NASA TECHNICAL NOTE **NASA TN D-4771**

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NASA TN D-4771

CONFORMAL MAPPING FOR STEADY TWO-DIMENSIONAL SOLIDIFICATION ON A COLD SURFACE IN FLOWING LIQUID

by Robert Siegel

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • AUGUST 1968

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ABSTRACT

Conformal mapping has been applied to determine the shape of two-dimensional solidified layers formed on a cold surface immersed in a flowing warm liquid. The frozen region is represented by a rectangle in a potential plane, and the rectangle is mapped into the physical plane to determine the frozen boundary configuration. The method is demonstrated for solidification on a cold plate of finite width that is insulated along its edges.

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SUMMARY

A method has been developed by which conformal mapping is applied to determine the shape of two-dimensional solidified layers formed on a cold surface immersed in flowing warm liquid. The liquid supplies heat by convection to the frozen interface, and the shape of the frozen layer adjusts so that this energy can be conducted through the layer to the cold surface. The interface shape is obtained by mapping the solidified region into a potential plane and by determining the conformal transformations between the potential and physical planes. The heat flow through the frozen material is also found. The method is applied specifically to solidification on a cold plate of finite width that is insulated on the edges.

INTRODUCTION

A method is described for computing steady-state frozen-layer configurations formed by solidification on a cooled plate immersed in a flowing liquid. The liquid is warmer than the freezing temperature, and consequently heat is transferred by convection to the frozen-layer - liquid interface. This energy is conducted through the layer to the cooled surface. The frozen layer will grow until the energy convected to the interface is in balance with that conducted through the layer. A steady-state frozen layer will thus be achieved.

A relatively small amount of analysis has been done on freezing with either a convective boundary condition or in a two-dimensional geometry. The convective boundary condition was considered in reference 1, where a one-dimensional transient solution was obtained for solidification of a liquid flowing over a flat plate. An example of two-dimensional solidification is reported in reference 2, which deals with freezing inside a square prism with the liquid always at the fusion temperature; for this condition there is no heat convected to the frozen interface. The time-varying frozen-layer configurations were determined approximately. References 3 and 4 considered steady and transient solidified layers in flow channels that are symmetric in cross section, being either a circular tube or a gap between parallel plates. The layers were two-dimensional since they varied in thickness along the channel length. However, axial conduction within the solidified layer was neglected so that the solidification portion of the analysis was one-dimensional at each axial location along the channel. The present analysis will treat only a steady-state situation, but it will deal with the complexities of a convective boundary condition and two-dimensionality.

In the present problem the free-stream temperature of the flowing liquid and the convective heat-transfer coefficient at the solidified-layer - liquid interface are assumed constant. Then, since the freezing temperature is also constant, the convective heat flux supplied to the frozen interface is constant. The interface is thus one of constant heat flux and constant temperature; the shape of the interface will form to meet these constraints.

A conformal mapping procedure is developed to determine the shape of the free boundary of the solidified layer. The boundary conditions on the frozen layer permit mapping it into a region of a potential plane. Conformal transformations are used to determine the frozen-layer shape from the temperature distribution represented by the region in the potential plane. The heat flow through the frozen layer can then be found.

SYMBOLS

A	dimensionless half width of plate, a/γ
a	half width of plate
b	intermediate parameter in mapping
C_1, C_2, C_3, C_4	constants of integration
h	heat transfer coefficient from flowing liquid to frozen interface
K	elliptic integral
k	thermal conductivity of solidified material
L	length of plate
N	dimensionless outward normal, n/γ
n	outward normal to interface
Q	heat flow rate through frozen layer
q	heat flux

S	dimensionless frozen-layer - liquid interface, s/γ
s	frozen-layer - liquid interface
T	dimensionless temperature, $(t - t_w)/(t_f - t_w)$
t	temperature
t_f	freezing temperature
t_l	liquid temperature
t_w	surface temperature of cold plate
u	intermediate mapping plane, $\xi + i\eta$
V	intermediate mapping plane, $\alpha + i\beta$
W	analytic function, $-T + i\psi$
X, Y	dimensionless coordinates x/γ, y/γ
x, y	Cartesian coordinates in physical plane
y_{thin}	thickness of thin layer
Z	dimensionless physical plane, $X + iY$
z	physical plane, $x + iy$
α, β	coordinates of V plane
γ	length scale parameter, $\frac{k}{h} \frac{t_f - t_w}{t_l - t_f}$
ζ	quantity defined as $-\frac{\partial T}{\partial X} + i \frac{\partial T}{\partial Y}$
θ	argument of ζ plane
ξ, η	coordinates of u plane
φ	dummy integration variable
ψ	coordinate in W plane
ω	mapping plane, $\log \lvert\zeta\rvert + i\theta$

ANALYSIS

This analysis will develop an analytical method which applies conformal mapping to determine the shape of two-dimensional steady-state frozen regions formed in a flowing liquid. The method will be applied to the specific geometry of freezing on a constant-

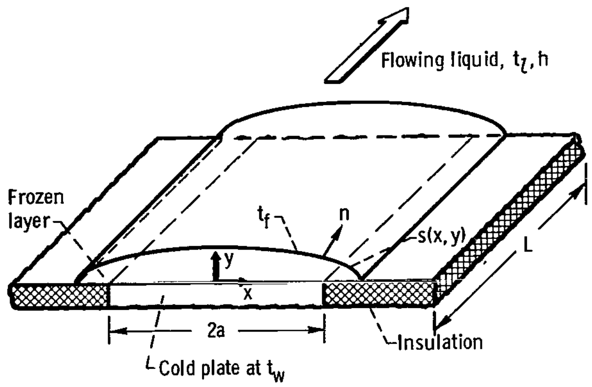

Figure 1. - Two-dimensional frozen layer formed on cold plate immersed in flowing liquid.

temperature plate of finite width as shown in figure 1. The plate, of width 2a, is insulated on its two sides; the length of the plate L is large, so that the configuration can be considered two dimensional. A liquid with a bulk temperature t_l that is higher than the freezing temperature t_f is flowing in the direction along the plate length. The flow provides a constant heat-transfer coefficient h between the liquid and the surface of the frozen layer at t_f.

Because h, t_l, and t_f are all constant, the heat flux transferred to the surface of the frozen layer is uniform over that surface. Thus, the boundary conditions on the frozen-layer - liquid interface are both uniform heat flux and uniform temperature:

$$q(s) = k \left. \frac{\partial t}{\partial n} \right|_s = h(t_l - t_f) \tag{1a}$$

$$t(s) = t_f \tag{1b}$$

The shape of the frozen-layer - liquid interface will adjust until the heat transferred to it by the flowing liquid is exactly balanced by the heat conducted through the frozen layer to the cooled plate.

When utilizing conformal mapping it is best to define dimensionless variables so that the dimensionless temperature and its normal derivative are unity at the frozen interface. Thus, the variable is defined as

$$T \equiv \frac{t - t_w}{t_f - t_w}$$

Then, at the plate surface

$$T = 0 \qquad (-a \leq x \leq a; \quad y = 0) \tag{2a}$$

and at the frozen-layer - liquid interface

$$T(s) = 1 \tag{2b}$$

The boundary-condition equation (1a) becomes

$$\left.\frac{\partial T}{\partial n}\right|_s = \frac{h}{k} \frac{t_l - t_f}{t_f - t_w}$$

A length scale parameter γ is defined as

$$\gamma \equiv \frac{k}{h} \frac{t_f - t_w}{t_l - t_f}$$

so that the temperature derivative can also be normalized to unity:

$$\left.\frac{\partial T}{\partial\left(\frac{n}{\gamma}\right)}\right|_s = \left.\frac{\partial T}{\partial N}\right|_S = 1 \tag{2c}$$

With all lengths nondimensioned by γ (i.e., $X = x/\gamma$, $Y = y/\gamma$, $A = a/\gamma$), the frozen layer with its associated boundary conditions is shown in figure 2 in the dimensionless physical plane $Z = X + iY$.

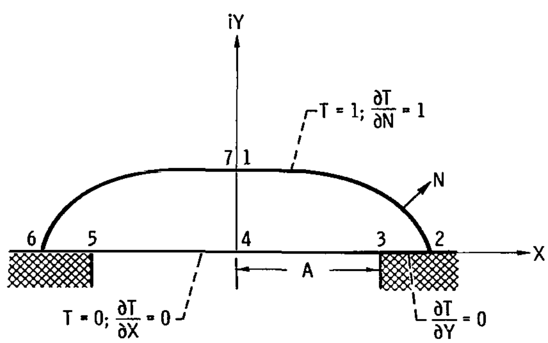

Figure 2. - Frozen layer in dimensionless physical plane, Z = X + iY.

The determination of the shape of the interface between the frozen layer and liquid is similar to the computation of the boundary of a free inviscid jet flowing into a constant-pressure environment. The velocity along the jet boundary is a constant as specified by Bernoulli's equation, but the jet boundary configuration is unknown. In the present case the frozen-layer - liquid interface geometry is again unknown, and the boundary conditions along the frozen-layer - liquid interface are known constants.

The conformal mapping procedure used for the freezing problem is based on the fact that the temperature distribution within the frozen layer is a solution to the Laplace heat conduction equation. Since the frozen region is simply connected, the temperature distribution can be taken as the real part of an analytic function W. In the case of heat conduction, the positive flow of heat is in the direction of the negative temperature gradient. As a result the potential function (real part of W) is -T, and the analytic function can be written as

$$W = -T + i\psi \tag{3}$$

The lines of constant T and constant ψ form an orthogonal net. By using conformal transformations between the W plane and the physical plane, the transformed functions will always be analytic and, hence, in the physical plane the temperatures will satisfy the Laplace equation. Since a function which is analytic in a given region is completely determined by its boundary values, it is only necessary to require that this mapping satisfy the boundary conditions.

The isotherms and adiabatic lines bounding the frozen layer are shown in figure 3, where the layer becomes a rectangle. Vertical lines in the W plane are isotherms, and horizontal lines are heat flow lines that are normal to the isotherms. Hence, for the present problem the segments 2-3 and 5-6 in the Z plane become horizontal lines in the W plane since along these lines $\partial T/\partial Y = 0$ and they are consequently normal to the isotherms. The height of the rectangle is not known but will be determined later to obtain the heat flow through the frozen layer.

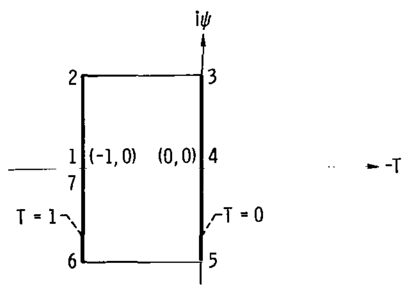

Figure 3. - Potential plane, $W = -T + i\psi$.

If the mapping between the W and Z planes is known, then the line 2-6 in the W plane can be transformed back to the physical plane to determine the interface configuration. After the mapping is completed, the heat flow through the frozen layer can be computed.

To find the relation between the W and Z planes, the condition is utilized that the value of the derivative of an analytic function is independent of direction. Consequently,

$$\frac{dW}{dZ} = -\frac{\partial T}{\partial X} + i\frac{\partial \psi}{\partial X} \tag{4}$$

Because the temperature distribution within the steady-state frozen layer is governed by the Laplace equation, the Cauchy-Riemann equations can be used to further relate derivatives of ψ and T. Using the relation

$$\frac{\partial \psi}{\partial X} = \frac{\partial T}{\partial Y} \tag{5}$$

equation (4) can be written as

$$\frac{dW}{dZ} = -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y} \tag{6}$$

Now a new quantity ζ is defined as

$$\zeta \equiv -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y} \tag{7}$$

Then equation (6) becomes

$$\frac{dW}{dZ} = \zeta$$

After integrating,

$$Z = \int \frac{1}{\zeta}\, dW + C_1 \tag{8}$$

where C_1 is a constant of integration. Equation (8) shows that the connection between the physical plane and the potential plane depends on the function ζ which must be related to W before the integration can be performed.

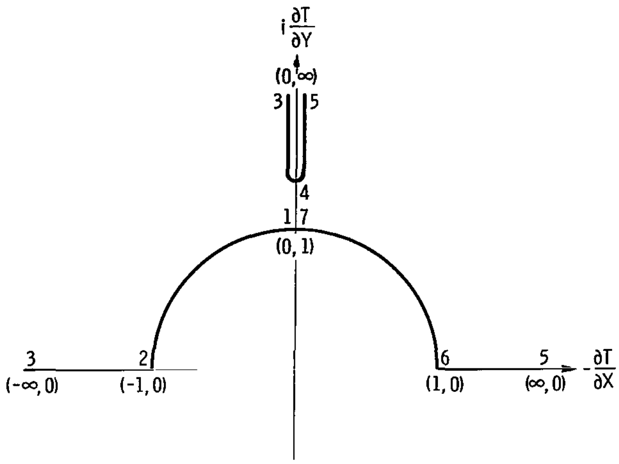

Figure 4. - Temperature derivative plane, $\zeta = -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y}$.

To examine the relation between ζ and W, a temperature derivative plane is constructed, yielding figure 4 for the present problem. The interface 2-1-7-6 has $\partial T/\partial N = 1$ all along it since the interface is at constant temperature and the heat flow is normal to the constant temperature lines. As a result the interface transforms to a unit semicircle on the ζ plane. Along 2-3 and 5-6, $\partial T/\partial Y = 0$ because the boundary is perfectly insulated, but $\partial T/\partial X$ varies from +1 and -1 at points 2 and 6 to $+\infty$ and $-\infty$ at the corners 3 and 5 where the heat flow lines have infinite curvature. Along 3-4-5, $\partial T/\partial X = 0$ since the boundary is at constant temperature. The $\partial T/\partial Y$ varies from ∞ at the corners 3 and 5 to a value at 4 that is unknown at this point in the analysis.

To map the region in the ζ plane into the W plane and thereby determine the relation between ζ and W, a logarithmic transformation is first applied:

$$\omega = \log \zeta = \log |\zeta| + i\theta \tag{9}$$

where θ is the argument of ζ. This maps the semicircle into a straight line and yields the configuration in figure 5.

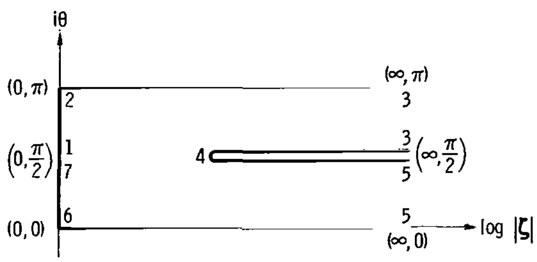

Figure 5. - Intermediate ω plane, $\omega = \log |\zeta| + i\theta$.

The region in figure 5 is in the form of a generalized rectangle which can be mapped into the positive half plane by using a Schwarz-Christoffel transformation. This transformation has sufficient degrees of freedom so that the points 3 and 5 can be mapped to -1 and +1 on the real axis. Points 2 and 6 are at -b and +b, where $|b| < 1$ and is a quantity that is to be determined. The transformation is

$$\frac{d\omega}{du} = \frac{C_2}{\sqrt{u - b}\sqrt{u + b}\,(u - 1)(u + 1)} \tag{10}$$

and the configuration in the u plane is shown in figure 6.

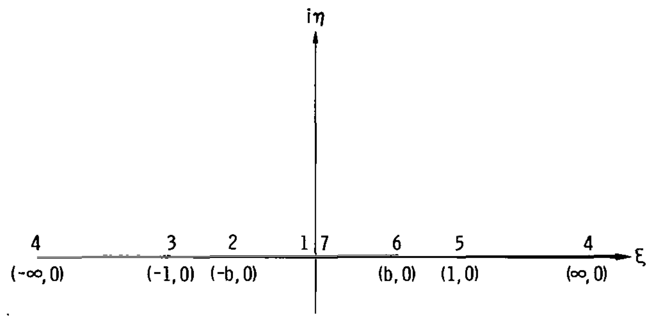

Figure 6. - Intermediate u plane, $u = \xi + i\eta$.

Equation (10) is integrated to yield

$$\omega = -\frac{C_2}{2\sqrt{1 - b^2}} \log \frac{\sqrt{u^2 - b^2} + u\sqrt{1 - b^2}}{\sqrt{u^2 - b^2} - u\sqrt{1 - b^2}} + C_3$$

To have point 1 at $\omega = i\pi/2$ correspond to $u = 0$, the constant $C_3 = i\pi/2$. Also, as u approaches b and -b, ω approaches 0 and $i\pi$, respectively; these conditions require that $C_2 = -\sqrt{1 - b^2}$. Then,

$$\omega = \frac{1}{2} \log \frac{\sqrt{u^2 - b^2} + u\sqrt{1 - b^2}}{\sqrt{u^2 - b^2} - u\sqrt{1 - b^2}} + \frac{i\pi}{2} \tag{11}$$

The upper half of the u plane of figure 6 is then mapped into a rectangle in the V

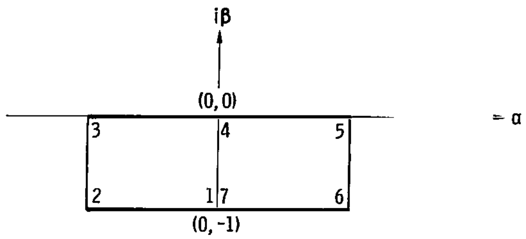

Figure 7. - Intermediate V plane, $V = \alpha + i\beta$.

plane as shown in figure 7. The mapping transformation is found from the Schwarz-Christoffel theorem as

$$\frac{dV}{du} = \frac{C_4}{\sqrt{u - b}\sqrt{u + b}\sqrt{u - 1}\sqrt{u + 1}} \tag{12}$$

The final mapping from the V plane to the W plane of figure 3 is a minus 90° rotation given by

$$W = -iV \tag{13}$$

With the mapping relations established between ζ and W, equation (8) can be used to transform to the dimensionless physical plane by writing the integral in the form

$$Z = \int \frac{1}{\zeta(u)} \frac{dW}{dV} \frac{dV}{du} du + C_1 \tag{14}$$

To find $\zeta(u)$, equation (9) is used in the form $\zeta = e^{\omega}$. Substituting ω from equation (11) and noting that $e^{i\pi/2} = i$ gives

$$\zeta = i\left(\frac{\sqrt{u^2 - b^2} + u\sqrt{1 - b^2}}{\sqrt{u^2 - b^2} - u\sqrt{1 - b^2}}\right)^{1/2} \tag{15}$$

Then with dV/du from equation (12) and $dW/dV = -i$, equation (14) becomes

$$Z = -\int \left(\frac{\sqrt{u^2 - b^2} - u\sqrt{1 - b^2}}{\sqrt{u^2 - b^2} + u\sqrt{1 - b^2}} \right)^{1/2} \frac{C_4}{\sqrt{u^2 - b^2}\,\sqrt{u^2 - 1}}\,du + C_1 \tag{16}$$

Multiplying both numerator and denominator of the term in parentheses by $\sqrt{u^2 - b^2} - u\sqrt{1 - b^2}$ yields

$$Z = -\int \frac{\sqrt{u^2 - b^2} - u\sqrt{1 - b^2}}{\left(u^2 - b^2 - u^2 + u^2b^2\right)^{1/2}} \frac{C_4}{\sqrt{u^2 - b^2}\,\sqrt{u^2 - 1}}\,du + C_1$$

This can be further simplified to

$$Z = -\frac{C_4}{b}\left(\int \frac{du}{u^2 - 1} - \sqrt{1 - b^2}\int \frac{u\,du}{\sqrt{u^2 - b^2}\,(u^2 - 1)}\right) + C_1 \tag{17}$$

Equation (17) is integrated to obtain

$$Z = \frac{C_4}{2b}\left(-\log\frac{u - 1}{u + 1} + \log\frac{\sqrt{u^2 - b^2} - \sqrt{1 - b^2}}{\sqrt{u^2 - b^2} + \sqrt{1 - b^2}}\right) + C_1 \tag{18}$$

where there is a branch cut between $-b$ and $+b$ along the real axis of the u plane.

The constants are evaluated by having $Z = 0$ at $u = \pm\infty$ and $Z = +A$ and $-A$ at $u = -1$ and $+1$, respectively. This gives

$$C_1 = 0$$

and

$$C_4 = \frac{Ab}{\log\sqrt{1 - b^2}}$$

Then,

$$\frac{z}{a} = \frac{Z}{A} = \frac{1}{2 \log \sqrt{1 - b^2}} \left(\log \frac{\sqrt{u^2 - b^2} - \sqrt{1 - b^2}}{\sqrt{u^2 - b^2} + \sqrt{1 - b^2}} - \log \frac{u - 1}{u + 1} \right) \tag{19}$$

The transformed frozen-layer - liquid interface extends between $\xi = +b$ and $\xi = -b$ on the real axis of the u plane in figure 6. If ξ is introduced for u in equation (19), where $|\xi| \leq b$, then the real and imaginary parts can be computed to obtain the x and y coordinates of the frozen-layer - liquid interface:

$$\left.\frac{x}{a}\right|_s = \frac{1}{2 \log \sqrt{1 - b^2}} \log \left(\frac{1 + \xi}{1 - \xi} \right) \tag{20a}$$

$$\left.\frac{y}{a}\right|_s = \frac{-1}{\log \sqrt{1 - b^2}} \tan^{-1} \sqrt{\frac{b^2 - \xi^2}{1 - b^2}} \tag{20b}$$

where $-b \leq \xi \leq b$.

The quantity b must now be related to the imposed physical conditions. It will be found that b is a function of only the single parameter $[(t_f - t_w)/(t_l - t_f)](k/ha)$. This will also lead to a relation between this parameter and the total amount of heat being transferred through the frozen layer.

The heat flow through the frozen layer and into the cold plate can be written in terms of the temperature gradient at the plate:

$$Q = 2L \int_4^3 k \frac{\partial t}{\partial y} dx \tag{21}$$

where the integration limits are the numbered positions in figure 2. Equation (21) is placed in dimensionless form:

$$\frac{Q}{2kL(t_f - t_w)} = \int_4^3 \frac{\partial T}{\partial Y} dX \tag{22}$$

By using equation (5), equation (22) can be expressed in terms of the stream function and then integrated:

$$\frac{Q}{2kL(t_f - t_w)} = \int_4^3 \frac{\partial\psi}{\partial X} dX = \psi(3) - \psi(4) \tag{23}$$

From figure 3, $\psi(3) = \psi(2)$ and $\psi(4) = \psi(1)$, so that

$$\frac{Q}{2kL(t_f - t_w)} = \psi(2) - \psi(1) \tag{24}$$

For the problem being considered here equation (24) is more convenient than equation (23) because points 2 and 1 are at finite values of u in figure 6, while point 4 is at $u = \pm\infty + i0$. This will make the integration limits finite in equation (25).

To find $\psi(2) - \psi(1)$ and thereby evaluate Q, equation (3) is used:

$$\psi(2) - \psi(1) = \mathcal{Im}\left[W(2) - W(1)\right] = \mathcal{Im}\int_{u=0}^{-b} \frac{dW}{dV}\frac{dV}{du} du \tag{25}$$

where the integration is performed for u along the real axis in figure 6. Then, using equations (12) and (13) and noting that $u = \xi$ on the real axis,

$$\frac{Q}{2kL(t_f - t_w)} = \int_{\xi=0}^{-b} \frac{C_4}{\sqrt{b^2 - \xi^2}\sqrt{1 - \xi^2}} d\xi \tag{26}$$

The constant C_4 was previously evaluated as

$$C_4 = \frac{Ab}{\log\sqrt{1 - b^2}} = \frac{ha}{k}\frac{t_l - t_f}{t_f - t_w}\frac{b}{\log\sqrt{1 - b^2}} \tag{27}$$

Inserting equation (27) into equation (26) yields

$$\frac{Q}{2kL(t_f - t_w)} = \frac{ha}{k}\frac{t_l - t_f}{t_f - t_w}\frac{b}{\log\sqrt{1 - b^2}}\int_0^{-b} \frac{d\xi}{\sqrt{b^2 - \xi^2}\sqrt{1 - \xi^2}} \tag{28}$$

A second relation for C_4 can be found by noting that in figure 3 the size of the rectangle must be such that $T(2) - T(3) = 1$. Then in a manner similar to the derivation of equations (25) and (26)

$$1 = -\left[W(2) - W(3)\right] = -\int_{-1}^{-b} \frac{dW}{dV}\frac{dV}{du}\,du = -\int_{\xi=-1}^{-b} \frac{C_4}{\sqrt{\xi^2 - b^2}\,\sqrt{1 - \xi^2}}\,d\xi$$

Solving for C_4 and inserting into equation (26) yields

$$\frac{Q}{2kL(t_f - t_w)} = -\frac{\displaystyle\int_0^{-b} \frac{d\xi}{\sqrt{b^2 - \xi^2}\,\sqrt{1 - \xi^2}}}{\displaystyle\int_{-1}^{-b} \frac{d\xi}{\sqrt{\xi^2 - b^2}\,\sqrt{1 - \xi^2}}} \tag{29}$$

By substituting equation (29) for Q into equation (28) the result can be rearranged into

$$\frac{k}{ha}\,\frac{t_f - t_w}{t_l - t_f} = -\frac{b}{\log \sqrt{1 - b^2}} \int_{-1}^{-b} \frac{d\xi}{\sqrt{\xi^2 - b^2}\,\sqrt{1 - \xi^2}} \tag{30}$$

The integrals in equations (29) and (30) are elliptic integrals that can be placed in the following forms by using trigonometric substitutions (ref. 5):

$$\frac{Q}{2kL(t_f - t_w)} = \frac{K(b)}{K\left(\sqrt{1 - b^2}\right)} \tag{31}$$

$$\frac{k}{ha}\,\frac{t_f - t_w}{t_l - t_f} = -\frac{b}{\log \sqrt{1 - b^2}}\,K\left(\sqrt{1 - b^2}\right) \tag{32}$$

where

$$K(b) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - b^2 \sin^2 \varphi}}$$

$$K\left(\sqrt{1 - b^2}\right) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - (1 - b^2) \sin^2 \varphi}}$$

Equations (20), (31), and (32) constitute the solution for the frozen layer being considered here. The physical quantities are related implicitly through the quantity b. When a value is chosen for b between 0 and 1, equations (20) give the configuration of the frozen layer, equation (31) gives the heat flow through the layer, and equation (32) gives the magnitude of the physical parameter $(k/ha)\left[(t_f - t_w)/(t_l - t_f)\right]$ associated with the layer.

RESULTS AND DISCUSSION

By using equation (32) the values of b corresponding to various $(k/ha)\left[(t_f - t_w)/(t_l - t_f)\right]$ can be found. Then these b values are used in equations (20) to obtain the coordinates of the frozen-layer - liquid interface. The frozen-layer configuration is shown in figure 8.

Thin layers result when the parameter $(k/ha)\left[(t_f - t_w)/(t_l - t_f)\right]$ is small (fig. 8(a)). This would be expected because a small value of the parameter occurs when the cooling of the plate is small and there is a large convective energy flux supplied to the frozen interface. The frozen layer forms to the proper configuration so that the convective flux can be conducted through it to the cold plate. For very thin layers, the layer approaches a constant thickness, and the heat flux through the layer is one dimensional. Then, letting the one-dimensional thickness be y_{thin},

$$h(t_l - t_f) = \frac{k(t_f - t_w)}{y_{thin}}$$

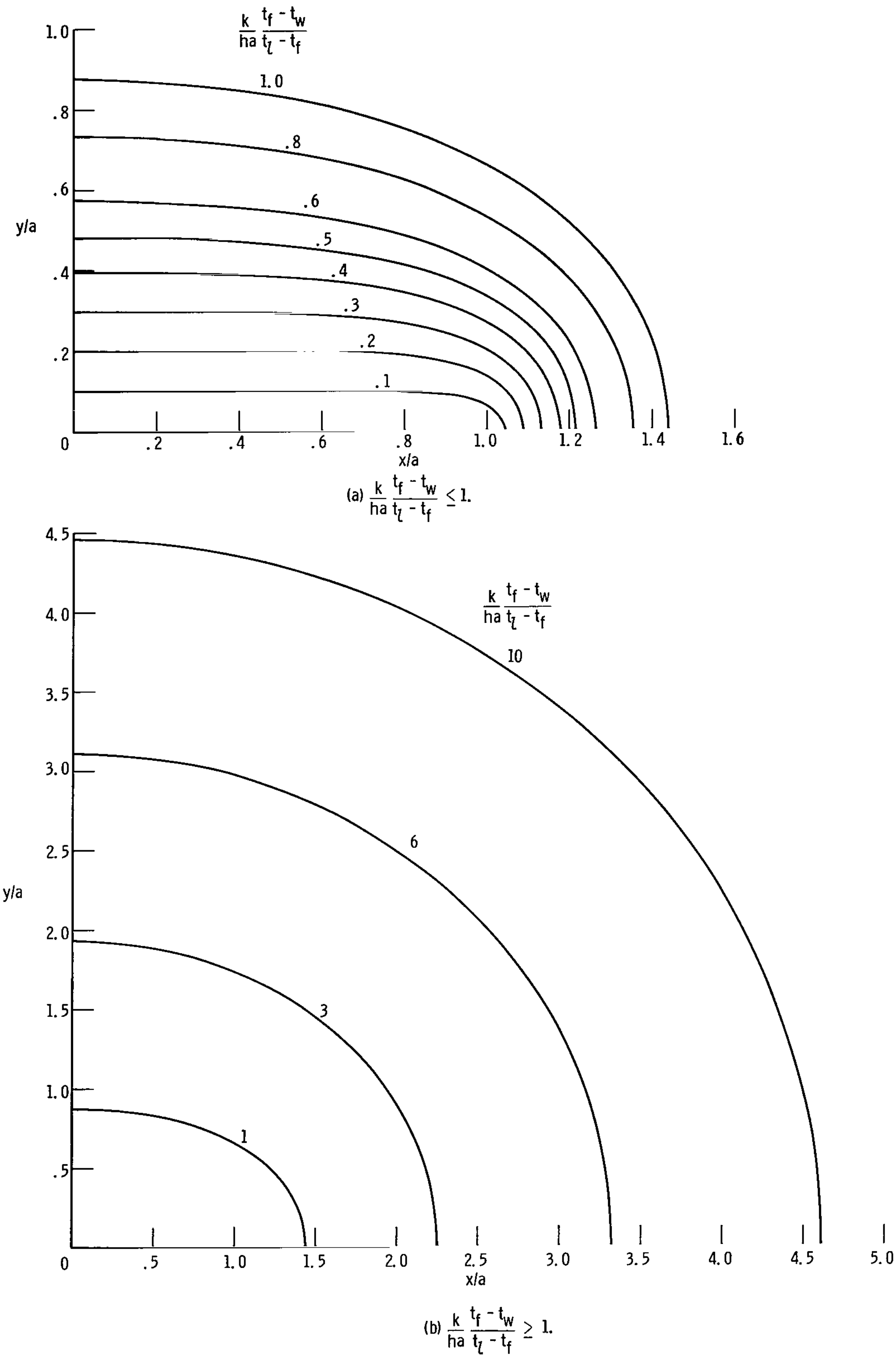

Figure 8. - Configuration of frozen layer for various values of the imposed conditions.

Solving for the dimensionless thickness,

$$\frac{y_{thin}}{a} = \frac{k}{ha} \frac{t_f - t_w}{t_l - t_f}$$

The thickness in figure 8(a) thus approaches the value of the parameter $(k/ha)\left[(t_f - t_w)/(t_l - t_f)\right]$ as the parameter becomes small.

For large values of $(k/ha)\left[(t_f - t_w)/(t_l - t_f)\right]$, as shown in figure 8(b), the frozen layer becomes quite thick and approaches a circular shape. In the limit of very large $(k/ha)\left[(t_f - t_w)/(t_l - t_f)\right]$ the width of the plate is small relative to the frozen-layer thickness, and the configuration approaches a two-dimensional sink with the heat flowing radially through the layer to the cold plate.

By inserting various b values into equations (31) and (32) the dimensionless heat flow through the layer $Q/2kL(t_f - t_w)$ can be evaluated in terms of the parameter $(k/ha)\left[(t_f - t_w)/(t_l - t_f)\right]$. This is shown in figure 9. As expected, the heat flow decreases

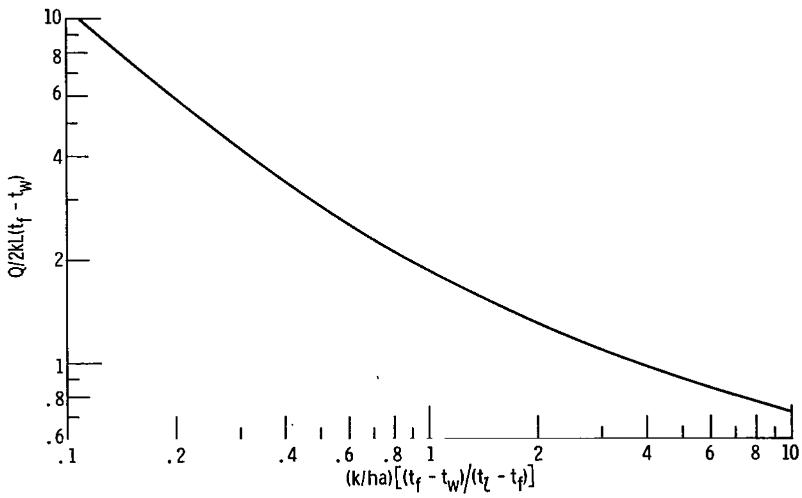

Figure 9. - Dimensionless heat flow through frozen layer as function of physical parameter $(k/ha)[(t_f - t_w)/(t_l - t_f)]$.

continuously as the frozen-layer thickness increases. For very thin layers the heat flow becomes one-dimensional and flows parallel to the y-axis. The surface area of the frozen interface approaches 2aL. Then the heat flow is $Q = h(t_l - t_f)2aL$, or, in dimensionless form,

$$\frac{Q}{2kL(t_f - t_w)} = \frac{ha}{k} \frac{t_l - t_f}{t_f - t_w}$$

CONCLUDING REMARKS

Conformal mapping has been applied for determining two-dimensional steady-state configurations of a solidified layer formed on a cooled plate immersed in a flowing liquid. The frozen layer is mapped into a potential plane $W = -T + i\psi$ where the negative of the dimensionless temperature is the potential function for heat flow. The potential plane is related to the physical plane by means of a function involving temperature derivatives:

$$\zeta = -\frac{\partial T}{\partial X} + i \frac{\partial T}{\partial Y}$$

The conditions at the frozen-layer boundary are used to determine the form of the ζ function. Then conformal mapping is applied to relate ζ to W, and the physical configuration is found from the integral

$$Z = \int \frac{1}{\zeta(W)} dW$$

The method was demonstrated specifically in the analysis of solidification on a cooled plate of finite width. An example for further application would be to determine the shape of solidified material inside noncircular liquid flow channels with cold walls. These types of channels are found in heat exchangers that are cooled cryogenically.

Lewis Research Center,
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