Behavior Near a Periodic Orbit of Functional Differential Equations

by

Jack K. Hale

*This research has been supported in part by the United States Air Force under Grant No. AFOSR-693-67, by the National Science Foundation under Grant No. GP-7471, by the United States Army under Grant No. DA-31-124-ARO-D-270 and in part by the National Aeromautics and Space Administration under Grant No. NGR-40-002-015.

Behavior near a periodic orbit of functional differential equations

by

Jack K. Hale

1. Introduction. Let $C = C([-r,0],E^n)$ be the space of continuous functions mapping the interval $[-r,0]$ into an n-dimensional real or complex vector space E^n. For any φ in C, define $\|\varphi\| = \sup_{-r\leq\theta\leq 0} |\varphi(\theta)|$ where $|x|$ is the norm of a vector x in E^n. If x is a continuous function mapping any interval $[a-r, a+A)$ into E^n, $A > 0$ and t is a given element of $[a, a+A)$, define x_t in C by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. If F is a function taking $R \times C$ into E^n, $R = (-\infty,\infty)$, then a functional differential equation is defined by the relation

$$\dot{x}(t) = F(t,x_t) \tag{1}$$

where $\dot{x}(t)$ denotes the right hand derivative with respect to t. A solution of (1) with initial value φ at σ is a continuous function $x = x(\sigma,\varphi)$ defined on $[\sigma-r, \sigma+A)$, $A > 0$ such that $x_\sigma = \varphi$ and x satisfies (1) for t in $[\sigma, \sigma+A)$. It is well known that $F(t,\varphi)$ continuous in t,φ and locally Lipschitzian in φ implies the existence and uniqueness of a solution of (1) through (σ,φ) for any σ,φ. Furthermore, the solution $x(\sigma,\varphi)(t)$ is continuous in (σ,φ,t) in its domain of definition. It will be assumed in the sequel that F is smooth enough to ensure this latter property and, in fact, $F(t,\varphi)$ is continuous in t,φ and has a continuous first derivative with respect to φ.

If $F(t,\varphi) = f(\varphi)$ is independent of t, system (1) is referred to as an autonomous functional differential equation

$$\dot{x}(t) = f(x_t). \tag{2}$$

In this case, it is no restriction to assume that $\sigma = 0$ and write the solution as $x(\varphi)$. An orbit $0^+(\varphi)$ of (2) through φ is defined in C and not in E^n as $0^+(\varphi) = \cup_{t \geqq 0} x_t(\varphi)$. The qualitative theory of autonomous functional differential equations is concerned with the study of the geometrical properties of the orbits of (2). We briefly outline below some of the results that have been obtained along this line for equilibrium points and periodic orbits.

An equilibrium or critical point of (2) is a constant function which satisfies (2). Without any loss in generality, we may assume this equilibrium point is $x = 0$; that is, $f(0) = 0$. The linear variational equation associated with $x = 0$ is

$$\dot{y}(t) = L(y_t) \tag{3}$$

where $L(\varphi)$ is the derivative of f at 0 evaluated at φ. The mapping $L: C \to E^n$ is a continuous linear operator and therefore has the form

$$L(\varphi) = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta) \tag{4}$$

where η is an $n \times n$ matrix with elements of bounded variation. The characteristic equation for (3) is

$$\det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta). \tag{5}$$

The solution $x = 0$ of (2) is called nondegenerate if $\lambda = 0$ is not a solution of (5) and elementary if no solution of (5) has zero real part.

If no solution of (5) has zero real part, then the space C can be decomposed as the direct sum of two linear subspaces, $C = S \oplus U$, which are invariant under the mapping induced by the solutions of (3) such that the solutions of (3) are decaying exponentials on S and growing exponentials on U. The set U is finite dimensional. The sets S and U are referred to as the stable and unstable manifolds, respectively, of (3). The set S may also be characterized as the set of initial values of those solutions of (3) which remain in a neighborhood of zero for all $t \geqq 0$. The set U is characterized in the same manner except for $t \leqq 0$.

The latter characterization of S and U may also be used to define the stable and unstable manifolds $\mathcal{S}$ and $\mathcal{U}$ for the solution $x = 0$ of (2). With this definition, it was shown by Hale and Perello [1] that an elementary critical point p of (2) exhibits in C a saddle point structure in the sense that the stable and unstable manifolds of p are locally homeomorphic to the stable and unstable manifolds of the linear variational equation.

Recently, Oliva [2] has formulated the concept of functional differential equations on a compact manifold. He also has shown that the family of all f which have a continuous bounded first derivative such that system (2) has only elementary (or nondegenerate) critical points is open and dense in the C^1-topology of all f having a continuous bounded first derivative.

If system (2) has a nonconstant ω-periodic solution p, then the linear variational equation relative to p is

$$
(6) \qquad \begin{aligned} \dot{z}(t) &= L(t,z_t) \\ L(t,\varphi) &= \int_{-r}^{0}[d_\theta \eta(t,\theta)]\varphi(\theta) \end{aligned}
$$

where $L(t+\omega,\varphi) = L(t,\varphi)$ is the derivative of f at p evaluated at φ. To the linear equation (6), one can associate a family of continuous linear operators $T(t,\sigma)$, $t \geqq \sigma$, taking C into C by defining

$$
(7) \qquad T(t,\sigma)\varphi = z_t(\sigma,\varphi)
$$

where $z(\sigma,\varphi)$ is the solution of (6) with initial value φ at σ. Stokes [3] has shown that the spectrum of the operator $T(\sigma+\omega,\sigma)$ consists only of point spectrum (except for $\{0\}$), is independent of σ and he justifiably calls the spectrum of this operator the <u>characteristic multipliers</u> of (7).

Since p is a nonconstant ω-periodic solution of (2), the derivative $\dot{p}$ is a nontrivial ω-periodic solution of (6) and therefore one of the characteristic multipliers of (6) is one. If $\dot{p}_\sigma$ is a generator for the generalized eigenspace of the multiplier one of $T(\sigma+\omega,\sigma)$, then $\dot{p}_\alpha$ is a generator for the generalized eigenspace of the multiplier one of $T(\alpha+\omega,\alpha)$ for any α. In such a case, we say the orbit Γ generated by the ω-periodic solution p is *non-degenerate*. If Γ is nondegenerate and also no other multipliers of (6) have modulus one, we say Γ is *elementary*. The theory of non-degenerate and elementary periodic orbits is not complete at the present time although a few results are available. For example, one can easily prove that a nondegenerate ω-periodic orbit Γ of (2) is isolated in the sense that there is a neighborhood V of Γ such that $V\backslash\Gamma$ contains no ω-periodic orbits.

Halanay [4] has proved the following more complicated statement: If the differential difference equation

$$\dot{x}(t) = g(x(t),x(t-r)) \tag{8}$$

has a nondegenerate periodic orbit Γ, then there is an $\epsilon_o > 0$ such that the equation

$$\dot{x}(t) = h(x(t),x(t-r),\epsilon) \tag{9}$$

has a nondegenerate periodic orbit Γ_ϵ for $|\epsilon| \leq \epsilon_o$ and $\Gamma_o = \Gamma$ provided that $h(x,y,\epsilon)$ is continuous in its arguments, continuously differentiable in x and y, and $h(x,y,0) = g(x,y)$.

We have obtained an extension of the result of Halanay to a rather general class of functional differential equations. Suppose

$x\colon R \to E^n$ is a continuous function, $t = (1+\beta)\tau$, $\beta > -1$ and $x(t) = y(\tau)$. Then

$$x_t(\theta) = y(\tau + \frac{\theta}{1+\beta}) \stackrel{\text{def}}{=} y_{\tau,\beta}(\zeta)$$

where $\zeta = \theta/(1+\beta)$ ranges over the interval $[-r_1, 0]$, $r_1 = r/(1+\beta)$. We say a function $f\colon C \to E^n$ is of <u>class (S)</u> is for any <u>continuously differentiable</u> function $x\colon R \to E^n$, the function $f(x_t) = f(y_{\tau,\beta})$ considered as a function of β is continuously differentiable. Any f of the form

$$f(\varphi) = F(\int_{-r}^{0}[d\eta_1(\theta)]g_1(\varphi(\theta)),\ldots,\int_{-r}^{0}[d\eta_k(\theta)]g_k(\varphi(\theta)))$$

where F and the g_j are continuously differentiable vector functions and the η_j are matrices of bounded variation is of class (S).

The following result is true: Suppose $G(\varphi,\epsilon)$ is of class (S), is continuous in φ,ϵ, continuously differentiable in φ and $G(\varphi,0) = f(\varphi)$. If system (2) has a nondegenerate periodic orbit Γ, then there is an $\epsilon_0 > 0$ such that the system

$$\dot{x}(t) = G(x_t,\epsilon) \tag{10}$$

has a nondegenerate periodic orbit Γ_ϵ for $|\epsilon| \leq \epsilon_0$ and $\Gamma_0 = \Gamma$.

The ideas in the proof of the above theorem are as follows. To obtain a periodic solution of (10) which for $\epsilon = 0$ coincides with the

ω-periodic solution of (10), let t by $(1+\beta)\tau$, $x(t) = y(\tau)$, to yield the equation

$$\frac{dy}{d\tau} = G(y_{\tau,\beta},\epsilon). \tag{11}$$

For a fixed ϵ and β sufficiently small, one shows there are a scalar function $B(\epsilon,\beta)$ and an ω-periodic function $p^*(\tau,\epsilon,\beta)$, $p^*(\tau,0,0) = p(\tau)$, satisfying

$$\frac{dy}{d\tau} = G(y_{\tau,\beta},\epsilon) - B(\epsilon,\beta)\dot{p}(\tau).$$

The hypotheses of the theorem imply that $B(0,0) = 0$ and $\partial B(\epsilon,\beta)/\partial\beta \neq 0$ for $\epsilon = 0$, $\beta = 0$. The implicit function theorem implies the existence of a $\beta(\epsilon)$, $\beta(0) = 0$, such that $B(\epsilon,\beta(\epsilon)) = 0$ and, therefore, the existence of a periodic solution of (10).

The discussion of an elementary periodic orbit is much more difficult. To the author's knowledge, the first result dealing with the stability properties of a periodic orbit of (2) is due to Stokes [5]. He proved the following theorem: If Γ is nondegenerate and all characteristic multipliers of (6) except one have modulus less than one, then there is a neighborhood V of Γ such that any solution of (2) with initial value in V must approach Γ exponentially as $t \to \infty$ and with asymptotic phase; that is, there are positive constants K,α such that for any φ in V, there is a constant γ with

$$\|x_{t+\gamma}(\varphi)-p_t\| \leqq Ke^{-\alpha t}, \ t \geqq 0.$$

For the case when some multipliers have modulus greater than one, some further information is also available. Suppose γ is a given constant and consider the linear variational equation for the solutions of (2) relative to $p(t+\gamma)$. There exist two subspaces of C, $S(\gamma)$ and $U(\gamma)$, such that any solution of the linear equation which $\to 0$ as $t \to \infty$ $(t \to -\infty)$ must have its initial value at $t = 0$ on $S(\gamma)$ $(U(\gamma))$. There are sets $S_\delta(\gamma)$, $U_\delta(\gamma)$ associated with the complete variational equation which are locally homeomorphic to $S(\gamma)$ and $U(\gamma)$ such that any solution which remains in a certain δ-neighborhood of zero for $t \geqq 0$ $(t \leqq 0)$ must have its initial value at $t = 0$ on $S_\delta(\gamma)$ $(U_\delta(\gamma))$. A set $\mathcal{S}$ is the stable manifold of a periodic orbit Γ relative to a neighborhood V of Γ if the orbit Γ is positively stable relative to initial values on $\mathcal{S}$ and any solution which remains in V for $t \geqq 0$ must have its initial value on $\mathcal{S}$ and approach Γ as $t \to \infty$. The unstable manifold is defined analogously for $t \leqq 0$. We have been able to prove that the sets $\mathcal{S} = \cup_{0 \leqq \gamma \leqq \omega}(p_\gamma + S_\delta(\gamma))$ and $\mathcal{U} = \cup_{0 \leqq \gamma \leqq \omega}(p_\gamma + U_\delta(\gamma))$ are stable and unstable manifolds, respectively, of Γ. Finally, the set $\mathcal{U}$ is either homeomorphic to $U_o(\delta) \times$ (a circle) or a generalized Mobius band with cross section $U_o(\delta)$. It has not been shown at this time that $\mathcal{S}$ is either homeomorphic to $S_o(\delta) \times$ (a circle) or a generalized Mobius band with cross section $S_o(\delta)$.

References

[1] Hale, J. K. and C. Perello, The neighborhood of a singular point of functional differential equations. Contrib. Diff. Eqs. 3(1964), 351-375.

[2] Oliva, W. M., Functional differential equations on compact manifolds and an approximation theorem. To appear in J. Diff. Eqs.

[3] Stokes, A., A Floquet theory for functional differential equations. Proc. Nat. Acad. Sci. USA, 48(1962), 1330-1334.

[4] Halanay, A., Differential Equations, Academic Press, 1966.

[5] Stokes, A., On the stability of a limit cycle of an autonomous functional differential equation. Contrib. Diff. Eqs. 3(1964), 121-140.