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THEORY OF A POTENTIAL FOR SATELLITE MOTION ABOUT A TRIAXIALLY ELLIPSOIDAL PLANET

by

Stephen J. Madden, Jr.

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ABSTRACT

This report presents a generalization of an analysis due to Vinti which provides an accurate reference orbit for perturbation theories for earth satellites. The general potential function which permits separation of the Hamilton-Jacobi equation in triaxially ellipsoidal coordinates is derived and specialized so that it satisfies the Laplace equation and qualifies as a gravitational potential. The theory of Stäckel systems is then applied to reduce the problem to quadratures.

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SECTION 1

INTRODUCTION

This report presents a generalization of the well-known analysis of Vinti which provides an accurate reference orbit for perturbation theories for the motion of a satellite in the gravitational field of the earth.

The organization of presentation followed here is somewhat unorthodox. Because of the algebraic complexity of some of the results and because some of the methods used are not readily available in the literature, the appendices have assumed an unusual importance. The main body of the report draws heavily on the appendices and cites results obtained therein by use of equation numbers prefaced by letters indicating the specific appendices referred to.

In particular, unless the reader is familiar with ellipsoidal coordinates and the theory of Stäckel systems, it is advisable that he turn first to Appendices A and D for a summary and references.

The idea to be pursued here is rather the reverse of the approach ordinarily used in satellite problems. Here we seek those mechanical systems which can be integrated by quadratures and examine them to find one which may be applicable to the case of satellite motion, rather than write down the equations of satellite motion and seek to solve them. The class of problems considered is restricted to soluble ones. That this is a profitable avenue for exploration has been shown by Vinti.

The characterization of systems will be via the Hamilton-Jacobi method; that is, we start with the Hamilton-Jacobi equation for motion of a particle and place successive sets of restrictions on the coordinates used and the form of the potential function to carry out the objective stated in the previous paragraph. The details of the approach are as follows: We find first the most general set of orthogonal coordinates that allow separation of the Hamilton-Jacobi equation, ellipsoidal coordinates [Weinacht].* Given these, we next find the restricted form of the potential function, V, which allows the equations of motion for a particle moving in the field generated by V to be solved by the separation of

*Names contained in brackets refer to references at the end of the report.

variables of the corresponding Hamilton-Jacobi equation. This function is then restricted further so that it satisfies the Laplace equation and qualifies as a gravitational potential. This doubly restricted function still contains free parameters, and these are then selected to eliminate any singularities in the region of interest outside the planets. Any remaining constants may be chosen to make the potential function agree with the actual gravitational potential outside the planet. There are enough parameters left to make this last step meaningful.

SECTION 2

THE STÄCKEL SYSTEM

Before progressing to the dynamical system, we must show that the proper prerequisites are satisfied by the ellipsoidal coordinate system. The most basic of these is that a Stäckel matrix exist for the particular ellipsoidal coordinates defined in Appendix A. These coordinates, λ, μ, and ν, listed in decreasing size, are the real roots of the cubic polynomial in t derived from the relation

$$\frac{x^2}{a^2+t}+\frac{y^2}{b^2+t}+\frac{z^2}{c^2+t}=1 \tag{1}$$

by multiplication by $(a^2+t)(b^2+t)(c^2+t)$ on both sides. The existence of this matrix, with the properties prescribed in Eqs. (D14) and (D15) is easily shown by displaying it.

Using the notation of Appendices A and D, it is [Eisenhart]

$$\Phi = \begin{vmatrix} \frac{\lambda^2}{4R^2} & \frac{1}{4R^2} & -\frac{\lambda}{4R^2} \\ -\frac{\mu^2}{4S^2} & -\frac{1}{4S^2} & \frac{\mu}{4S^2} \\ \frac{\nu^2}{4T^2} & \frac{1}{4T^2} & -\frac{\nu}{4T^2} \end{vmatrix} \tag{2}$$

Note that the i-th row of Φ depends only on the i-th coordinate as required, i.e., the first row depends on λ only, etc.

A direct computation yields the determinant needed to compute the inverse matrix of Φ,

$$\det|\Phi| = \frac{1}{64R^2S^2T^2}(\lambda-\mu)(\mu-\nu)(\nu-\lambda) \tag{3}$$

The inverse matrix, $\boldsymbol{\Psi}$, is given by

$$\Psi = \Phi^{-1} = \begin{vmatrix} \dfrac{1}{h_\lambda^2} & \dfrac{1}{h_\mu^2} & \dfrac{1}{h_\nu^2} \\ \\ \dfrac{\mu\nu}{h_\lambda^2} & \dfrac{\lambda\nu}{h_\mu^2} & \dfrac{\lambda\mu}{h_\nu^2} \\ \\ \dfrac{(\mu+\nu)}{h_\lambda^2} & \dfrac{(\lambda+\nu)}{h_\mu^2} & \dfrac{(\lambda+\mu)}{h_\nu^2} \end{vmatrix} \tag{4}$$

As required, the first row consists of the squares of the reciprocals of the metric coefficients. Thus the ellipsoidal coordinates possess the required matrices. The coordinates will allow separation of variables in the Hamilton-Jacobi equation.

The next step is to consider the potential function, it also must be chosen to allow separability. In consonance with Eq. (D16), the potential must be the form

$$V = \frac{\phi(\lambda)}{(\lambda-\mu)(\lambda-\nu)} + \frac{\psi(\mu)}{(\mu-\lambda)(\mu-\nu)} + \frac{\omega(\nu)}{(\nu-\lambda)(\nu-\mu)} \tag{5}$$

where ϕ, ψ, and ω are arbitrary functions. These functions are restricted by forcing them to be such that V satisfies the Laplace equation

$$\frac{1}{h_\lambda h_\mu h_\nu}\left[\frac{\partial}{\partial\lambda}\left(\frac{h_\mu h_\nu}{h_\lambda}\frac{\partial V}{\partial\lambda}\right) + \frac{\partial}{\partial\mu}\left(\frac{h_\lambda h_\nu}{h_\mu}\frac{\partial V}{\partial\mu}\right) + \frac{\partial}{\partial\nu}\left(\frac{h_\lambda h_\mu}{h_\nu}\frac{\partial V}{\partial\nu}\right)\right] = 0$$

The work involved in this step is considerable and it has been carried out in Appendix B where we find from Eq. (B19) that

$$\begin{aligned} \phi(\lambda) &= R(\lambda)\int\frac{p_4(\lambda)}{R^3(\lambda)}d\lambda + K_1R(\lambda) \\ \\ \psi(\mu) &= S(\mu)\int\frac{p_4(\mu)}{S^3(\mu)}d\mu + K_2S(\mu) \\ \\ \omega(\nu) &= T(\nu)\int\frac{p_4(\nu)}{T^3(\nu)}d\nu + K_3T(\nu) \end{aligned} \tag{6}$$

In these equations K_1, K_2, K_3 are free constants and p_4 is an arbitrary polynomial of the fourth order. The equations are shown to be necessary in Weinacht and sufficient in Appendix B.

The final step before applications are considered, as given in the introduction, is the elimination of singularities. This step is carried out in Appendix C, and the result is that we must have

$$\psi(\mu) = \omega(\nu) = 0 \tag{7}$$

and

$$\phi(\lambda) = -ER(\lambda)$$

where E is the product of the gravitational constant and the mass of the planet. Hence the resulting separable potential is of the form

$$V = -E\frac{R(\lambda)}{(\lambda-\mu)(\lambda-\nu)} \tag{8}$$

It is now apparent that the satellite motion in ellipsoidal coordinates, with a potential function given by Eq. (8) gives rise to a Stäckel system, since the necessary matrices have been given in Eqs. (2) and (4) and since Eq. (8) is of the form of Eq. (5), with Eq. (7) holding.

SECTION 3

APPLICATION TO THE ELLIPSOIDAL PROBLEM

The major concern in this section is the reduction of the relevant Hamilton-Jacobi equation to quadratures. With use of Eq. (A18), the result in Eq. (8) becomes

$$V = -E \frac{\sqrt{(\lambda + a^2)(\lambda + b^2)(\lambda + c^2)}}{(\lambda - \mu)(\lambda - \nu)} \tag{9}$$

and since from Eq. (A11)

$$\frac{1}{h_\lambda^2} = \frac{4R^2(\lambda)}{(\lambda - \mu)(\lambda - \nu)}$$

we see that

$$V(\lambda, \mu, \nu) = -\frac{E}{4R(\lambda)} \frac{1}{h_\lambda^2} \tag{10}$$

At this point it is possible to apply the theory of Stäckel systems as described in Eq. (D16) with

$$\xi_1 = \lambda, \quad \xi_2 = \mu, \quad \xi_3 = \nu,$$

and

$$V_1 = -\frac{E}{4R(\lambda)}, \quad V_2 = V_3 = 0$$

Thus the quadratures derived from Eq. (D17) are given by a solution of

$$\begin{aligned} \frac{1}{2}\left(\frac{\partial W_\lambda}{\partial \lambda}\right)^2 &= \frac{1}{4R^2(\lambda)} \left\{ \alpha_1 \lambda^2 + \alpha_2 - \lambda \alpha_3 \right\} + \frac{E}{4R(\lambda)} \\ \frac{1}{2}\left(\frac{\partial W_\mu}{\partial \mu}\right)^2 &= \frac{1}{4S^2(\mu)} \left\{ -\alpha_1 \mu^2 - \alpha_2 + \mu \alpha_3 \right\} \\ \frac{1}{2}\left(\frac{\partial W_\nu}{\partial \nu}\right)^2 &= \frac{1}{4T^2(\nu)} \left\{ \alpha_1 \nu^2 + \alpha_2 - \nu \alpha_3 \right\} \end{aligned} \tag{11}$$

The constants α_1, α_2, α_3 can be derived from initial conditions by use of the integrals of the system in Eq. (D22)

$$\left(\frac{p_\lambda^2}{2} - \frac{E}{4R(\lambda)}\right)\frac{1}{h_\lambda^2} + \frac{p_\mu^2}{2}\frac{1}{h_\mu^2} + \frac{p_\nu^2}{2}\frac{1}{h_\nu^2} = \alpha_1 \tag{12}$$

$$\left(\frac{p_\lambda^2}{2} - \frac{E}{4R(\lambda)}\right)\frac{\mu\nu}{h_\lambda^2} + \frac{p_\mu^2}{2}\frac{\lambda\nu}{h_\mu^2} + \frac{p_\nu^2}{2}\frac{\lambda\mu}{h_\nu^2} = \alpha_2 \tag{13}$$

$$\left(\frac{p_\lambda^2}{2} - \frac{E}{4R(\lambda)}\right)\frac{(\mu+\nu)}{h_\lambda^2} + \frac{p_\mu^2}{2}\frac{(\lambda+\nu)}{h_\mu^2} + \frac{p_\nu^2}{2}\frac{(\lambda+\mu)}{h_\nu^2} = \alpha_3 \tag{14}$$

Namely, given Cartesian coordinates for the initial velocity and position we can employ Eqs. (A8) and (D6) to find λ, μ, ν, p_λ, p_μ, p_ν for the initial point and hence α_1, α_2, α_3 from Eqs. (12), (13), (14). In theory, this specifies the right-hand sides in Eq. (11) and leads to W_λ, W_μ, W_ν and hence to $W(\lambda, \mu, \nu, \alpha_1, \alpha_2, \alpha_3)$ from Eq. (D12). The equations (D11) then lead to the solution in terms of the quadratures given by Eq. (11) or by Eq. (D25).

Some further information can be extracted from these integrals for the satellite case where λ is greater than zero. The satellite case is given by prescribing that the energy, α_1, be negative to obtain bounded orbits. The integral of Eq. (12) then shows that we must have

$$\frac{p_\lambda^2}{2} - \frac{E}{4R(\lambda)} < 0 \tag{15}$$

since if this condition does not hold everywhere, then α_1 must be positive, giving a contradiction. We can use Eq. (15) in the second integral, Eq. (13), along with the knowledge of the ellipsoidal coordinates to show that

$$\alpha_2 < 0$$

this follows from an examination of the individual terms. The final integral does not give a definite result unless λ is large and positive enough to make $\lambda + \nu$ and $\lambda + \mu$ positive also. In this case we can say that

$$\alpha_3 > 0$$

This information does lead to knowledge useful in the evaluation of the elliptic integrals that result from Eq. (11). However, in the case of near-earth satellites it is possible to avoid these integrals and perform an approximate inversion.

SUMMARY

This report has presented the theoretical aspects of the separable ellipsoidal potential, its derivation and specification and its use, in principle, to solve for the motion of a mass particle in the force field derived from it.

The present report has been written at a natural break-point in the theory of this potential. While it is of theoretical interest because it is the most general potential which separates the Euclidean Hamilton-Jacobi equation in orthogonal coordinates, in practice the thrust of the development from this point on is away from generality and towards more specialization. Therefore it is advisable to terminate the general work at this stage and continue the specialization in another report.

The process of specialization consists mainly of two steps. The ellipsoidal coordinates, λ, μ, and ν have been employed until this point because of their symmetry and mathematical convenience. However, once the potential function has been found, there is no need to remain with them, and a first step is to define new coordinates which relate more directly to spherical coordinates and express the potential in terms of them. The second step is to investigate and determine methods for an approximate inversion of the elliptic integrals as can be done if a suitable small parameter can be found.

These steps can be carried out to yield results which are confirmed by a comparison with numerical results due to application of the method of Vinti.

APPENDIX A

THE ELLIPSOIDAL COORDINATES

A coordinate system in three dimensions associates a set of three numbers with each point in space. In physical problems there is often a particular coordinate system which makes this association in an especially appropriate way due to certain symmetries or analytic simplifications that it may bring about.

In this respect the present problem is no exception. The coordinates to be used are ellipsoidal coordinates.

Let each point in space be assigned the coordinates (x, y, z) in a fixed Cartesian reference frame. The corresponding ellipsoidal coordinates, (λ, μ, ν) are defined as the three real solutions of the cubic polynomial in t obtained from the equation

$$\frac{x^2}{a^2+t} + \frac{y^2}{b^2+t} + \frac{z^2}{c^2+t} = 1 \tag{A1}$$

We define λ as the largest root, μ as the next largest, and ν as the smallest root.

That there are three real roots can be verified by considering the cubic polynomial

$$f(t) = \gamma(t) - x^2(b^2+t)(c^2+t) - y^2(a^2+t)(c^2+t) - z^2(a^2+t)(b^2+t) \tag{A2}$$

where

$$\gamma(t) = (a^2+t)(b^2+t)(c^2+t) \tag{A3}$$

Without loss of generality we order the constants a, b, c so that in the general case

$$a^2 > b^2 > c^2 > 0 \tag{A4}$$

The real character of the roots of Eq. (A2) can be shown by examining the sign of f(t) for five particular values of t. By virtue of Eq. (A4) we have,

$$f(+\infty) = +\infty$$

$$f(-c^2) = -z^2(a^2 - c^2)(b^2 - c^2) < 0$$

$$f(-b^2) = +y^2(a^2 - b^2)(b^2 - c^2) > 0 \tag{A5}$$

$$f(-a^2) = -x^2(a^2 - b^2)(b^2 - c^2) < 0$$

$$f(-\infty) = -\infty$$

The sign changes indicate that the following restrictions hold for λ, μ, and ν,

$$-a^2 \leq \nu \leq -b^2 \leq \mu \leq -c^2 \leq \lambda < \infty \tag{A6}$$

If we successively substitute λ, μ, and ν into Eq. (A1) and use these inequalities, we find that the resulting equations with λ, μ, ν held constant represent confocal ellipsoids, hyperboloids of one sheet, and hyperboloids of two sheets, respectively, when x, y, and z vary. With some special exceptions, the members of each family of surfaces are orthogonal to all of the members of each of the others and thus the coordinate system is orthogonal.

By their definition, we have that λ, μ, ν satisfy

$$(t - \lambda)(t - \mu)(t - \nu) = f(t) \tag{A7}$$

and this leads us to expressions for x^2, y^2, and z^2 in terms of the new coordinates. If we successively put $t = -a^2$, $-b^2$, $-c^2$, we find

$$x^2 = \frac{(a^2 + \lambda)(a^2 + \mu)(a^2 + \nu)}{(a^2 - b^2)(a^2 - c^2)}$$

$$y^2 = \frac{(b^2 + \lambda)(b^2 + \mu)(b^2 + \nu)}{(b^2 - a^2)(b^2 - c^2)} \tag{A8}$$

$$z^2 = \frac{(c^2 + \lambda)(c^2 + \mu)(c^2 + \nu)}{(a^2 - c^2)(b^2 - c^2)}$$

These formulas, together with the next one to be stated, allow the comparison of the potential found in this report with the more standard spherical harmonic expressions. Addition of the three members of Eq. (A8) gives,

$$r^2 = x^2 + y^2 + z^2 = a^2 + b^2 + c^2 + \lambda + \mu + \nu \tag{A9}$$

Special Values

For later use certain facts must be established about the cases when the equalities are attained in Eq. (A6); this will be important in consideration of possible singularities in the potential function and in consideration of the case of double roots of the polynomial f(t) in Eq. (A2). Special values of λ, μ, and ν arise in consideration of the coordinate planes and coordinates axes in the underlying frame of reference. It should be noticed that the following results are specialized to the case of orbital motion by the following consideration. In the end result, a, b, and c will approximate principal radii of the earth and of course for satellite motion the radius vector to the satellite must have a magnitude greater than any of these. Thus, for convenience in the following, the assumption is made that x, y, and z are such that λ is positive.

Consider first the three coordinate axes.

1. The x-axis is given by:

$$\begin{aligned} \lambda &= x^2 - a^2 > 0 \\ \mu &= -c^2 \\ \nu &= -b^2 \end{aligned} \tag{A10}$$

2. The y-axis is given by:

$$\begin{aligned} \lambda &= y^2 - b^2 > 0 \\ \mu &= -c^2 \\ \nu &= -a^2 \end{aligned} \tag{A11}$$

3. The z-axis is given by:

$$\begin{aligned} \lambda &= z^2 - c^2 > 0 \\ \mu &= -b^2 \\ \nu &= -a^2 \end{aligned} \tag{A12}$$

These results follow from detailed examination of f(t) in Eq. (A2) in each case.

The problem of the coordinate planes is slightly more complicated, since here f(t) reduces to a quadratic equation rather than a linear one as in the case of the coordinate axes.

For the x-y plane (z = 0), the cubic equation for λ, μ, and ν becomes

$$0 = (c^2 + t)[(a^2 + t)(b^2 + t) - x^2(b^2 + t) - y^2(c^2 + t)]$$

It is immediately apparent that one root is $t = -c^2$. Furthermore, since λ is assumed positive, by Eq. (A6) we must have the following result. The x-y plane is characterized by

$$\mu = -c^2 \tag{A13}$$

A similar analysis for the y-z plane, (x = 0), shows that it is characterized by

$$\nu = -a^2 \tag{A14}$$

The case of the x-z plane, (y = 0), is more complicated. Here one root is given by $t = -b^2$. This is in the admissible range of both μ and ν by Eq. (A6). The question is whether μ or ν is equal to $-b^2$ for this case. An examination shows that for one part of the x-z plane $\mu = -b^2$ and for another $\nu = -b^2$. Thus the x-z plane is characterized by both μ and ν becoming equal to $-b^2$ somewhere in it. Moreover, there is a curve in this plane,

$$\frac{x^2}{a^2 - b^2} - \frac{z^2}{b^2 - c^2} = 1$$

where $\lambda = \mu = -b^2$. Thus, for example, for the potential to be considered later to be analytic everywhere outside r = a we must consider this double root case. In summary, the x-z plane is characterized by

$$\mu = -b^2 \quad \text{or} \quad \nu = -b^2$$

or

$$\mu = \nu = -b^2 \tag{A15}$$

Other properties we will need will be stated without detailed derivation; further information may be found in one of the references [Hildebrand, Kellogg, Mason and Weaver].

The metric in this coordinate system is

$$ds^2 = h_\lambda^2 d\lambda^2 + h_\mu^2 d\mu^2 + h_\nu^2 d\nu^2 \tag{A16}$$

where

$$\begin{aligned} h_\lambda &= \frac{1}{2}\frac{\sqrt{(\lambda - \mu)(\lambda - \nu)}}{R(\lambda)} \\ h_\mu &= \frac{1}{2}\frac{\sqrt{(\lambda - \mu)(\mu - \nu)}}{S(\mu)} \\ h_\nu &= \frac{1}{2}\frac{\sqrt{(\nu - \mu)(\nu - \lambda)}}{T(\nu)} \end{aligned} \tag{A17}$$

The functions $R(\lambda)$, $S(\mu)$, and $T(\nu)$ are defined as

$$
\begin{aligned}
R(\lambda) &= \sqrt{(\lambda+a^2)(\lambda+b^2)(\lambda+c^2)} \\
S(\mu) &= \sqrt{-(\mu+a^2)(\mu+b^2)(\mu+c^2)} \\
T(\nu) &= \sqrt{(\nu+a^2)(\nu+b^2)(\nu+c^2)}
\end{aligned}
\tag{A18}
$$

The metric coefficients of Eq. (A17) permit the derivation of the Laplacian,

$$
\begin{aligned}
\nabla^2 V = \frac{4}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)} \Bigg\{ & R(\lambda)(\mu-\nu) \frac{\partial}{\partial\lambda}\left[R(\lambda)\frac{\partial V}{\partial\lambda}\right] \\
& + S(\mu)(\lambda-\nu) \frac{\partial}{\partial\mu}\left[S(\mu)\frac{\partial V}{\partial\mu}\right] \\
& + T(\nu)(\lambda-\mu) \frac{\partial}{\partial\nu}\left[T(\nu)\frac{\partial V}{\partial\nu}\right] \Bigg\}
\end{aligned}
\tag{A19}
$$

The gradient of a function $g(\lambda, \mu, \nu)$ follows from Eq. (A17) since

$$
\nabla g = \left(\frac{1}{h_\lambda}\frac{\partial g}{\partial\lambda}, \frac{1}{h_\mu}\frac{\partial g}{\partial\mu}, \frac{1}{h_\nu}\frac{\partial g}{\partial\nu} \right) \tag{A20}
$$

Finally, it follows that $\lambda \longrightarrow r^2$ as $\lambda \longrightarrow \infty$. This is seen from Eq. (A9), where division by λ yields

$$
\frac{r^2}{\lambda} = 1 + \frac{a^2+b^2+c^2+\mu+\nu}{\lambda}
$$

Since μ and ν are restricted by Eq. (A6), we have the desired result,

$$
\lim_{\lambda\to\infty} \frac{r^2}{\lambda} = 1 \tag{A21}
$$

APPENDIX B

GENERATION OF THE SOLUTIONS

In this appendix, we indicate how the solutions of the Laplace equation, which were displayed before, are obtained. Not a small part of the motivation for this appendix comes from the fact that unless a proper route is followed the required algebraic manipulations become enormous.

As has been stated, the solution of the Laplace equation must be of a certain form in order for the Hamilton-Jacobi equation to separate. The form is

$$V(\lambda,\mu,\nu) = \frac{\phi(\lambda)}{(\lambda-\mu)(\lambda-\nu)} + \frac{\psi(\mu)}{(\mu-\lambda)(\mu-\nu)} + \frac{\omega(\nu)}{(\nu-\lambda)(\nu-\mu)} \tag{B1}$$

The functions ϕ, ψ, and ω must be chosen to satisfy the Laplace equation, and in order to obtain the simplest expressions we make the choices in two steps.

For the first step, we ignore the ranges of λ, μ, and ν and place the main emphasis on finding a solution. This allows a symmetric form of the Laplacian to be used. The form follows simply by moving the minus sign outside the radical $S(\mu)$ in Eq. (A19) as the imaginary quantity i and by assuming that the polynomial remaining under the square root is positive. The resulting Laplace equation is, using the first expression of Eq. (A18),

$$\begin{aligned}\nabla^2 V = \frac{4}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)} \Bigg\{ & R(\lambda)(\mu-\nu)\, \frac{\partial}{\partial\lambda}\left[R(\lambda)\, \frac{\partial V}{\partial\lambda}\right] \\ & + R(\mu)(\nu-\lambda)\, \frac{\partial}{\partial\mu}\left[R(\mu)\, \frac{\partial V}{\partial\mu}\right] \\ & + R(\nu)(\lambda-\mu)\, \frac{\partial}{\partial\nu}\left[R(\nu)\, \frac{\partial V}{\partial\nu}\right] \Bigg\}\end{aligned} \tag{B2}$$

This equation is invariant under cyclic permutations of λ, μ, ν and will be convenient in that once a solution is found, two others may be generated by applying two successive permutations of λ, μ, ν to it. Of course, straightforward computation can avoid this step by guessing two new solutions and verifying them in Eq. (A19). We look for solutions of the form

$$V = \frac{GR(\lambda)}{(\lambda-\mu)(\lambda-\nu)} \tag{B3}$$

where G is a constant. When this is substituted into Eq. (B2), for convenience the terms in braces are displayed one at a time and the factor multiplying the terms in the braces is neglected, we make the assumption that it is nonzero.

Since $R(\lambda)$ contains a radical, it will be helpful to work in terms of $R^2(\lambda)$ and its derivatives, denoted here by primes. After some algebra, we find

$$R(\lambda)(\mu - \nu)\frac{\partial}{\partial\lambda}\left[R(\lambda)\frac{\partial V}{\partial\lambda}\right] = \frac{GR(\lambda)(\mu-\nu)}{(\lambda-\mu)^3(\lambda-\nu)^3}\times$$

$$\left[\frac{1}{2}R^2(\lambda)''(\lambda-\mu)^2(\lambda-\nu)^2 - \frac{3}{2}R^2(\lambda)'\left\{(\lambda-\mu)(\lambda-\nu)^2 - (\lambda-\mu)^2(\lambda-\nu)\right\}\right.$$

$$\left. + 2R^2(\lambda)\left\{(\lambda-\nu)^2 + (\lambda-\nu)(\lambda-\mu) + (\lambda-\mu)^2\right\}\right] \tag{B4}$$

$$R(\mu)(\nu-\lambda)\frac{\partial}{\partial\mu}\left[R(\mu)\frac{\partial V}{\partial\mu}\right] = \frac{-GR(\lambda)}{(\lambda-\mu)^3}\left\{2R^2(\mu) + \frac{1}{2}R^2(\mu)'(\lambda-\mu)\right\} \tag{B5}$$

and

$$R(\nu)(\lambda-\mu)\frac{\partial}{\partial\nu}\left[R(\nu)\frac{\partial V}{\partial\nu}\right] = \frac{GR(\lambda)}{(\lambda-\nu)^3}\left\{2R^2(\nu) + \frac{1}{2}R^2(\nu)'(\lambda-\nu)\right\} \tag{B6}$$

By adding Eqs. (B4), (B5), and (B6) we obtain, after grouping over a common denominator, a factor

$$\frac{GR(\lambda)}{(\lambda-\mu)^3(\lambda-\nu)^3}$$

which multiplies a polynomial in λ, μ, ν. This polynomial vanishes identically when an arbitrary cubic polynomial is substituted for the function R^2. The cancellations are quite spectacular. Since $R^2(\lambda)$ can be considered arbitrary, its roots can be considered arbitrary also. This means that Eq. (B3) provides a solution for arbitrary values of a, b, and c. Thus Eq. (B3) is a solution, and

$$\begin{aligned} \phi(\lambda) &= G_1R(\lambda) \\ \psi(\mu) &= G_2R(\mu) \\ \omega(\nu) &= G_3R(\nu) \end{aligned} \tag{B7}$$

give solutions also, with G_1, G_2, and G_3 arbitrary constants.

We now return completely to the notation of Appendix A and find a more complex set of solutions which follows from the three just found. To proceed, define the three basic solutions

$$S_\lambda = \frac{R(\lambda)}{(\lambda-\mu)(\lambda-\nu)}, \quad S_\mu = \frac{S(\mu)}{(\lambda-\mu)(\mu-\nu)}, \quad S_\nu = \frac{T(\nu)}{(\nu-\lambda)(\nu-\mu)} \tag{B8}$$

where Eq. (A18) has been used. Since the Laplacian is a linear operator,

$$\nabla^2 S_\lambda = \nabla^2 S_\mu = \nabla^2 S_\nu = 0 \tag{B9}$$

and, employing this fact, we seek a solution in the form

$$V = S_\lambda H(\lambda) + S_\mu J(\mu) + S_\nu K(\nu) \tag{B10}$$

Hence

$$\nabla^2 V = \nabla^2(S_\lambda H) + \nabla^2(S_\mu J) + \nabla^2(S_\nu K)$$

and by Eq. (B9)

$$\begin{aligned} \nabla^2 V = {} & S_\lambda \nabla^2 H + S_\mu \nabla^2 J + S_\nu \nabla^2 K \\ & + 2\nabla S_\lambda \cdot \nabla H + 2\nabla S_\mu \cdot \nabla J + 2\nabla S_\nu \cdot \nabla K \end{aligned} \tag{B11}$$

Because H, J and K are functions of only one variable, the notation of Eq. (A17) gives

$$\nabla H = \left(\frac{1}{h_\lambda}\frac{\partial H}{\partial \lambda}, \ 0, \ 0\right)$$

$$\nabla J = \left(0, \ \frac{1}{h_\mu}\frac{\partial J}{\partial \mu}, \ 0\right)$$

$$\nabla K = \left(0, \ 0, \ \frac{1}{h_\nu}\frac{\partial K}{\partial \nu}\right)$$

With the use of these, the dot products in Eq. (B11) can be simplified to give

$$\begin{aligned} 2\nabla S_\lambda \cdot \nabla H &= \frac{2}{h_\lambda^2}\frac{\partial S_\lambda}{\partial \lambda}\frac{\partial H}{\partial \lambda} \\ 2\nabla S_\mu \cdot \nabla J &= \frac{2}{h_\mu^2}\frac{\partial S_\mu}{\partial \mu}\frac{\partial J}{\partial \mu} \\ 2\nabla S_\nu \cdot \nabla K &= \frac{2}{h_\nu^2}\frac{\partial S_\nu}{\partial \nu}\frac{\partial K}{\partial \nu} \end{aligned} \tag{B12}$$

Likewise

$$S_\lambda \nabla^2 H = \frac{S_\lambda}{h_\lambda h_\mu h_\nu} \frac{\partial}{\partial \lambda}\left[\left(\frac{h_\mu h_\nu}{h_\lambda}\right)\frac{\partial H}{\partial \lambda}\right]$$

$$S_\mu \nabla^2 J = \frac{S_\mu}{h_\lambda h_\mu h_\nu} \frac{\partial}{\partial \mu}\left[\left(\frac{h_\lambda h_\nu}{h_\mu}\right)\frac{\partial J}{\partial \mu}\right] \tag{B13}$$

$$S_\nu \nabla^2 K = \frac{S_\nu}{h_\lambda h_\mu h_\nu} \frac{\partial}{\partial \nu}\left[\left(\frac{h_\lambda h_\mu}{h_\nu}\right)\frac{\partial K}{\partial \nu}\right]$$

The three relations in Eq. (A17) now can be used to replace the h's in terms of R, S, T, and λ, μ, ν. When this is done, Eqs. (B12) and (B13) may be summed to give

$$0 = \nabla^2 V = \frac{\partial}{\partial \lambda}\left(S_\lambda^2 R \frac{\partial H}{\partial \lambda}\right) + \frac{\partial}{\partial \mu}\left(S_\mu^2 S \frac{\partial J}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(S_\nu^2 T \frac{\partial K}{\partial \nu}\right) \tag{B14}$$

application of the definitions of Eq. (B8) allows the reduction of Eq. (B14) to

$$0 = \frac{\partial}{\partial \lambda}\left\{\frac{F_\lambda}{(\lambda-\mu)^2(\lambda-\nu)^2}\right\} + \frac{\partial}{\partial \mu}\left\{\frac{F_\mu}{(\lambda-\mu)^2(\mu-\nu)^2}\right\} + \frac{\partial}{\partial \nu}\left\{\frac{F_\nu}{(\nu-\lambda)^2(\nu-\mu)^2}\right\} \tag{B15}$$

In this equation

$$F_\lambda = R^3(\lambda) \frac{\partial H}{\partial \lambda}$$

$$F_\mu = S^3(\mu) \frac{\partial J}{\partial \mu} \tag{B16}$$

$$F_\nu = T^3(\nu) \frac{\partial K}{\partial \nu}$$

The equation (B15) contains all the necessary information, the form of the functions F_λ, F_μ, F_ν follows simply. Integration of Eq. (B15) with respect to λ yields

$$\frac{F_\lambda}{(\lambda-\mu)^2(\lambda-\nu)^2} - \frac{\partial}{\partial \mu}\left\{\frac{F_\mu}{(\lambda-\mu)(\mu-\nu)^2}\right\} + \frac{\partial}{\partial \nu}\left\{\frac{F_\nu}{(\nu-\lambda)(\nu-\mu)^2}\right\} + C(\mu,\nu) = 0$$

Here $C(\mu,\nu)$ is an arbitrary function of μ and ν. A multiplication by the factor

$$(\lambda-\mu)^2(\lambda-\nu)^2$$

leads to the result that

$$F_\lambda + p_4(\lambda, \mu, \nu) = 0$$

where F_λ is a function of λ alone and $p_4(\lambda, \mu, \nu)$ is a fourth order polynomial in λ whose coefficients depend on μ and ν. However, since λ, μ, ν are independent variables, each of the coefficients must be a constant [Kellogg, pg. 205]. Thus

$$F_\lambda = p_4(\lambda) \triangleq C_4\lambda^4 + C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 \tag{B17}$$

with C_0, C_1, C_2, C_3, C_4 constant.

A similar analysis shows that F_μ and F_ν must both be quartic polynomials. If arbitrary coefficients are assigned to each polynomial and if they are substituted into Eq. (B15), we find that the coefficients must be the same for F_λ, F_μ, F_ν.

Thus from Eq. (B16)

$$\begin{aligned} H(\lambda) &= \int \frac{p_4(\lambda)}{R^3(\lambda)}\, d\lambda + K_1 \\ J(\mu) &= \int \frac{p_4(\mu)}{S^3(\mu)}\, d\mu + K_2 \\ K(\nu) &= \int \frac{p_4(\nu)}{T^3(\nu)}\, d\nu + K_3 \end{aligned} \tag{B18}$$

where p_4 is an arbitrary fourth-order polynomial, given by Eq. (B17), and the K's are constants.

If we use Eq. (B10), then our solutions are given by Eq. (B1), with

$$\begin{aligned} \phi(\lambda) &= R(\lambda) \int \frac{p_4(\lambda)d\lambda}{R^3(\lambda)} + K_1 R(\lambda) \\ \psi(\mu) &= S(\mu) \int \frac{p_4(\mu)d\mu}{S^3(\mu)} + K_2 S(\mu) \\ \omega(\nu) &= T(\nu) \int \frac{p_4(\nu)d\nu}{T^3(\nu)} + K_3 T(\nu) \end{aligned} \tag{B19}$$

These solutions include those given by Eq. (B7) as a special case.

APPENDIX C

EXAMINATION AND SPECIALIZATION OF THE SOLUTION

While the general solution for the separable potential given in Eq. (B19) is not needed here, we give, for reference, the reduction necessary to place the integrals in a form compatible with standard references [Byrd and Friedman]. A typical integral is

$$\int \frac{p_4(x)dx}{[(x+a^2)(x+b^2)(x+c^2)]\,[(x+a^2)(x+b^2)(x+c^2)]^{1/2}} \tag{C1}$$

To treat this, and reduce it to a sum of elliptic integrals, we must find the partial fraction expansion of

$$\zeta = \frac{p_4(x)}{(x+a^2)(x+b^2)(x+c^2)}$$

After computation we find that

$$\zeta = C_4 x + [C_3 - (a^2+b^2+c^2)C_4] + \frac{M}{x+a^2} + \frac{N}{x+b^2} + \frac{P}{x+c^2} \tag{C2}$$

The coefficients M, N, P display a pleasing symmetry. Their values are

$$M = \frac{C_4 a^8 - C_3 a^6 + C_2 a^4 - C_1 a^2 + C_0}{(b^2-a^2)(c^2-a^2)}$$

$$N = \frac{C_4 b^8 - C_3 b^6 + C_2 b^4 - C_1 b^2 + C_0}{(a^2-b^2)(c^2-b^2)} \tag{C3}$$

$$P = \frac{C_4 c^8 - C_3 c^6 + C_2 c^4 - C_1 c^2 + C_0}{(a^2-c^2)(b^2-c^2)}$$

After use of this expansion, there are only three types of integrals to be evaluated. These are of the following forms

$$\int \frac{x\,dx}{\sqrt{(x+a^2)(x+b^2)(x+c^2)}} \tag{C4}$$

$$\int \frac{dx}{\sqrt{(x+a^2)(x+b^2)(x+c^2)}} \tag{C5}$$

and

$$\int \frac{dx}{(x+a^2)\sqrt{(x+a^2)(x+b^2)(x+c^2)}} \tag{C6}$$

There is one integral of forms Eqs. (C4) and (C5) and there are three of the form Eq. (C6).

From this point on, we start specification of the free constants in Eq. (B19) to eliminate singularities in the region of interest outside a planet. There is a certain element of choice in the order of presentation due to the limits which must be prescribed for the integrals. However, the results of the imposition of conditions followed here will be the same regardless of the order chosen. These conditions are necessary in order that the solution V qualify as a gravitational potential.

The basic requirement is that the integrals exist for the permissible ranges of the variables λ, μ, ν. First, the region $\lambda \longrightarrow \infty$ will be examined. From Eq. (A21) it can be seen that $\lambda \longrightarrow r^2$ as $\lambda \longrightarrow \infty$ so that in this limit, since μ and ν are bounded by Eq. (A6), the dominant term becomes

$$V \approx \frac{\phi(\lambda)}{(\lambda-\mu)(\lambda-\nu)} \tag{C7}$$

with $\phi(\lambda)$ given in Eq. (B19). Hence

$$V \approx \frac{R(\lambda)}{(\lambda-\mu)(\lambda-\nu)}\left[\int_{\lambda}^{a} \frac{p_4(\xi)d\xi}{R^3(\xi)} + K_1\right]$$

Here a is an arbitrary constant, $-c^2 \leq a < \infty$. If this integral is to exist as $\lambda \longrightarrow \infty$, we must set $C_4 = 0$. Because K_1 is arbitrary, we may set the limits on the integral so that $\phi(\lambda)$ becomes

$$\phi(\lambda) = R(\lambda)\left[\int_{\lambda}^{\infty} \frac{p_3(\xi)d\xi}{R^3(\xi)} + K_1\right] \tag{C8}$$

where $p_3(\xi)$ is identical with $p_4(\xi)$ with $C_4 = 0$.

In order to have agreement with classical theory, V must approach

$$U = -\frac{E}{\sqrt{\lambda}}$$

where $\mu = \nu = -b^2$ lie on a hyperbola and thus are found arbitrarily far away from the origin. The terms that must be analyzed are

$$\frac{\psi(\mu)}{(\mu - \lambda)(\mu - \nu)} + \frac{\omega(\nu)}{(\nu - \lambda)(\nu - \mu)}$$

The difference between the functions $\psi(\mu)(\nu - \lambda)$ and $(\mu - \lambda)\omega(\nu)$ must vanish more rapidly than $(\mu - \nu)$ does regardless of the way in which μ and ν approach $-b^2$. If we set

$$\mu = -b^2 + \epsilon$$

$$\nu = -b^2 - \delta$$

$$\delta, \epsilon > 0$$

then an examination of the problem for ϵ and δ as they approach zero indicates that

$$K_2 = K_3 = 0$$

and the limits in $\psi(\mu)$ and $\omega(\nu)$ must be chosen so that

$$\psi(\mu) = S(\mu) \int_{-b^2}^{\mu} \frac{C\, d\xi}{S(\xi)} \tag{C14}$$

$$\omega(\nu) = T(\nu) \int_{\nu}^{-b^2} \frac{C\, d\xi}{T(\xi)} \tag{C15}$$

If we examine the existence of the potential only, as has been done to this point, the constant C is not determined. However, if the gradient of the potential is also considered, the balance of terms found by the choice of the limits in Eqs. (C14) and (C15) fails to hold, and in order to eliminate the singularities in this case we must choose

$$C = 0 \tag{C16}$$

Thus the final potential becomes

$$V = \frac{-E\, R(\lambda)}{(\lambda - \mu)(\lambda - \nu)} \tag{C17}$$

As a check, this potential can be found to reproduce the potential of Vinti, to be found in the references, in the limit as $b \rightarrow a$ and $c \rightarrow 0$.

APPENDIX D

EUCLIDEAN STÄCKEL SYSTEMS

This appendix presents a brief summary of the results of Stäckel which are pertinent to the present investigation. It is included primarily for reference purposes and to introduce notation. More detailed expositions of the theory are contained in the references at the end of the report [Eisenhart, Iszak, Stäckel].

Consider the motion of a particle as described by a set of Cartesian coordinates x_1, x_2, x_3, with the forces acting on the particle derivable from a potential function $V(x_1, x_2, x_3)$. The Hamiltonian function for such a system is

$$H = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - V(x_1, x_2, x_3) = \alpha_1 \tag{D1}$$

Here the dots over the coordinates denote time differentiation and α_1 is a constant, the energy. With the use of H, the Hamiltonian equations of motion may be written as

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial x_i} \quad , \quad (i = 1,2,3) \tag{D2}$$

where p_i is the momentum conjugate to x_i,

$$p_i = \dot{x}_i$$

Let ξ_1, ξ_2, ξ_3 be a set of orthogonal curvilinear coordinates introduced into the Euclidean system by the transformation

$$x_i = x_i(\xi_1, \xi_2, \xi_3) \quad , \quad (i = 1,2,3) \tag{D3}$$

Small changes in the coordinates ξ_1, ξ_2, ξ_3 will move a point in space through a distance as given by [Hildebrand].

$$ds^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2 + h_3^2 d\xi_3^2 \tag{D4}$$

The coefficient functions h_1, h_2, h_3 in general depend on all of the coordinates and will be referred to here as the metric coefficients. These coefficient functions will play an important role in the succeeding results.

At this point it is possible to write down the Hamiltonian and the equations of motion in terms of the new coordinates. The new Hamiltonian is

$$H = \frac{1}{2} \sum_{i=1}^{3} \frac{1}{h_i^2} p_{\xi_i}^2 + V(\xi_1, \xi_2, \xi_3) = \alpha_1 \tag{D5}$$

as before, p_{ξ_i} will be the momentum conjugate to ξ_i and we find

$$p_{\xi_i} = h_i \dot{\xi}_i \tag{D6}$$

The Hamiltonian equations of motion are

$$\dot{\xi}_i = \frac{\partial H}{\partial p_{\xi_i}} \ , \quad \dot{p}_{\xi_i} = - \frac{\partial H}{\partial \xi_i} \ , \quad (i = 1,2,3) \tag{D7}$$

This system, together with initial conditions which determine the six constants of integration, will give a complete description of the motion once the coordinate transformations of Eq. (D3) are specified.

Consider now the central question to be considered in this appendix. This is to find the most general set of coordinates of Eq. (D3) which permit the solution of the system of Eq. (D7) to be reduced to quadratures and to find the form of the potential function which allows this reduction to be carried out. This question was considered by Stäckel in 1891, in terms of the Hamilton-Jacobi equation for the system in Eq. (D7). This is

$$\frac{1}{2} \sum_{i=1}^{3} \frac{1}{h_i^2} \left(\frac{\partial W}{\partial \xi_i} \right)^2 + V(\xi_1, \xi_2, \xi_3) = \alpha_1 \tag{D8}$$

The function

$$W = W(\xi_1, \xi_2, \xi_3, \alpha_1, \alpha_2, \alpha_3) \tag{D9}$$

plays the fundamental role in this theory (α_2, α_3 are arbitrary constants of integration), for if the partial differential equation (D8) can be solved for W the solution of the system in Eq. (D7) is given implicitly. The relations which permit this are

$$p_{\xi_i} = \frac{\partial W}{\partial \xi_i} \ , \quad (i = 1,2,3) \tag{D10}$$

and

$$\frac{\partial W}{\partial \alpha_1} = t + \beta_1 \ , \quad \frac{\partial W}{\partial \alpha_2} = \beta_2 \ , \quad \frac{\partial W}{\partial \alpha_3} = \beta_3 \tag{D11}$$

In Eq. (D11), β_1, β_2, β_3 are three additional arbitrary constants. These three constants and the constants α_1, α_2, α_3 give the six necessary to match given initial conditions. The solution process can be summarized as follows:

1. Solve Eq. (D8) for $W(\xi_1, \xi_2, \xi_3, \alpha_1, \alpha_2, \alpha_3)$.
2. Differentiate as in Eq. (D11) to find the ξ_i given implicitly in terms of the α_i, β_i, and t.
3. Solve for ξ_i explicitly as functions of the α_i, β_i and t and insert them in Eq. (D10) to find the p_{ξ_j}. This last step is optional if only position information is of interest. It can also be bypassed by time differentiation of the final expressions for the ξ_i and use of Eq. (D6) if this course of action is simpler.

In principle, and with proper mathematical restrictions, these steps can be carried out. However, the matter of the solution of Eq. (D8), the initial step, is usually as difficult as the solution of the original dynamical system, Eq. (D7).

Stäckel looked at the case where W can be found in the form

$$W = \sum_{i=1}^{3} W_i(\xi_i, \alpha_1, \alpha_2, \alpha_3) \tag{D12}$$

the separable case, where each function W_i depends only on ξ_i and the constants α_i. He sought each function W_i in the form of an integral of a function of ξ_i and α_i alone,

$$W_i = \int^{\xi_i} f_i(\zeta, \alpha_1, \alpha_2, \alpha_3)\, d\zeta \tag{D13}$$

In this case both steps one and two of the solution can be carried out with at most an integration left to be done. Solutions of this desirable form do not exist in general, and assumptions must be made upon the system to assure their existence. Stäckel's results are given as follows:

1. Assume that a matrix, depending on the coordinate system only, called a Stäckel matrix, exists. This matrix,

$$\Phi = \{\phi_{ij}\} = \{\phi_{ij}(\xi_i)\} \tag{D14}$$

is such that the i-th row contains functions depending on the i-th curvilinear coordinate only, as indicated.

2. Assume that the inverse of the matrix Φ, Ψ, exists and has the property that

$$\Psi = \Phi^{-1} = \{\psi_{ij}\} \ , \quad \psi_{1j} = \frac{1}{h_j^2} \tag{D15}$$

the first row is made up of the reciprocals of the squares of the metric coefficients.

3. Assume that the potential function is of the form

$$V = \sum_{j=1}^{3} \frac{V_j(\xi_j)}{h_j^2} \tag{D16}$$

where the function V_j depends on the coordinate ξ_j only.

Systems which satisfy these three assumptions are called Stäckel systems.

The results, under these assumptions, are that the most general coordinates that allow separation are triaxially ellipsoidal coordinates as given in Appendix A [Weinacht] and that the solution is given by Eq. (D12) where we find the function W_j from

$$\frac{1}{2}\left(\frac{\partial W_j}{\partial \xi_j}\right)^2 = \sum_{k=1}^{3} \alpha_k \phi_{jk} - V_j \tag{D17}$$

The quantities, ϕ_{jk} and V_j are as defined in Eqs. (D14) and (D16) and the α_k are arbitrary constants with α_1 equal to the energy in the case considered here.

Before commenting on the assumptions and results, we first verify that specification of W_j by means of Eq. (D17) leads to a solution of the Hamilton-Jacobi equation.

The first step simply involves a reformulation of the Hamilton-Jacobi equation in terms of the inverse of the Stäckel matrix and the potential functions V_j. From Eqs. (D15), (D16) and (D8) we find

$$\frac{1}{2} \sum_{j=1}^{3} \psi_{1j}\left[\left(\frac{\partial W}{\partial \xi_j}\right)^2 + V_j\right] = \alpha_1 \tag{D18}$$

It follows immediately from Eq. (D12) that

$$\frac{\partial W}{\partial \xi_j} = \frac{\partial W_j}{\partial \xi_j} \tag{D19}$$

and the Hamilton-Jacobi equation is

$$\frac{1}{2} \sum_{j=1}^{3} \psi_{1j}\left[\left(\frac{\partial W_j}{\partial \xi_j}\right)^2 + V_j\right] = \alpha_1 \tag{D20}$$

The verification follows simply by substitution using Eq. (D17), and Eq. (D20) becomes

$$\sum_{j=1}^{3} \psi_{1j} \sum_{k=1}^{3} \alpha_k \phi_{jk} = \alpha_1$$

or

$$\sum_{k=1}^{3} \alpha_k \left\{ \sum_{j=1}^{3} \psi_{1j} \phi_{jk} \right\} = \alpha_1$$

Since Ψ and Φ are inverse matrices, the summation in braces is zero when $k \neq 1$ and one when $k = 1$. This gives the result that

$$\alpha_1 = \alpha_1$$

Thus it has been verified that the Stäckel assumptions are sufficient to produce a separable solution to the Euclidean Hamilton-Jacobi equation. The necessity of these assumptions can be found in the original Stäckel Habilitationsschrift.

In addition to the solution by quadratures, Stäckel systems possess a set of quadratic integrals of the equations (D7). These follow from Eq. (D17) with the use of Eq. (D10). The integrals are given by

$$\frac{1}{2} p_{\xi_j}^2 - \sum_{k=1}^{3} \alpha_k \phi_{jk} + V_j = 0 \quad , \quad j = 1,2,3 \tag{D21}$$

These are useful in determining the constants α_k in terms of the initial conditions. It will be possible, in the case considered in the body of this report, to find the signs of the constants α_k.

A more convenient form of Eq. (D21) is obtained by multiplication by ψ_{mj} and summation of the result with respect to j. The final result is

$$\sum_{j=1}^{3} \psi_{mj} \left[\frac{1}{2} p_{\xi_j}^2 + V_j \right] = \alpha_m \quad , \quad m = 1,2,3 \tag{D22}$$

This expression permits the α_m to be determined directly from initial conditions.

In conclusion, it should be pointed out that the derivatives in Eq. (D11), the equations to be solved for the coordinates, can be written down explicitly with quadratures remaining to be done. From Eq. (D17), the function W_j is given by

$$W_j = \pm \int \left\{ 2 \sum_{k=1}^{3} \alpha_k \phi_{jk} - 2V_j \right\}^{1/2} d\xi_j \tag{D23}$$

The plus sign is to be used if ξ_j is increasing and vice versa. Let us assume for our convenience that the sign is absorbed into $d\xi_j$ and perform the differentiations to find

$$\frac{\partial W_j}{\partial \alpha_m} = \int \frac{\phi_{jm}}{\sqrt{2 \sum_{k=1}^{3} \alpha_k \phi_{jk} - 2V_j}} \, d\xi_j \tag{D24}$$

where both j and m range between one and three. By summing Eq. (D24) we obtain the final result

$$\frac{\partial W}{\partial \alpha_m} = \sum_{j=1}^{3} \int \frac{\phi_{jm}(\xi_j)}{\sqrt{2 \sum_{k=1}^{3} \alpha_k \phi_{jk}(\xi_j) - 2V_j(\xi_j)}} \, d\xi_j \tag{D25}$$

Thus, in the general case, ξ_j is to be found by inverting a function which is a sum of three terms each representing an integral. The difficulty of this task depends on the potential function and on the Stäckel matrix and must be assessed in each case. In practice approximations must be used to do the inversion.

LIST OF REFERENCES

1. Byrd, P. F. and Friedman, M. D., Handbook of Elliptic Integrals for Engineers and Physicists, Springer-Verlag, Berlin, 1954.

2. Cook, A. H., "Exact Solutions in the Theory of Orbits", Monthly Notices Royal Astronomical Society, V. 134, pgs. 253-268.

3. Eisenhart, L. P., "Separable Systems of Stäckel", Annals of Mathematics, V. 35, No. 2, April, 1934, pages 284-305.

4. Hildebrand, F. B., Advanced Calculus for Applications, Prentice-Hall, New Jersey, 1964.

5. Iszak, I. G., "A Theory of Satellite Motion about an Oblate Planet", Smithsonian Astrophysical Observatory Special Report No. 52, November, 1960.

6. Kellogg, O. D., Foundations of Potential Theory, Dover, New York, 1953.

7. Mason, M. and Weaver, W., The Electromagnetic Field, Dover, New York.

8. Pars, L. A., Analytical Dynamics, Wiley, New York, 1965.

9. Stäckel, P., "Uber Die Integration Der Hamilton-Jacobischen Differentialgleichung", Habilitationsschrift, Halle, 1891.

10. Stäckel, P., "Sur L'Intégration de L'Equation Différentielle de Hamilton", Academie des Sciences Comptes Rendus, V. 121, 1895, pgs. 489-492.

11. Vinti, J. P., "New Method of Solution for Unretarded Satellite Orbits", Journal of Research, National Bureau of Standards V. 63B, 1959, pgs. 105-116.

12. Vinti, J. P., "Theory of an Accurate Intermediary Orbit for Satellite Astronomy", Journal of Research, National Bureau of Standards, V. 65B, 1961, pgs. 169-201.

13. Weinacht, J., "Uber die Bedingt-Periodische Bewegung eines Massenpunktes", Mathematische Annalen, V. 91, 1924, pgs. 279-299.