-.001t}) - 1.645K\ [5.6e^{-0.002t} + 6.4e^{-0.001t} + 4]^{1/2}$$

$$- 30K = 0.$$

By graphing the left member of the above equation one can estimate the time t at which the curve crosses the axis and hence obtain a more exact solution analytically if desired.

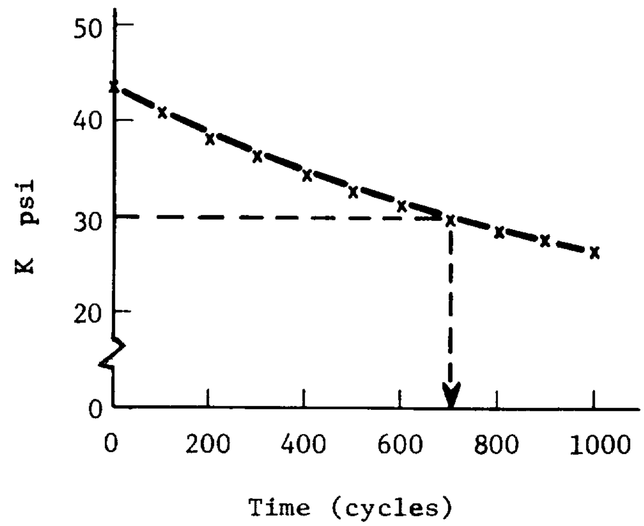


Figure 5-7 Strength Versus Elapsed Time

The number of cycles is estimated to be 700. It should be emphasized that in this example the initial strength and final strength were assumed to be highly correlated and that was considered in the analysis.

6.0 Numerical Index Values

Guidance is given in this section for obtaining numerical values to be used for the various single item reliability indexes which were introduced in Secs. 4 and 5. Numerical index values (or data as they are sometimes referred to) result from actual measurements, either from operational use or from testing. Reliability index measurement is at best difficult. Most attempts at it suffer from lack of precision in the failure criteria, in recording the operating conditions, and in knowledge of the history of use of the item. It is desirable to keep in mind such loose conditions under which the data for most reliability indexes was obtained so as not to exaggerate their accuracy.

6.1 Comprehensive Guide

A recent Navy-sponsored effort to identify reliability data sources gives elaboration on sources of reliability data and specific information regarding where to direct inquiries [Ref. 26]. This is a comprehensive guide and identifies many sources for direct reliability indexes as well as for supporting reliability data.

6.2 Reliability Measure Sources

Index values for failure rates and other reliability measures as were identified in Sec. 4 are treated here. Almost universally these data are for an assumed constant failure rate for nonrepairable items and for an assumed mean-time-between-failure of the Poisson process for repairable items.

MIL HDBK 217A. This is a widely used and generally available source. Typically the failure rate of the nonrepairable item is for an electronic part and is shown graphically as a function of several stresses, with additional multipliers to be used for different classes of operational use. The latest revision of this is dated December 1, 1965, and is revision A [Ref. 27]; however, as of this date another revision is in process.

MTBF Estimating Relationships. Simple MTBF estimating relationships have been developed for electronic equipment and are quite useful for preliminary predictions. Here the independent variables may simply be the number of active elements and the usage class of the equipment [Refs. 2 and 27].

FARADA. The focal point of reliability data is the Failure Rate Data (FARADA) program which is sponsored by the Tri-service and NASA in cooperation with qualified government contractors. This program is currently conducted by the Naval Fleet Missile Systems Analysis and Evaluation Group (FMSAEG) at Coronado, California [Ref. 28]. Data inputs from hundreds of sources are collected, compiled and distributed. The primary distribution is in the form of a series of handbooks.

Reliability Analysis Central. The Air Force Rome Air Development Center is currently developing a Reliability Analysis Central which is planned to become fully operational by mid 1969 [Ref. 29]. The Central is to serve as the Air Force focal point for reliability data.

Non-electronic Data. The data sources noted to this point in Sec. 6.2 are primarily electronic in nature. Generally there is more electronic data than for other commodities and failure causes. Some compilation of non-electronic reliability data are Refs. 30 and 31 sponsored respectively by the Navy and Air Force. The Air Force-sponsored work is still in progress.

6.3 Bound-Crossing Data

Distribution information to be used with the bound-crossing type of reliability measure of Sec. 5 is commented on below. The degradation type of failure mode is often not explicitly considered in reliability predictions for electronic items, and there are few established data sources for this failure mode. The FARADA program and the developing RADC Reliability Analysis Central include degradation and drift data under their scope of activity, though not much data are yet included. Equipment and system producers who perform degradation studies have, of course, compiled some numerical information. Sometimes this is made publicly available to others [Ref. 32]. Non-electronic reliability predictions of the bound-crossing variety are principally the stress-strength problem. An Air Force-sponsored compilation of appropriate data for such predictions has been recently published [Ref. 33]. The data here are primarily for distributions of fatigue strength of various mechanical material.

6.4 Remarks

An undesirable feature of currently available data is that too often it is a matter of collecting and passing on what has been reported without very much analysis. One reason for this is that the inputs coming into these collection points are often lacking in supporting information so that analysis is not possible. As the previously mentioned FARADA program continues to progress and as the Reliability Analysis Central becomes established, it can be expected that there will be more screening and analysis on what is ultimately made generally available. An example of the type of data collecting and analysis which is desirable was recently performed for NASA and was concerned with historical reliability data from inflight spacecraft [Ref. 34].

Many equipment suppliers are currently collecting and analyzing reliability data on equipment which they have produced. This sort of data collection effort is extremely desirable and is to be encouraged. If the samples from which such data are drawn are of sufficient quantity, the opportunity exists for developing data that can be drawn on by the equipment suppliers to give more precise results.

The person who is interested in reliability data should keep his eye open in the general literature. Occasionally a paper or report will contain preliminary data of the sort which is not in the established data sources. As an example, consider the human failure mode. A recent paper remarked that a certain percentage of actual failures were found to be caused by human error [Ref. 35]. Certain reliability predictions would be better off to include such failure modes with best available index values rather than to omit them.

Part III: Multi-Item Problems

Various approaches for developing reliability prediction equations for system reliability as functions of item reliabilities and other variables are presented. These are the conventional and classical ones which are suitable for practical applications. Inputs to these equations are the single item reliability definitions from Part I.

Section 7 covers logic models, time is explicitly brought into consideration in Sec. 8, and Sec. 9 covers the influence of environments which are known probabilistically and bound-crossing problems. This material will, to varying extents, be old-hat to experienced reliability analysts. However, some of it is not stressed in existing reliability analysis handbooks or books; including the following: In Sec. 7 are the use of cuts and paths for developing prediction models for complex configurations and the problem of models for multi-phase missions. In Sec. 8 the extreme value approach is discussed in a general manner and a general reliability prediction model is derived (first known publication). In Sec. 9 are discussions of the influence of environment which is known probabilistically and a specific application of this to the multi-item stress-strength problem.

Reliability prediction equations have the apparent use of providing a numerical reliability prediction index for a proposed system configuration. Although the details contained in this report explicitly cover only this use, it is well to be aware of other applications. These include: Using the model for sensitivity studies in order to study the results of changes in input parameters by either limit or probabilistic approaches. An approach could involve application of the method of moments such as cited in Sec. 9.3.1 for a different problem. Another use is to provide part of the equations needed for the application of literal optimization approaches to reliability allocation problems. This use prompted the derivation of the general redundancy model of Sec. 8.4. Yet another use is that certain practical engineering guidelines can be gleaned from studying the models. An instance of this is the outline at the end of Sec. 9.2 for multi-item stress-strength problems.

It should also be noted that the discussions of Secs. 7, 8, and 9 are oriented mainly toward bringing items together into a system. These modeling concepts are, of course, the same ones which would be utilized for bringing detailed failure-modes together, where an item might have multiple failure modes such as were acknowledged in Secs. 4.1 and 4.3. Some treatment of modes will be given in Sec. 7.6 on N-state logic models and in Sec. 9.1 on the general question of the influence of environment. Part IV will pursue detailed consideration of failure modes.

7. Logic Models

The purpose of this section is to develop prediction models for multi-item systems using logic modeling approaches. The system being modeled could be a completely general one. Conventionally the system model includes only hardware, but the model could be extended to include human operators, environments, signals, loads, or other factors which may affect the achievement of system success. Although the techniques given are applicable to large complex systems as well as to lower level equipments, the discussion will be about systems containing only a limited number of items so that it can be followed readily.

Probability of item success or failure for the logic based system models would come from the appropriate measure of Part II. However, attention must be given to probabilistic independence assumptions. The approaches in Sec. 7 are for the situation where operating conditions (or environments) are assumed to be known, or if they are unknown, item reliabilities are independent of this uncertainty. There still could be dependence among the probability of success for items at fixed operating conditions, and if it exists then it must be reflected in the logic models.

The reliability logic diagram such as shown in Fig. 7-1 and throughout Sec. 7 is a useful starting place for the discussion. In the reliability block diagram each block is a two-state item (non-failed or failed). The manner in which the blocks are connected describes the non-failed system in terms of the items comprising the system.

Organization of this section is to introduce first the basic set operations in Sec. 7.1 and then to apply them to various system configurations throughout the remainder of this section.

7.1 Basic Set Operations and Calculus of Probability

In order to predict the reliability of a system given the reliability logic diagram and the probabilities of success (or failure) of the individual items, it is necessary to understand basic set operations and the associated calculus of probabilities. See Appendices A.4 and A.5 for a brief introduction to these techniques and a summary of basic results. For example, suppose that the system under consideration is composed of three items, A, B, and C in a series logic as illustrated in Fig. 7-1 below.

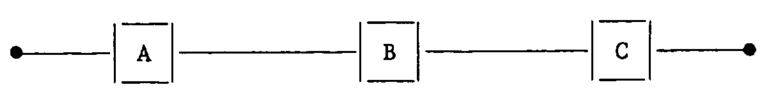


Figure 7-1 Series Logic

The successful operation of the system is equivalent to each of the items operating or not failing. Let A denote the event that item A is operating, B and C similarly denote successful operation of items B and C. In this terminology A represents the event of successful item operation. The event that all three items operate is denoted by the logical intersection of the events A, B, and C and is denoted by

$$A \cap B \cap C$$

or simply

$$ABC.^{*}$$

Now let the probability that item A operates under stated conditions be P(A), and similarly for B and C. The probability that all three items perform successfully is given by

$$P(A)\ P(B)\ P(C),$$

<u>provided</u> the events A, B, and C are independent, that is, that the occurrence of A does not in any way alter the probability that B occurs, etc. with respect to the other events. Further discussion concerning the notion of independence is given in Appendix A. 4 . If the events are not independent the probability may be written as

$$P(A)\ P(B|A)\ P(C|AB)$$

where $P(B|A)$ is the probability that B occurs given that A has occurred. In this section the independence assumption will be used very liberally because of the resulting complexities in not using this assumption, and also because the items can sometimes be defined such that the assumption of independence is reasonable. However, the user of the techniques should not automatically assume independence without some self-questioning.

If the system consists of three items as illustrated below

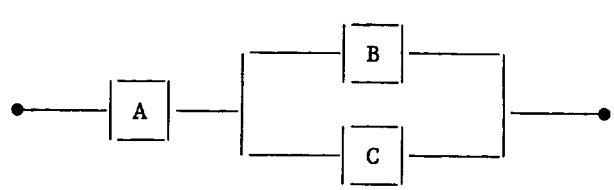


Figure 7-2 Mixed Logic

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* The simpler notation i.e. ABC will be used, and the alternate notation i.e. $A \cap B \cap C$ is cited in this introductory discussion so that the reader will be aware of it, as this alternate notation is also widely used.

then the components B and C are said to be in parallel. The successful operation of the system is equivalent to the operation of (A and B) or (A and C); expressed in another way it may be stated as the operation of A and (B or C). Thus the logical rules may be stated as

$$(A \cap B) \cup (A \cap C) \quad \text{or} \quad AB + AC$$

or

$$A \cap (B \cup C) \quad \text{or} \quad A(B + C).$$

The symbols for the intersection or product are $A \cap B$ or AB as were used above for series logic and for the sum or union are $A \cup B$ or $A + B$. The first expression may be obtained from the latter by performing the logical multiplication of A with the union of B and C. The probability that the system performs successfully is given by the

$$P[A(B + C)]$$

or

$$P(A)\ P(B + C)$$

if A, B, and C are independent. The probability of B + C is the probability that one or the other or both of the events B and C are successful. One rule for finding the probability of B + C is

$$P(B + C) = P(B) + P(C) - P(BC).$$

This can easily be seen by using the following Venn diagram. Let B and C be denoted by the overlapping events as shown below.

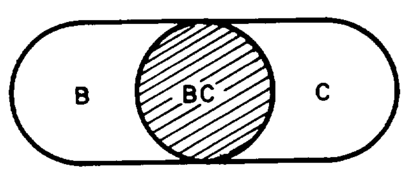


Figure 7-3 Overlapping or Intersecting Events

The shaded portion represents the intersection of B and C and if one obtains the P(B) and adds the P(C) one sees that the P(BC) is counted twice, thus it must be subtracted from the added probabilities to obtain the P(B + C), which is the probability associated with the occurrence of all events enclosed by the boundaries of the events B and C.

Another way in which one can obtain the probability of the successful operation of B + C is to use the fact that failure occurs only if <u>both</u> B and C fail, i.e. $\bar{B}\,\bar{C}$. Thus

$$P(B + C) = 1 - P(\bar{B}\,\bar{C})$$

$$= 1 - P(\bar{B})\,P(\bar{C})$$

assuming independence. Note that since B and $\bar{B}$ are complementary events, that is, one or the other of these events must occur, then

$$P(B) + P(\bar{B}) = 1$$

or

$$P(\bar{B}) = 1 - P(B).$$

The following numerical examples are given for illustrating the notions introduced so far.

<u>Example 7-1</u>

Let the system be as follows:

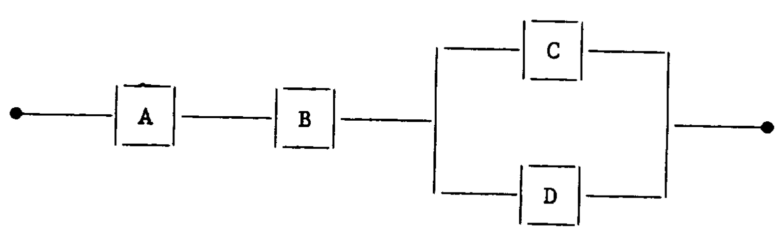


Figure 7-4 Logic Diagram for Example 4-1

Let $P(A) = .99$, $P(B) = .95$, $P(C) = .90$, and $P(D) = .95$ and assume that the events are independent under the given operating conditions. Then the successful operation of the system is given by

$$\begin{aligned} P(S) &= P[AB(C + D)] \\ &= P(AB)\ [P(C) + P(D) - P(CD)] \\ &= 0.9405\ [0.90 + 0.95 - 0.855] \\ &= 0.9405\ [0.995] \\ &\simeq 0.936 \text{ (rounded to 3 decimal places).} \end{aligned}$$

The same result is obtained by using the complementary event and thus

$$\begin{aligned} P(S) &= P(AB)\ [1 - P(\bar{C})\ P(\bar{D})] \\ &= 0.9405\ [1 - (.10)(.05)] \simeq 0.936 \text{ as before.} \end{aligned}$$

The latter way is usually simpler and will be used throughout this section with few exceptions. Again the reader is cautioned that the use of the above formulas implies independence of the events A, B, C, and D.

The set of items A, B, and C may be considered a success path or <u>path</u> (success understood) and likewise A, B, and D constitutes a second success path. The system will fail if either A, B, or C and D fail, and these three sets of items constitute <u>cuts</u> of the equipment. In Sec. 7.4 the notions of paths and cuts will be used to obtain bounds on the probability of success (or failure).

Certain diagrams may be used to aid in the probability calculations and interpretation. Consider the use of a <u>tree diagram</u> for Ex. 4.1 above.

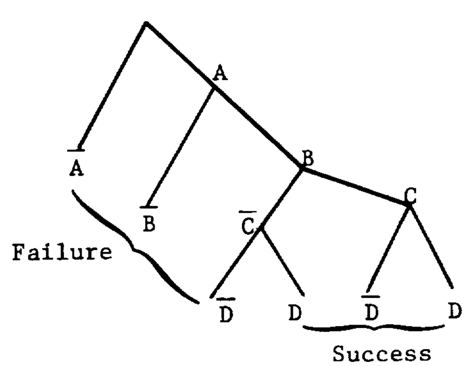


Figure 7.5 Tree Diagram for Example 4.1

The probability of success is given by

$$P(A)\ P(B)\ [P(C\bar{D}) + P(D\bar{C}) + P(CD)].$$

Such tree diagrams can be easily sketched with experience and the probability expressions written down by hand. However, such techniques would be limited to relatively simple systems. It will be assumed here that for very complex systems one will use a computer program solution. However, the needs exist for a basic understanding of the techniques in order not to incorrectly apply a particular technique. Ref. 36 presents a more detailed discussion of the tree diagram approach.

Another approach which can be applied to relatively simple systems is that of using Boolean algebra, an algebra of sets. Just as one can perform operations of addition with sets or events as above. Ref. 37 presents a complete discussion of this approach. A brief discussion of Boolean algebra is given in Appendix A.5.

7.2 Applications to Various System Configurations

In this section the concepts of Sec. 7.1 will be applied to logic configurations where the model can be written by simple visual inspection.

7.2.1 Series Configuration

If the items of the system are in a series configuration, that is, if each item must operate in order that the system will successfully perform its function, then the probability of success is given by

$$P(S) = P(A_1 A_2 \cdots A_n)$$

where there are n components in series configuration as indicated in Fig. 7-6:

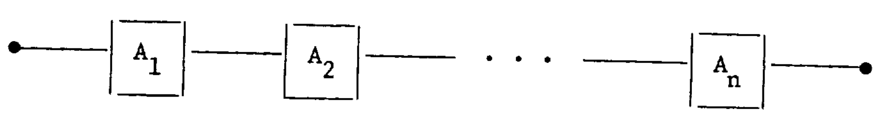


Figure 7-6 Series Configuration

If the events $A_1, A_2, \ldots, A_n$ are independent then

$$P(S) = P(A_1)\ P(A_2) \cdots P(A_n)$$

$$= \prod_{i=1}^{n} P(A_i). \tag{7-1}$$

If all the items have very high reliability a useful approximation is that

$$P(S) \simeq 1 - \sum_{i=1}^{n} P(\bar{A}_i). \qquad (7\text{-}2)$$

If all the items are identical then Eq. 7-1 becomes

$$P(S) = [P(A)]^n = [1 - P(\bar{A})]^n. \qquad (7\text{-}3)$$

In fact it can be shown that the approximation Eq. 7-2 is a lower bound to P(S), i.e.,

$$P(S) \geq 1 - n\,P(\bar{A}),$$

where all the items are identical.

7.2.2 Parallel Configuration

If several items are in parallel, that is, the system operates if one or more of the items operate, then the probability of successful operation is given by

$$P(S) = 1 - P(\text{all components fail})$$

$$= 1 - \prod_{i=1}^{n} P(\bar{A}_i), \qquad (7\text{-}4)$$

where a parallel configuration is illustrated below.

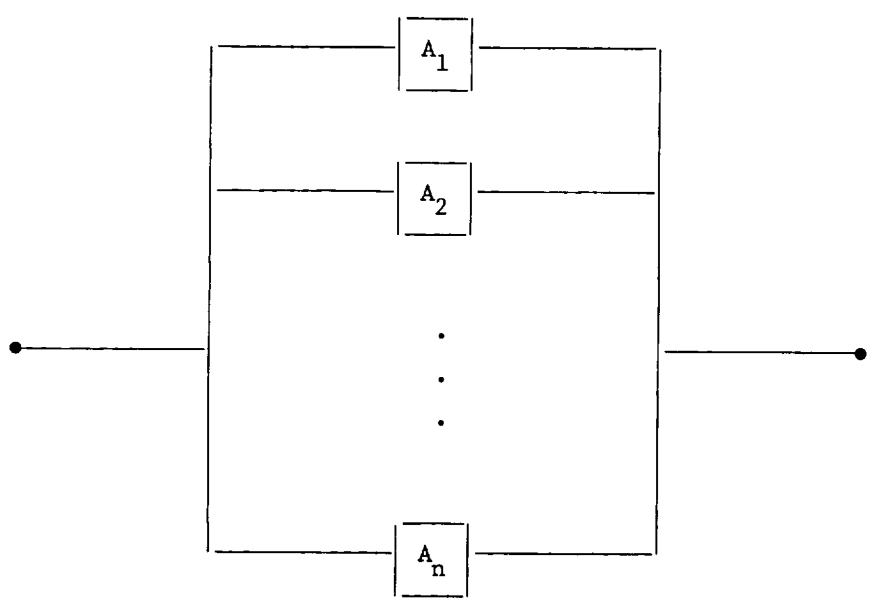


Figure 7-7 Parallel Configuration

Another configuration might be one which requires at least k out of n successful items in a parallel configuration in order for the successful system operation. In this case the probability of success is given by the following if all the items are identical.

$$P(S) = \sum_{i=k}^{n} \binom{n}{i} P^{i}(A)\ [1 - P(A)]^{n-i} \text{ (if all itmes are identical)} \tag{7-5}$$

or

$$P(S) = 1 - \sum_{i=0}^{k-1} \binom{n}{i} P^{i}(A)\ [1 - P(A)]^{n-i}, \tag{7-6}$$

where $\binom{n}{i}$ is the number of combinations of i items taken from n items, that is,

$$\binom{n}{i} = \frac{n!}{i!\ (n-i)!} . \tag{7-7}$$

Eq. 7-6 would be easier to apply if the k were small compared to n. A similar expression may be written in the case of non-identical items, however, the case of identical items is more typical. Such formulas are useful for a system such as a nuclear reactor in which one needs only a certain number of control rods to shut down the reactor, or in the case of an airplane which needs only two engines of four in order to take off, or in majority voting logic schemes.

It is important to note that independence is assumed in the above approaches. In particular, if all the items were subjected to a critical environment during the mission, then the events of failure may not be independent as assumed above. Similarly, if failure of one item increases the stress and thus the probability of failure of another item, the independence assumption may not be correct.

7.2.3 Mixed Configurations

Parallel-Series. A parallel-series configuration is as shown below in Fig. 7-8.

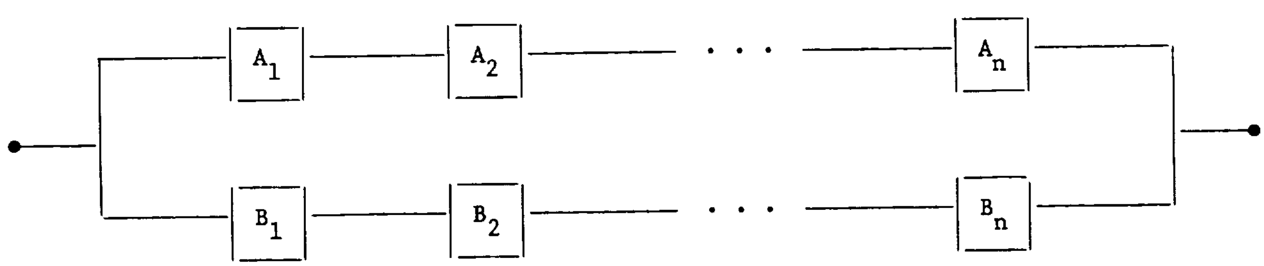


Figure 7-8 Parallel-Series Configuration

The probability of success is given by using the fact that either $A_1, \ldots, A_n$ must all operate or $B_1, \ldots, B_n$ must all operate or both. The simpliest approach is to first apply the series formula replacing $A_1, \ldots, A_n$ by A and $B_1, \ldots, B_n$ by B, thus reducing to the more simplified versions shown in Fig. 7-9.

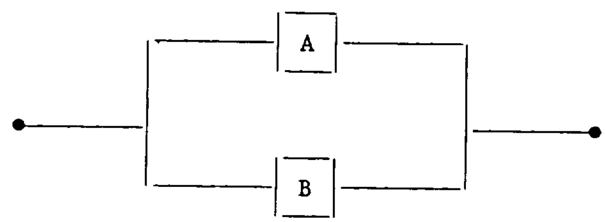


Figure 7-9 Reduction of Configuration in Fig. 7-8

Thus

$$P(A) = \prod_{i=1}^{n} P(A_i)$$

and

$$P(B) = \prod_{i=1}^{n} P(B_i).$$

Then one uses Eq. 7-4 for parallel configurations to obtain

$$P(S) = 1 - P(\bar{A})\, P(\bar{B})$$

or in expanded form

$$P(S) = 1 - [1 - P(A)]\, [1 - P(B)]$$

$$= 1 - [1 - \prod_{i=1}^{n} P(A_i)]\, [1 - \prod_{i=1}^{n} P(B_i)]. \quad (7\text{-}8)$$

In this approach one has a tool for simplifying complex circuits of systems step-by-step until it is reduced to a relatively simple logic configuration. The same approach will be applied to some other examples below.

<u>Series-parallel</u>. Let there be a subsystem of m items in parallel and n of these subsystems in series as shown in Fig. 7-10.

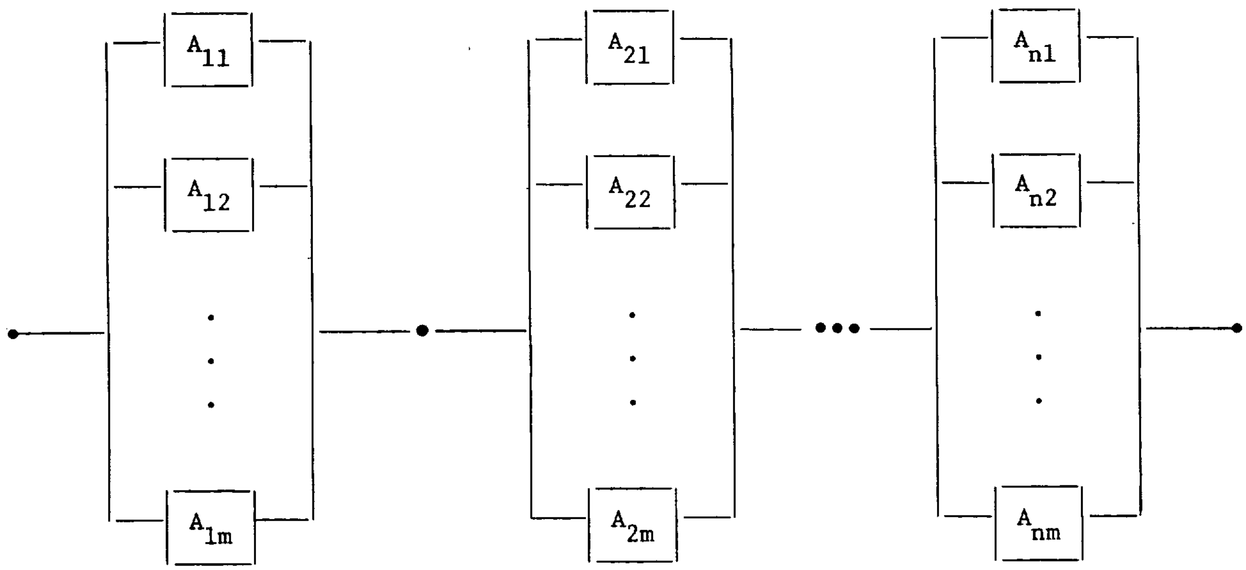


Figure 7.10 Series-Parallel Configuration

The probability of success for the ith subsystem A_i containing m identical items in parallel is given by

$$P(A_i) = 1 - \prod_{j=1}^{m} P(\bar{A}_{ij})$$

and the new simplified configuration becomes that shown in Fig. 7-11.

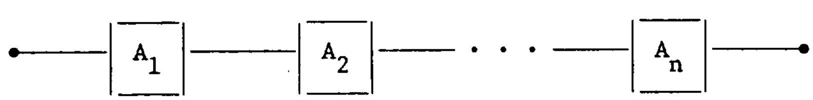


Figure 7.11 Reduction of Configuration in Fig. 7.10

As the A_i are in a series configuration

$$P(S) = \prod_{i=1}^{n} P(A_i) = \prod_{i=1}^{n} (1 - \prod_{j=1}^{m} P(\bar{A}_{ij})). \quad (7\text{-}9)$$

It is not necessary to treat the m_i as being equal to m for all i, and the above formula could be generalized by replacing the m by m_i, $i = 1, \ldots, n$. Many configurations can be treated by one of the above configurations. Two examples are given below to demonstrate some of the formulas, although the examples are worked from basic considerations.

Example 7-2

Let the configuration be as shown in Fig. 7-12.

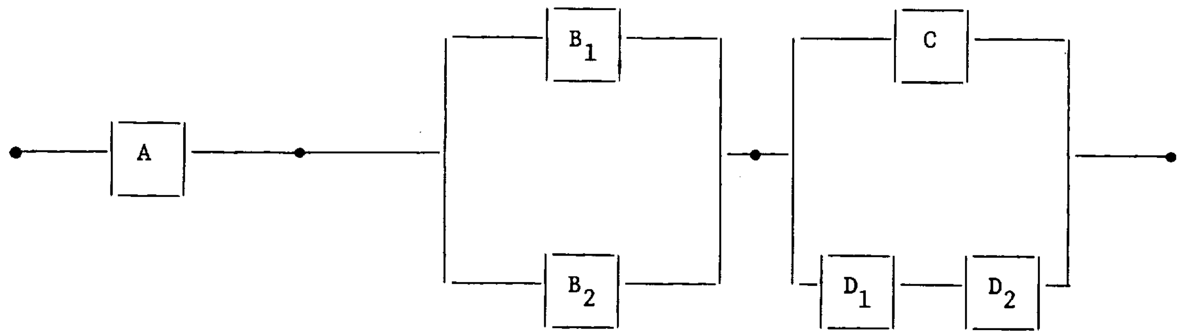


Figure 7-12 Configuration for Ex. 7-2

where

$$P(A) = 0.99$$

$$P(B_1) = P(B_2) = .90$$

$$P(C) = 0.95$$

$$P(D_1) = P(D_2) = .98.$$

and the events are assumed to be independent.

Now

$$P(S) = P(A)\ P(B)(1 - P(\bar{C})\ P(\bar{D}))$$

where

$$P(B) = 1 - P(\bar{B}_1)\ P(\bar{B}_2)$$

$$P(\bar{D}) = 1 - P(D) = 1 - P(D_1)\ P(D_2).$$

Note that one cannot write $P(\bar{D}) = P(\bar{D}_1)\ P(\bar{D}_2)$, that is the event D fails is not equivalent to D_1 and D_2 both failing to operate, but that either one or the other or both failing to operate. Substituting the numerical results yields

$$P(S) = (0.99)\ [(1 - (0.10)(0.10)]\ [1 - (0.05)(1 - .98^2)]$$

$$= 0.933.$$

Example 7-3

Let the configuration be as shown in Fig. 7-13

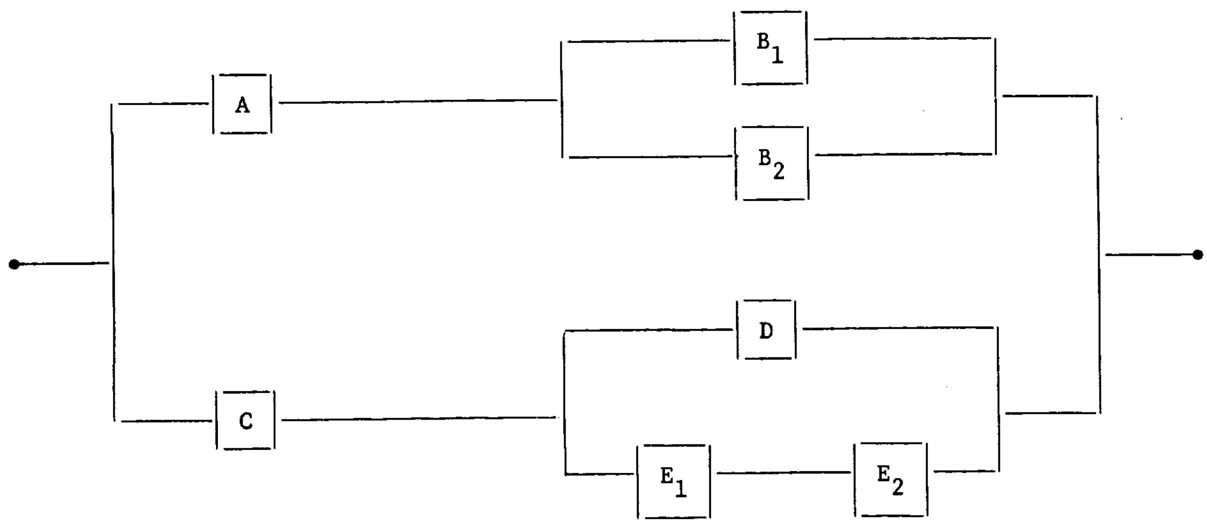


Figure 7-13 Configuration for Ex. 7-3

and the associated probabilities be

$$P(A) = 0.95,\ P(C) = 0.98,\ P(B_1) = P(B_2) = .95,$$

$$P(D) = 0.90,\ P(E_1) = P(E_2) = 0.90.$$

Then if the above is replaced by

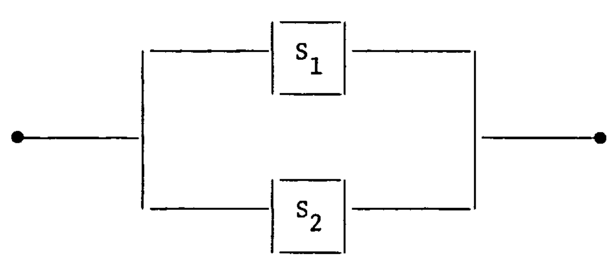


Figure 7-14 Reduction of Configuration for Ex. 7-3

$$P(S_1) = P(A)(1 - P(\bar{B}_1)\, P(\bar{B}_2))$$

$$= (0.95)(1 - (.05)^2) = 0.9476$$

$$P(S_2) = P(C)(1 - P(\bar{D})\, P(\bar{E}))$$

$$= (0.98)(1 - (0.10)(1 - (.90)^2)) = 0.961.$$

7.3 Conditional Probabilistic Approach

We have seen from the above discussion that when the reliability logic diagram consists of series, parallel, and/or mixed configurations, the mathematical logic model can be written directly and easily. However, complex systems cannot always be reduced to a convenient configuration as stated above. In such cases it may be convenient to use the fact that the probability of success of the system given a particular state of the subsystem (which may be for either one item or a collection of items or an environmental state) multiplied by the probability that the subsystem is in the particular state. This result applies when the states B_i, $i = 1, \ldots, n$ of the subsystem are exhaustive and mutually exclusive, that is

$$P(B_1 + B_2 + B_3 + \ldots + B_n) = 1$$

(the B's include all possible events or occurrences)

and

$$B_i B_j = 0$$

(the B's are mutually exclusive or have no common occurrences).

Hence the system success probability P(S) is given by

$$P(S) = \Sigma\, P(S|B_i)\, P(B_i). \qquad (7\text{-}10)$$

The proper selection of the B_i, $i = 1, \ldots, n$ can aid in the solution of the problem. Essentially one wishes to select the states B_i such that the logic diagram reduces to a form for which the probabilistic model can be written easily. See Ref. 38.

Example 7-4

Consider a system of five (5) items functionally arranged in the configuration shown below. The success paths flow from left to right, and there are no right to left portions in a success path. Success paths are A_1A_5, $A_2A_3A_5$, and A_2A_4, but $A_1A_3A_4$ is not a success path.

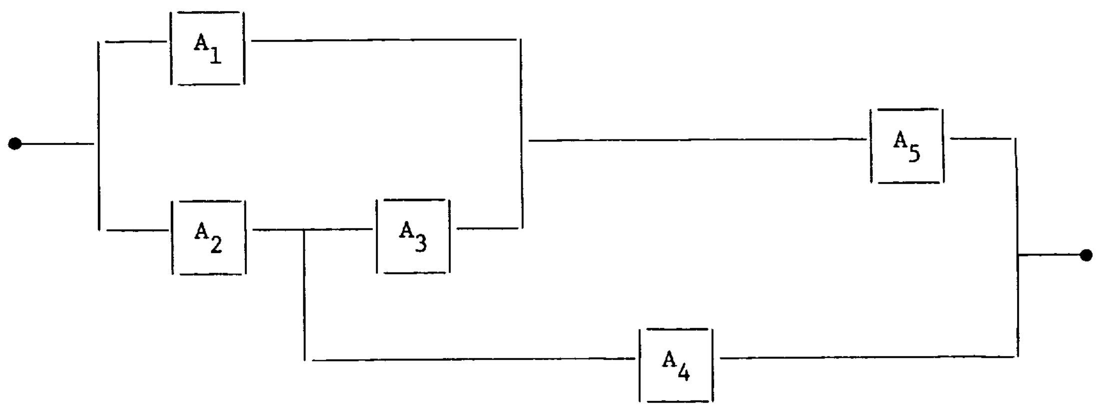


Figure 7-15 Functional Diagram for Ex. 7-4

The solution using conditional probabilities is given first and then a Boolean algebra approach is shown in order to illustrate the difference.

<u>Using Conditional Probabilities</u>

Select events $B_1 = A_2A_5$, $B_2 = A_2\bar{A}_5$, $B_3 = \bar{A}_2A_5$, $B_4 = \bar{A}_2\bar{A}_5$ which are disjoint (mutually exclusive) and exhaustive. The selection of items is quite arbitrary. One could just as easily write the probabilistic model using other items. Now

$$P(B_i\ B_j) = 0 \qquad i \neq j, \qquad i, j = 1, 2, 3, 4$$

and
$$P(B_1 + B_2 + B_3 + B_4) = 1$$

or

$$P(A_2A_5 + A_2\bar{A}_5 + \bar{A}_2A_5 + \bar{A}_2\bar{A}_5) = 1.$$

The reliability logic diagram can be simplified as indicated below for the various states of items A_2 and A_5. The conditional probabilities of success given the various states B_i are given in the last column of Table 7-1 where p_i and q_i denote the probabilities of successful operation and failure under stated conditions of the respective items A_i, $i = 1, \ldots, 5$. The system success probability may be expressed as

$$\begin{aligned} P(S) &= p_2p_5(1 - q_1q_3q_4) + p_2q_5(p_4) + q_2p_5(p_1) + q_2q_5(0) \\ &= p_2p_5 - p_2p_5q_1q_3q_4 + p_2q_5p_4 + q_2p_5p_1 \end{aligned}$$

and after some algebraic reduction using p = 1 - q

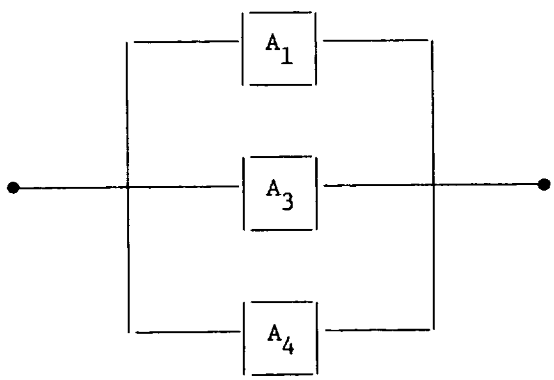
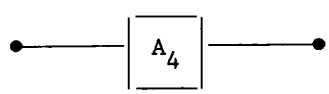
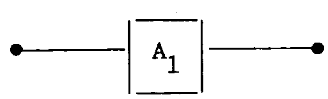
$$P(S) = p_1p_5 + p_2p_3p_5 + p_2p_4$$

$$- p_1p_2p_3p_5 - p_1p_2p_4p_5 - p_2p_3p_4p_5$$

$$+ p_1p_2p_3p_4p_5.$$

Table 7-1

Conditional Logic Diagrams and Associated Probabilities

State	Logic Diagram	Conditional Probability of Success
$B_1 \equiv A_2A_5$		$1 - q_1q_3q_4$
$B_2 \equiv A_2\bar{A}_5$		p_4
$B_3 \equiv \bar{A}_2A_5$		p_1
$B_4 \equiv \bar{A}_2\bar{A}_5$	No Success Path	0

Using Boolean Algebra

The Boolean algebra success model is

$$P(S) = P\{A_1A_5 + A_2[A_3A_5 + A_4]\}$$

$$= P\{A_1A_5 + A_2A_3A_5 + A_2A_4\}.$$

and expanding using Theorem 4 of Appendix 4 and substituting the item success probabilities p will yield the same results as were obtained above using conditional probabilities.

7.4 Models Using Cuts and Paths

The concept of cut sets and success paths (or tie sets) offers another approach to the development of reliability prediction models for systems having complexities. In particular this approach is advantageous where the same item may appear more than once in the reliability block diagram. Such a situation could arise where a system must perform a number of functions but some items are used in more than one function. Here a different reliability logic diagram could be prepared for each function where the same item will appear more than once. Another situation could arise where different functions are to be performed by the system during subsequent mission phases, thus leading to a different reliability logic diagram for each phase where the same item will appear more than once. The cuts and paths approach can be used to obtain an exact model, but this will usually be quite involved and the advantage is that an approximate model can be readily developed. The more important results are given in this section as derived in Ref. 39. The system reliability is defined as the probability of successful function of all of the items in at least one tie set or the probability that all cut sets are good. A tie set or success path is a directed path from input to output as indicated in the simple system in Fig. 7-16A. The tie sets or success paths are 2, 5; 1, 3, 5; and 1, 4, 5, respectively. A cut set is a set of items which literally cuts all success paths or tie sets. One is normally interested in the minimal cut set; i.e., the smallest or minimal set of items such that the elimination of any one item would no longer make it a cut. This is because a nonminimal cut set corresponds to more item failures than are required to cause system failure. In the above example the minimal cut sets are 1, 2; 2, 3, 4; 5. Note that 1, 5 is not a minimal cut set since 5 is already a cut set and is a subset of 1, 5. A cut set cuts the line of communication between input and output. A cut set is good if at least one of its elements is operative. The system failure probability or system unreliability is the probability that all tie sets are bad (a tie set is bad if at least one item fails) or the probability that at least one cut set is bad (that is, all its items are bad). Hereafter, cut set will usually mean minimal cut set.

Let T_i, i = 1, ..., I denote the tie sets, I in number; and C_j, j = 1, ..., J denote the cut sets, J in number. The above statement for system reliability R can be expressed as follows.

$$R = P\{T_1 + T_2 + \cdots + T_I\} = P\{\text{at least one tie set is good}\} \qquad (7\text{-}11)$$

or

$$R = P\{C_1 C_2 \cdots C_J\} = P\{\text{all cut sets are good}\}. \qquad (7\text{-}12)$$

The sets C_j, j = 1, ..., J contain common items and thus $R \neq \prod_{j=1}^{J} P\{C_j\}$.

Equivalently the unreliability is expressed as

$$1 - R = P\{\overline{T}_1 \overline{T}_2 \cdots \overline{T}_I\} = P\{\text{all tie sets are bad}\} \qquad (7\text{-}13)$$

or

$$1 - R = P\{\overline{C}_1 + \overline{C}_2 + \cdots + \overline{C}_J\} = P\{\text{at least one cut set is bad}\}. \qquad (7\text{-}14)$$

Similarly the tie sets $\overline{T}_i$, i = 1, ..., I contain common items and thus

$$1 - R \neq \prod_{i=1}^{I} P\{\overline{T}_i\}.$$

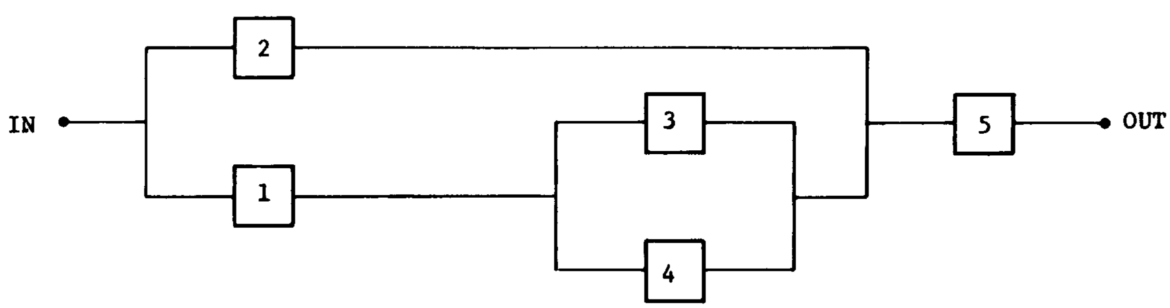


Figure 7-16A Simple Reliability Logic Diagram

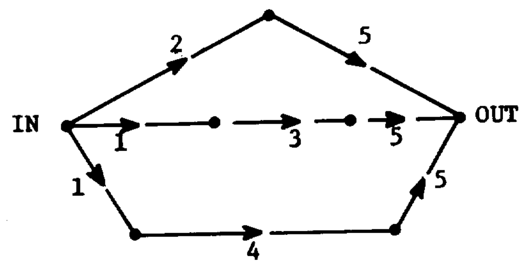


Figure 7-16B Reliability Graph Corresponding to Functional Logic Diagram

Formulas 7-12 and 7-13 are not convenient for computation as the cut and tie sets contain common items. The probability that all cut sets are good (or that all tie sets are bad) cannot be obtained by multiplying the individual probabilities that the cut sets are good (or that the tie sets are bad). The "good" (or "bad") cases must be enumerated in order to perform the required computation and the corresponding probabilities added. However, this approach does not lend to a computerized approach. The formulas 7-11 and 7-14 can be expanded into a sum of probabilities associated with one set, two sets, etc. as shown in standard probability texts. These expanded forms can then be "chopped off" at desired points to obtain bounds to the system reliability. The above are exact formulas for the system reliability and unreliability. Bounds can be obtained by using the basic probabilistic inequalities given below. A computer program, which is described in Vol. II - Computation, has been developed for Eqs. 7-20 and 7-21 and for further generalizations of these bounds.

$$R = P\{T_1 + T_2 + \cdots T_I\} \leq \Sigma P\{T_i\}, \tag{7-15}$$

$$R = P\{T_1 + T_2 + \cdots T_I\} \geq \Sigma P\{T_i\} - \sum_{i_1<i_2} P\{T_{i_1} T_{i_2}\}, \text{etc.} \tag{7-16}$$

Thus an upper bound R_{U1} and a lower bound R_{L1} to the reliability are respectively

$$R_{U1} = \Sigma P\{T_i\} \tag{7-17}$$

$$R_{L1} = \Sigma P\{T_i\} - \sum_{i_1<i_2} P\{T_{i_1} T_{i_2}\}. \tag{7-18}$$

In the same manner another upper bound is obtained,

$$R_{U2} = \Sigma P\{T_{i_1}\} - \sum_{i_1<i_2} P\{T_{i_1}T_{i_2}\} + \sum_{i_1<i_2<i_3} P\{T_{i_1}T_{i_2}T_{i_3}\}. \quad (7\text{-}19)$$

The summations are over all possible combinations of the subscripts taken 2 at-a-time, 3 at-a-time, etc.

Similarly the inequalities of Eqs. 7-15 and 7-16 can be applied to the cut-set form of the equation for unreliability of Eq. 7-14 to obtain

$$1 - R \leq \Sigma P\{\overline{C}_j\}$$

or

$$R \geq 1 - \Sigma P\{\overline{C}_j\} = R_{L2} \quad (7\text{-}20)$$

and by using two terms

$$R \leq 1 - \Sigma P\{\overline{C}_j\} + \sum_{j_1<j_2} P\{\overline{C}_{j_1}\overline{C}_{j_2}\} = R_{U3}. \quad (7\text{-}21)$$

Example 7-5

Consider the reliability graph given in Fig. 7-11. Assume independence between items and let the probabilities of success for each of the items be $p_1 = 0.93$, $p_2 = 0.86$, $p_3 = 0.92$, $p_4 = 0.95$, $p_5 = 0.98$. The probabilities for the ties and cuts are as follows:

$$P\{T_1\} = P\{2\ 5\} = 0.8428$$

$$P\{T_2\} = P\{1\ 3\ 5\} = 0.8385$$

$$P\{T_3\} = P\{1\ 4\ 5\} = 0.8658,$$

and

$$P\{C_1\} = 1 - P\{\overline{1}\ \overline{2}\} = 1 - .0098 = 0.9902$$

$$P\{C_2\} = 1 - P\{\overline{2}\ \overline{3}\ \overline{4}\} = 1 - 0.00056 = 0.99944$$

$$P\{C_3\} = 1 - P\{\overline{5}\} = 1 - 0.02 = 0.98.$$

Upper and lower bounds for the reliability are given by using Eqs. 7-17, 7-18, 7-19, 7-20, and 7-21, respectively,

$$R_{U1} = P\{T_i\} > 1 \text{ (not useful as } R_{U1} \leq 1.)$$

$$R_{L1} = 0.843 + 0.838 + 0.866 - P\{1\ 2\ 3\ 5\} - P\{1\ 2\ 4\ 5\} - P\{1\ 3\ 4\ 5\}$$

$$= 0.2848$$

$$R_{U2} = 0.2848 + 0.6850 = 0.9698 = R \text{ (This result is equal to the system reliability)}$$

$$R_{L2} = 1 - P\{\overline{C}_j\} = 1 - 0.03036 = 0.96964$$

$$R_{U3} = 1 - \Sigma\, P\{\overline{C}_j\} + \Sigma\, P\{\overline{C}_{j_1}\overline{C}_{j_2}\} = 1 - 03036 + 0.00024 = 0.96988.$$

$$= 0.96988.$$

As stated by Messinger [Ref. 39] the bounds based on the cut sets are best in the high reliability region and those based on the tie sets are best in the low reliability region. Hence the bounds R_{L2} and R_{U3} are the preferred bounds in the above example and R_{U2} in this case saves no computation as it is the exact probability of system success, as there are only three tie sets and the bound uses all combinations of tie sets up to and including three sets.

In more general problems in which there are J cut sets the number of terms to be obtained in the lower and upper bounds computations are J and J(J-1)/2 respectively. This is compared to 2^J-1 terms obtained by expanding either Eq. 7-11 or 7-14 using tie sets or cut sets respectively.

7.5 Multi-Phase Mission

The approaches given thus far in Sec. 7 are applicable to a given mission phase and more general treatment must be given to certain multi-phase missions. This approach is useful for the type of situation as experienced in a lunar orbit mission or lunar landing and return mission in which the environment and the configuration changes with the successive phases of the mission. In such a mission an item used in several phases may have different probabilities associated with each phase. Consider the following configuration for example.

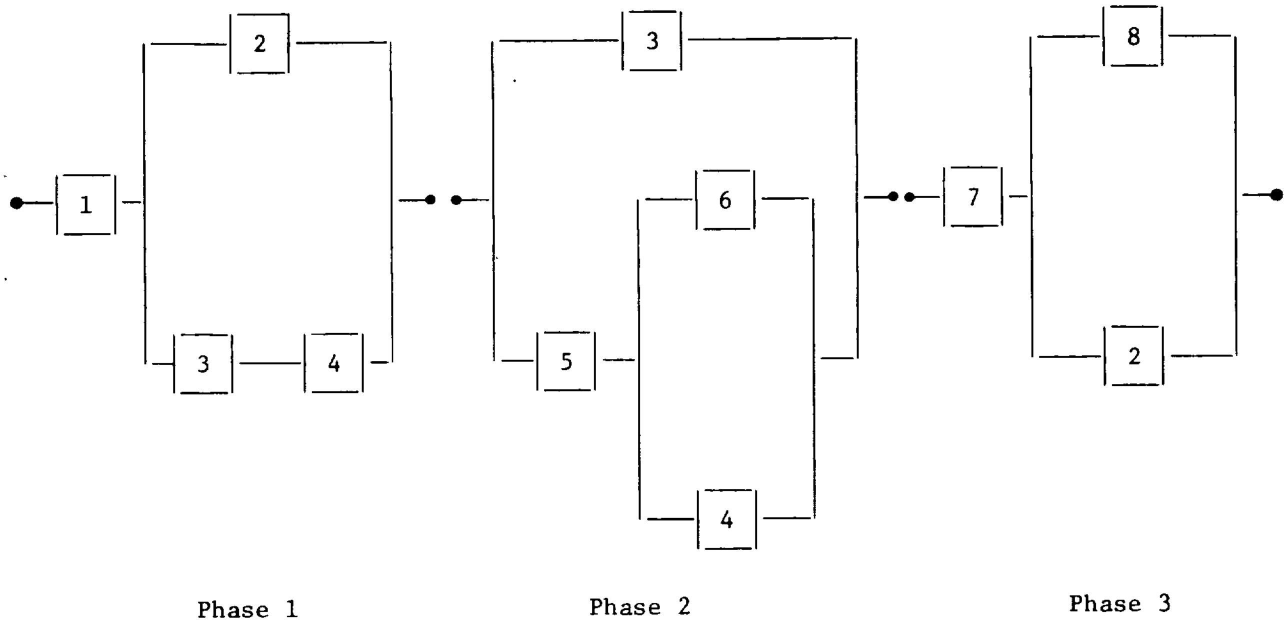


Figure 7-17 Multi-phase Configuration for Ex. 7-6

If the event of success in phase 1 is denoted by $P(S_1)$, and similarly for phases 2 and 3 by $P(S_2)$ and $P(S_3)$, then the probability of mission success P(S) is given by the following relationship

$$P(S) \;=\; P(S_1|E_1)\; P(S_2|E_2;S_1)\; P(S_3|E_3;S_1,S_2),$$

which can be written in general form for p phases

$$P(S) \;=\; P(S_1|E_1) \cdots P(S_p|E_p;S_1,S_2, \ldots, S_{p-1}). \tag{7-22}$$

These formulations guide the computational procedure so as to include the effects of environmental stresses in the jth phase and the previous stress history in phases j-1, j-2, ..., 1, having obtained the probabilities for the items in each of the phases. Usually one has to enumerate all of the conditions for each phase and sum the products of the conditional probabilities over possible combinations of conditions. One approach is patterned after that given in Sec. 7.3 and is illustrated in the following example.

Example 7-6

Consider the example given in Fig. 7-17. Let the probabilities of success for the various items in phases 1, 2, and 3 of the mission be as given in the following table.

	Phase 1	Phase 2	Phase 3
1	$P(1\|E_1) = 0.99$	-	-
2	$P(2\|E_1) = 0.95$	-	$P(2\|E_3,S_1,S_2) = 0.92$
3	$P(3\|E_1) = 0.94$	$P(3\|E_2,S_1) = 0.96$	-
4	$P(4\|E_1) = 0.98$	$P(4\|E_2,S_1) = 0.99$	-
5	-	$P(5\|E_2,S_1) = 0.97$	-
6	-	$P(6\|E_2,S_1) = 0.94$	-
7	-	-	$P(7\|E_3,S_1,S_2) = 0.97$
8	-	-	$P(8\|E_3,S_1,S_2) = 0.96$

For this example consider the various ways in which success in Phase 1 can occur. They are:

1) all items (1, 2, 3, and 4) operate for Phase 1,
2) items 1, 3, and 4 operate and 2 fails,
3) items 1, 2, and 4 operate and 3 fails,
4) items 1, 2, and 3 operate and 4 fails, and
5) items 1 and 2 operate and 3 and 4 fail.

All other combinations of successes and failures will result in failure of Phase 1. For each of the above conditions it is necessary to obtain the conditional probability of success in Phase 2, and similarly in Phase 3. There is a slight simplification in this example in that no common items are contained in Phases 2 and 3, hence it is not necessary to consider all of the possibilities in Phase 2 prior to obtaining the conditional probabilities in Phase 3. Consider now the conditional probabilities for Phase 2 for each of the conditions given above and in environment E_2.

Case 1) $$P(S_2|S_1, E_2) = P(5 \text{ or } 5\cdot 6 \text{ or } 5\cdot 4|E_2)$$

$$= 1 - P(\overline{3})\ [1 - P(5)\ \{1 - P(\overline{6})\ P(\overline{4})\}]$$

where $P(\overline{3})$ indicates the probability of failure of component 3, P(5) success of item 5 in Phase 2, etc.

Case 2) Same as for case 1 as item 2 does not appear in Phase 2; however, Phase 3 is altered.

Case 3) $$P(S_2|S_1, E_2) = P(5\cdot 6 \text{ or } 5\cdot 4)$$

$$= P(5)\ P(6) + P(5)\ P(4) - P(4)\ P(5)\ P(6)$$

Case 4) $$P(S_2|S_1, E_2) = P(3 \text{ or } 5\cdot 6|E_2)$$

$$= P(3) + P(5)\ P(6) - P(3)\ P(5)\ P(6)$$

Case 5) $$P(S_2|S_1, E_2) = P(5\cdot 6|E_2)$$

$$= P(5)\ P(6)$$

Similarly one can analyze Phase 3 subject to the five (5) conditions of success in Phase 1. The corresponding conditional probabilities are as follows:

Case 1) $$P(S_3|S_1S_2E_3) = P(7\cdot 8 \text{ or } 7\cdot 2)$$

$$= P(7)\ P(8) + P(7) - P(2)\ P(7)\ P(8)$$

Case 2) $$P(S_3|S_1S_2E_3) = P(7\cdot 8) = P(7)\ P(8)$$

Case 3) Same as Case 1.

Case 4) Same as Case 1.

Case 5) Same as Case 1.

Hence the overall mission reliability P(S) can be obtained by summing the products of the conditional probabilities for the respective cases 1) through 5). Thus

$$
\begin{aligned}
P(S) &= (0.86639)(.99877)(.9669)\\
&+ (0.04560)(.99877)(.9312)\\
&+ (0.05530)(.96942)(.9669)\\
&+ (0.017681)(.99647)(.9669)\\
&+ (0.001129)(.9118)(.9669)\\
&\simeq 0.949.
\end{aligned}
$$

The above approach uses the conditional probabilistic approach of Sec. 7.3. The approach can be rather tedious as it usually would be necessary to list all of the conditions for each phase and hence the number of different cases would be the product of the number of conditions in each phase.

Because the above approach can be lengthy and tedious, an approximation to the mission reliability is possible by use of the method of paths and cuts as described in Sec. 7.4. In this approach the reliabilities of the components would be taken to be the reliability up to the end of the last phase in which they are used. If the probability of failure is assumed to be zero (0) for the phases in which a component is not used then, the input reliabilities would be equal to the product of the separate conditional probabilities for each phase.

7.6 N-State Logic Model

The considerations thus far in Sec. 7 have been based on a two-state model for each item, one failed state and a non-failed or successful state. In this section we consider a case in which some of the items may be considered as having two or more failed states, such as opening, shorting, noisy, drift, etc. No additional tools are needed to solve a problem of this type; however, the analysis does become more complex. One might need to perform such an analysis in order to make correct decisions between subsystem configuration. For example, See Ref. 40 which uses a two-state and a three-state analysis of a particular circuit. As an example consider a diode-quade with a shorting bar as shown in Fig. 7-18. The circuit fails if two shorts occur in series (e.g., diodes 1 and 2 or 1 and 4) or if two opens occur in one end (e.g., diodes 2 and 4 or 1 and 3). Otherwise the system performs successfully. The probability of a diode opening is denoted by p_0 shorting by p_s. Another technique will be used below to obtain the probability of success or failure. It is certain that an individual diode will either perform, or short, or open (assuming no other mode of failure for this analysis). Hence

$$p + p_0 + p_s = 1 \tag{7-23}$$

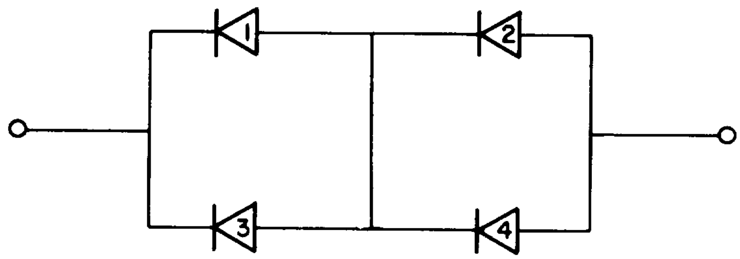


Figure 7-18 Diode-Quad with Shorting Bar

As there are four diodes consider the expansion of Eq. 7-23 to the fourth power. Thus

$$\begin{aligned} 1 &= (p + p_0 + p_s)^4 = p^4 + 4p^3(p_0 + p_s) + 6p^2(p_0 + p_s)^2 + 4p(p_0 + p_s)^3 + (p_0 + p_s)^4 \\ &= p^4 + 4p^3p_0 + 4p^3p_s + 6p^2p_0^2 + 12p^2p_0p_s + 6p^2p_s^2 + 4p(p_0^3 + 3p_0^3p_s + 3p_0p_s^2 + p_s^3) \\ &\qquad + p_0^4 + 4p_0^3p_s + 6p_s^2p_s^2 + 4p_sp_s^3 + p_s^4. \end{aligned}$$

This expression yields all the various combinations of shorts, opens, and no failures for the quad configuration given. The coefficients yield the number of ways in which a certain combination can occur. For example, consider the term

$$12p^2p_0p_s;$$

these are 12 ways of obtaining 1 open, 1 short, and 2 operating diodes. That is, there are 4 ways of selecting the shorted diode, 3 ways of selecting the open diode from the remaining 3 diodes, and the last two can be selected in only 1 way. Thus $4 \times 3 = 12$ ways of obtaining this particular combination. If there is only one short and only one open, a failure cannot occur according to the above statement of failure. Hence, this term is put into the success probability in the following formula. Similarly each term can be treated to determine which portion of the combinations of opens and shorts contribute to failure or success.

$$\begin{aligned} 1 &= (p^4 + 4p_0p^3 + 4p^3p_s + 4p^2p_0^2 + 12p^2p_0p_s + 2p^2p_s^2 + 8pp_0^2p_s + 4pp_0p_s^2) \\ &\quad + (2p^2p_0^2 + 4p^2p_s^2 + 4pp_0^2 + 4pp_0^2p_s + 8pp_0p_s^2 + 4pp_s^3 + p_0^4 + p_0^3p_s + 6p_0^2p_s^2 + 4p_0p_s^3 + p_s^4) \\ &= P(S) + P(F) \text{ respectively,} \end{aligned}$$

where P(S) and P(F) are given in parentheses above. Because p_0 and p_s are very small compared to p the above expressions can be approximated by the following.

$$P(S) \simeq p^4 + 4p_0p^3 + 4p^3p_s,\ P(F) \simeq 2p^2p_0^2 + 4p^2p_s^2 \simeq 2p_0^2 + 4p_s^2$$

and the actual probability of success is bounded by

$$p^4 + 4p_0p^3 + 4p^3p_s < P(S) < 1 - (2p_0^2 + 4p_s^2).$$

Example 7-7

Suppose for the diode-quad given in Fig. 7-18 above

$$p = 0.99$$

$$p_0 = 0.0080$$

$$p_s = 0.0020$$

$$.9606 + 0.0388 < P(S) < 1 - (0.000128 + 0.000016)$$

$$0.9994 < P(S) < 0.99986.$$

It must be emphasized that independence of the events has been assumed throughout the above analysis. If the diode-quad were exposed to a critical environment in its mission life or if failure of one diode increased the probability of failure of another diode, then the probability of success would be altered by the appropriate conditional probabilities of failure under the given conditions.

The above discussion just touches on an important topic area such as an N-state analysis. In actual practice an analysis which takes the possible modes of failure of each component into consideration and which gives the subsystem behavior for each failure mode would result in an extremely large number of cases to examine. This can be true even for a two state analysis. Hence one must cope with the dimensionality problem by first identifying the more likely weaknesses of the equipment and then to perform a detailed analysis on these components such as an analysis of a particular redundant configuration as for the diode-quade. The logical operations and the probability analysis for the N-state situation are more complex than that for a two-state analysis but the same basis techniques are applicable.

8. Models Considering Time

In this section the explicit use of time is considered for multi-item problems. A straightforward approach is to develop a logic model as in Sec. 7 where item success and failure probabilities are expressed probabilistically as attributes, and then to substitute for each attribute the appropriate time measure as described in Sec. 4.2. This will be discussed first in Sec. 8.1. For some problems the substitution approach is not applicable, and a more involved convolution approach is discussed in Sec. 8.2 for these problems. The approaches presented in Secs. 8.1 and 8.2 can be used for the first time to failure of a system where the individual items can have many possible time to failure distributions such as gamma or log-normal. However, most often it is assumed that all the items have exponential failure distributions. Where this assumption is made, the system reliability prediction models of Sec. 8.1 and 8.2 are applicable regardless of how much operating time has been accumulated and if it is known that all items in a system are non-failed. Further, if the exponential failure distribution is assumed for all items, then the methods of continuous Markov processes and difference equations can be used to develop reliability models without first developing a logic model. This approach is acknowledged in Sec. 8.3, along with other approaches which are somewhat specialized. The final Sec. 8.4 contains the development of a general redundancy equation which is suitable for general reliability prediction and which also may be used for reliability allocation decisions.

8.1 Logic Model Substitution

The logic form of reliability prediction models can be readily extended to explicitly consider time. This is done by simply substituting the applicable probabilities of success or failure as functions of time, R(t) or F(t), for each item, into the multi-item logic model. Such a substitution is possible where the R(t) or F(t) for each item is applicable for the time t of interest, which means that reliability prediction models for certain systems such as the classical standby redundancy and the rope models cannot be developed by this method. The following sections will treat this and considerations other than logic based model substitutions. In this section several of the logic based models from Sec. 7 will be extended via examples to consider time explicitly. Those not treated can be readily developed and are shown in most reliability books and handbooks. Exponential failure time distributions for items will be used because of its conventional emphasis. Other distributions can be readily substituted for first-failure time models, but as they lead to complications if later failures are explicitly considered they are not so widely used.

Series System. If all the items of a system must operate in order for the system to perform its intended function, then the items are said to be in a series system. In Sec. 7.2.1 it was stated that the probability that n items $A_1, A_2, \ldots, A_n$ operate, assuming independence, is given by

$$P(s) = \prod_{i=1}^{n} P(A_i).$$

If item A_i has a mean time between failures (MTBF) of θ_i or a failure rate $\lambda_i (=1/\theta_i)$ and if the mission time is T_M, then

$$R(A_i) = P(\text{component } A_i \text{ survives time } T_M) = e^{-\lambda_i T_M} = e^{-T_M/\theta_i}.$$

provided the λ_i is constant throughout the entire mission. If λ_i changes with the mission phases one must perform the computations for each phase separately for non-serial systems. Some simplification can be made in this procedure. The mission success probability is given by

$$P(S) = \prod_{i=1}^{n} e^{-\lambda_i T_M} = e^{-T_M \Sigma \lambda_i}, \qquad (8\text{-}1)$$

that is, the failure rates can be added for the n components in series to obtain an overall system failure rate. If some failure time distribution other than the exponential is appropriate the $R(A_i)$ can be expressed as the appropriate integral of the density function. These integrals are tabulated for almost all density functions of interest in many standard statistics texts.

Parallel Configuration. If n items are in parallel then system success is equivalent to at least one item operating. Another way of stating system success is that the items do not all fail. Using this logical form the following result is obtained

$$\begin{aligned} P(S) &= 1 - \prod_{i=1}^{n} P(\bar{A}_i) \\ &= 1 - \prod_{i=1}^{n} (1-e^{-\lambda_i T_M}). \qquad (8\text{-}2) \end{aligned}$$

For small values of $\lambda_i T_M$ (all i) the following approximation can be used to simplify the above calculations.

$$\begin{aligned} 1 - e^{-\lambda_i T_M} &= 1 - (1 - \lambda_i T_M + \frac{\lambda_i^2 T_M^2}{2!} - \ldots) \\ &\simeq \lambda_i T_M. \end{aligned}$$

Using this approximation

$$P(S) \approx 1 - T_M \prod_{i=1}^{n} \lambda_i \text{ for } \lambda_i T_M \text{ very small.} \tag{8-3}$$

8.2 Standby and Rope Models

Development of reliability prediction models for some systems cannot be accomplished by substitution in logic models. Such systems are those where all the items are not used throughout the time interval of interest (standby redundancy) and where the probability of success for some items change at uncertain times in the time interval(rope redundancy) of interest. For these situations prediction models can be developed using the convolution concept.

8.2.1 Standby Redundancy

Case 1-Perfect Switch.

Suppose that a system consists of m items, m-1 on standby, for use when one of the items fails in use as indicated in Fig. 8-1.

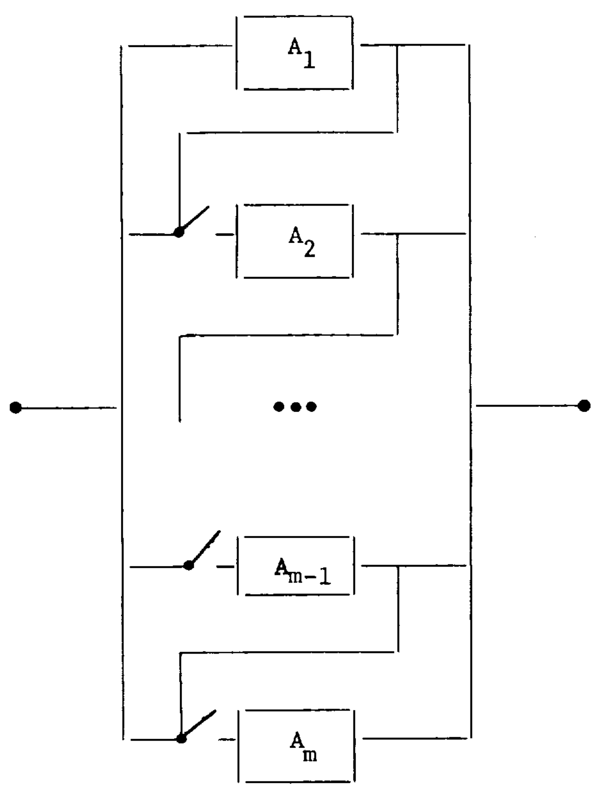


Figure 8-1 Standby System With m-1 Items on Standby

It is assumed for the present that the switchover devices are 100 percent reliable (i.e. that the failure rate is zero in the standby operation), and that each item has an exponential failure time distribution with failure rate λ_i. Let T_M be the mission time. Now if the system is to perform its function for the time T_M, the total of the operating times must exceed T_M. If $t_1, t_2, \ldots, t_m$ are the times to failure of each of the respective components, then the probability of successful operation of the system P(S) is equivalent to the probability that the cumulated failure times of the m components exceeds T_M, or

$$P(S) = P(t_1 + t_2 + \ldots + t_m \geq T_M).$$

Consider this problem for the case m = 2, in which it is necessary to obtain the probability that $t = t_1 + t_2 \geq T_M$. Now

$$p(t_1) = \lambda_1 e^{-\lambda_1 t_1},\ 0 \leq t_1 < \infty$$

$$p(t_2) = \lambda_2 e^{-\lambda_2 t_2},\ 0 \leq t_2 < \infty$$

and the probability that $t_1 + t_2 \geq T_M$ is given by the double integral

$$P(S) = 1 - P(F) = 1 - P(t_1 + t_2 < T_M)$$

$$= 1 - \lambda_1\lambda_2 \int_0^t \int_0^{t-t_1} e^{-\lambda_1 t_1} e^{-\lambda_2 t_2}\, dt_2 dt_1 \qquad (8\text{-}4)$$

where the region of integration is shown in Fig. 8-2.

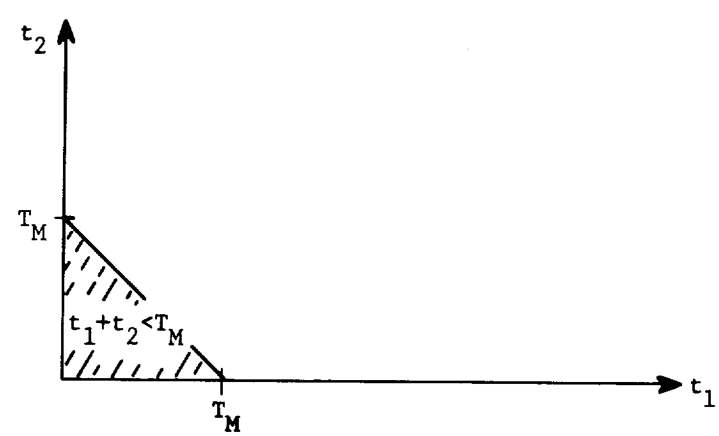


Figure 8-2 Region of Integration

Integration of the equation (8-1) yields

$$P(S) = \begin{cases} e^{-\lambda_1 T_M} + \dfrac{\lambda_1}{\lambda_2-\lambda_1}[e^{-\lambda_1 T_M} - e^{-\lambda_2 T_M}], & \text{If } \lambda_2 \neq \lambda_1 \\ e^{-\lambda_1 T_M}[1 + \lambda_1 T_M], & \text{If } \lambda_2 = \lambda_1. \end{cases} \tag{8-5}$$

The above formula can be interpreted as the probability that item 1 survives the entire mission time plus the probability that item 1 fails in time t_1 but that $t_2 \geq T_M - t_1$ (the contribution of the second term).

In case the items are all identical and perfect switching exists then the probability that a system of m components (m-1 standby components) survives T_M is given by

$$\begin{aligned} P(S) &= P(t_1 + t_2 + \ldots + t_m \geq T_M) \\ &= e^{-\lambda T_M}\left(1 + \lambda T_M + \frac{\lambda^2 T_M^2}{2!} + \ldots + \frac{\lambda^{m-1} T_M^{\ m-1}}{(m-1)!}\right). \end{aligned} \tag{8-6}$$

Note that this formula gives the probability of 0, 1, 2, ..., m-1 failures for a variable having the Poisson distribution with mean number of failures given by λT_M.

Case 2- Imperfect Switch. If imperfect switching were taken into consideration the second term in the above would have to be multiplied by the probability that the switch-over occurs, P(sw) say, and hence

$$P(S) = e^{-\lambda_1 T_M} + \frac{\lambda_1}{\lambda_2-\lambda_1} P(sw)\ [e^{-\lambda_1 T_M} - e^{-\lambda_2 T_M}]. \tag{8-7}$$

See Ref. 41 for a statement of the above result when several standby components are allowed. Also see Sec. 8.4 for a more general formula for combinations of redundancy.

8.2.2 Rope Model

In some physical situations system failure does not occur until all or k out of n items fail (for example, as strands in a rope), but the failure of some of the items increases the stresses on the remaining items and thereby decreases their reliability.

<u>Case 1</u>: Suppose that the load on a system is constant and that initially n items are sharing the load. As the elements fail the remaining load is equally shared by the remaining elements. Thus if the original stress per element is S_0/n, then the subsequent stresses increase $S_0/(n-1)$ for 1 failure, $S_0/(n-2)$ for two failures, etc. The increase in stress on each item will usually result in a corresponding increase in the failure rate for the items as the ratio of the operating stress to the rated stress S_r increases. Let the stress ratio be h as given by

$$h = \frac{\text{operating stress}}{\text{rated stress}} = \frac{S_0/(n-f)}{S_r} = \frac{S_0}{s\ S_r}$$

where f is the number of failures and $s(=n-f)$ is the number of survivors. If the rated stress is exceeded by S_0/s then the system is assumed to fail. Let the maximum number of failures be $n - k$, or k be the number of minimum number of items for operation. Thus for non-failed operation the stress ratio must be less than unity, i.e.

$$h = \frac{S_0/k}{S_r} < 1,$$

or

$$s = n-f \geq k.$$

Now suppose that the failure rate for an individual item at time t for stress ratio h is denoted by $\lambda(t;\ h)$.

In this first case assume that

$$\lambda(t;\ h) = \lambda_0 h,$$

that is, λ increases linearly with h, λ_0 is a constant. The failure rate for the system λ_S is given by

$$\lambda_S = s\lambda_0 h$$

where s is the number of non-failed or successful items. Now

$$h = \frac{S_0}{s\ S_r}$$

and thus

$$\lambda_S = s \cdot \frac{\lambda_0 S_0}{s\ S_r} = \frac{S_0 \lambda_0}{S_r}.$$

which is constant. Ref. 42 treats this case and Ref. 43 has included it as a special case of finding the reliability of a parallel redundant system when the item failure rate is $\lambda = \lambda(h)$, a general function of the stress ratio.

Thus in this case of constant system failure rate the time to failure of the system is given by

$$T_S = t_1 + t_2 + \ldots + t_f$$

where f is the number of failures. Now if each t_i is assumed to have the exponential failure time density function, i.e.

$$p(t_i) = \lambda_S \exp(-\lambda_S\, t_i),$$

the distribution of T_S is given by the f fold convolution of $p(t_i)$,

$$p(T_S) = p(t_1) * p(t_2) * \ldots * p(t_f).$$

For n = 2 items,

$$\begin{aligned} p(T_S) &= p(t_1) + p(t_2) \\ &= \int_{t_1=0}^{T_S} p(t_1)\, p(T_S - t_1)\, dt_1 \\ &= \lambda_S^2 T_S\, e^{-\lambda_S T_S}. \end{aligned}$$

Similarly for the f fold convolution one obtains

$$p(T_S) = [p(t)]^{f^*} = \frac{\lambda_S^f T_S^{\,f-1} \exp\{-\lambda_S T_S\}}{\Gamma(f)},\ T_S > 0. \tag{8-8}$$

This is the gamma density function with shape parameter f and the same scale parameter λ as for the exponential distribution. See Ref. 44 for further details in the derivation of the distribution. For f = n-k+1 yields

$$p(T_S) = \frac{\lambda_S^{\,n-k+1}\, T_S^{\,n-k} \exp\{-\lambda_S T_S\}}{\Gamma(n-k+1)}$$

where T_S is the time the n-k+1st failure occurs.

Case 2: Suppose that the failure rate of an individual item is of the general form

$$\lambda = \lambda(h)$$

of the stress ratio h, where $\lambda(h)$ is not necessarily linear as indicated in Fig. 8-3. The result is given in Ref. 42 in the form of a complex integral with values of the residues to be determined.

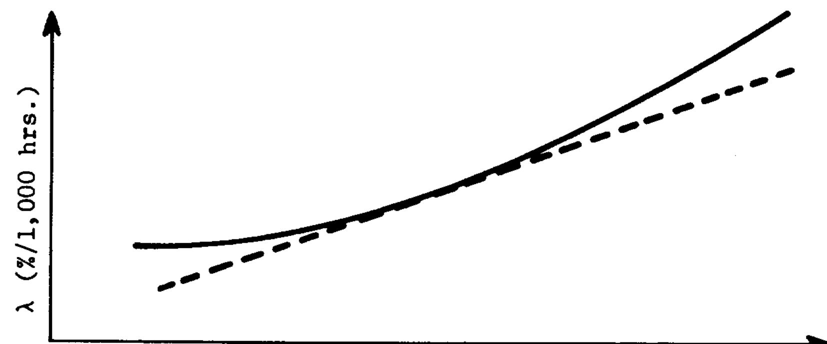


Figure 8-3 Dependency of Failure Rate λ on h

8.3 Additional Approaches

Several other approaches which are used for deriving reliability models explicitly concerning time are briefly identified.

8.3.1 Continuous Markov Process

Another method of deriving conventional reliability models when all items in the system have an exponential distribution is to use the approach of a first order Markov process and difference equations. A text [Ref. 3] is devoted mainly to the derivation of models based on this approach. A space-state diagram relates the possible transitions between the possible system states. The postulate is applied: the probability of a state change during (t, t+dt) is λdt plus terms of smaller order than dt and the probability that more than one change occurs is smaller than dt. This approach leads to a set of linear homogeneous differential equations, which can be solved for the probability of success as a continuous function of time. Thus it is the approach used in Sec. 4.5 for the development of the Poisson process. Different system configurations (e.g. series, active-parallel, and standby-parallel) lead to different success probability functions, which are identical to those obtained from the approach in the preceding Secs. 8.1 and 8.2.

The Markov process approach can be readily extended to include maintenance, which is really the advantage of this type of model formulation. Here the state-space transition diagram is expanded from only failure transitions to include both failure and repair transitions. The same postulate can be applied to repair as was applied to failure, resulting in an expanded set of differential equations. These can be solved for availability formulas. This Markov process formulation is thus best suited for system level modeling where both maintainability and reliability are to be explicitly considered, but where the operational profile and the system are not so complex that an analytical approach becomes unwieldy.

8.3.2 Extreme Value Theory

An approach for obtaining certain prediction equations can be based on concepts of order statistics when the lifetime distribution of all items are identical and independent. Here the probability density function is derived for the particular item which, when it fails, will fail the system. For each of the following systems this item is:

(1) Series: Shortest lifetime pdf from n series items.

(2) Parallel: Largest lifetime pdf from ℓ parallel items.

(3) Series Strings in Parallel: Largest lifetime pdf from ℓ items from the shortest lifetime from n items.

(4) Parallel in Series String: Smallest lifetime pdf from n items from the largest lifetime from ℓ items.

As in most practical problems all items do not have identical pdf's, the general applicability of this approach is restricted.

<u>Example 8-1</u>

If a system consists of n items in series, e.g., linked together in the form of a chain, the lifetime of the chain cannot be more than that of the weakest link. The life length distribution of the chain would be that of the shortest life length. Ref. 43 treats this problem. The probability that the shortest life is less than t is given by

$$F_1(t) = 1 - \text{Prob}\{\text{lives of all n items are greater than or equal to t}\}$$

$$= 1 - [1 - F(t)]^n = 1 - R^n(t)$$

$$p_1(t) = n[1 - F(t)]^{n-1}\, p(t).$$

where $R(t) = 1 - F(t)$ is the reliability of a single item

$p(t) =$ lifetime density function of a single item.

Suppose that the distribution function for the lifetime of each member of the chain is Weibull, i.e.

$$F(t) = 1 - \exp\{- (\frac{t-\gamma}{\eta})^{\beta} \}, \text{ for } x \geq \gamma,$$

then

$$F_1(t) = 1 - \exp\{- n(\frac{t-\gamma}{\eta})^{\beta} \}$$

and the

$$p_1(t) = \frac{n\beta}{\eta} (\frac{t-\gamma}{\eta})^{\beta_1} \exp\{-n(\frac{t-\gamma}{\eta})^{\beta} \}.$$

8.3.2 Flowgraphs

Flowgraphs are a graphical method of representing simultaneously a set of equations which have been applied to electronic and other engineering problems. They augment a classical mathematical approach. There has been some exploratory application of flowgraph techniques to the development of reliability prediction models [Refs. 45 and 46] but this approach is not widely used. An advantage of a flowgraph approach would be that if one is already skilled in their use for engineering problems then this may be a ready method for learning about the development of reliability equations.

8.4 General Redundancy Model

Three of the redundancy models which have been introduced are those for:

(1) all items functioning, i.e. Eq. 8-2 which will be referred to as items in parallel,

(2) standby redundancy where there is a "perfect" switch, i.e. Eq. 8-6, which will be referred to as spares, and

(3) standby redundancy where there is a switch, i.e. Eq. 8-7, which will continue to be referred to as standby redundancy.

Interest is with a general reliability model for parallel arrangements of identical items of any of these three redundancy approaches where the failure criterion can be one or more items must work. In addition to reliability prediction this model can also be used for the general allocation problem concerning optimum selection of a redundant configuration. This model is an input for a general reliability cost tradeoff program (RECTA) which is covered in Volume II - Computation.

In this section the following notation is used:

n	identical items in parallel,
m	identical spares,
r	identical items in standby redundancy,
n_0	number of items that must work,
p	switch reliability,
s	the number of switches which work,
t	the mission time, and
λ	the failure rate.

The general formulas for the cases in which (a) $n_0 = 1$ and (b) $n_0 > 1$ are derived separately. Although the first case is a special case of the latter, case (b), it is useful to derive the simpler case first for a better understanding of the more general formula.

8.4.1 Reliability of a System for $n_0 = 1$

In this section the general formula is derived for the situation that only one item must work.

The probability that s switches work is given by the binomial formula

$$\binom{r}{s} p^{s}(1-p)^{r-s}.$$

If s switches work then the m spares plus the s items in standby result in m+s items on "standby", (manual or automatic). Thus the reliability is given by the formula

$$R_e = \sum_{s=0}^{r} \left[\binom{r}{s} p^{s}(1-p)^{r-s} R_{\ell}(n,\ m+s;\ t)\right],$$

where $R_{\ell}(n,\ m+s;\ t)$ is the reliability for a mission of length t given s switches work, n active items and m spares are available, and hence m+s standby items. The reliability R_{ℓ} is given by three cases:

<u>Case 1</u>: $m+s = 0$. In this case the reliability is given simply by the probability that at least one of the n active items survives time t, that is,

$$R_{\ell} - 1 - [1-e^{-\lambda t}]^{n}.$$

<u>Case 2</u>: m+s = 1. In this case the reliability is the probability that the n active items plus the one (1) standby item survive time t, or

$$R_\ell = 1 - \sum_{j=0}^{n} \binom{n}{j} (-1)^j e^{-j\lambda t} A(j),$$

where

$$A(j) = \frac{1}{(m+s-1)!} \int_0^t e^{-\lambda t_2(1-j)} (\lambda t_2)^{m+s-1} d(\lambda t_2), \text{ general formula,}$$

$$= \int_0^t e^{-\lambda t_2(1-j)} d(\lambda t_2) \text{ for } m+s = 1$$

and for specific values of j we obtain,

$$A(0) = 1-e^{-\lambda t},$$

$$A(1) = \lambda t, \text{ and}$$

$$A(j) = (e^{(j-1)\lambda t}-1)/(j-1), \; j = 2, \ldots, n.$$

<u>Case 3</u>: m+s = 2, 3, ..., ∞ or m+s is a positive integer larger than 1. If m+s is larger than one (m+s>1) the formula for R_ℓ is the same as the above with the exception that the A(j) are given by the following formulas which include Case 2 as a special case.

$$A(0) = 1 - \frac{-e^{-\lambda t}}{(m+s-1)!} [(\lambda t)^{m+s-1} + (m+s-1)(\lambda t)^{m+s-2}$$

$$+ \ldots + (m+s-1)!],$$

$$A(1) = (\lambda t)^{m+s}/(m+s)!, \text{ and}$$

$$A(j) = \frac{e^{(j-1)\lambda t}}{(m+s-1)!(j-1)^{m+s}} \{[(j-1)\lambda t]^{m+s-1} - (m+s-1) [(j-1)\lambda t]^{m+s-2}$$

$$+ \ldots + (-1)^{m+s-1} (m+s-1)!\} + \frac{(-1)^{m+s}}{(j-1)^{m+s}}, \; j = 2, 3, \ldots, n.$$

Derivation. The derivation of the expression for R_ℓ is given in the following discussion. First consider the probability that a system of n items in parallel (all active) will survive time t. Let $t_{(n)}$ be the longest time of survival for the n items and hence what is required is the probability that $t_{(n)} > t$. The probability that all n items fail in the interval (0, t) is given by

$$F_1(t) = P\{t_{(n)} \leq t\} = F^n(t) \tag{8-9}$$

and the probability that at least one item survives time t is given by $1 - F_1(t)$ or

$$P\{t_{(n)} > t\} = 1 - [1-e^{-\lambda t}]^n. \tag{8-10}$$

The probability density function for $t_{(n)}$ is given by differentiating $F_1(t)$ to yield

$$p_1(t_1) = n[1-e^{-\lambda t_1}]^{n-1} e^{-\lambda t_1}\lambda.$$

where t_1 is substituted for $t_{(n)}$ for convenience.

It is now desired to find the time to failure distribution for the m+s "spares" in order to find the total survival time for active and spare items. It is assumed that the n parallel active items have all failed at time t_1 and then the m+s spares will be used one-at-a-time until all have failed. Thus we want the probability density of the time to failure of these m+s spares with the assumption that one of them is used immediately, at time zero for the spares. The survival time is the sum of m+s-1 times each of which has an exponential failure time density function. Hence the frequency function for the sum (t_2) is the Gamma distribution

$$p_2(t_2) = \lambda e^{-\lambda t_2} (\lambda t_2)^{m+s-1}/(m+s-1)!$$

where t_2 is used to denote the survival time of m+s "spares", automatic and/or manual. The reliability R_ℓ is given by the probability that the sum of the two survival times as described above, $t_1 + t_2$, is larger than or equal to t, i.e.,

$$P(t_1 + t_2 \geq t).$$

The probability that the sum is less than t is given by the convolution integral

$$\int_{t_2=0}^{t} p_2(t_2)\, F_1(t-t_2)dt_2,$$

where $F_1(t-t_2)$ is obtained by substitution in Eq. 8-9 above.

$$\int_0^t [\lambda e^{-\lambda t_2} \frac{(\lambda t_2)^{m+s-1}}{(m+s-1)!}][1-e^{-\lambda(t-t_2)}]^n \, dt_2$$

$$= \int_0^t [e^{-\lambda t_2} \frac{(\lambda t_2)^{m+s-1}}{(m+s-1)!}][\sum_{j=0}^{n} \binom{n}{j} (-1)^j e^{-j\lambda(t-t_2)}] \, d(\lambda t_2)$$

$$= \sum_{j=0}^{n} \binom{n}{j} (-1)^j e^{-j\lambda t} \frac{1}{(m+s-1)!} \int_0^t e^{-\lambda t_2} e^{j\lambda t_2} (\lambda t_2)^{m+s-1} \, d(\lambda t_2).$$

Hence, for $m+s \geq 1$,

$$1-R_\ell(n,m+s,t) = P(t_1 + t_2 \leq t) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j e^{-j\lambda t} A(j), \tag{8-11}$$

where

$$A(j) = \frac{1}{(m+s-1)!} \int_0^t e^{-\lambda t_2(1-j)} (\lambda t_2)^{m+s-1} \, d(\lambda t_2).$$

<u>If $m+s = 0$</u>. There is no need for $A(j)$, $j = 0, 1, \ldots, n$, and we use Eq. 8-10 for n items in active parallel.

<u>If $m+s = 1$</u>. For $j \neq 1$ but an integer greater than or equal to zero

$$A(j) = \int_0^t e^{-\lambda t_2(1-j)} \, d(\lambda t_2)$$

$$= \frac{1}{(1-j)} [1 - e^{-\lambda t(1-j)}],$$

and for $j = 1$

$$A(1) = \int_0^t d(\lambda t_2) = \lambda t.$$

<u>If $m+s > 1$</u>.

$$A(0) = \frac{1}{(m+s-1)!} \int_0^t e^{-\lambda t_2} (\lambda t_2)^{m+s-1} \, d(\lambda t_2)$$

$$= 1-e^{-\lambda t} [(\lambda t)^{m+s-1} + (m+s-1)(\lambda t)^{m+s-2} + \ldots + (m+s-1)!]/(m+s-1)!$$

$$A(1) = \frac{1}{(m+s-1)!} \int_0^t (\lambda t_2)^{m+s-1} \, d(\lambda t_2) = \frac{(\lambda t)^{m+s}}{(m+s)!}$$

$$A(2) = \frac{1}{(m+s-1)!} \int_0^t e^{\lambda t_2} (\lambda t_2)^{m+s-1} \, d(\lambda t_2)$$

$$A(2) = \frac{1}{(m+s-1)!} e^{\lambda t} [(\lambda t)^{m+s-1} - (m+s-1)(\lambda t)^{m+s-2}$$

$$+ \ldots + (-1)^{m+s-1} (m+s-1)!] + (-1)^{m+s}$$

or in general for $j \geq 2$

$$A(j) = \frac{e^{(j-1)\lambda t}}{(m+s-1)!(j-1)^{m+s}} \Big[[(j-1)\lambda t]^{m+s-1} - (m+s-1) \, [(j-1)\lambda t]^{m+s-2}$$

$$+ \ldots + (-1)^{m+s-1} (m+s-1)! \Big] + \frac{(-1)^{m+s}}{(j-1)^{m+s}} .$$

Having obtained all A_j, for $j = 0, 1, \ldots, n$ the results are substituted into Eq. 8-11 to obtain

$$P(t_1 + t_2 \leq t),$$

and then the desired probability is the reliability R_ℓ, that is

$$R_\ell(n,m+s,t) = 1 - P(t_1 + t_2 \leq t).$$

This result must be obtained for each possible s and used in the formula for the reliability of an item,

$$R_e = \sum_{s=0}^{r} \left[\binom{r}{s} p^s (1-p)^{r-s} R_\ell(n,m+s,t) \right] . \qquad (8\text{-}12)$$

8.4.2 Reliability of a System for $n_0 \geq 1$

Suppose that n_0 items must operate in order for a system to properly perform its function. In the previous derivation $n_0 = 1$ and the distribution of life with n items in active parallel was given by the maximum life for the n items. In this case the time to failure is given as the time to failure of the $n - n_0 + 1\underline{th}$ item. The probability that the $n - n_0 + 1\underline{th}$ item fails in the interval (t, t+dt) is given by

$$p_3(t)dt = \frac{n!}{(n_0-1)!(n-n_0)!} \cdot [F(t)]^{n-n_0} [1-F(t)]^{n_0-1} p(t)dt,$$

where the probability density function and distribution function for a single item are

$$p(t) = \lambda e^{-\lambda t},$$

$$F(t) = 1 - e^{-\lambda t}.$$

Thus

$$p_3(t) = C(n, n_0)\lambda e^{-\lambda t(n_0-1+1)} [1 - e^{-\lambda t}]^{n-n_0}$$

$$= C(n, n_0)\lambda e^{-\lambda t n_0} \sum_{k=0}^{n-n_0} \binom{n-n_0}{k} (-1)^k e^{-k\lambda t}, \qquad (8\text{-}13)$$

where

$$C(n,n_0) = n!/[(n_0-1)!\ (n-n_0)!].$$

The distribution function of the time to the $n-n_0 + 1\underline{th}$ failure can be obtained by integration of $p_3(t)$,

$$F_3(t) = C(n, n_0) \int_0^t \sum_{k=0}^{n-n_0} \binom{n-n_0}{k} (-1)^k e^{-\lambda t(k+n_0)} \lambda dt,$$

$$= C(n, n_0) \sum_{k=0}^{n-n_0} \binom{n-n_0}{k} (-1)^{k+1} \frac{1}{(k+n_0)} [e^{-\lambda t_3(k+n_0)} - 1],$$

where t_3 is used to denote the life-length of the n active items. Hence

$$F_3(t_3) = C(n-n_0) \sum_{k=0}^{n-n_0} \binom{n-n_0}{k} (-1)^{k+1} \frac{1}{(k+n_0)} e^{-\lambda t_3(k+n_0)} + B(n, n_0)$$

where

$$B(n,\ n_0) \quad = \quad C(n,\ n_0) \quad \sum_{k=0}^{n-n_0} \left[\binom{n-n_0}{k} \quad (-1)^k \frac{1}{(k+n_0)} \right] \tag{8-14}$$

It is now possible to derive the distribution of $t_2 + t_3$ where t_2 is the time to failure of the m+s "spares" and t_3 is the time of failure of the n active items in parallel of which n_0 must survive. Hence, by convolution of these two distributions the distribution of $t = t_2 + t_3$ is given by the integral

$$\int_{t_2=0}^{t} p_2(t_2)\ F_3(t - t_2)dt_2$$

or

$$P\{t_2 + t_3 \leq t\} \quad = \quad \int_0^t \left[\lambda e^{-\lambda t_2} \frac{(\lambda t_2)^{m+s-1}}{(m + s - 1)!} \right] \cdot \left[C(n,\ n_0) \sum_{k=0}^{n-n_0} \binom{n-n_0}{k} \right.$$

$$\left. \cdot \quad (-1)^{k+1} \frac{1}{(k+n_0)} \quad \cdot \quad e^{-\lambda(t-t_2)(k+n_0)} + B(n,\ n_0) \right] dt_2.$$

or

$$P\{t_2 + t_3 \leq t\} \quad = \quad C(n,n_o) \sum_{k=0}^{n-n_0} (-1)^k \frac{1}{k+n_0} \binom{n-n_0}{k} \left[A(0) \ - \ e^{-\lambda t(k+n_0)} \ A(k+n_0) \right]$$

where $A(0)$ and $A(k+n_o)$ are obtained by using the previously derived equations for $A(j)$ for $j = 0$ and for $j = k+n_0$.

9. Environment and Bound-Crossing Problems

In this section various approaches which have been covered thus far in Parts II and III are brought together. Mainly the material is concerned with a multi-item system which is to operate in an environment which is known probabilistically or which is comprised of functionally related items. In particular, practical conclusions which can result from the reliability prediction analyses are noted at the ends of Secs. 9.2 and 9.3. This is the final section of this report concerned with analytical detail which has immediate practical significance. The following sections of Part IV mainly present the results of investigations on approaches for bringing into the analysis more detailed information bearing on the dependence question.

9.1 Environment Described Probabilistically

System reliability logic models such as those developed in Sec. 7 when all items are independent can be expressed in functional notation as

$$R = R(\underline{R}),\ \underline{R} = (R_1, \ldots, R_j, \ldots, R_n)$$

where R is the reliability for a system and each R_j, $j = 1, \ldots, n$ is the reliability of a single item. If each R_j is conditional on environment $R_j|\underline{s}$ and if the probability density of the environments $p(\underline{s})$ is known, then the unconditional system reliability is the expected value,

$$E(R) = \int_{\underline{s}} R(\underline{R})\ p(\underline{s})d\underline{s}. \tag{9-1}$$

This is the extension to multiple item models of the approach noted by Eq. 4-15 for single items.

Eq. 9-1 would be applied to the situation in which each item is used at the same environment and the environment is described by a probability density $p(\underline{s})$. Note that this means that the average reliability of each item cannot be obtained separately (using Eq. 4-15) and this average reliability substituted into the system reliability equation $R = R(\underline{R})$. That is

$$\int_{\underline{s}}(R_1|\underline{s})(R_2|\underline{s})\ p(\underline{s})d\underline{s} \neq \int_{\underline{s}}(R_1|\underline{s})\ p(\underline{s})d\underline{s} \int_{s}(R_2|\underline{s})\ p(\underline{s})d\underline{s}$$

for the simple case of two serial items. Whether or not using the incorrect separate approach yields conservative or optimistic results depends on the details of the particular problem. Some generalized statements of this sort have been developed for certain multi-item stress-strength problems and they will be noted in the following section.

The system reliability model $R = R(\underline{R})$ may result from any of the configurations and approaches noted in Sec. 7. The item reliability conditional on environment could result from testing. An item reliability measure could be of the form of the various measures of Sec. 4, or it could be based on the bound-crossing concepts of Sec. 5. For bound-crossing concepts where the bound is fixed, such as in Sec. 5.2, the performance attribute y and the environment $\underline{s}$ need to be dependent, i.e., $p(y,\underline{s}) \neq p(y)\, p(\underline{s})$, in order for the item reliability to be conditional on environment. Application of the fixed bound would give

$$\int_{y_\ell}^{y_u} p(y|\underline{s})\,dy = R|\underline{s} \; .$$

This resulting bound-crossing based reliability measure can then be readily inserted into Eq. 9-1 as an $R_j|\underline{s}$ along with other reliability measures for a multi-item system.

An expanded reliability definition which is essentially an elaboration on Eq. 9-1 has been proposed in Ref. 47 where the orientation was for catastrophic and drift failure modes for an item in a probabilistic environment. Eq. 9-1 is thus the basis of an approach where there is a probabilistic environment if the orientation is for separate physical items where there is a reliability measure for each item such as has been the viewpoint throughout Part III, or where the orientation is for separate failure modes where multiple modes are specifically identified as in Secs. 4.1 and 4.3, in Ref. 47, and developed in Part IV.

For stress-strength problems where the bound is a distribution such as in Sec. 5.3 the item reliability is always conditional on the environment (stress). Extreme value approaches cited in Sec. 8.3.2 for obtaining system reliability models which explicitly considered time are also applicable for certain multi-item stress strength problems.

The following section will expand on the multi-item stress-strength problem using detailed illustrations.

9.2 Stress-Strength Problems

Multi-item stress-strength problems considered here will demonstrate an application of the more general remarks made in Sec. 9.1. The general problem area is the extension of the single item stress-strength reliability measure of Sec. 5.3 to a multi-item system. The potential mistake in reliability prediction here is to obtain separately the reliability of each item in a system such as using Eq. 4-15, and then to substitute this into system logic models as those from Sec. 7. The correct development of multi-item stress-strength problems is presented in Section 9.2.1 and this is followed with some important practical conclusions in Section 9.2.2.

9.2.1 Prediction Approaches

Two basic approaches are used: (1) the calculation of the conditional probability that the strength exceeds a given stress and then integrating this result over the assumed stress distribution, and (2) the derivation of the probability density function (pdf) for the smallest (or largest) value of strength in a sample of n items, and then using the joint distribution of this density with that of stress to obtain the desired probability. Mathematically the first computation can be expressed as follows:

(1) Obtain the probability that $y > s_0$ for a single item, i.e.

$$P(y > s_0) = \int_{s_0}^{\infty} p(y)dy ,$$

where s_0 is the fixed stress level, and $p(y)$ is the pdf of strength. The examples will use uniform distributions and systems with few items, but the approaches are of course applicable to different distributions and systems with many items as well as with complex configurations.

(2) Obtain a general expression for the system reliability R in terms of the item reliabilities, knowing the system configuration. For each item, substitute the result of (1) into the system reliability model to obtain a system model as a function of the stress $R(s)$.

(3) Integrate the above reliability model over the stress pdf, $p(s)$, i.e.

$$\int_{s=0}^{\infty} R(s)\ p(s)ds ,$$

where $R(s)$ is the system reliability as given by (2) above.

The second computation follows the procedure described below:

(1) Obtain the distribution of the smallest strength in the case of a series logic (or largest strength in the case of parallel logic in which only one item must operate). For example, the probability that the smallest item in n selected at random from a distribution function $F(y)$ has a strength less than y is given by

$$P(y_s \leq y) = 1 - [1 - F(y)]^n$$

that is, 1 minus the probability that they (the n strengths) are all larger than y.

(2) The result in (1) is the distribution function for the smallest observation and it must be differentiated to obtain the pdf for the smallest strength y_s, $p_1(y_s)$.

(3) The joint pdf of strength y_s and stress s is given by

$$p_1(y_s)\ p(s)$$

and it must be integrated over the region $y_s \geq s$ to obtain the probability that the strength is adequate to withstand the imposed stress, i.e.

$$\iint_{y_s \geq s} p_1(y_s)\ p(s) dy_s\ ds.$$

The examples given below will illustrate these two approaches.

Example 9-1

First consider a single element with strength between 80 and 100 psi and stress between 60 and 85 psi. If the density functions are uniform on the respective intervals and the stress and strengths are independent the following two-dimensional plot indicates the region of inadequate strength.

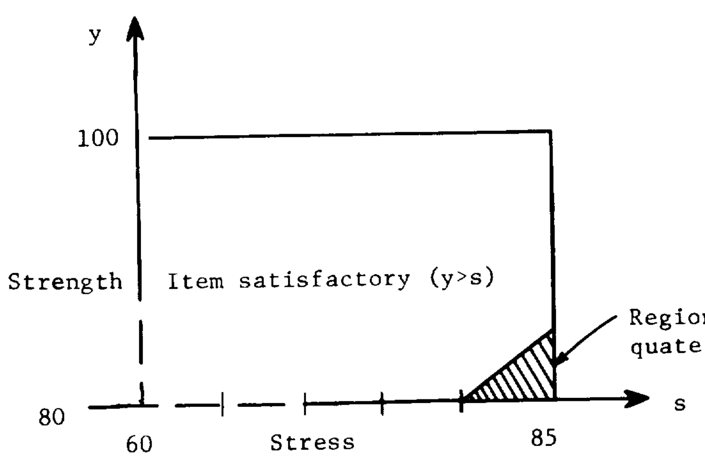


Figure 9.1 Region of Inadequate Strength

Since the two-dimensional distribution is uniform the probability that y exceeds s is given by 0.975, i.e.,

$$= \int_{y>s} \frac{1}{25}\, ds \,\frac{1}{20}\, dy$$

$$= 1 - \frac{1}{500} \int_{80}^{85}\int_{y}^{85} ds\, dy \;=\; 1 - \frac{1}{500}\int_{80}^{85} [85 - y] dy$$

$$= 1 - \frac{1}{500}\left[85y - \frac{y^2}{2}\right]_{80}^{85} = 1 - 0.025 = 0.975 \ .$$

Thus the item reliability is 0.975.

Example 9-2

Now consider a serial system with n items and suppose that each item has the same strength distribution, the items are selected at random, and that they are all exposed to the same stress given by the stress density function above. Thus the probability that this system will be adequate is equivalent to the probability that all items are adequate; that is, each of the strengths will exceed the stress value.

Approach 1: Now suppose that the stress is considered to be known or fixed at s_0, then the probability that an item selected at random has strength exceeding s_0 is given by

$$P(y > s_0) \;=\; \begin{cases} 1 \text{ if } s_0 < 80 \\ \dfrac{100 - s_0}{20} \text{ if } 80 \le s_0 \le 85. \end{cases}$$

Hence the probability that all n items have strength exceeding s_0 is given by

$$P^n(y > s_0) \;=\; \begin{cases} 1 \text{ if } s_0 < 80 \\ \dfrac{(100 - s_0)^n}{20} \text{ if } 80 \le s_0 \le 85. \end{cases}$$

The expected value of $P^n(y > s_0)$ for the uniform stress distribution is the unconditional probability of no failure

$$E[P^n(y > s_0)] \;=\; \int_{80}^{85} \left(\frac{100 - s}{20}\right)^n \frac{1}{25}\, ds + \int_{60}^{80} 1 \cdot \frac{1}{25}\, ds$$

$$= \frac{1}{n+1}\left(\frac{100 - s}{20}\right)^{n+1} \left(- \frac{1}{20 \times 25}\right)\Bigg]_{80}^{85} + \frac{4}{5} \ .$$

For n = 2 this probability is 0.95416.

Approach 2: Consider the second approach as described above for the same problem. The probability that the strength of a serial system is adequate is equivalent to stating that the minimum strength of n strengths selected at random from the strength distribution will exceed the stress value. The probability that the smallest value of y_i, i = 1, ..., n (say $y_{(1)}$) is less than y is given by

$$
\begin{aligned}
P(y_s \leq y) &= 1 - (\text{probability all values are greater than } y) \\
&= 1 - (1 - F(y))^n
\end{aligned}
$$

where

F(y) is the distribution function for y as shown below.

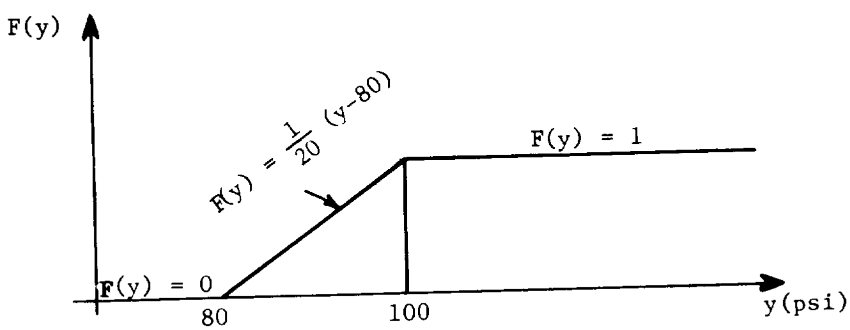


Figure 9-2 Strength Distribution Function

Hence

$$
F(y) = \begin{cases} 0 & y < 80 \text{ psi} \\ \frac{1}{20}(y - 80), & 80 \leq y \leq 100 \text{ psi} \\ 1, & y > 100 \text{ psi}. \end{cases}
$$

Now the probability density function for the smallest observation is

$$
p(y_s) = n[1 - F(y)]^{n-1}\, p(y).
$$

Thus the joint density function for y and s is

$$p(y_s)\ p(s)dy_s\ ds$$

$$= n[1 - F(y)]^{n-1}\ p(y) \cdot p(s)\ dy_s\ ds\ .$$

Using the fact that y and s are uniform probability density functions

$$p(y) = \frac{1}{20}\ ,\ p(s) = \frac{1}{25}\ .$$

Hence the probability that $y_s \leq s$ (that is, a failure occurs) is given by

$$\iint_{y \leq s} n[1 - \frac{1}{20}(y_s - 80)]^{n-1} \frac{1}{20} \cdot \frac{1}{25}\ ds\ dy_s$$

$$= \int_{80}^{85} n[1 - \frac{1}{20}(y_s - 80)]^{n-1} \frac{1}{20} \cdot \frac{1}{25}[85 - y_s]dy_s$$

For n = 2 this reduces to 0.04582 and thus the probability that $y_1 > s$ is 0.95416.

Example 9-3

Suppose there are three items in parallel and that at least one must work (strength exceed stress). Let the strength distributions be identical and uniform as given above and let the stress distribution be the same as above.

Approach 1: Using the first approach we obtain the probability that the strength of a single item exceeds a specified stress s_0 and then integrate this result over the stress distribution to obtain the unconditional probability. The probability that for a single element, strength exceeds stress s_0 is given by

$$P(y > s_0) = \begin{cases} 1 \text{ if } s_0 < 80 \\ \dfrac{100 - s_0}{20} \text{ if } 80 \leq s_0 \leq 85. \end{cases}$$

The probability that at least one of three exceeds the value s_0 is

$$1 - P[\text{all three have strength less than } s_0]$$

$$= 1 - [1 - P(y > s_0)]^3,$$

and hence the unconditional probability that the system is adequate is

$$= \int_{60}^{85} \left\{1 - [1 - P(y > s_0]^3\right\} \frac{1}{25}\, ds$$

$$= \int_{60}^{80} 1 \cdot \frac{1}{25}\, ds + \int_{80}^{85} \left\{1 - [1 - \left(\frac{100 - s_0}{20}\right)]^3\right\} \frac{1}{25}\, ds$$

$$= \frac{4}{5} + \int_{80}^{85} [1 + \left(\frac{80 - s}{20}\right)^3] \frac{1}{25}\, ds = 0.999218.$$

Approach 2: Using this approach the density function must first be obtained for the largest strength. The probability that the largest of three strengths exceeds y is given by

1 - P(all three strengths are less than y)

and the probability that the largest is less than or equal to y is

$$F^3(y)$$

where

$$F(y) = \begin{cases} 0 & , \quad y < 80 \\ \frac{1}{20}(y - 80), & 80 \leq y \leq 100 \\ 1 & , \quad y > 100 \end{cases}.$$

Thus by differentiating $F^3(y)$ the pdf of the largest strength is obtained, i.e.

$$p_1(y) = \begin{cases} 1 & , \quad y < 80 \\ \frac{3}{20}\left(\frac{1}{20}(y - 80)\right), & 80 \leq y \leq 100 \\ 0 & , \quad y > 100 \end{cases},$$

and thus the probability that y exceeds s is given by

$$1 - \int_{80}^{85} \left(\int_{y}^{85} \frac{3}{20} [\frac{1}{20}(y - 80)]^2 \frac{1}{25}\, ds \right) dy = 0.999218.$$

9.2.2 Practical Results

The results of the examples in Sec. 9.2.1 will be used to illustrate the error introduced by incorrectly treating probabilistic dependence. Recall that the single item reliability from Ex. 9-1 was 0.975. In Ex. 9-2 for two series items, the correct approach resulted in R = 0.95416. If the (incorrect) approach was used of treating these as two independent items in series was used then $R = 0.975^2 = 0.950525$. Thus the incorrect approach resulted in an unwarranted pessimistic reliability for a series system. Further, in Ex. 9-3 for three parallel items the correct approach resulted in $R = 0.999218$. If the incorrect approach of treating these as three independent items had been used, then $R = 1 - 0.025^3 = 0.99925$. Thus for a parallel system the incorrect approach resulted in an unwarranted optimistic reliability. Although the magnitude of these errors for these examples is not very large, it should be recognized that only few items were considered. The errors in the above examples illustrate the results of more extensive analyses in Refs. 48 and 49. These references show these results with greater elaboration for certain situations where each item is identical and at the same stress:

(1) Serial System. Obtaining the reliability of each item separately and then substituting these into a series system model of multiplying item reliabilities will yield pessimistic system reliability predictions.

(2) Parallel System. Obtaining the reliability of each item separately and then substituting this into a logic reliability model will yield optimistic system reliability predictions.

These results have been shown for situations where the stress-strength distributions are normal [Ref. 48] and where they are rectangular [Ref. 49]. Some practical guidelines gleaned from these results and expanded on in these Refs. are:

Serial Systems

(1) Mount items so they experience the same environment, i.e., a compact unit.

(2) Use consectively manufactured items in the same system, i.e., same manufacturer and lot.

(3) Select items with similar failure modes.

Parallel Systems

(1) Mount items so they experience different environments, i.e., different planes and location.

(2) Use items in the same system from different manufacturers and lots.

(3) Select items with different failure modes.

9.3 Functionally Related Variables

A class of multi-item bound-crossing reliability prediction problems are those where there is no meaningful reliability measure for each item in the system. In the multi-item stress-strength problem of Sec. 9.2 (where in the more general terminology the item strengths were item performance characteristics and the stress was the interface characteristic) it was appropriate to have a reliability measure for each item and a multi-item or system reliability measure. The problem being considered here is a more general one where there is not a performance attribute for each item, but the performance attributes are only for the system. That is, the functional relationships between the system performance attributes and the item and interface characteristics are such that any possible variation in any characteristic can always be compensated for by some possible variation in a different characteristic. The problem here is to obtain the distribution of the performance attributes from the distributions of the item and interface characteristics. Then the bounds are applied to the system performance attributes for the system reliability prediction. We will not be concerned here in Sec. 9.3 with mixtures of this more general problem with those of Secs. 9.1 and 9.2. The reader interested in such complexities is referred to Part IV. Some practical problems which have been widely treated in reliability analysis are those for performance variation analysis of electronic circuits and of systems in general [Ref. 7].

The basic procedure for reliability prediction of functionally related variables is as follows:

(1) Select the performance attributes of interest. These most often are functional outputs.

(2) Develop the deterministic mathematical models at nominal conditions relating the performance attributes to item and interface characteristics.

(3) Estimate the variability of the item and interface characteristics. For electronic parts these typically reflect the initial (manufacturing) variations, aging effects, and the influence of environmental inputs.

(4) Compute the following:

a. The expected variability of and possibly the correlation between the performance attributes.

b. Identify sources of performance attributes variability. Possible sources include contributions from the linear, nonlinear, and interaction behavior of the deterministic models, and from variations and correlation between the independent variables.

c. Predict the probability of successful performance by assigning limits to the expected performance attribute variations.

The more practical benefits are using the results of (4) for identifying designs which are susceptible to failure, and for providing redesign guidance. They are also useful for comparing alternate design approaches, and for aiding the assignment of specification limits. Normally the prediction of the probability of acceptable performance that can be obtained from a performance variations analysis is not highly precise because the approach is an approximate one, but more so because of the lack of precision in the data on part and interface characteristic behavior.

A solution to the problem in closed form is almost never possible, but mainly provides a better understanding of what an approximate approach is attempting. (See the discussion in Section 11 concerning Mode 2 for identification of an approach in closed form.) What is usually done in practice is use an approximate approach such as the method of moments (sometimes called the propagation of errors). Other approaches are identified in Ref. 7. The method of moments approach is presented below as an illustration.

9.3.1 Method of Moments

In the moments approach the functional relationship is expanded in a Taylor series. Higher order terms may be used, although most applications only use the linear terms. Measures of location and variability of the item and interface characteristics, which are the independent variables, are described by means and central moments. The degree of association which might exist between two independent variables is described by the correlation coefficient. The mean and central moments of the dependent variables are obtained from the application of expected value theory, which gives the mean and central moments of the dependent variable as functions of terms obtained from the Taylor series expansion and the mean and central moments of the independent variables. The distribution of the performance variables is then obtained by either assuming a distribution, or by fitting a distribution by the method of equating moments, for example. Correlation between the various performance attributes can also be obtained by this approach, but this is not usually noted or developed in reliability applicatons of this technique.

For simpler problems, requiring the use of only first order terms, it is possible to use this technique without a computer. Conversion of the functional model to a Taylor series yields sensitivity and possible interaction terms which readily provide information on variability sources. When the problem becomes more complex, as an involved functional relationship and high order moments, a computer is required. Advantages of this approach are simplicity for easier problems, and resultant information on sources of variability.

Mathematically the method of moments for a single performance attribute is as follows:

If the relationship

$$y = y(x_1, x_2, \ldots, x_n)$$

can be approximated by a linear function

$$y \simeq c_0 + c_1 x_1 + \dots + c_n x_n,$$

it is possible to approximate the distribution of y for certain distributions of the variables x_i, i = 1, ..., n. For example, if x_i is normally distributed with mean μ_i and standard deviation σ_i and if the correlation between x_i and x_j is r_{ij}, then the distribution of y is approximately normally distributed with mean

$$\mu\{y\} \simeq c_0 + c_1\mu_1 + \dots + c_n\mu_n,$$

and standard deviation

$$\sigma\{y\} \simeq [c_1^2\sigma_1^2 + \dots + c_n^2\sigma_n^2 + 2c_1c_2\sigma_1\sigma_2r_{12} + \dots$$

$$+ 2c_{n-1}\, c_n\sigma_{n-1}\sigma_n r_{n-1,n}]^{1/2} .$$

Example 9-4

Model

The linear amplifier, for which the circuit is shown in Fig. 9-3 is used here to illustrate a reliability prediction analysis using the method of moments.

For audio frequency applications, the transistor is adequately described by the hybrid or h-parameters. See Ref. 50 for further details on the circuit description and the derivation of the mathematical model. From circuit analysis the model for current gain is as follows:

$$A_i = \frac{R3}{R3 + R4}\;\frac{h_{fe}}{1 + h_{oe}U_2}\;\frac{U_1}{U_1 + \dfrac{(\Delta^h e)U_2 + h_{ie}}{1 + h_{oe}U_2}}$$

where

$$U_1 = \frac{R1\ R2}{R3 + R4}, \qquad U_2 = \frac{R3\ R4}{R3 + R4},$$

$$\Delta^{h}{}^{e} = h_{ie}\,h_{oe} - h_{re}\,h_{fe} .$$

Part Characteristics

The means and standard deviations of the part characteristics are contained in Table 9-1.

Table 9-1

Linear Amplifier Circuit Component
Part Parameters-Means and Standard Deviations

Parameter	Mean	Standard Deviation
R1	47.05K ohm	0.97K ohm
R2	7.03K ohm	0.17K ohm
R3	380.9 ohm	8.54 ohm
R4	468.7 ohm	11.14 ohm
h_{fe}	102	11.1
h_{re}	576×10^{-6}	0.46×10^{-6}
h_{oe}	556×10^{-6} mhos	68.6×10^{-6} mhos
h_{ie}	254	24.9

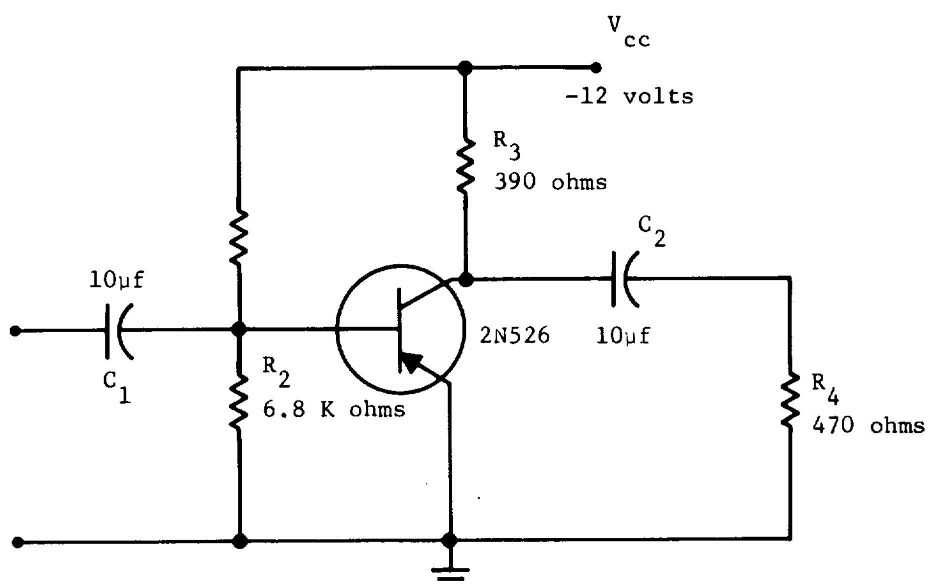


Figure 9-3 Linear Amplifier Circuit

The following matrix contains the correlation coefficients r_{ij} between pairs of the equivalent circuit transistor parameters. The resistances are sampled at random from separate distributions and are uncorrelated with each other and with the h-parameters.

$$\begin{array}{cc} & \begin{array}{cccc} h_{fe} & h_{oe} & h_{ie} & h_{re} \end{array} \\ \begin{array}{c} h_{fe} \\ h_{oe} \\ h_{ie} \end{array} & \left[\begin{array}{cccc} 1 & 0.595 & 0.912 & 0.165 \\ & 1 & 0.608 & 0.400 \\ \text{(by symmetry)} & & 1 & 1 \end{array}\right] \end{array}$$

Analysis

As suggested in the proposed approach one first performs a sensitivity analysis and checks the function $A_i = A_i(\quad)$ for non-linearity and for interaction. Because the function is essentially linear, the first and second moments of the performance can be obtained from the linear approximation to the performance, i.e.

$$\begin{aligned} A_i &\simeq c_0 + c_1 h_{fe} + c_2 h_{ie} + \ldots + c_8 R4 \\ &= 39.38 + 0.387 h_{fe} + 118.3 h_{re} - 0.742 \times 10^4 h_{oe} \\ &\qquad - 0.00619 \Delta h_{ie} + 0.416 \times 10^{-5} \Delta R1 + 0.186 \times 10^{-3} \Delta R2 \\ &\qquad + 0.0512 \Delta R3 - 0.0502 \Delta R4. \end{aligned}$$

Output

The estimated mean and standard deviation of A_i are given by $\hat{\mu}\{A_i\} = 39.38$ and

$$\begin{aligned} \hat{\sigma}\{A_i\} &= [(0.387)^2 s^2\{h_{fe}\} + \ldots + (-0.0502)^2 s^2\{R4\} + \\ &\quad + 2(0.387)(118.3)\, s\{h_{fe}\}\, s\{h_{re}\}\, r\{h_{fe}, h_{re}\} + \ldots \\ &\quad + 2(-0.742 \times 10^4)(-0.00619)\, s\{h_{oe}\}\, s\{h_{ie}\}\, r\{h_{oe}, h_{ie}\}]^{1/2} \\ &= 3.91. \end{aligned}$$

Remark 1. A reliability prediction is obtained by assigning a desired limit and then by obtaining a numerical value from any normal distribution table.

Remark 2. If the function could not be approximated by a linear function higher order moments and/or distributions of the part characteristics would be required.

Remark 3. The standard deviations and means used in the above analysis were inherent variations in the part characteristics. Variation as a result of operation environment, inputs, stresses, loads, and/or aging were not included. The analysis would be the same except that the total standard deviations would be larger than the above. In addition, correlations between the behavior of the parts characteristics may be introduced as a result of changes in a third variable, such as temperature, affecting two or more part characteristics.

Part IV. Refinements of Prediction Models

Some material concerning the structuring of reliability prediction models is presented in this part. The intent here is to provide insight somewhat beyond the current conventional practices. On occasion a hue and cry is raised as to whether or not current conventional practices are appropriate. Anyone who has performed reliability predictions and who has given serious consideration to the appropriateness of these predictions has likewise on occasion felt a bit uncomfortable. Yet, to many persons it is not obvious how to go beyond conventional practices.

The material presented in Part IV is directed toward those who are concerned with the development of reliability prediction models. A frame of reference is presented which will fit together details of certain reliability prediction problems. There are strong limitations on the extent to which these notions can be applied to real problems, with the main limitation being data.

To develop an approach to structuring certain features of reliability prediction models which reflects more detail is a stumbling point. The difficulties may eventually turn out to be elementary in hindsight, but documentation providing guidance on the type of problem considered in the following sections is rare. Remarks will be made freely in the hope that some may be of help in overcoming these difficulties. The following questions introduce some possible stumbling points and questions of interest.

(1) What is to be done if the conventional assumption of probabilistic independence is not made? What are sources of dependence and how are they reflected in structuring the problem?

(2) What are the features of a failure mode? How are variables treated which are probabilistic but which do not have values that always cause a failure?

(3) What is the pertinence of the typical engineering deterministic equations used for obtaining performance and stress.

(4) What is the relationship between degradation or catastrophic failure at the source (point of repair) and the manner in which system performance will be affected?

(5) How are the above considerations brought together?

(6) What are the implications of replies to these questions on real-world reliability predictions and on other reliability analyses?

Two examples are considered in some detail in Secs. 10 and 11 to introduce the notation and to formulate or structure the reliability prediction problem. Sec. 10 is a discussion of an example which is intended to illustrate features concerning catastrophic and degradation failures. A number of related problems are simultaneously treated in a different example presented in Sec. 11. The purpose is to structure the problems and not to obtain numerical solutions. Next in Sec. 12 the points made in Secs. 10 and 11 are expressed in general notation which results in detailed reliability prediction models. The above questions are replied to individually in Sec. 13 based on the contents of Secs. 10, 11, and 12, serving as concluding remarks for Part IV.

This material is somewhat related to earlier efforts at RTI supported by NASA ORQA [Ref. 47].

10. Catastrophic and Degradation Failures

There are two broad classes of failure modes which are popularly cited; these are a catastrophic failure and a degradation (or drift) failure. A degradation failure is an unsatisfactory level of a performance attribute, and a catastrophic failure is an abrupt change in a performance attribute, usually culminating in no meaningful measure of the performance attribute.

The question here is, "Is there a unique relationship between the classification of system failure into catastrophic or degradation and a similar classification of the source (point of repair) of system failure?" The answer will be developed by considering some failures associated with an electronic transmitter.

Catastrophic failure at the source:

(1) A part within a system opens or shorts. The result could be an immediate catastrophic failure of an output performance attribute, thus a catastrophic failure; or, the result could be a degradation failure of an output performance attribute. For example, the heater winding of a temperature control oven opens and the carrier frequency of a transmitter drifts. The oven winding open is an illustration where the system would not immediately fail, but rather results in an increased probability or later system failure.

(2) An input such as a supply voltage is completely lost. This results in the complete loss of all performance attributes.

Degradation failure (or conditions) at the source:

(1) An output performance attribute crosses a bound and is considered to have failed. Here there is some value of the performance attribute present, but it is outside of the desired range. This type of failure may have no single cause, as there may be several different parts which could be changed in order to correct the failure. There may be several items considered as failures according to the bounds on the performance characteristic in each part's specification, or there may be no part considered as having failed according to these criteria. Here there would be a functional relationship between the output performance attribute and the characteristics of the parts.

(2) An internal performance attribute crosses a bound, which causes an output performance attribute to fail catastrophically. An example is an oscillator ceasing to oscillate because of part characteristic value changes, with the result that an output performance attribute fails catastrophically. This type of failure is similar to the above as it may have no single cause.

The above examples illustrate that there is no unique correspondence between catastrophic and degradation failure modes at the detailed level (source) to that at the system output performance attribute level. That is, a catastrophic failure at the point of repair may show up at the system performance level as either a catastrophic or degradation failure, and similarly for vice-versa. Further, a degradation failure may not have any unique point of repair. The examples cited above for degradation failures within a system were for cases where there was not a unique repair point. It is possible there could be, such as where an output performance attribute is only a transformation of a single part characteristic. A more specific illustration drawing on the above example discussion is shown as Table 11.1 to assist in summarizing these relations.

Table 11.1

Illustrating Output-Source and Catastrophic-Degradation Failure Mode Relations for a Transmitter

System output performance attribute behavior	Source of failure within the system	
	Degradation	Catastrophic
Degradation, e.g., carrier frequency drift	Oscillator drifted, may not have a unique source.	Open winding of temperature control oven.
Catastrophic, e.g., no output	Oscillator ceases to oscillate, may not have unique source.	Supply voltage lost.

Whether or not a failure is catastrophic or degradation will not be a dominating consideration in the ensuing discussion. That is, it is not absolutely necessary that identification of one or the other failure modes be maintained in the model. This is a key point in structuring a detailed reliability prediction model, as there is a tendency to carry along too much detail in the notation which culminates in side-tracking. Introducing any detailed failure mode at the source in a particular problem may utilize any of several description methods which will be covered in the following sections. The method depends on the form of the given information. Of course there are certain forms which could prevail, as for certain commodities. The notation which will be used does not specifically identify the type of failure as to catastrophic or degradation. Of course, the person setting up the problem will have a classification in mind for each mode which is introduced.

11. A Detailed Prediction Example

In order to aid the reader in becoming oriented a somewhat realistic problem will be considered. Figure 11-1 illustrates an electrical circuit, a regulated voltage divider using a zener diode as a reference. The typical electrical notation are first shown in capital letters and the equivalent functional notation which will be used in Part IV is given in parentheses in small letters. This functional notation will be defined and discussed in a more general vein in Sec. 12.

The notation which appears in Fig. 11-1 plus several additional terms are discussed below.

Z	Zener reference diode, assumed to be a constant voltage source over the current range of interest,
R_1, R_2, R_3	Resistors, each with a deterministic temperature relationship,
$E_i = x_1$	Input voltage,
$E_Z = x_2$	Zener reference voltage,
x_3	Ambient temperature in °C,
x_4	Resistance of R_1 at 25°C,
x_5	Resistance of R_2 at 25°C,
x_6	Resistance of R_3 at 25°C,
$E_{01} = y_1$	Output voltage,
$E_{02} = y_2$	Output voltage,
k_i, $i = 1,2,3$	Linear temperature coefficient for the ith resistor,
$R_1 = y_3$	Resistance of R_1 at a specific temperature, $y_3 = x_4 + (x_3 - 25)k_1$
$R_2 = y_4$	Similar to y_3, $y_4 = x_5 + (x_3 - 25)k_2$,
$R_3 = y_5$	Similar to y_3, $y_5 = x_6 + (x_3 - 25)k_3$,
$I_1 = u_1$	Current as designated in Fig. 11-1,
$I_2 = u_2$	Current as designated in Fig. 11-1
$W_Z = u_3$	Power of zener diode,

		u_4	Ambient temperature of the circuit,
W_{R_2}	=	u_5	Power of R_2, and
W_{R_3}	=	u_6	Power of R_3.

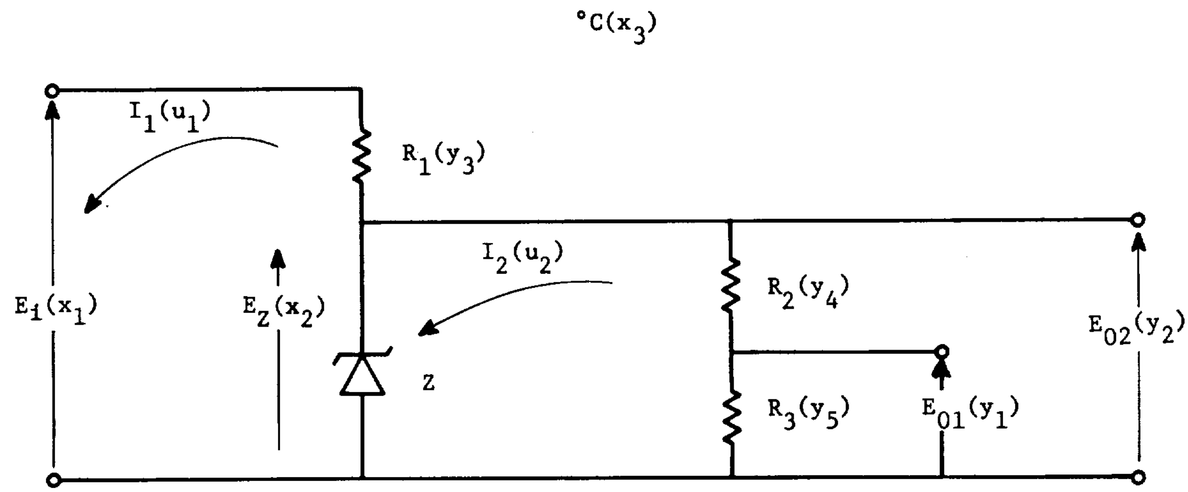


Figure 11-1 Regulated Voltage Supply

Note: The symbols in parentheses correspond to those used in the problem formulation in this section.

Some of the variables are continuous and have known or assumed probability density functions (pdf's). These are:

Known pdf	Comment
$p(x_1\|m_1)$	x_1, or E_i, has a probability density which is conditional on no complete loss of E_{01}, which is designated m_1.
$p(x_2,x_3)$	x_2, or E_Z, has a probabilistic dependence with x_3, or °C. That is, temperature effect on the reference diode is not known deterministically.
$p(x_3)$	The probability density of temperature, which is the marginal density of $p(x_2,x_3)$.
$p(x_4)$	Nominal value of resistor R_1 is independent.
$p(x_5,x_6)$	Resistors R_2 and R_3 are of the same nominal value, and have a probabilistic dependence; when one is high, the other also tends to be high.

The circuit is designated as having seven different failure modes:

$j = 1$	Complete loss of E_i,
$j = 2$	Drift of E_{01} (or y_1) outside of an acceptable interval of values Γ_{y_1},
$j = 3$	Drift of E_{02} (or y_2) outside of an acceptable interval of values Γ_{y_2},
$j = 4$	Catastrophic failure of Z,
$j = 5$	Catastrophic failure of R_1,
$j = 6$	Catastrophic failure of R_2,
$j = 7$	Catastrohpic failure of R_3, and
m_j	Event that the jth failure mode does not occur.

Catastrophic failures noted above are those which might occur as influenced by the internal stresses. It is known that each item is not initially catastrophically failed. Additional known information concerns each catastrophic failure mode, m_4 though m_7. Relationships between the probability that these failure modes will not occur and appropriate environments are known; thus $P(m_j|\ldots\text{environment(s)}\ldots) = m_j(\ldots\text{environment(s)}\ldots)$ are available for m_4 through m_7. Note that conventional graphs for failure rate versus stresses such as those found in MIL-HDBK-217A [Ref. 27] could provide this type of relationship.

The functional notation for deterministic relations will be such as $y_3 = y_3(x_3,x_4)$, $P(m_4|u_3,u_4) = m_4(u_3,u_4)$, and $u_1 = u_1(x_1,x_2,x_3,x_4)$.

The question is how to structure the problem for the probability that none of these failure modes occur, where no assumption of independence is made involving the features noted above. Each of the failure modes will be treated separately and then they are brought together into a composite model.

Mode m_1

Mode reliability $P(m_1)$ is some known value between 0 and 1.

Mode m_2

Electrical equations are conventional engineering deterministic ones and are used for obtaining the performance attribute E_{01} (or y_1), i.e.

$$E_{01} = y_1 = \frac{y_5 x_2}{y_4 + y_5} = y_1(x_2, x_3, x_5, x_6)$$

where y_4 and y_5 are known functions of x_3, x_5, and x_6. The zener reference voltage x_2 is dependent on x_3 the ambient temperature and the joint pdf $p(x_2,x_3)$ has been obtained. The mode reliability $P(m_2)$ is the probability that y_1 falls within the interval of acceptable values, Γ_{y_1}. If $p(y_1)$ denotes the pdf of y_1, then

$$P(m_2) = \int_{\Gamma_{y_1}} p(y_1) dy.$$

However a difficult problem is implied by the above integration, that of obtaining $p(y_1)$ using the functional relationship given above.

In some few problems a transformation can be defined relating the new variable y_1 to the original variables x_2, x_4, x_5, x_6 and the distribution of the new variable obtained from that of the original variables by means of the Jacobian of the transformation. (See Ref. 51 for a description of the method.)

Usually the above approach is tedious or the integral cannot be obtained in a closed form. In such cases, which is the usual situation, one has to use some other approach. Often the method of moments is used in which y_1 is expanded in a Taylor Series using the first order terms (higher order terms may be used but seldom are) and obtaining the moments of y_1 (first and second order) in terms of the moments of x_2, x_3, x_5, and x_6. Hence the distribution of y_1 is approximated by the method of moments.

Another procedure is to evaluate the integral

$$P(m_2) = \int_{\Gamma_{y_1}} p(x_2,x_3)\ p(x_5,x_6) dx_2 dx_3 dx_5 dx_6$$

where Γ_{y_1} determines a region of integration of x_2, x_3, x_5, and x_6. This is still difficult but some approximations may be possible and a Monte Carlo simulation could be used to obtain the estimate. However the latter approach would require a very large number of trials if $P(m_2)$ is near 1 and a high precision of the estimate is desired.

Although each of the above procedures is suitable for estimating the reliability of this mode, the latter one will be required for bringing together all modes for a single circuit reliability model. This will be treated later.

Mode m_3

The performance attribute E_{02} (or y_2) is known:

$$E_{02} = E_Z, \; y_2 = x_2.$$

Mode reliability is defined to be the probability that y_2 (or x_2) falls within Γ_{y_2}, i.e.

$$P(m_3) = \int_{x_3} \int_{\Gamma_{y_2}} p(x_2, x_3)\, dx_2 dx_3.$$

Mode m_4

Electrical equations needed here are the conventional ones for the power stress W_Z.

$$W_Z = u_3 = E_Z(I_1 - I_2) = x_2(u_1 - u_2)$$

where

$$I_1 = u_1 = \frac{E_i - E_Z}{R_1} = \frac{x_1 - x_2}{y_3}$$

$$I_2 = u_2 = \frac{E_Z}{R_2 + R_3} = \frac{x_2}{y_4 + y_5}.$$

The mode reliability $P(m_4)$ is defined as the probability of no catastrophic failure of the zener reference diode Z. The relationship of $P(m_4)$ to fixed levels of the stresses of temperature °C and power W_Z is known:

$$P(m_4|W_Z, °C) = P(m_4|u_3, u_4) = m_4(u_3, u_4).$$

The power u_3 is a function of $\underline{x}$ (all the x's) denoted by $u_3 = u_3(\underline{x})$. The ambient temperature u_4 is known, $°C = x_3 = u_4$. Now $\underline{x}$ has a joint pdf denoted by $p(\underline{x})$. Hence the mode reliability is the expected value

$$E[P(m_4|u_3,u_4)] = \int_{\underline{x}} m_4[x_3,u_4(\underline{x})]\ p(\underline{x})d\underline{x}.$$

The reason for obtaining the expected value of the probability of no failure given the stress is the fact that some or all of the x's are probabilistic variables having pdf's. If the failure occurred when the stress exceeded a particular value then the probability would be obtained in a manner similar to that of m_2 and m_3.

Mode m_5

No electrical equations are used as the relation of $P(m_5)$ contains only a single stress, temperature, °C:

$$P(m_5|°C) = P(m_5|u_4) = m_5(u_4) \text{ where } u_4 = x_3.$$

Mode reliability is the expected value

$$E[P(m_5|u_4)] = \int_{x_3} m_5(x_3)\ p(x_3)dx_3.$$

Mode m_6

Electrical equations are for the power stress W_{R_2}.

$$W_{R_2} = u_5 = I_2^2\ R_2 = u_2^2\ y_4$$

where u_2 is noted in m_4. Mode reliability is similar to m_4:

$$P(m_6|°C,W_{R_2}) = P(m_6|u_4,u_5) = m_6(u_4,u_5)$$

where $u_4 = x_3$ and $u_5 = u_5(x_2,x_3,x_5,x_6) = u_5(\underline{x}')$, say where $\underline{x}' = x_2,x_3,x_5,x_6$.

$$E[P(m_6|u_4,u_5)] = \int_{\underline{x}'} m_6[x_3,u_5(\underline{x}')]\ p(\underline{x}')d\underline{x}'.$$

Mode m_7

The development here is similar to m_6, where y_4 for m_6 become y_5 for m_7. The m_7 reliability in functional notation is identical to that for m_6.

Composite Reliability

The reliability of the circuit is the probability that none of the failure modes occur;

$$R = P(m_1 m_2 \ldots m_7)$$

Although this probability can be expressed as the product of conditional reliabilities,

$$R = P(m_1)\, P(m_2|m_1) \cdots P(m_7|m_1, \ldots, m_6),$$

this does not aid in the evaluation of the reliability in this example due to the commonality of the variables $\underline{x}$ to the various modes. Thus the mode reliabilities which were formulated individually in the previous discussion cannot be multiplied together to obtain the overall reliability. The circuit reliability is obtained by the evaluation of a multiple integral which simultaneously considers the probabilities of non-failure of the seven modes. Thus

$$R = P(m_1) \int_{\{x_1, x_4\}} \int_{\left\{\begin{array}{c}\underline{x}' \subset \Gamma_{y_1} \\ \text{and} \\ \underline{x}' \subset \Gamma_{y_2}\end{array}\right\}} m_4(\underline{x})\, m_5(x_3)\, m_6(\underline{x}')\, m_7(\underline{x}')\, p(\underline{x}|m_1)\, d\underline{x}$$

where all terms are as developed in the preceding discussion. The region of integration is a restricted one for only certain values of $\underline{x}'$, that is, those contained in Γ_{y_1} and Γ_{y_2}, is there a success. In words the reliability of the circuit is a multiple integral over the acceptable regions of the variables defined by bounds. The integral contains the product of the conditional probabilities of non-failure of those modes, conditioned on the environment distributions.

The above reliability expression is rather formidable, indicating that consideration of dependence resulting from correlation between variables and from the effect of the same basic variables on more than one mode reliability yields a complex relationship.

A numerical integration would be tedious and require a computerized solution. It would not seem possible to provide a single computer program to treat a very wide class of these problems although specific subroutines are available to perform numerical integrations. Thus one must use an approximate numerical solution. The simplest approach would seem to be a Monte Carlo simulation. Numerical computation is discussed later in Sec. 12.2.

12. General Model Development

The features which were contained in the example problem of Sec. 11 are brought together and are expressed in general notation. This notation explicitly allows much detail (known or starting information), but only part of this detail would be expected in a specific problem. Keep in mind that the example of Sec. 11 was an illustration of this generalization.

In Sec. 12.1 a prediction model is developed for the series situation where occurrence of any failure mode will imply system failure. Numerical solution approaches are briefly covered in Sec. 12.2 for the series situation model. Next, Sec. 12.3.1 briefly comments on extending the series model to include the explicit treatment of time. The final Sec. 12.3.2 comments on the extension of the series situation model to a parallel situation where some failure modes can occur but the system remains unfailed.

12.1 Series Situation Model

A detailed reliability prediction model is developed for the situation where the occurrence of any failure mode results in system failure. This will be referred to as the series situation model. However, the reader is cautioned not to expect that the final composite model will literally be a product of individual probabilities of non-failure of each mode. Explicit consideration of mode dependencies results in the final composite model being of a different form than a product.

12.1.1 Notation

Much of the material in Secs. 10 and 11 pertains to the selection of notation. Seeing how to structure the detailed reliability prediction problem considered here is aided by an approach which leans toward using common notation for mathematically similar descriptions rather than using different symbols for the different physical features having common mathematical descriptions.

As conventionally used:

t	Time,
$y=y(x)$, $w=w(\underline{x})$	Functional relationship,
$\underline{x}$	Vector, i.e., $\underline{x} = (x_1, x_2, \ldots, x_n)$,
$P(A)$	Probability of the event A,
$p(x)$	Probability density function (pdf) of x, and
Γ	Bounds (region of acceptable values),

Additional notation which is not so conventional and which will be explained in the following sections.

d_i	Event that a failure mode which will be referred to as direct-fixed does not occur. $i = 1, 2, \ldots, \ell$. The event that this failure mode does occur is $\bar{d}_i$.

e_j Event that a failure mode which will be referred to as direct-variable does not occur, $j = 1, 2, \ldots, m$. The event that this failure mode does occur is $\bar{e}_j$.

b_k Event that a failure mode which will be referred to as bound-crossing does not occur, $k = 1, 2, \ldots, n$. The event that this failure mode does occur is $\bar{b}_k$.

The d, e, and b will replace the m used in the example problem of Sec. 11 as the failure modes illustrated there are now being classified into the three types of mathematical descriptions which were used. Additionally,

x_s Common variable,

y_v Performance attribute,

u_w Environment,

The shorter expression as noted below will be used to indicate the joint occurrence of events.

Conventional Form: $P(d_1, d_2, \ldots, d_i, \ldots, d_\ell) = P(d_1)\, P(d_2|d_1) \cdots P(d_i|d_1, d_2, \ldots, d_{i-1}) \cdots P(d_\ell|d_1, d_2, \ldots, d_{\ell-1})$

Shorter Form: $P(\underline{d}) = \prod_{i=1}^{\ell} P(d_i|\underline{d}')$ where $\underline{d}'$ thus indicates appropriate conditional events.

12.1.2 Common Variables

There are common variables $\underline{x}$ which influence the probability of certain failure modes. The common variables may be deterministic or probabilistic; this discussion emphasizes them as probabilistic. The complete probability density $p(\underline{x})$ of all probabilistic common variables is given information, including any dependence. No special acknowledgement is made in the $p(\underline{x})$ notation for those common variables which are deterministic. Examples in the problem of Sec. 11 of common variables which were probabilistic were all x's, for example, x_1, input voltage; x_3, ambient temperature; and x_4, resistance of R_1 at 25°C. Thus common variables could be interface characteristics such as supply voltage, load, or temperature. Also they could be internal characteristics of parts such as resistance or beta.

The common variables appear in functional relationships for obtaining performance attributes, $y_v = y_v(\underline{x})$ for all v and environments $u_w = u_w(\underline{x})$ for all w as in the conventional engineering equations where all variables are deterministic. Examples of performance attribute equations were those for the y's in the problem of Sec. 11, and examples of environmental (or stress) equations were those for the u's. Probability densities of the performance attributes $\underline{y}$ and the environments $\underline{u}$ will be needed and are not usually known. They can be determined (in concept) from known probability densities of the common variables $\underline{x}$ and the functional relationships $y(\underline{x})$'s. When the probability density of a performance attribute or an environment is known, including any known dependence with common variables, they will be initially classified as common variables. Thus if a $p(y, \underline{x})$ or a $p(u, \underline{x})$ is initially known, the y or u will be introduced into the composite problem structure as $y = x$ or $u = x$. This is done for two reasons. The first is to avoid additional special notation for what are "special cases" in the context of the more complex composite model being formulated. The second reason is to assist in insuring that some of the more devious correlation effects are included, such as the following examples. In the problem of Sec. 11 zener reference voltage x_2 was directly a performance attribute in mode m_2 where $y_2 = x_2$, and also appeared in several environment- and performance-equations. The performance attribute probability densities which are obtained directly from testing, either by necessity (its $y = y(\underline{x})$ not known) or for convenience, would tend to be of this nature. The temperature x_3 also appeared in several environment- and performance-equations, including that for m_5 where it was directly an environment $u_4 = x_3$.

12.1.3 Modes

An undesired event which may or may not occur is a failure mode, e.g., the loss of an input voltage, the opening of a resistor, or the drift of a performance attribute outside prescribed bounds. There may be several failure modes for a single item, e.g., the opening or shorting of the resistor, or a mode may involve more than one item, e.g., an output voltage of an amplifier comprised of multiple items. A mode may be a feature of other than hardware, e.g., physical shock impulse or a human error.

Thus, in general, a failure mode can be some undesired feature of a part within a system, an input to a system, or an output of a system, including human features. Further, what is physically a single item at the smallest level of repair may have more than one mode associated with it, and a mode may involve more than one physical item. The problem treated is primarily concerned with the non-occurrence of a failure mode. The probability that a failure mode will not occur is either known or can be determined from functional relationships and probabilistic methods.

Modes are classified below according to the manner in which they are treated in the analysis. System and part failures are commonly thought of as catastrophic or degradation, where a catastrophic failure is an abrupt change in some characteristic, and a degradation (or drift) failure is a characteristic value outside of some bounds. In general, each of the mode description types which are noted below may be for an event which would commonly be considered as either a catastrophic or degradation failure. That is, there is not necessarily a unique form of the mathematical description for either a catastrophic or a degradation failure. Catastrophic and drift failures are thus just a different method of classifying failure modes from that which is developed. The example discussed in Sec. 10 illustrates this point.

Direct-Fixed Mode. The reliabilities of these modes are fixed values which are known. These modes may or may not be dependent on each other, and the dependencies are known. An example of this mode in the problem of Sec. 11 is m_1, the loss of the input voltage. If the possibility had been considered of the parts being initially catastrophically failed so that no circuit operation was ever possible, these would also have been modes of this type.

The reliability of a single mode, d, is

$$0 < P(d) < 1 \; .$$

The reliability of all direct-fixed modes is

$$R = P(\underline{d}) = \prod_{i}^{\ell} P(d_i|\underline{d}') \qquad (12\text{-}1)$$

In general these failure modes would be interface events and internal part events which preclude the existence of some common variable. Thus in the example cited the occurrence of $\bar{m}_1$, complete loss of the input voltage, will mean that some value of the input voltage (and common variable) x_1, will not be possible, thus $p(x_1|m_1)$. Also direct-fixed modes could be events completely aside from all common variables.

Direct-Variable Mode. Reliability of each of these modes is conditional on some environment level, where there might be dependence between mode reliabilities at fixed environment levels. Each environment is a function of the common variables. Examples of direct-variable modes in the problem of Sec. 11 were m_4 through m_7 which were for the non-catastrophic failure of the parts.

Reliability of a single mode is

Given: $P(e|\underline{u}) = e(\underline{u})$, $u_w = u_w(\underline{x})$ for all w, and $p(\underline{x})$

Obtain: $P(e|\underline{x}) = e(\underline{x}) = e[\underline{u}(\underline{x})]$

$$R = P(e) = \int_{\underline{x}} e(\underline{x}) \; p(\underline{x}) d\underline{x} \; .$$

Thus, mode reliability is obtained by an averaging, the expected value operation. Figure 12-1 illustrates the development of this type of mode reliability description.

Reliability for multiple modes is

Given: $P(e_j|\underline{e}', \underline{u}) = e_j(\underline{u})$, $u_w = u_w(\underline{x})$ for all w, and $p(\underline{x})$

Obtain: $P(e_j|\underline{e}', \underline{x}) = e_j(\underline{x}) = e_j[\underline{u}(\underline{x})]$

$$R = P(\underline{e}) = \int_{\underline{x}} [\prod_{j}^{m} e_j(\underline{x})] \; p(\underline{x}) d\underline{x} \qquad (12\text{-}2)$$

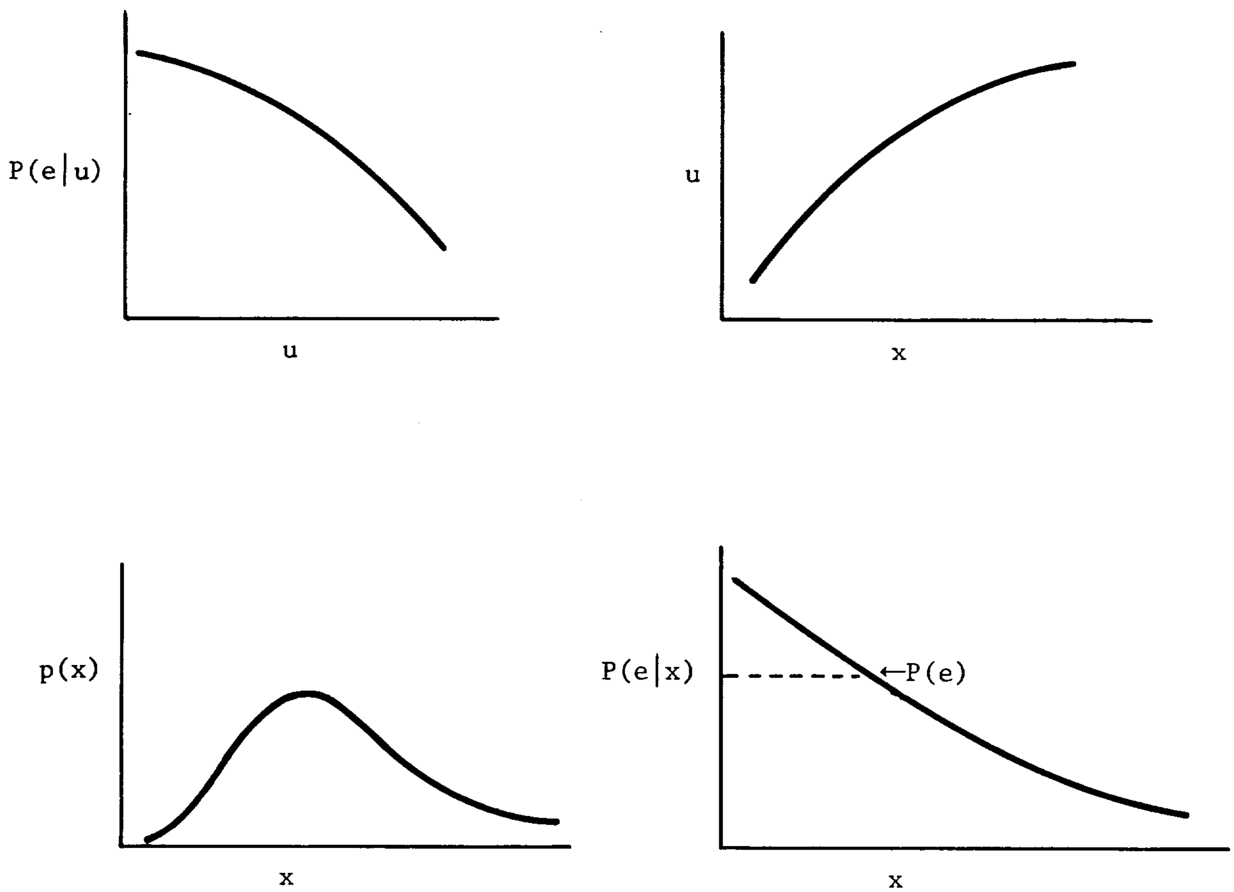


Figure 12-1 Direct-Variable Mode

Environments $\underline{u}$ in these modes for electronic parts typically would be stresses such as current, power, or temperature. The deterministic equations $u_w = u_w(\underline{x})$ could be conventional electronic equations for obtaining stresses. Note that it is possible the mode reliability may be conditional on an environment where the environment is also a common variable, or some $u_w = x_s$. This was the situation in the problem of Sec. 11 where the reliability of m_5 was conditional only on temperature, and temperature also appeared in environment and performance euqations. The direct-variable modes are where the type of environment information presented in MIL-HDBK-217A [Ref. 27] would be applicable, but note that this reference infers explicit treatment of time which has not yet been introduced here and it always assumes mode independence at fixed environment levels, as did the problem of Sec. 11.

Bound-Crossing Mode. The reliability of a single bound-crossing mode is the probability that a performance attribute y remains within designated bounds Γ_y. Bounds are established either on the basis of judgment or on a more theoretical basis such as a condition for oscillation of an electronic circuit oscillator or for a stress-strength problem. Each performance attribute is a function of the common variables $\underline{x}$. Examples of this mode in the problem of Sec. 11 were modes m_2 and m_3 for the output voltages.

Reliability of a single mode:

Given: $y = y(\underline{x}),\ p(\underline{x}),\ \Gamma_y = \Gamma_\ell < y < \Gamma_u$

Obtain: $R = P(b) = \int_{\Gamma_y} p(\underline{x})d\underline{x}$.

Thus, the region in $\underline{x}$ such that $\Gamma_\ell < y(\underline{x}) < \Gamma_u$ is the probability of success. Fig. 12-2 illustrates the development of this type of mode description.

Reliability of multiple modes is:

$$R = P(\underline{b}) = \int_{\Gamma_{\underline{y}}} p(\underline{x})d\underline{x}\ , \qquad (12\text{-}3)$$

where

$$y_v = y_v(\underline{x}) \text{ and } \Gamma_\ell < y_v < \Gamma_u \text{ for all v.}$$

An important point to note for the bound-crossing mode is that treatment of this mode does not involve the expected value operation. Rather, the bounds on the performance attributes $\Gamma_{\underline{y}}$ yields two complementary regions in the common variables $\underline{x}$, with probability density $p(\underline{x})$; values of $\underline{x}$ in one region will result in failure and values of $\underline{x}$ in the other region will result in acceptable performance.

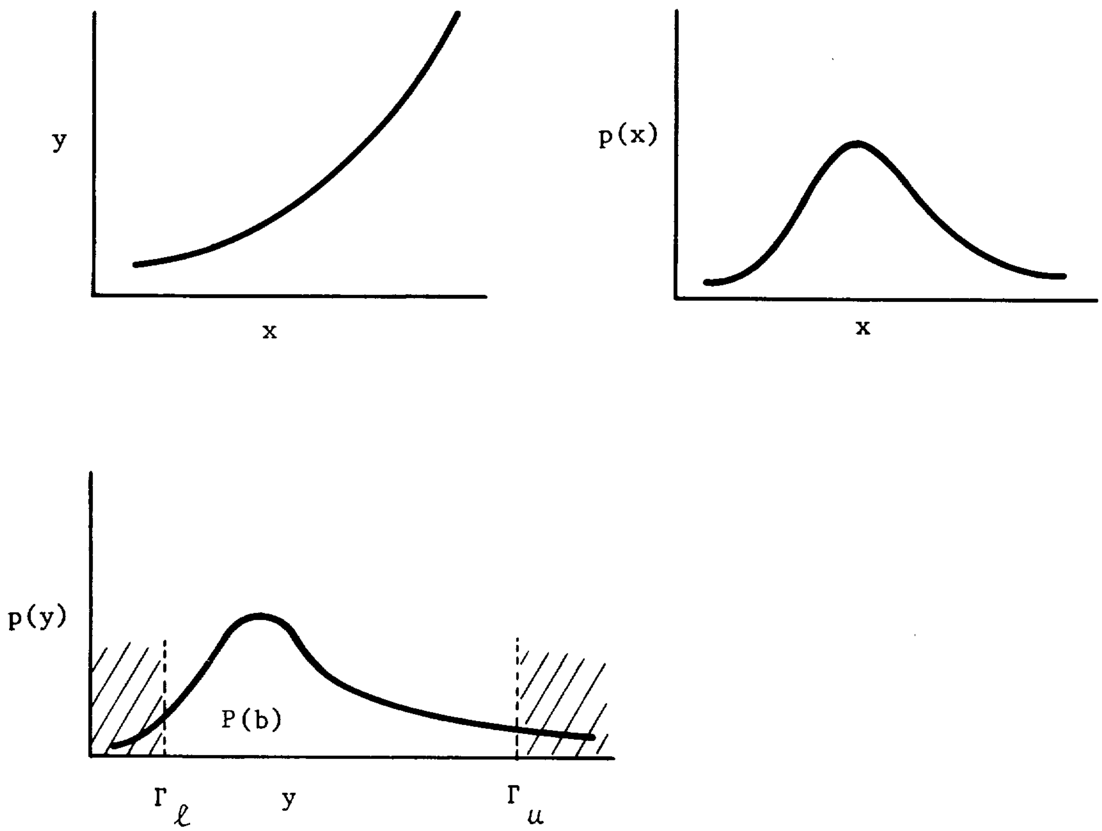


Figure 12-2   Bound-Crossing Mode

Discussion of computation, i.e., transformation, method of moments, and Monte Carlo for obtaining the reliability of this type of mode description was contained in mode m_2 in the problem of Sec. 11.

12.1.4 Composite Model

The three types of failure mode descriptions of Sec. 12.1.3 are brought together into a composite series model where the occurrence of any failure mode will mean system failure. Consider first the direct-fixed modes.

$$P(\underline{d}) = \prod_{i=1}^{\ell} P(d_i|\underline{d}') \qquad \text{where } 0 < P(d_i|\underline{d}') < 1 .$$

If there were no dependencies between any of these modes, then the resulting product of mode reliabilities would be the simple model which is so widely assumed for reliability of items in serial logic.

Bring in the direct-variable modes:

$$P(\underline{d}, \underline{e}) = P(\underline{d})\, P(\underline{e}|\underline{d})$$

$$= \prod_{i=1}^{\ell} P(d_i|\underline{d}') \int_{\underline{x}} \prod_{j=1}^{m} P(e_j|\underline{d}, \underline{e}', \underline{u}) \quad p(\underline{x}|\underline{d})d\underline{x}$$

where

$$P(e_j|\underline{d}, \underline{e}', \underline{u}) = e_j(\underline{u}),\; u_w = u_w(\underline{x}) \text{ for all } w .$$

The multiple mode descriptions above are expressed conditionally on other direct-variable modes. A reason is that several different modes may apply to the same physical item. For example, if a two terminal electronic part has the open and short failure modes explicitly treated, then the part can either fail by (open) or (short|no open) or vice versa. This possibility was not explicitly treated in the problem of Sec. 11. Introduce the bound-crossing modes for the complete model

$$R = P(\underline{d}, \underline{e}, \underline{b}) = P(\underline{d})\, P(\underline{e}, \underline{b}|\underline{d})$$

$$= \prod_{i=1}^{\ell} P(d_i|d') \int_{\{\underline{x}'\}}\int_{\{\underline{x}''\subset \Gamma_{\underline{y}}\}} \prod_{j=1}^{m} P(e_j|\underline{d},\underline{e}', \underline{u}) \quad p(\underline{x}|\underline{d})d\underline{x} \qquad (12\text{-}4)$$

where $\underline{x} = (\underline{x}', \underline{x}'')$, $y_v = y_v(\underline{x}'')$, $\underline{x}'$ do not appear in any $y_v = y_v(\underline{x}'')$, and $\Gamma_{\ell} < y_v < \Gamma_u$ for all v , and the supporting information noted above for the direct-variable modes still applies.

Equation 12-4 is the composite reliability prediction model. This is the general functional notation counterpart of the specific problem functional notation reliability prediction model which was formulated in Sec. 11.

12.2 Numerical Calculation

An approach for numerical calculation of the composite reliability model using Monte Carlo simulation is shown in Fig. 12-3. Step (3) in Fig. 12-3 can be omitted by using n instead of c in the denominator of step (4) if no estimate is wanted of the reliability of bound-crossing failures. It is also possible to obtain an estimate of the dispersion of the distribution of reliabilities, although this is not shown on Fig. 12-3.

Another approach for numerical calculation would be to use a discrete approximation of the complete region of the common variables $\underline{x}$ instead of sampling the region. Figure 12-4 shows this approach. A grid network would be established covering the complete region and resulting in discrete cells. This approach would be useful where some of the input information would be obtained directly from testing at the nodes of the grid network. A discrete approximation approach would most likely be applied to a limited number of common variables. In an experimental application of notions similar to these in Part IV to a tilt-stabilization platform, temperature and input voltage were considered as common variables [Ref. 24]. A discrete approximation approach was used of the region of temperature and input voltage, where testing was conducted at each node to obtain input information for a bound-crossing failure mode.

Any realistic application of the concepts in Part IV would utilize a modern digital computer. It is not felt that numerical computation would be the most limiting factor in realistic applications.

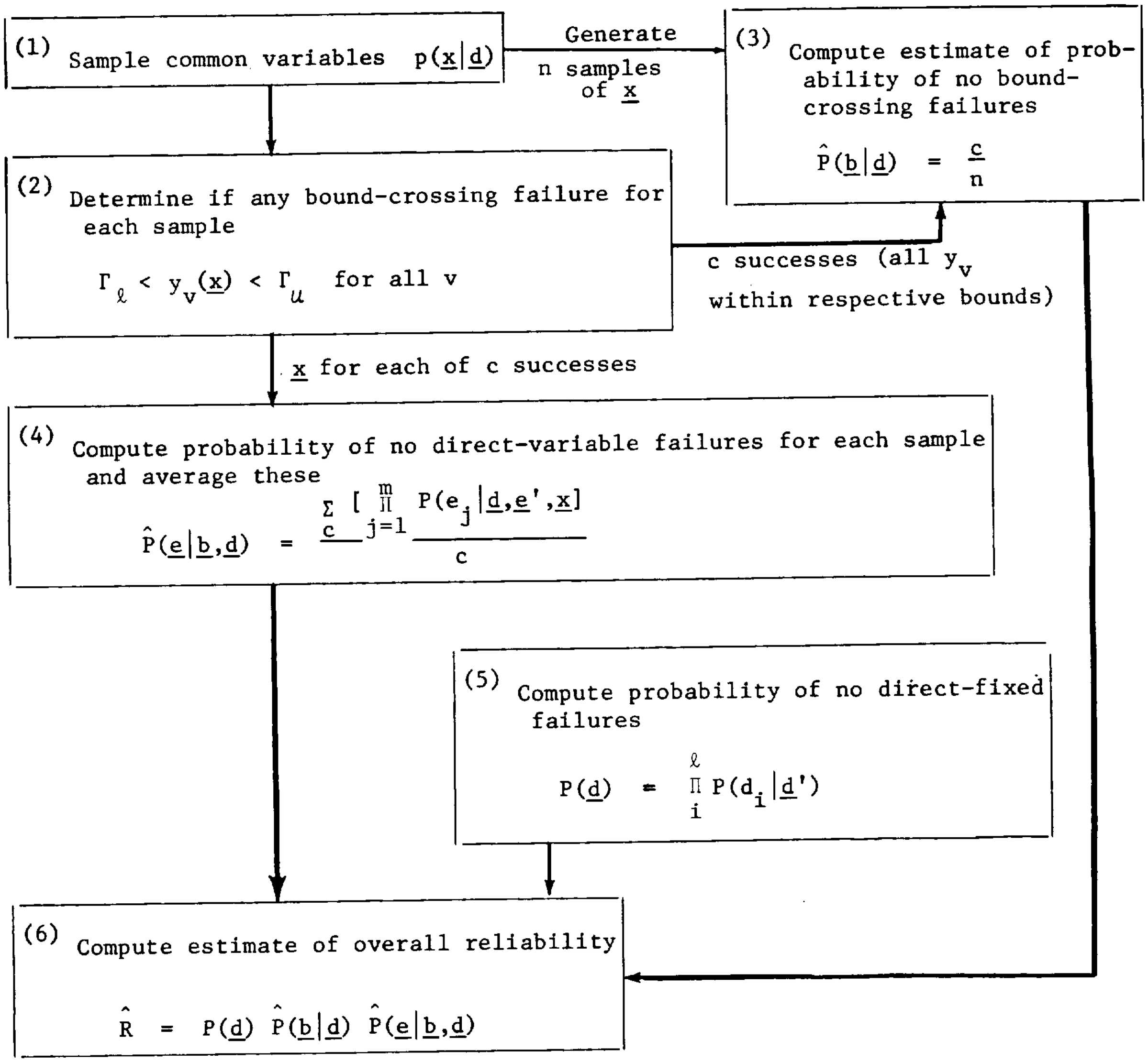


Figure 12-3 Monte Carlo Simulation for Approximate Numerical Calculation

Approximate common variable joint density

$$\int_{\underline{x}} p(\underline{x}|\underline{d})\,d\underline{x} = 1 \approx \sum_{z} P_z(\underline{x}|\underline{d}) = 1$$

where z = cell

↓

Determine if any bound-crossing failure at $\underline{x}$ for each z

$$\Gamma_\ell < y_v(\underline{x})\ \Gamma_u \quad \text{for all } v$$

→ yes, drop ($\underline{x}$) for these z

↓ no, call z'

Compute probability of no bound-crossing and direct-variable modes

$$P(\underline{b},\underline{e}|\underline{d}) = \sum_{z'}\left\{\left[\prod_{j=1}^{m} P(e_j|\underline{d},\underline{e}',\underline{x}_{z'})\right] P_{z'}(\underline{x})\right\}$$

↓

Compute overall reliability by considering direct-fixed modes

$$R = P(\underline{b},\underline{e},\underline{d}) = \left[\prod_{i=1}^{\ell} P(d_i|\underline{d}')\right] P(\underline{b},\underline{e}|\underline{d})$$

Figure 12-4 Approximate Numerical Calculation Using Discrete Approximation of Common Variables

12.3 Additional Considerations

Questions naturally arise concerning extensions to explicitly treating time and to parallel situations. The essence of the approach has been illustrated thus far, and brief comments are given below on these questions. The comments are brief and qualitative because mathematical notation becomes even more complex. Some key features are noted which would be useful to one seriously pursuing the problem. That is, one will have to develop the detail and these features are noted for guidance.

12.3.1 Explicitly Treating Time

As might be expected, explicitly treating time in the series situation model is a reasonably straightforward extension of the approach used in Sec. 12.1. The concept of common variables and of the three types of mode descriptions remains unchanged. Mathematical descriptions of reliability measures as functions of time for the direct modes would be as described in Secs. 4.2 and 4.3 and of time-varying probability density functions for common variables and performance attributes would be as described in Sec. 5.4. Thus, for common variables the time dependence is denoted by $p(\underline{x}; t)$ and for a particular variable $x_s(t) = x_s(\underline{a}; t)$ for all s, where $p(\underline{a})$ is the pdf of the constants, $\underline{a} = (a_1, a_2, \ldots, a_n)$.

Nothing unique exists about the direct-fixed mode. There would be great practical difficulty in obtaining the direct-variable mode reliability descriptions conditional on a time-varying environment

$$P[e_j(t) | \underline{d}, \underline{e}', \underline{u}(t)].$$

This would result from the general situation that a large variety of possible forms of $\underline{u}(t)$ are possible. It is, therefore, difficult to develop tables or other standard information for general use with time-varying environments. A situation which is more practical is where the direct-variable mode reliabilities are functions of time, but the probability densities of the common variables and thus the time varying environments are not functions of time, i.e., conventional failure rate graphs of Ref. 27. The bound-crossing mode will have the unique feature that there could be a specific failure time for each possible value of $\underline{a}$ (where $\underline{a}$ is that noted above for the common variables). This will have the effect of entering into the composite model as truncations on the reliability time-functions of the direct-variable modes. Both monotonic and some non-monotonic performance attribute variations could be treated, as they both become first-crossing problems.

12.3.2 Parallel Models

The situation where the occurrence of certain failure modes does not mean system failure will be referred to as the parallel one. Before proceding to more involved considerations it is pertinent to note that the expected value operation as cited in Sec. 9.1 can be applied to conventional reliability models for redundancy where the environment is a probabilistic variable. This straightforward approach is useful in certain practical problems. Where the complexities of Sec. 12.1 are present, branching modeling concepts could be used. First, the series situation model of 12.1 would be structured where there are no failure modes. For other non-failed system states some of the input information may be different than for the system-state where there are no failure modes. For instance, some common variables could take on values of zero, and performance attribute and environment equations could change. System-state change sequences would have to be traced, and a detailed reliability prediction model developed for each sequence. Where time is explicitly treated, the time that system-state changes will occur is an explicit variable, and time-wise convolutions could be used in tracing through the detail. Thus, explicit treatment of time in redundancy situations would significantly increase complexity.

13. Concluding Remarks for Part IV

Each number below refers to the corresponding question noted in the introduction to Part IV. Replies to these questions are intended to serve as concluding remarks to Part IV. The replies take into consideration mainly the material presented in Part IV.

(1) The first question concerns the conventional assumption of probabilistic independence in reliability models. The probabilistic dependence treated in Part IV results from various sources. Consider first the series model: (a) the most straightforward dependence occurs in the given conditional probabilities in the direct-fixed and the direct-variable modes. (b) the common variables can bring about dependence among direct-variable modes in addition to that noted in (a), dependence among bound-crossing modes, and dependence between direct-variable and bound-crossing modes.

Also to be noted here is to avoid confusing probabilistic dependence with error in structuring the problem. Where there is a deterministic relation $y = y(\underline{x})$ between a performance attribute and common variables, and if some of the common variables were forced to be treated as modes in that some judgment-based bound was put on each of these common variables and the functional relationship ignored, then simply the erroneous reliability prediction would be obtained. Additional dependence is introduced when a general parallel model is developed as features of the series model may be conditional on the system state.

(2) A key feature in structuring the composite reliability prediction models of Sec. 12 is recognition of the distinction between failure modes, common variables, performance attributes, and environments. This distinction is of the sort which tends to be obvious in hind-sight but was not beforehand. A distraction seems to be a tendency to want to treat separately different real-world features which are really mathematically similar in the sense of structuring the composite reliability prediction model. For example, for electrical equipment there is a tendency to separate the internal part characteristics from the interface characteristics.

(3) The typical deterministic equations of engineering have been divided into two categories, those concerned with performance attributes and those concerned with environments. The performance attribute equations are used for the bound-crossing mode. A performance attribute may be either an output of the system or it may be some internal performance of the system. The latter is not of interest to the system user, but there may be certain bounds within which the internal performance must remain or else the output(s) of the system, which are of interest to the user, will be affected. The environment equations are used in the direct-variable mode.

4) The discussion in Sec. 10 illustrated that there is no 1-to-1 correspondence between classification of a failure at a point of repair (source) into catastrophic or degradation and a similar two-way classification of the manner in which the system performance is affected. The models which are developed in Sec. 12 are based on a classification system concerning the mathematical manner by which an individual failure mode is described, and do not emphasize the classification system of catastrophic and degradation failures.

(5) The composite models developed and presented in Sec. 12 show how the various features are brought together. These composite models do not resemble the more familiar prediction models which are widely used. Bringing the various failure mode types and common variables together for even a simple series situation is shown to be complex. Note also that the composite model includes many of the single-item reliability measures of Part II and the conventional reliability models of Part III.

(6) The complexity of the composite reliability models in Sec. 12 and the general lack of necessary input data combine to support the current practices of using simpler models. It is possible that there may be problems where some features of the composite model would offer some return which would be worth the effort. Certain relatively simple systems which have high safety implications might warrant more complex analyses. An example might be relatively simple devices concerned with explosives, such as detonation circuits. The need for high reliability might justify the efforts necessary to develop the appropriate data. Another possible application area would be at a systems level with regard to redundancy. This would be as discussed in Sec. 12.3.2 concerning the use of the expected value operation for redundancy where some features of the composite model are dropped. In some situations there may be some value in using features of the composite model in efforts to achieve balance in design with regard to efforts to reduce various failure modes. In such cases little emphasis would be given to the absolute numerical value of the reliability prediction number, but rather the values would be compared for different design approaches. Also note that in a real-world problem, it may be that only a small number of the variables present in the problem will require treatment in the depth implied by the detailed model.

Generally speaking, experimental applications and further investigation are necessary in order to determine if more complex reliability prediction models along these lines have anything to offer in a practical sense. Of course, before such investigations can be attempted it is necessary that an approach to structuring the problem be developed, and this necessary first step was an objective of the investigations reported in Part IV.

Motivation for pursuing detailed reliability prediction models includes maintainability objectives as well as reliability ones. A detailed reliability model might eventually be useful for maintainability improvements concerning automated predictive maintenance, test point selection, and repair procedure development. These potential maintainability uses would generally require models for parallel configurations.

APPENDIX

Mathematics of Prediction - Probability

The calculus of probability plays an important part in the prediction of system reliability. The basic definitions and distribution theory (discrete and continuous-univariate and multivariate) play a basic role. The Boolean algebra and the calculus of probabilities provide the appropriate analytical tools for manipulating these probabilistic inputs. In the calculation of a probability of system behavior there is little choice in the simplifying approaches that can be taken except to use some of the reliability bounds and approximations. Even to use these techniques requires a formal introduction to the basic methods and a thorough understanding of the assumptions implied in their use.

In order to make the written material as brief as possible summary tables have been prepared to cover specific topics such as continuous variables, Boolean algebra, calculus of probabilities, etc. Supporting each of these tables are cited references, discussions and examples demonstrating the techniques in the corresponding table. Appendix references are contained along with all other references in the single reference section.

A.1 Continuous Random Variables and Distributions

A continuous random variable is typically one that can take on any value in an interval. For example, the lifetime of a transistor under a certain set of test conditions could be any time greater than zero. For a very large population of transistors one would expect the lives to be scattered or distributed over a large interval of time. Continuous distribution functions are used to describe such statistical behavior. Table A.1-1 summarizes basic concepts concerning continuous random variables and their distributions. A summary of several common distributions is presented in Table A.1-2. Given a density function $p(x)$ the characteristics can be evaluated by application of the formulae in Table A.1-1.

Central Limit Theorem

One of the most important results in statistics is the central limit theorem (CLT) which states that if $x_1, x_2, \ldots, x_n$ are independent random variables all having the same distribution function $F(x)$ with mean μ and standard deviation σ, then the sum

$$s = \sum_{i=1}^{n} x_i$$

is asymptotically Normally distributed with mean $n\mu$ and standard deviation $\sigma\sqrt{n}$, i.e.,

$$P(s < s_0) = \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{n}} \int_{-\infty}^{s_0} \exp\{\frac{-1}{2\sigma^2 n}(s - n\mu)^2\}ds$$

for n sufficiently large. This result is true under very general conditions on $F(x)$; if all variables have the same distribution then it is sufficient that the second moment of x be finite. A more general form of the CLT and additional discussion of the above case appear in Ref. 52.. An important aspect of the theorem is how large n must be before the normal approximation applies. Clearly this dependence on n is conditioned by the shape of the distribution. Sums of variables having highly skewed distributions would tend to Normality more slowly than for those having symmetrical or more nearly Normal distributions. In the latter case sums of variables with n larger than 25 or 30 are very closely approximated by the Normal distribution.

Table A.1-1

Continuous Random Variables And Distributions

1. x is a random variable (r.v.) having density function p(x) and (cumulative) distribution function F(x).*

2. $F(x) = \int_{-\infty}^{x} p(t)dt$ (t is a dummy variable) and

$$p(x) = \frac{dF(x)}{dx} .$$

3. Property: $F(-\infty) = 0$; $F(\infty) = 1$.

4. Specifically for the range R over which x is defined

$$\int_R p(x)dx = 1.$$

5. Probability:
$$P(x \leq a) = \int_{-\infty}^{a} p(x)dx = F(a)$$

$$P(a \leq x \leq b) = \int_{a}^{b} p(x)dx = F(b) - F(a).$$

6. Expectation: For any function g(x),

$$E[g(x)] = \int_R g(x)\ p(x)dx.$$

7. Mean of x (first moment about the origin):

$$E(x) = \int_R x\, p(x)dx = \nu_1.$$

8. Mean square of x (second moment about the origin):

$$E(x^2) = \int_R x^2 p(x)dx = \nu_2.$$

9. k-th moment of x with respect to the origin:

$$E(x^k) = \int_R x^k p(x)dx = \nu_k.$$

---

*In precise mathematical notation, X is used to denote a random variable, then $F(x) = P(X \leq x)$, and for a continuous variable $p(x)dx = P(x \leq X \leq x+dx)$.

10. Variance of x (second moment about the mean):

$$E\{[x-E(x)]^2\} = \sigma^2(x) = \int_R [x-E(x)]^2 p(x)dx = \mu_2.$$

11. k-th moment of x about the mean:

$$E\{[x-E(x)]^k\} = \int_R [x-E(x)]^k p(x)dx = \mu_k.$$

12. Relationship between the first four moments:

$$\mu_0 = \nu_0 = 1$$

$$\mu_1 = 0$$

$$\mu_2 = \nu_2 - \nu_1^2, \; \nu_1 = \text{mean value of } x$$

$$\mu_3 = \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3$$

$$\mu_4 = \nu_4 - 4\nu_3\nu_1 + 6\nu_2\nu_1 - 3\nu_1^4.$$

13. Truncated distribution, $F_T(x)$, of $F(x)$:

$$F_T(x) = \begin{cases} F(x)/F(T) & x \leq T \\ 1 & x > T. \end{cases}$$

<u>Example</u>

Let x be a random variable with density function

$$p(x) = \lambda e^{-\lambda x}, \qquad \lambda > 0, \; x \geq 0.$$

This is the well-known Weibull density function with $\theta = 1/\lambda$ and $k = 1$ or the negative exponential density function.

Distribution: $$F(x) = \int_0^x \lambda e^{-\lambda t} dt = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}.$$

Probability: $P(1 \leq x \leq 2) = \int_1^2 \lambda e^{-\lambda x} dx = e^{-\lambda} - e^{-2\lambda}$

or

$$F(2) - F(1) = (1-e^{-2\lambda}) - (1-e^{-\lambda}) = e^{-\lambda} - e^{-2\lambda}.$$

Mean: $E(x) = \int_0^\infty \lambda x e^{-\lambda x} dx = \frac{\Gamma(2)}{\lambda} = \frac{1}{\lambda}, \ (\Gamma(k) = (k-1)!).$

Variance: $\sigma^2(x) = \int_0^\infty (x - 1/\lambda)^2 \lambda e^{-\lambda x} dx = \frac{1}{\lambda^2}.$

k-th moment about the origin:

$$\nu_k = \int_0^\infty x^k \lambda e^{-\lambda x} dx = \frac{(k-1)!}{\lambda^k}.$$

Table A.1-2

Continuous Density Functions and Associated Characteristics

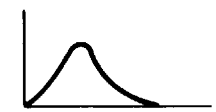
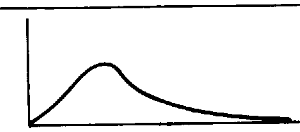
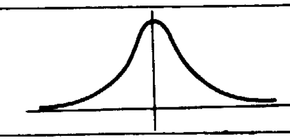
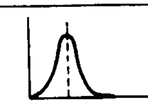
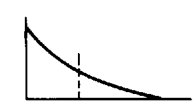
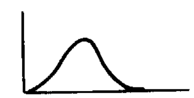
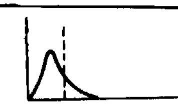
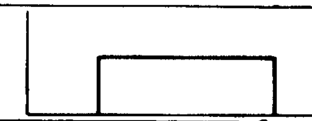
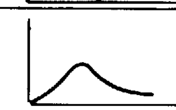
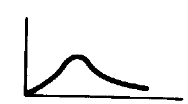
Type Distribution	Density Function f(x)	Mean E(x) and Variance V(x)	Graph of Typical p(x) or Standard Form
Beta (Standard Form)	$p(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \ 0 \leq x \leq 1$ $= 0$ elsewhere $\alpha, \beta > 0$	$E(x) = \frac{\alpha}{\alpha+\beta}$ $V(x) = \frac{\alpha \cdot \beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
F	$p(x) = \left(\frac{m}{n}\right)^{n/2} x^{(m/2)-1} / B\left(\frac{m}{2}, \frac{n}{2}\right)\left[1 + \left(\frac{m}{n}x\right)\right]^{(m+n)/2}, \ 0 \leq x \leq \infty$ $= 0$ elsewhere	$E(x) = \frac{n}{n-2}$ $V(x) = \frac{n^2[(n-2) - m\ (n-4)]}{m(n-2)^2\ (n-4)}$	
t	$p(x) = \left(\frac{1}{n}\right)^{1/2} \frac{1}{B\left(\frac{1}{2}, \frac{n}{2}\right)} \cdot \frac{1}{(1+\frac{1}{n}x^2)^{(n+1)/2}}, -\infty \leq x \leq \infty$	$E(x) = 0$ $V(x) = \frac{n}{n-2}$	
Standard Normal (or Gaussian)	$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-(x-\mu)^2/\sigma^2], \ -\infty \leq x \leq \infty$	$E(x) = \mu$ $V(x) = \sigma^2$	
Exponential	$p(x) = \lambda e^{-\lambda x}, \ x \geq 0, \ \lambda > 0$ or $p(x) = \frac{1}{\theta}e^{-x/\theta}, \ x \geq 0, \ \theta > 0$ where $\theta = 1/\lambda.$	$E(x) = 1/\lambda = \theta$ $V(x) = 1/\lambda^2 = \theta^2$	

Table A.1-2 (Continued)

Type Distribution	Density Function f(x)	E(x), V(x)	Graph of Typical p(x) or Standard Form
Weibull	$p(x) = \frac{kx^{k-1}}{\theta^k} \exp\left[-\left(\frac{x}{\theta}\right)^k\right], \ x \geq 0$	$E(x) = \theta\Gamma(\frac{1}{k}+1)$ $V(x) = \theta^2\ \Gamma(\frac{2}{k}+1) - \theta^2\ \Gamma^2(\frac{1}{k}+1)$	
Lognormal	$p(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\{-[(\ell nx-\mu)^2/2\sigma^2\}, \ 0 \leq x < \infty$	$E(x) = \exp[\mu+\sigma^2/2]$ $V(x) = \exp[2\mu+\sigma^2]\ (e^{\sigma^2}-1)$	
Uniform	$p(x) = \frac{1}{h}, \ 0 \leq x \leq h$	$E(x) = h/2$ $V(x) = h^2/12$	
Gamma	$p(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \ x \geq 0$	$E(x) = \beta\alpha$ $V(x) = \beta^2\alpha$	
Generalized Gamma	$p(x;\alpha,\beta,\gamma) = \frac{\gamma}{\beta^\alpha\ \Gamma(\alpha/\gamma)} x^{\alpha-1}\ \exp[-(x/\beta)^\gamma],$ $0 \leq x < \infty, \alpha, \beta, \gamma > 0$	$E(x) = \beta\Gamma((\alpha+1)/\gamma)/\Gamma(\alpha/\gamma)$ $V(x) = \beta^2\Gamma((\alpha+1)/\gamma)/\Gamma(\alpha/\gamma)$ $- \beta^2\Gamma^2((\alpha+1)/\gamma)/\Gamma^2(\alpha+\gamma).$	

A.2 Discrete Random Variables and Distributions

A discrete random variable is one that takes on a finite or a countably infinite number of values. For example, a binomial variable takes on two values corresponding to a success or a failure, such as tossing a coin and the occurrence of a head being a success. On the other hand, the number of telephone calls on a given line for a specified time may be approximated by a Poisson variable for time intervals of "constant density". The number of calls might be considered to take on any one of a countably infinite number of values, 0, 1, 2, ..., etc.

Table A.2-1 summarizes the definitions and notation for the characteristics of distributions of discrete random variables. Table A.2-2 contains some of the common discrete distributions and the means and the variances. Ref. 53 contains a complete discussion of many discrete random variables and the pertinent characteristics.

<u>Example</u>

Suppose that it is desired to obtain the probability of three or fewer failures in a time interval of length t where an item upon failure is replaced by a new item. Suppose further that the exponential failure time distribution is applicable. Let the failure rate be $\lambda = 0.01$/hour and the time be 200 hours.

From the above information the mean or expected number of failures is 2 items. Furthermore the probability of x failures is given by the Poisson formula and thus for three or fewer failures the probability is expressed as

$$P(x \leq 3) = \frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-2}2^2}{2!} + \frac{e^{-2}2^3}{3!}$$

$$= 0.8569.$$

Table A.2-1

Basic Definitions Concerning Discrete Distributions

1.	Random variable	$x_i,\ i = 0, 1, 2, \ldots, \infty\ (x_i < x_j,\ i < j)$	(may also be finite)
2.	Probability density function	$p(x_i),\ i = 0, 1, \ldots, \infty$	
3.	Cumulative distribution function	$P(x_i) = \sum_{j=0}^{i} p(x_j)$	
4.	Expected value (mean) of x	$E[x] = \sum_{i=0}^{\infty} x_i\, p(x_i)$	
		$= \sum_{i=0}^{n} x_i\, p(x_i)$	if finite number of possibilities. In the following results the index of summation is omitted and the summation over all possible values (finite or infinite) is implied.
5.	Variance of x	$\sigma^2[x] = E[x^2] - E^2[x]$	
		$= \Sigma x_i^2\, p(x_i) - [\Sigma x_i\, p(x_i)]^2$	
	or	$\sigma^2[x] = E[x(x-1)] - E^2[x] + E[x]$	using factorial moments for an integral valued discrete variable where $E[x(x-1)] = \Sigma x_i(x_i-1)\, p(x_i)$
6.	Expected value of g(x)	$E[g(x)] = \Sigma g(x_i)\, p(x_i)$	
7.	k-th moment about the origin	$\nu_k = E(x^k) = \Sigma x_i^k\, p(x_i)$	
8.	k-th moment about the mean	$\mu_k = E[(x-\nu_1)^k]$	
		$= \Sigma (x_i-\nu_1)^k\, p(x_i)$	

Table A.2-2

Discrete Frequency Functions, Means and Variances

	Type Distribution	P(x occurrences)	Mean	Variance
1.	Binomial, $B(x; n, p)$	$\binom{n}{p} p^x (1-p)^{n-x}$, $0 \le x \le n$.	np	npq
2.	Poisson, $P(x; \mu)$	$e^{-\mu}\mu^x/x!$, $x \ge 0$.	μ	μ
3.	Hypergeometric, $H(x; n, N_1, N_2)$	$\binom{N_1}{x}\binom{N_2}{n-x} \Big/ \binom{N_1+N_2}{n}$, $0 \le x \le N_1$.	$N_1 n/(N_1+N_2)$	$\frac{N_1 N_2 \; n(N_1+N_2-n)}{(N_1+N_2)^2 \; (N_1+N_2-1)}$
4.	Geometric, $G(x; p)$	$(1-p)^{x-1} p$, $x \ge 1$.	$1/p$	$\frac{1-p}{p^2}$
5.	Negative Binomial, $NB(x; p, r)$	$\binom{x-1}{r-1} p^r (1-p)^{x-r}$, $x \ge r$.	r/p	$\frac{r(1-p)}{p^2}$

A.3 Multivariate Distributions (Emphasis on Bivariate Case)

Consider the situation in which two or more measurements on a part are being obtained, e.g. the equivalent h-parameters of a transistor. These two measurements would have a joint probability density function (pdf) p(x, y), say, where x and y denote the respective measurements. If the two variables are statistically independent then

$$p(x, y) = p_1(x)\, p_2(y),$$

and hence the joint density functions can be written down knowing the individual pdf's. If the variables are not independent the multivariate density function can be obtained by assuming a particular form such as the Normal density function and estimating the unknown parameters from available data.

Most of the properties of bivariate (two-variate) distributions are straightforward generalizations of the univariate distributions given earlier. The new concepts are those of conditional and marginal distributions, covariance and correlation. The generalization of these results to multivariate distributions is easily made and one should see Ref. 51 for these results.

Independent Random Variables. If two variables x and y are independent then the covariance of x and y, denoted by Cov(x, y) is

$$\mathrm{Cov}(x, y) = \iint (x-E(x))\, p_1(x)(y-E(y))\, p_2(y)\,dx\,dy = 0.$$

However the inverse is not true, i.e. two variables may have zero covariance (or zero correlation i.e. $\rho(x, y) = 0$) but not be independent. For example, suppose that u and v are independent variables, and let $x = u + v$, $y = u - v$. Then

$$E(xy) = E(u^2) - E(v^2) = 0,\ E(y) = 0, \text{ and}$$

$$\mathrm{Cov}(x, y) = 0 \text{ and } \rho(x, y) = 0.$$

However, x and y are dependent. See Ref. 17 for additional examples. Thus the correlation is not a general measure of dependence but rather a measure of linear dependence of two variables in physical terms; the correlation coefficient is a dimensionless covariance.

Table A.3-1

Bivariate Distributions

1. Let x, y be a pair of random variables having the joint distribution function $F(x, y)$ and density function $p(x, y)$.

2. $\iint_R p(x, y)\, dxdy = 1$, where R is the region over which x and y are defined.

3. $\int_{-\infty}^{y} \int_{-\infty}^{x} p(u, v) dudv = F(x, y)$.

4. $F(-\infty, -\infty) = 0$, $F(\infty, \infty) = 1$.

5. $p(x, y) = \frac{\partial F(x, y)}{\partial x \partial y}$.

6. $$P(a < x < b, c \leq y \leq d) = \int_c^d \int_a^b p(x, y) dx\, dy$$

$$= F(b, d) + F(a, c) - F(a, d) - F(b, c).$$

7. $E(g(x, y)) = \iint_R g(x, y)\, p(x, y) dxdy$.

8. $E(x) = \iint_R x p(x, y) dxdy$.

9. If x and y are independent random variables (r.v.'s) then

$$p(x, y) = p_1(x)\, p_2(y) \text{ and}$$

$$E(x) = \int x\, p_1(x) dx \text{ and } E(y) = \int y\, p_2(y) dy.$$

10. $E(xy) = \iint_R xy\, p_1(x)\, p_2(y) dxdy$

$= E(x)\, E(y)$ if x and y are independent r.v.'s.

11. $E(x - E(x))^2 = \sigma^2(x)$, $E(y - E(y))^2 = \sigma^2(y)$.

12. $E\{(x - E(x))(y - E(y))\} = Cov(x, y) = \text{Covariance of } x \text{ and } y$

$$= \iint [x - E(x)][y - E(y)]\, p(x, y)dxdy.$$

13. Correlation of x and y $= \rho\{x, y\} = Cov(x, y)/\sigma(x)\, \sigma(y)$
where

$$\sigma(x) = [\sigma^2(x)]^{1/2} \text{ and } \sigma(y) = [\sigma^2(y)]^{1/2}.$$

14. Marginal distribution of x is given by

$$p_1(x) = \int_{R_y} p(x, y)dy.$$

15. The conditional distribution of y for given x is given by

$$p(y|x) = \frac{p(x, y)}{p_1(x)}$$

$$= p_2(y) \text{ if x and y are independent r.v.'s.}$$

Example

Let x and y have a bivariate density function

$$p(x, y) = \frac{1}{2\pi\sqrt{1-c^2}} \exp\{- \frac{1}{2(1-c^2)} (x^2 - 2cxy + y^2)\}.$$

First of all note that

$$\iint_R p(x, y)dxdy = 1$$

since by completion of the square of the exponent

$$p(x, y) = \frac{1}{2\pi\sqrt{1-c^2}} \iint \exp\{- \frac{1}{2(1-c^2)} (x^2 - 2cxy + c^2y^2)$$

$$+ \frac{(c^2 - 1)}{2(1-c^2)} y^2\}dxdy.$$

If the variables are transformed as follows:

$$u = (x - cy)/\sqrt{1-c^2}$$

$$v = y$$

then

$$p(u, v) = \frac{1}{2\pi} \iint \exp\{-(\frac{u^2}{2} + \frac{v^2}{2})\} du\, dv, \tag{A.3-1}$$

using the fact that the Jacobian of the transformation is given by

$$1/ \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 1/ \begin{vmatrix} 1/\sqrt{1-c^2} & -c/\sqrt{1-c^2} \\ 0 & 1 \end{vmatrix} = \sqrt{1-c^2} .$$

(A-3.1) can be written as the product of the integrals

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du \bullet \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} dv .$$

Since each is the integral of the standard Normal density function the above product is unity.

Next the marginal distribution of y is given by

$$\begin{aligned} p_2(y) &= \int p(x, y) dx \\ &= \frac{1}{\sqrt{2\pi}} \exp\{ - \frac{y^2}{2} \} \quad . \end{aligned}$$

Hence the conditional distribution of x given y is

$$p(x|y) = \frac{p(x, y)}{p(y)} = \frac{1}{\sqrt{2\pi(1-c^2)}} \exp\{- \frac{1}{2(1-c^2)} (x - cy)^2\}.$$

Mean, Variance and Covariance Formulas

Let $x_1, x_2, \ldots, x_n$ be n random variables with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ respectively and correlations ρ_{12} $(= \rho(x_1, x_2))$, $\rho_{13}, \ldots, \rho_{n-1, n}$. The following results are true independent of the distributions of the variables. Let y be a linear combination of the variables given by

$$y = c_0 + c_1x_1 + c_2x_2 + \ldots + c_nx_n.$$

Then the mean and variance of y are denoted by μ_y and σ_y^2 and are given by

$$\mu_y = c_0 + c_1\mu_1 + c_2\mu_2 + \dots + c_n\mu_n = c_0 + \sum_{i=1}^{n} c_i\mu_i$$

$$\sigma_y^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \dots + c_n^2\sigma_n^2$$

$$+ 2\rho_{12}c_1c_2\sigma_1\sigma_2 + \dots + 2\rho_{n-1,n}c_{n-1}c_n\sigma_{n-1}\sigma_n$$

or

$$\sigma_y^2 = \sum_{i=1}^{n} c_i^2\sigma_i^2 + 2\sum_{i<j}\sum c_ic_j\rho_{ij}\sigma_i\sigma_j .$$

where σ_i is the standard deviation of the i-th variable. The above formulas are true in general and one notes that the mean μ_y of y does not involve the correlations.

Now if the variables are uncorrelated (if they are independent as indicated previously) the formula for the variance reduces to

$$\sigma_y^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \dots + c_n^2\sigma_n^2 .$$

Now consider two functions

$$y = c_0 + c_1x_1 + \dots + c_nx_n$$

$$w = \ell_0 + \ell_1x_1 + \dots + \ell_nx_n ,$$

then the covariance of y and w is given by

$$\mathrm{Cov}\{y, w\} = c_1\ell_1\sigma_1^2 + \dots + c_n\ell_n\sigma_n^2 + \sum_{j=1}^{n}\sum_{i=1}^{n} \ell_jc_i\sigma_j\sigma_i\rho_{ij} .$$

If the functions are not linear it is often possible to use a Taylor series expansion of the function $f(\underline{x})$ and then apply the mean and variance computations to this form. These formulas must be used with care, e.g. by checking the magnitude of the errors which may result in using them. Thus if

$$y = f(\underline{x})$$

then

$$y \simeq f(\underline{\mu}) + \sum \frac{\partial f}{\partial x_i}\bigg|_{\underline{\mu}} \Delta x_i + \frac{1}{2}\sum \frac{\partial^2 f}{\partial x_i^2}\bigg|_{\underline{\mu}} \Delta x_i^2$$

$$+ \frac{1}{2} \sum_{i \neq j} \sum \frac{\partial^2 f}{\partial x_i \partial x_j} \bigg|_{\underline{\mu}} \Delta x_i \Delta x_j , \quad \Delta x_i \;=\; x_i - \mu_i ,$$

and hence using only the first order terms

$$\mu_y \;\approx\; f(\underline{\mu})$$

$$\sigma_y^2 \;\approx\; \sum \left(\frac{\partial f}{\partial x_i} \bigg|_{\underline{\mu}} \right)^2 \sigma_i^2 ,$$

where

$$\underline{\mu} \;=\; (\mu_1, \mu_2, \ldots, \mu_n),$$

and where $\frac{\partial f}{\partial x_i}\Big|_{\underline{\mu}}$ denotes the evaluation of the derivative at $\underline{\mu}$.

The above results are summarized in the following table.

Table A.3-2

Mean, Variance, and Covariance Formulas

General Case for Single Function.

If $y \;=\; c_0 + c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$,

then $\mu_y \;=\; c_0 + c_1 \mu_1 + c_2 \mu_2 + \ldots + c_n \mu_n \;=\; c_0 + \sum_1 c_i \mu_i$

and $\sigma_y^2 \;=\; \sum_i c_i^2 \sigma_i^2 + \sum_{\substack{i \\ i \neq j}} \sum_j c_i c_j \sigma_i \sigma_j \rho_{ij}$.

Variables Uncorrelated.

$$\mu_y \;=\; c_0 + c_1 \mu_1 + \ldots + c_n \mu_n \;=\; c_0 + \sum_1^n c_i \mu_i$$

$$\sigma_y^2 \;=\; \sum_1^n c_i^2 \sigma_i^2 .$$

General Case for two functions.

If $y = c_0 + \sum c_i x_i$

and $w = \ell_0 + \sum \ell_j x_j$

then

$$\mathrm{Cov}(y,w) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i \ell_j \sigma_i \sigma_j \rho_{ij}.$$

If x_i and x_j are uncorrelated, i.e. $\rho_{ij} = 0$ for $i \neq j$, then

$$\mathrm{Cov}(y, w) = \sum c_i \ell_i \sigma_i^2.$$

General Case for single nonlinear function.

If $y = y(\underline{x})$, $\underline{x} = (x_1, \ldots, x_n)$

then using only a first order approximation

$$\mu_y \simeq y(\underline{\mu}), \ \underline{\mu} = (\mu_1, \ldots, \mu_n), \text{ vector of means,}$$

and

$$\sigma_y^2 \simeq \sum \left(\left. \frac{\partial y}{\partial x_i} \right|_{\underline{\mu}} \right)^2 \sigma_i^2.$$

A.4 Calculus of Probabilities

Starting with certain definitions and axioms several useful theorems of the calculus of probabilities can be derived. Eight such theorems are given in Table A.4-1 A major difficulty in reliability literature stems from the notion of statistical independence, often called just independence. This notion is basic to many reliability calculations, for it is often assumed in these calculations that the failure of one item in a system is independent of the failure of all other items in the system. In non-probability language, two events are said to be independent if knowledge of the outcome of one in no way affects the outcome of the other event. The simplest example of independent events is perhaps two tosses of a coin - the result of the first toss in no way affects the outcome of the second toss, so the events are independent. As a more pertinent example, suppose two amplifiers are selected at random from a collection of 100 amplifiers from a production process. Does knowledge concerning the value of current gain for the first amplifier alter in any way the probability that the current gain for the second amplifier falls in any given interval? If the answer is that it does not, then the two observations of current gain are independent. In applications these results would usually be treated as independent because if the current gain distribution were $F(x)$, the observation of an x_1 for the first amplifier would not aid in locating the value x_2 for the second one as it presumably could fall anywhere on the defined region R for x with the same probability distribution $F(x)$ as that for the first observation.

Consider as another example the measuring of the current gain y_c and the voltage gain y_v of a single amplifier. Does knowledge of the value of current gain alter information concerning the voltage gain? Chances are that it would because high values of y_c may correspond to higher than average (or lower than average) values of y_v and vice versa. Thus it is normally assumed that such variables may be dependent unless data analyses imply otherwise.

Similar examples can be considered in the reliability prediction area. If the event of failure of one of two items in parallel in no way affects the failure behavior of the other item the two events are independent. On the other hand if failure of one would alter the probability distribution of failure time of the second item the two events of failure are dependent.

In probabilistic terms the above discussion can be summarized as follows. A and B are independent if

$$P(B|A) = P(B)$$

and hence

$$P(AB) = P(B|A)\,P(A) = P(B)\,P(A),$$

where $P(B|A)$ is read "the probability of the event B given the event A has occurred."

Another consideration with respect to independence is that of statistical independence and conditional independence which has been denoted as physical independence in Ref. 54.

Two events A and B are said to be conditionally (physically) independent if and only if they are statistically independent under environment E_i, that is,

$$P(XY|E_i) = P(X|E_i)\, P(Y|E_i)$$

Physical (conditional) independence does not necessarily imply statistical independence of the unconditional events X and Y. In order to compute the reliability of a system one usually obtains the conditional probabilities (that is, given the environments) and then obtains the weighted average of these conditional probabilities using the $P(E_i)$ as the weights. In mathematical terms

$$P(XY) = \sum_i P(XY|E_i)\, P(E_i),$$

or

$$P(XY) = \sum_i P(X|E_i)\, P(Y|E_i)\, P(E_i).$$

Frequently in reliability prediction the mission is subdivided into phases in each of which the environment is essentially constant throughout the entire phase. Hence one uses a formula such as the above. A more complete discussion of the concepts of physical and statistical independence appears in Ref. 48 and Ref. 54.

Table A. 4-1

Calculus of Probabilities

Definitions and Axioms

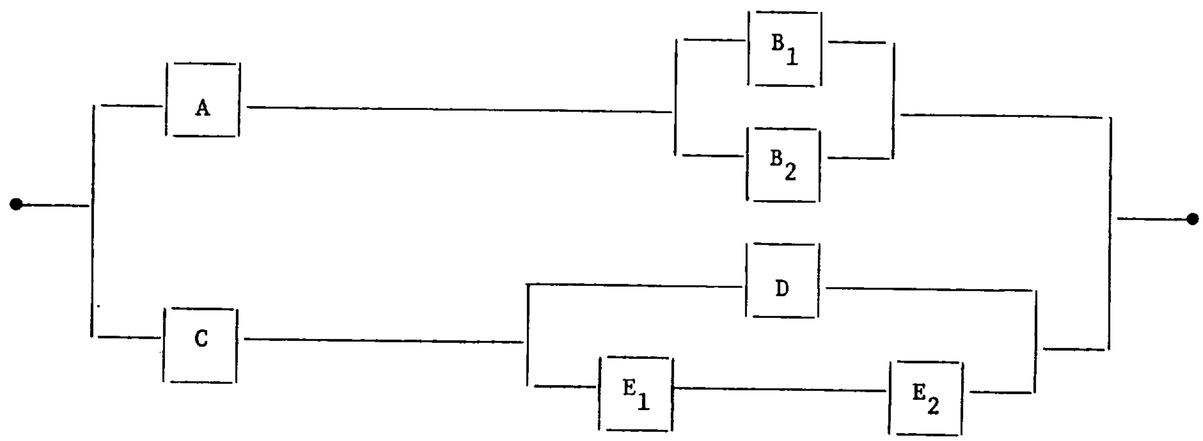
1. $P(A) = \text{Probability of } A$
2. $0 \leq P(A) \leq 1$
3. $P(\text{Sure Event}) = P(I) = 1$
4. $P(\text{An Impossible Event}) = P(\phi) = 0$
5. $P(A+B) = P(A) + P(B) - P(AB)$
6. $P(B|A) = $ Probability of B on the hypothesis that A has occurred
7. $P(AB) = P(A)\,P(B|A)$

Theorems

1. $P(A+\overline{A}) = P(A) + P(\overline{A}) = 1$
2. $P(A) = P(AB) + P(A\overline{B})$
3. $P(A+B) = 1 - P(\overline{AB})$
4. $$P(A_1+A_2+\cdots+A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) - P(A_1A_2) - P(A_1A_3) - \cdots - P(A_{n-1}A_n) + P(A_1A_2A_3) + \cdots + (-1)^{n-1} P(A_1A_2 \cdots A_n)$$
5. $P(A_1A_2 \cdots A_n) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1A_2) \cdots P(A_n|A_1 \cdots A_{n-1})$
6. If $A_1, \cdots, A_n$ are all mutually independent events, then
$$P(A_1A_2 \cdots A_n) = \prod_{i=1}^{n} P(A_i)$$
7. If $A_1, \cdots, A_n$ are pairwise mutually exclusive i.e. $A_iA_j = \phi$ (null set) for all pairs $i, j = 1, \ldots, n,\ i \neq j$, then
$$P(A_1+A_2+\cdots+A_n) = \sum_{i=1}^{n} P(A_i).$$
8. Bayes Rule - Let $B_1, B_2, \cdots$ be a collection of events which are mutually exclusive and exhaustive, i.e. $B_i \cdot B_j = \phi$ for $i \neq j$, and $B_1 + B_2 + \cdots = I$, then
$$P(B_i|A) = \frac{P(B_iA)}{P(A)} = \frac{P(A|B_i)\,P(B_i)}{\Sigma P(A|B_i)\,P(B_i)}.$$

Example

Consider the system reliability logic diagram shown below where the symbols assigned to each element represent the event of success for that element.



Let the associated probabilities be $P(A) = P(B_1) = P(B_2) = 0.95$,

$$P(C) = 0.98,\ P(D) = 0.90,\ P(E_1) = P(E_2) = 0.90.$$

Let P_1 denote the path A, B_1

P_2 A, B_2

P_3 C, D

P_4 C, E_1, E_2.

Then the probability of success is the probability that at least one of the paths $P_1, \ldots, P_4$ is "good", that is,

$$\begin{aligned}
P(S) &= P\{P_1 + P_2 + P_3 + P_4\} \\
&= P\{P_1\} + P\{P_2\} + P\{P_3\} + P\{P_4\} \\
&\quad - P\{P_1P_2\} - P\{P_1P_3\} - P\{P_1P_4\} - P\{P_2P_3\} \\
&\quad - P\{P_3P_4\} - P\{P_2P_4\} \\
&\quad + P\{P_1P_2P_3\} + P\{P_1P_2P_4\} + P\{P_1P_3P_4\} + P\{P_2P_3P_4\} \\
&\quad - P\{P_1P_2P_3P_4\}.
\end{aligned}$$

Now $P\{P_1\} = P\{AB_1\} = P\{A\}\, P\{B_1\}$ assuming independence of the events A and B_1, hence

$$P\{P_1\} = (0.95)(0.95) = 0.9025$$

Similarly the remaining probabilities are obtained.

$$P\{P_2\} = 0.9025$$

$$P\{P_3\} = 0.8820$$

$$P\{P_4\} = 0.7938$$

$$P\{P_1P_2\} = P\{AB_1AB_2\} = P\{AB_1B_2\} = 0.8574$$

$$P\{P_1P_3\} = P\{AB_1CD\} = 0.7960$$

$$P\{P_2P_3\} = P\{AB_2CD\} = 0.7960$$

$$P\{P_2P_4\} = P\{AB_1CE_1E_2\} = 0.7164$$

$$P\{P_1P_4\} = P\{AB_1CE_1E_2\} = 0.7164$$

$$P\{P_3P_4\} = P\{CDCE_1E_2\} = 0.7144$$

$$P\{P_1P_2P_3\} = P\{AB_1AB_2CD\} = P\{AB_1B_2CD\} = 0.7562$$

$$P\{P_1P_2P_4\} = P\{AB_1AB_2CE_1E_2\} = P\{AB_1B_2CE_1E_2\} = 0.6806$$

$$P\{P_1P_3P_4\} = P\{AB_1CDCE_1E_2\} = P\{AB_1CDE_1E_2\} = 0.6448$$

$$P\{P_2P_3P_4\} = P\{AB_2CDCE_1E_2\} = P\{AB_2CDE_1E_2\} = 0.6448$$

$$P\{P_1P_2P_3P_4\} = 0.6125$$

Hence

$$P\{S\} = 0.9981.$$

It should be noted that a particular kind of system redundancy is implied for the success probability to be computed in the above way. Specifically, independence is required of all failure events, which usually implies active redundancy in the system. A computer program for performing a reliability prediction such as that above is described in Vol. II - Computation.

A.5 Boolean Algebra

In predicting the reliability of a system we are concerned with various events, such as a performance measure lies between two given values, no failures in an interval of time t, less than three defects in a sample of ten items from a lot of material, etc. Such events will be devoted by capital letters A, B, C, etc. An event is the result of an experiment and can be considered to be a collection of possible outcomes of an experiment within the space of all possible outcomes. For example, the selection of three or more good items from a lot of five items is an event. The space of all possible outcomes contains $2^5 = 32$ points in the sample space, correspond to all items bad, only one item good (5 points), two items good (10 points), etc., ..., five items good (1 point). The event of three or more good items corresponds to 16 of these 32 sample points. The basic definitions, operations, and properties of Boolean Algebra are summarized in Table A. 5-1. This summary covers only those introductory topics included in the first few chapters of a text on the subject. Ref. 37 gives a complete discussion of Boolean Algebra and its applications.

<u>Example</u>

Simplify the following expression:

$$[(\overline{\bar{A}\bar{B}}) + C]\ \overline{(A+\bar{C})}$$

Applying the dualization law,

$$[(A+B) + C]\ (\bar{A}C).$$

The distributive law yields

$$(\bar{A}CA) + (\bar{A}CB) + (\bar{A}CC)$$

or

$$\phi + (\bar{A}CB) + (\bar{A}C)$$

which is

$$(\bar{A}CB) + (\bar{A}C).$$

Since $\bar{A}CB \subset \bar{A}C$, the above is equivalent to $\bar{A}C$.

Table A. 5-1

Boolean Algebra
(Algebra of Classes)

Notation, Definitions, and Logical Operations:

1. A <u>sure event</u>, an event which always occurs when an experiment or observation is made, denoted by I.
2. An <u>impossible event</u>, an event which never occurs as an outcome of an experiment, denoted by ϕ.
3. The <u>complementary event</u> or <u>complement</u> of A, is the event that A does not occur, denoted by $\bar{A}$.
4. The <u>sum</u> or <u>union</u> of A and B, denoted by A + B or $A \cup B$, is the event that at least one of A and B occurs.
5. The <u>product</u> or <u>intersection</u> of A and B, denoted by AB or $A \cap B$ is the event that both A and B occur.
6. If occurrence of B implies the occurrence of A, then $B \subset A$.
7. If AB = ϕ, then A and B are disjoint.

Let F_0 be a family of events which includes I and which is used with respect to the sum and product logical operations. Then events belonging to the <u>field F_0</u> satisfy the following relations:

$$A + AA = AA = A$$
$$A + B = B + A, \quad AB = BA$$
$$(A+B+C) = A + (B+C), \quad (AB)C = A(BC)$$
$$A(B+C) = AB + AC$$
$$A + \bar{A} = I, \quad A\bar{A} = \phi$$
$$A + I = I, \quad AI = A$$
$$A + \phi = A, \quad A\phi = \phi.$$

Dualization Laws:

$$\overline{A + B} = \bar{A}\bar{B}$$
$$\overline{AB} = \bar{A} + \bar{B}.$$

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