QUANTUM THEORY OF AN ELECTRON GAS

WITH

ANOMALOUS MAGNETIC MOMENTS

IN

INTENSE MAGNETIC FIELDS

Hong-Yee Chiu*

and

Vittorio Canuto**

Institute for Space Studies
Goddard Space Flight Center, NASA
New York, New York

and

Laura Fassio-Canuto

Department of Physics
C.I.E.A., Ap. Post. 14740
Mexico 14, D.F.

*Also with Physics Dept. and Earth & Space Sciences Dept., State University of New York, Stony Brook, N.Y.

**On leave of absence from Dept. of Physics, C.I.E.A., Ap. Post. 14740, Mexico 14, D.F.

A B S T R A C T

In this paper we have continued our investigation of the properties of an electron gas in intense magnetic fields.[1,2,3,4] We have obtained expressions for the thermodynamic energy density particle density, the magnetic moment, and pair density (created in equilibrium with the system) for a gas of electrons with an anomalous magnetic moment, using the exact eigenvalues of the Dirac-Pauli equation for an electron with an anomalous magnetic moment, obtained by Ternov _et al_.[7] According to the solution of Ternov _et al_, the lowest energy states of the electrons can be zero at certain field strengths. We have shown that pair creation does not occur spontaneously at the expense of the magnetic field energy, but only at the expense of thermodynamic energy of other particles of the system. Exact expressions for the pair density is given.

I. Introduction.

In several previous papers[1-4] we have derived thermodynamic properties for an electron gas in a magnetic field, using the exact solution of the Dirac equation for an electron gas in a magnetic field.

. We have found that the gas becomes very anisotropic. Further, because the lowest energy state of the electron is unaltered, we concluded that spontaneous pair creation will not take place even when the classical spin energy $\mu_B \vec{\sigma} \cdot \vec{H}$ (where $\mu_B = \frac{|e|\hbar}{2mc}$ is the magnetic moment of the electron and H is the magnetic field) exceeds mc^2.

However, it is a well known[5] fact that the electron possesses an anomalous magnetic moment amounting to $\mu = \frac{\alpha}{2\pi}\mu_B$, in addition to its Dirac moment $\frac{|e|\hbar}{mc}$. The question then naturally arises: will this small amount of additional magnetic moment change our conclusions reached previously? In this paper we shall study the effect of the anomalous magnetic moment on the ~~thermodynamic properties~~ of matter. In general we shall consider field strengths of the order of $\frac{4\pi}{\alpha}\frac{m^2c^3}{e\hbar} \sim 4 \times 10^{16}$ gauss. Admittedly this field strength is rather high and may not be realized in nature. However, our main purpose is to investigate if spontaneous pair creation will really take place when the effect of the anomalous magnetic moment is included. It is shown later that in the presence of an anomalous magnetic moment, in certain cases the separation energy between the positive and negative energy states will become zero. However, we shall show that spontaneous pair creation still will not take place at the expense of magnetic field energy.

II. The Anomalous Magnetic Moment and the Dirac-Pauli Equation.

It is well known that the interaction of an electron with an anomalous magnetic moment within any external field A_μ can be described by the Dirac-Pauli equation [5]

$$\left[\gamma_\mu\left(\partial_\mu + \frac{i|e|}{\hbar c}A_\mu\right) - \mu\frac{i}{2\hbar c}F_{\mu\nu}\gamma_\mu\gamma_\nu + \frac{mc}{\hbar}\right]\psi = 0 \tag{1}$$

where μ is taken to give the correct values of the magnetic moment and it turns out that:[6]

$$\mu = \frac{\alpha}{2\pi}\frac{|e|\hbar}{2mc}$$

$F_{\mu\nu}$ is the electromagnetic tensor:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

Eq.(1) has been solved by Ternov *et al.*[7], giving the following energy eigen values ($x \equiv p_z/mc$, $H_c = m^2c^3/e\hbar$)

$$E(x,n,s) = mc^2\left[x^2 + \left\{\sqrt{1+\frac{H}{H_c}(2n+s+1)} + s\frac{\alpha}{4\pi}\frac{H}{H_c}\right\}^2\right]^{1/2} \tag{2}$$

$n = 0, 1, \ldots \qquad s = \pm 1.$

We have checked Ternov *et al.*[7] calculations and have confirmed their results.

III. The Properties of the Energy States.

The behavior of Eq.(2) is schematically shown in Figure 1 for the three cases: (1), the case of the spin equal to zero (i.e., no magnetic moment); (2), the magnetic moment is identical to the Dirac magnetic moment $\frac{|e|\hbar}{2mc}$; and (3), when the anomalous magnetic moment is included, in addition to the Dirac magnetic moment considered in (2). It is seen that the presence of a magnetic field in the z-direction is to cause the energy in the x and y directions to become quantized but there is a degeneracy between the cases with the principal quantum number n and the spin parallel to the field (s = +1) and the case with the principal quantum number n+1 and the spin antiparallel to the field(s= -1), when the magnetic moment of the electron takes the value of the Dirac magnetic moment. This degeneracy is, however, removed when the anomalous magnetic moment is included. The two degenerate levels then split , in much the same way as the removal of degeneracy in the Zeeman splitting of atomic spectral lines.

The lowest energy states of the electron in the absence of z-momentum ($p_z = x\,mc = 0$) are zero when the field strengths are given by

$$H = H_c\left(\frac{4\pi}{\alpha}\right)^2\left[\eta + \sqrt{\eta^2 + \left(\frac{\alpha}{4\pi}\right)^2}\,\right] \rightarrow \begin{cases} \simeq 2\eta H_c\left(\frac{4\pi}{\alpha}\right)^2 & (\eta \neq 0) \\ = \frac{4\pi}{\alpha} H_c & (\eta = 0) \end{cases} \tag{3}$$

where $\eta = 0,1,2,3,\ldots\infty$. For the case x=0, Eq. (2) becomes:

$$E(0,n,s) = \pm mc^2\left[\left\{1 + \frac{H}{H_c}(2n+s+1)\right\}^{1/2} + s\,\frac{\alpha}{4\pi}\,\frac{H}{H_c}\right] \tag{4}$$

We have calculated the first five energy states as functions of field strengths as shown in Figure 2.

Consider the case of n = 0 and s = -1. The energy state of an electron is then $mc^2\left[x^2+\left(1-\frac{\alpha}{4\pi}\frac{H}{H_c}\right)^2\right]^{1/2}$. If it were possible for the z-momentum to vanish identically, then the energy state of the electron in the state n=0 and s= -1 would be

$$E(0,1,-1) = + mc^2\sqrt{\left(1-\frac{\alpha}{4\pi}\frac{H}{H_c}\right)^2} = mc^2\left(1-\frac{\alpha}{4\pi}\frac{H}{H_c}\right) \tag{5}$$

which changes signs after the magnetic field is increased beyond the critical value $\frac{H}{H_c} = \frac{4\pi}{\alpha}$. Does this mean that electrons will become positrons at this field strength? This is only a fallacy as the value of x can never be zero. As long as there is one electron in the Universe the Fermi energy of the electron is non-zero, and the operation in Eq.(5) is not permissible and the sign of the energy state is an invariant property of the electron or the positron. This also means that the positive and negative energy levels of the electrons will never cross each other (non-crossing property). Naturally this implies that spontaneous pair creation in a magnetic field without an external energy source is forbidden.

It may be pointed out that the condition

$$\frac{H}{H_c} = \frac{4\pi}{\alpha} \tag{6}$$

can also be written as

$$H = \frac{4\pi e}{r_0^2} \tag{7}$$

where $r_o = e^2/mc^2$ is the classical radius of the electron. Eq.(7) is independent of $\hbar$ and is a classical result.[8] Similarly Eq.(3) can be written as

$$H = \frac{4\pi e}{r_0^2} \cdot \frac{4\pi}{\alpha} \left[\eta + \sqrt{\eta^2 + \left(\frac{\alpha}{4\pi}\right)^2} \right] \qquad (8)$$

III. The Magnetic Moment.

In previous papers[1-4] we have shown that spontaneous magnetization cannot take place in a noninteracting electron gas. This conclusion is not altered by the inclusion of the anomalous magnetic moment of the electron, as expected. The magnetic moment of an electron gas with an anomalous magnetic moment is obtained in a similar way as we have done previously; here we will not repeat the intermediate steps but will give the results ($\phi = kT/mc^2$, μ is the chemical potential in mc^2 units):

$$\begin{aligned}\frac{\mathcal{M}}{\mathcal{M}_0} = {} & \frac{1}{2} b_0^2 C_2\left(\frac{\phi}{b_0}, \frac{\mu}{b_0}\right) + \frac{1}{2} b_0(1-b_0) C_1\left(\frac{\phi}{b_0}, \frac{\mu}{b_0}\right) + \\ & + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 C_2\left(\frac{\phi}{a_n}, \frac{\mu}{a_n}\right) + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 C_2\left(\frac{\phi}{b_n}, \frac{\mu}{b_n}\right) - \\ & - \frac{H}{H_c}\sum_{n=1}^{\infty} \frac{n a_n}{a_n+b_n} C_1\left(\frac{\phi}{a_n}, \frac{\mu}{a_n}\right) - \frac{H}{H_c}\sum_{n=1}^{\infty} \frac{n b_n}{a_n+b_n} C_1\left(\frac{\phi}{b_n}, \frac{\mu}{b_n}\right) \\ & - \frac{1}{4}\sum_{n=1}^{\infty} a_n(a_n-b_n) C_1\left(\frac{\phi}{a_n}, \frac{\mu}{a_n}\right) + \frac{1}{4}\sum_{n=1}^{\infty} b_n(a_n-b_n) C_1\left(\frac{\phi}{b_n}, \frac{\mu}{b_n}\right)\end{aligned} \qquad (9)$$

where

$$\mathcal{M}_0 = \frac{2}{\pi^2}\frac{\mu_B}{\lambda_c^3}\,; \quad \lambda_c = \frac{\hbar}{mc}\,; \quad a_n = \left(1+2n\frac{H}{H_c}\right)^{1/2} + \frac{\alpha}{4\pi}\frac{H}{H_c}\,; \quad b_n = \left(1+2n\frac{H}{H_c}\right)^{1/2} - \frac{\alpha}{4\pi}\frac{H}{H_c} \qquad (10)$$

The functions $C_k(\phi, \mu)$ are defined by the following equations:

$$C_1(\phi,\mu) = \int_0^\infty [1+v^2]^{-1/2} \left\{ 1+\exp\left[\frac{(1+v^2)^{1/2}-\mu}{\phi}\right]\right\}^{-1} dv \quad ; \qquad C_2(\phi,\mu) = C_3(\phi,\mu) - C_1(\phi,\mu)$$

$$C_3(\phi,\mu) = \int_0^\infty [1+v^2]^{1/2} \left\{ 1+\exp\left[\frac{(1+v^2)^{1/2}-\mu}{\phi}\right]\right\}^{-1} dv \tag{11}$$

$$C_4(\phi,\mu) = \int_0^\infty \left\{ 1+\exp\left[\frac{(1+v^2)^{1/2}-\mu}{\phi}\right]\right\}^{-1} dv$$

The properties of the C_k functions ($k = 1,2,3,4$) have been studied extensively previously and we will not elaborate their properties here.

In Figure 3 we have plotted the magnetic moment of an electron gas as a function of density for the following values of field strengths: $\frac{H}{H_c} = 10^{-1} \frac{4\pi}{\alpha}$, $\frac{1}{2}\frac{4\pi}{\alpha}$ and $\frac{4\pi}{\alpha}$. Generally the induced magnetization is only 10^{-3} of the impressed field. This conclusion is almost identical to that obtained previously. We therefore conclude that the inclusion of the anomalous magnetic moment does not change our main conclusion that spontaneous magnetization cannot take place in a dense electron gas.

IV. Thermodynamic Properties.

The energy density U and particle density N of an electron gas can be obtained directly from the energy eigen values obtained earlier. In analogy to calculations presented in previous papers, we have:

$$\mathcal{U} = \frac{1}{2} \mathcal{U}_o \, \theta \left[\mathcal{U}_1 + \mathcal{U}_2 \right] \tag{12}$$

$$N = \frac{1}{2} N_o \, \theta \left[N_1 + N_2 \right] \tag{13}$$

where

$$\mathcal{U}_o = \frac{1}{\pi^2} \frac{mc^2}{\lambda_c^3} \quad ; \quad N_o = \frac{1}{\pi^2} \frac{1}{\lambda_c^3} \quad ; \quad \theta = \frac{H}{H_c} \tag{14}$$

and

$$\mathcal{U}_1 = \sum_{n=1}^{\infty} \int_0^{\infty} F_1(x,n) \, E_1(x,n) \, dx \tag{15}$$

$$\mathcal{U}_2 = \sum_{n=0}^{\infty} \int_0^{\infty} F_2(x,n) \, E_2(x,n) \, dx \tag{16}$$

$$N_1 = \sum_{n=1}^{\infty} \int_0^{\infty} F_1(x,n) \, dx \tag{17}$$

$$N_2 = \sum_{n=0}^{\infty} \int_0^{\infty} F_2(x,n) \, dx \tag{18}$$

$$F_r(x,n) = \left\{ 1 + \exp \frac{E_r(x,n) - \mu}{\phi} \right\}^{-1} \tag{19}$$

$$E_1(x,n) = \left\{ x^2 + \left[\sqrt{1+2n\theta} + \frac{\alpha}{4\pi}\theta \right]^2 \right\}^{1/2} \tag{20}$$

$$E_2(x,n) = \left\{ x^2 + \left[\sqrt{1+2n\theta} - \frac{\alpha}{4\pi}\theta \right]^2 \right\}^{1/2} \tag{21}$$

Introducing the C_k functions, Eqs.(15)-(18) can be reduced into convenient form with the transformation in the integration variables:

$$V = \frac{x}{a_n} \qquad \text{or} \qquad V = \frac{x}{b_n} \tag{22}$$

The results are:($b_n \neq 0$)

$$\mathcal{U}_1 = \sum_{n=1}^{\infty} a_n^2 \, C_3\left(\frac{\phi}{a_n}, \frac{\mu}{a_n} \right) \tag{23}$$

$$\mathcal{U}_2 = \sum_{n=0}^{\infty} b_n^2 \, C_3\left(\frac{\phi}{b_n}, \frac{\mu}{b_n} \right) \tag{24}$$

$$N_1 = \sum_{n=1}^{\infty} a_n \, C_4\left(\frac{\phi}{a_n}, \frac{\mu}{a_n} \right) \tag{25}$$

$$N_2 = \sum_{n=0}^{\infty} b_n \, C_4\left(\frac{\phi}{b_n}, \frac{\mu}{b_n} \right) \tag{26}$$

b_n vanishes when the lowest energy state of the electron is zero. When this happens for some values of n = m, we have:

$$\mathcal{U}_2 = \sum_{\substack{n=0 \\ n\neq m}}^{\infty} b_n^2\, C_3\left(\frac{\phi}{b_n}, \frac{\mu}{b_n}\right) + \mathcal{F}_1(\phi,\mu) \tag{27}$$

$$N_2 = \sum_{\substack{n=0 \\ n\neq m}}^{\infty} b_n\, C_4\left(\frac{\phi}{b_n}, \frac{\mu}{b_n}\right) + \mathcal{F}_0(\phi,\mu) \tag{28}$$

where

$$\mathcal{F}_0(\phi,\mu) = \int_0^{\infty} \left\{1 + \exp\left(\frac{x-\mu}{\phi}\right)\right\}^{-1} dx = \phi \ln\left(1 + \exp\frac{\mu}{\phi}\right) \tag{29}$$

$$\mathcal{F}_1(\phi,\mu) = \int_0^{\infty} \left\{1 + \exp\left(\frac{x-\mu}{\phi}\right)\right\}^{-1} x\, dx \tag{30}$$

V. Pair Creation.

We shall now study the most important case of interest, namely the pair creation phenomena when the field strength is close to $\frac{4\pi}{\alpha} H_c$ or $2\eta H_c \left(\frac{4\pi}{\alpha}\right)^2$, $\eta = 1,2,3,\ldots$. At these field strengths the lowest energy state of the electron is zero. It will be of great interest to see if pairs can be created spontaneously.

The equation which governs pair creation equilibrium is:

$$\mu_- + \mu_+ = 0 \tag{31}$$

or

$$\mu_- = -\mu_+ \equiv \mu \tag{32}$$

where μ_- and μ_+ are the chemical potential of the electron and positron respectively. The number density of positrons, n_+, is given by

$$n_+ = N_0 \left[N_1(-\mu, \phi) + N_2(-\mu, \phi) \right] \tag{33}$$

Charge neutrality condition requires that

$$n_- - n_+ = n_0 \tag{34}$$

where n_0 is some constant. Consider the case of vacuum, then $n_0 = 0$, and $\mu = 0$. In most cases of interest, all terms in n_- and n_+ are negligible except those with $b_m = 0$. We then find from Eq.(28) that

$$n_- = n_+ = N_0 \phi \ln \left[1 + \exp \frac{\mu}{\phi} \right] \tag{35}$$
$$\cong N_0 \phi \ln 2$$

which is proportional to the temperature. In the presence of matter and in the vicinity of some field strengths such that $b_m = 0$, we find that

$$n_- = N_0 \left[N_1(\mu, \phi) + N_2(\mu, \phi) + \phi \ln \left\{ 1 + \exp \frac{\mu}{\phi} \right\} \right] \tag{36}$$

$$n_+ \simeq N_0 \phi \ln \left[1 + \exp\left(-\frac{\mu}{\phi}\right) \right] \tag{37}$$

It is easily seen that the number of pairs vanishes at zero temperature. This means that the energy of pairs created comes from thermodynamic energy from other particles and not from the energy of the magnetic field.

However, the number of pairs created is directly proportional to the temperature in the nondegenerate limit ($\mu - 1 \gg \phi$) and is proportional to $\phi \exp(-\mu/\phi)$ in the degenerate limit. Additional processes like the pair annihilation process [9] $e^- + e^+ \longrightarrow \nu + \bar{\nu}$ can take place more favorably when a strong magnetic field is present. This will aid energy dissipation. However, this will not occur until the field strength exceeds $H = \frac{4\pi}{\alpha} H_c \simeq 10^{16}$ gauss. It is not known whether such a strong field may exist in nature.

VI. Conclusion.

We have discussed the thermodynamic properties of an electron gas in a magnetic field with an anomalous magnetic moment. According to the solutions of the Dirac equation with an anomalous magnetic moment, the lowest energy states of an electron can become zero when the field strengths exceed 10^{16} gauss. We have shown that spontaneous pair creation cannot take place at the expense of the field energy. We have also derived expressions for the magnetic moment, the energy density, the number density, and the pair density as functions of temperature and the chemical potential.

VII. Acknowledgements.

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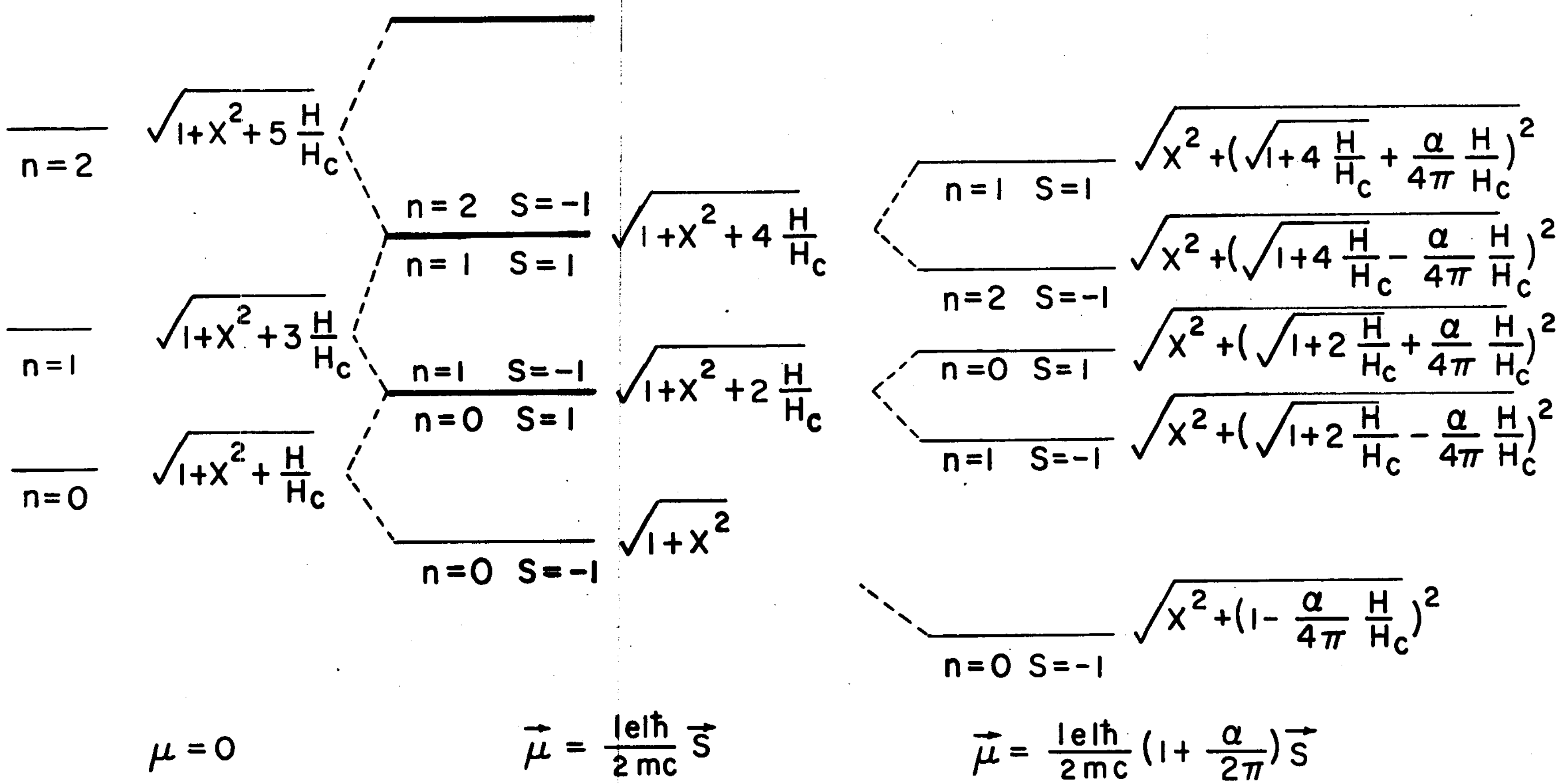
Figure Captions.

Figure 1. The behavior of the energy eigenvalues Eq.(2) in units of mc^2 for three cases explained in Section II. It is clear that the main effect of the anomalous magnetic moment is to remove the degeneracy between the levels with quantum numbers n and s=1, and n+1 and s=-1. Arbitrary scale used.

Figure 2. The energy levels of Eq.(2) (in units of mc^2) as a function of the parameter $\frac{H}{H_c}\frac{\alpha}{4\pi}$ for the case x=0. The lowest level n=0 and s=-1 is shown to reduce to zero for $\frac{H}{H_c}\frac{\alpha}{4\pi} = 1$.

Figure 3. The relation between the magnetic moment M ($M = \mathcal{M}/\mathcal{M}_0$, $\mathcal{M}_0 = 2\mu_B \pi^{-2} \lambda_c^{-3}$) and the electron density N/N_0 ($N_0 = \pi^{-2} \lambda_c^{-3}$) for the degenerate case. V. M exhibits the same undulating behavior as in the case without an anomalous magnetic moment (See reference 3). The various peaks correspond to the excitation of different magnetic states. At a given density the magnetic moment decreases as the field strength increases, until the first magnetic quantum state is excited.



n=2
$\sqrt{1+X^2+5\frac{H}{H_c}}$
n=1
$\sqrt{1+X^2+3\frac{H}{H_c}}$
n=0
$\sqrt{1+X^2+\frac{H}{H_c}}$
n=2 S=-1
n=1 S=1
$\sqrt{1+X^2+4\frac{H}{H_c}}$
n=1 S=-1
n=0 S=1
$\sqrt{1+X^2+2\frac{H}{H_c}}$
n=0 S=-1
$\sqrt{1+X^2}$
n=1 S=1
$\sqrt{X^2+\left(\sqrt{1+4\frac{H}{H_c}}+\frac{\alpha}{4\pi}\frac{H}{H_c}\right)^2}$
n=2 S=-1
$\sqrt{X^2+\left(\sqrt{1+4\frac{H}{H_c}}-\frac{\alpha}{4\pi}\frac{H}{H_c}\right)^2}$
n=0 S=1
$\sqrt{X^2+\left(\sqrt{1+2\frac{H}{H_c}}+\frac{\alpha}{4\pi}\frac{H}{H_c}\right)^2}$
n=1 S=-1
$\sqrt{X^2+\left(\sqrt{1+2\frac{H}{H_c}}-\frac{\alpha}{4\pi}\frac{H}{H_c}\right)^2}$
n=0 S=-1
$\sqrt{X^2+\left(1-\frac{\alpha}{4\pi}\frac{H}{H_c}\right)^2}$
$\mu=0$
$\vec{\mu}=\frac{|e|\hbar}{2mc}\vec{S}$
$\vec{\mu}=\frac{|e|\hbar}{2mc}\left(1+\frac{\alpha}{2\pi}\right)\vec{S}$

Fig. 1

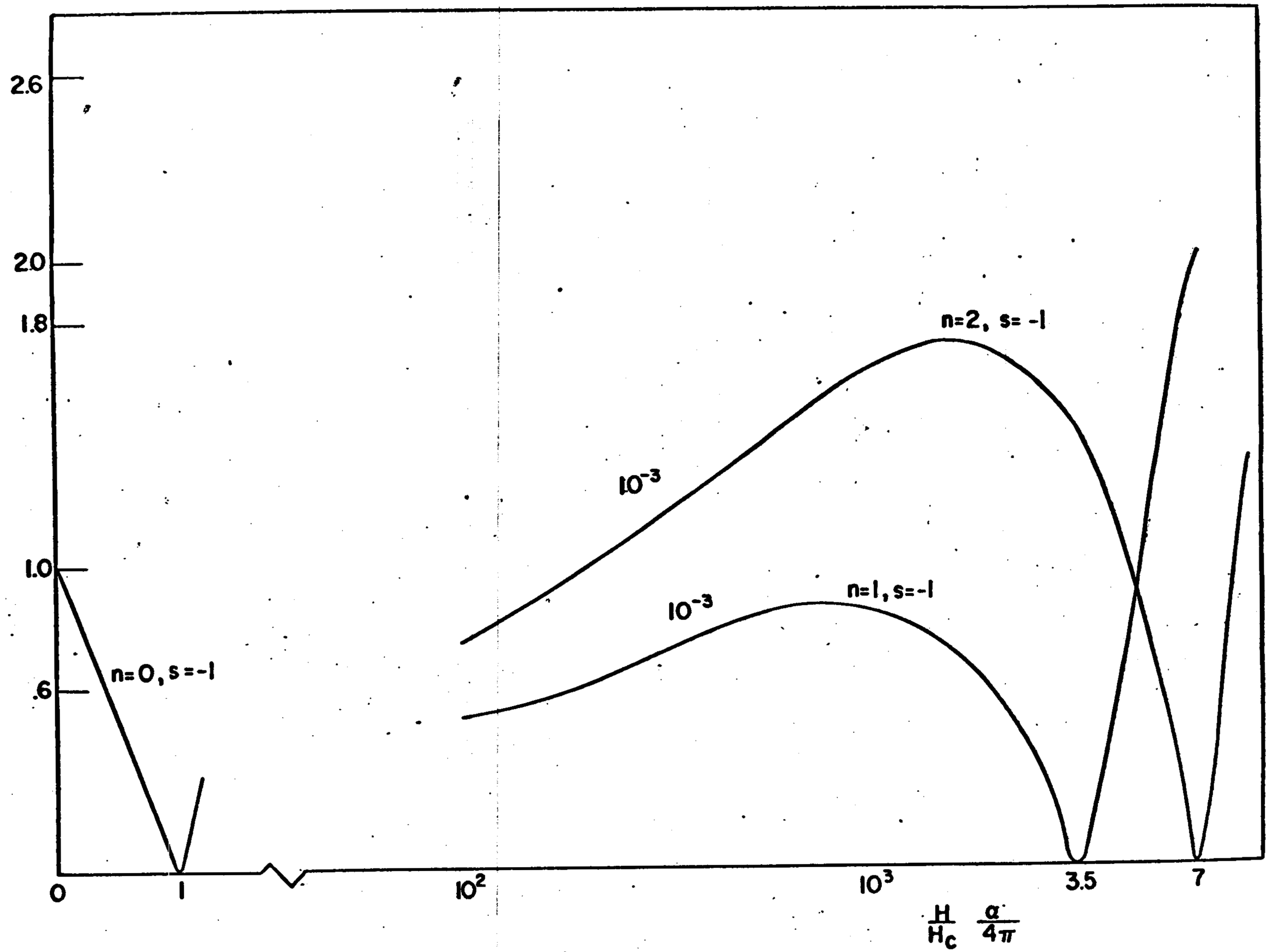
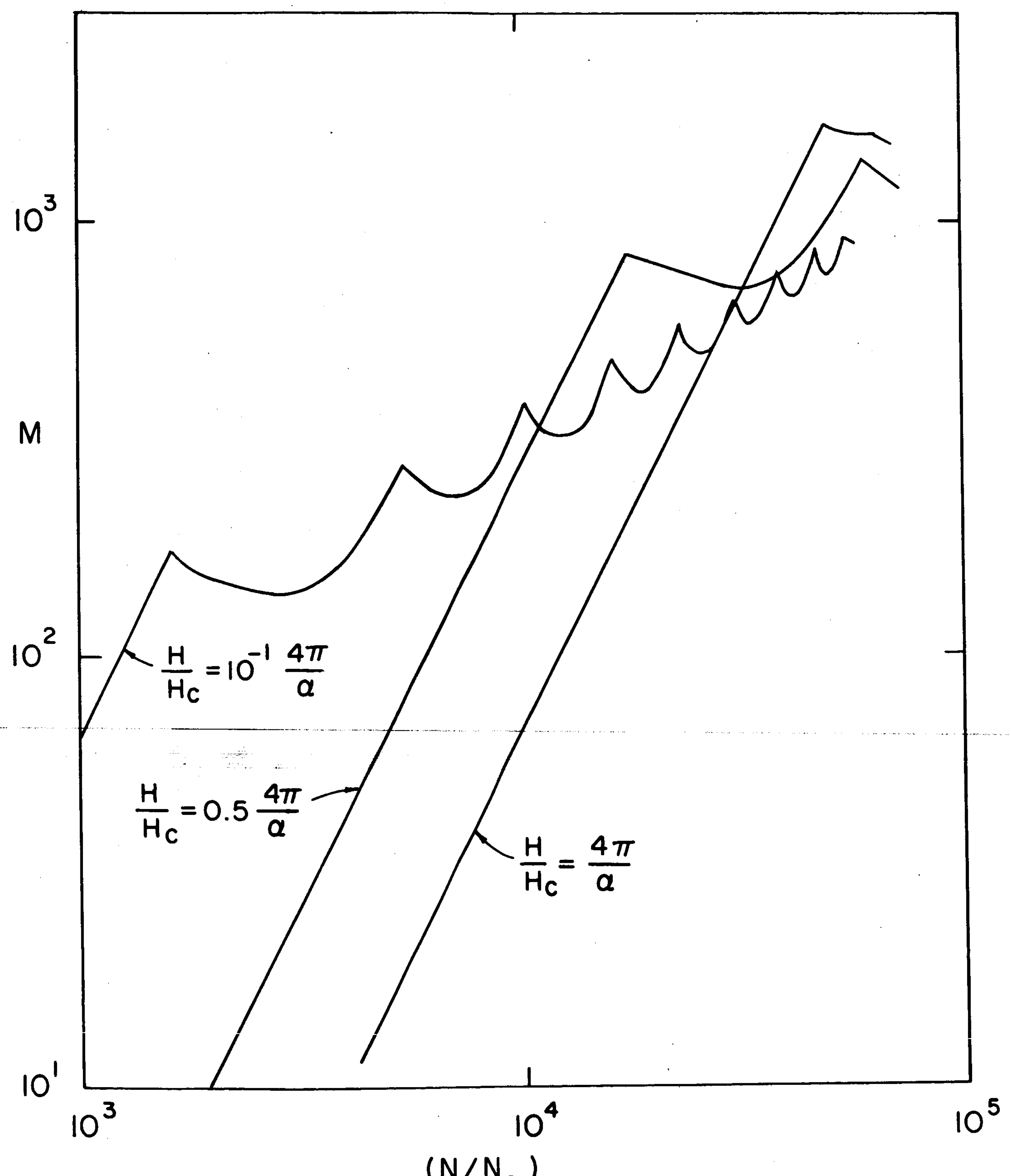

2.6
2.0
1.8
1.0
.6
n=0, s=-1
n=2, s= -1
n=1, s=-1
10^{-3}
10^{-3}
0
1
10^2
10^3
3.5
7
$\frac{H}{H_c}$ $\frac{\alpha}{4\pi}$

FIG 2



M
10^3
10^2
10^1
$\frac{H}{H_c} = 10^{-1}\,\frac{4\pi}{\alpha}$
$\frac{H}{H_c} = 0.5\,\frac{4\pi}{\alpha}$
$\frac{H}{H_c} = \frac{4\pi}{\alpha}$
10^3
10^4
10^5
(N/N_o)

Fig. 3