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A MATRIX METHOD FOR STUDYING MOTIONS OF SPACECRAFT CONSISTING OF INTERCONNECTED RIGID BODIES

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SUMMARY

A system of matrix equations has been developed which describes the interaction between two rigid bodies joined at a point. The relative motion of the bodies is treated by projection matrices. A projection matrix is found which determines the angular acceleration allowed by relative angular motion about a common attachment point. Another projection matrix is found which determines the components of relative angular acceleration of the two bodies caused by physical constraints on their relative angular motion. By using these matrices, an equation is found that relates the motions of the two bodies where physical rotational constraints are automatically applied. This equation, the translational momentum equation, the angular momentum equation, and the translational constraint equation form a set of equations which can be used to add rigid rotating bodies to a dynamical system to achieve a new dynamical system of any desired complexity. This method is useful when constraint forces and torques must be calculated.

An example dynamical system which consists of a central hub with four rigid wings connected to the hub at four different attachment points is studied. The equations of motion for this system were integrated numerically, and some parameters which describe the motion of the system are presented graphically.

INTRODUCTION

Many spacecraft must erect or deploy in space panels such as solar cell panels and heat radiators or they must deploy booms for antennas or experiment mountings. The Pegasus type satellites, which deployed meteoroid penetration detectors with areas in excess of 200 meters^2 are typical examples of these spacecraft. It is impossible to test these spacecraft in the zero gravity environment in which they must finally operate. Also since a spacecraft's physical geometry is usually determined by evolution from the original design to the final working design, it is too expensive and time consuming to build dynamic scale models for each evolutionary step. Thus, a need exists for an analytical method to calculate reliably the motions expected for such spacecraft while operating in space.

Analytical studies of these large relative motion problems found in the literature usually fall into two groups. The first group contains those studies which use energy methods with emphasis being placed on the symmetry of the initial angular velocity of the dynamical system as well as on the physical symmetry of the system. References 1 and 2 are examples of such studies. The results of these analyses are applicable only to dynamical systems having equivalent spin modes and physical geometry. The second group of studies are those that use vector mechanics. (See refs. 3, 4, and 5.) This group has results that are valid for general initial angular velocities. However, references 4 and 5 are valid only for satellites which deploy simple booms (booms consisting of single lumped mass points). The results of reference 3 are valid only when the motion of the central body is known. Thus neither group of studies gives results which are generally valid for the calculation of motions for satellites of a general geometry.

A method for calculating the motions of spacecraft consisting of interconnecting rigid bodies is presented in this report. The method can be applied to structures of any general configuration or changing configuration such as a spacecraft that deploys components. The method allows the motions of the rigid bodies to be calculated as well as the interaction forces and torques between the bodies. The number of bodies in a single system which can be studied by this method is limited only by the speed and storage of modern digital computers. Matrix notation is used because it is thereby possible to cast the governing equations in terms of variables common to all such dynamic systems. The resulting equations can therefore be used (as is) to study any such dynamical system.

An application of the method is presented in which the motion of a proposed NASA micrometeoroid spacecraft deploying four large panels is studied. Some motion time histories are generated for this dynamic system which consists of five rigid bodies – the central spacecraft body plus four rigid bodies representing the deploying panels. Two means of deploying the panels are investigated. In the first case the panels are assumed to be deployed by torsion bars with viscous damping. Motions are calculated for situations for two damping rates and for situations in which one panel is allowed to deploy before the other three. The second case studied assumed that the panels are symmetrically deployed by centrifugal force resulting from spin of the spacecraft. This second case serves as a check on the method since the angular momentum of the five-body system is conserved. The analysis given here is valid up to panel locking only, but it is possible to extend the method to cover such cases as panel locking.

SYMBOLS

The notation shown is similar to the Dirac notation used in reference 6.

a   acceleration of center of mass of body, meters/second^2

A	moment of inertia about center of mass of body, kilogram-meters^2
$A(i,j)$	a set of equations governing motion of two interconnected rigid bodies (body i and body j) when no rotational constraints are present
B	moment of inertia about reference point of a body which does not coincide with its center of mass, kilogram-meters^2
B_m	similar to B and nonsingular (see eq. (27)), kilogram-meters^2
C	coordinate frame
C_Y	inertial coordinate frame
$D(i)$	a set of equations governing motion of body i
e	a unit base for an orthogonal coordinate frame, dimensionless
$E(i,j)$	a set of equations governing motion of body j relative to body i
f	net force on a body, newtons
F	interaction constraint force, newtons
G	force on a body, newtons
H	component of angular momentum, kilogram-meters^2/second
i,j	denotes specific bodies
$[I]$	identity matrix
L	interaction torque, newton-meters
L_a	applied interaction torque, newton-meters
L_c	constraint interaction torque, newton-meters
m	mass of a body, kilograms

M	transformation matrix, dimensionless
N	torque on a body, newton-meters; also integer
P_a	projection matrix for applied interaction torque, dimensionless
P_c	projection matrix for constraint interaction torque, dimensionless
q	inertial acceleration of reference point on a body, meters/second^2
R_1,R_2,R_3	rotation matrices (see eqs. (29) to (31))
$S(1,2,\ldots,N)$	a set of equations governing the motion of an N body dynamical system which is composed of interconnected rigid bodies
t	time, sec
T	total torque applied to a body about its center of mass, newton-meters
V^1,V^2	vector in C_1 and C_2 representation
$V(P)$	velocity of reference point P in inertial space, meters/second
W	damping rate
X	matrix made up of components of three vectors
Y,Z	elastic torque rate constants, newton-meters/radian
$[0\rangle$	column vector with all components being zero
α	vector from a body's reference point to its center of mass, meters
β	vector between reference points of two bodies, meters
$\delta_{i,j}$	Kronecker delta
ψ,θ,γ	modified Euler rotation angles (see fig. 5), radians

η	unit vector
ω	angular velocity, radians/second
Ω	position coinciding with α, meters
ζ	a vector which is reciprocal to η, dimensionless

Notation:

$\langle\,]$	row vector
$[\,\rangle$	column vector
$\lvert[\,\rangle\rvert$	length of $[\,\rangle$
$[\,]$	square matrix
$\lfloor\,\rceil$	antisymmetric square matrix
$[\,]^T$	transpose of $[\,]$
$[\,]^{-1}$	inverse of $[\,]$
$(\bar{\ }) \times (\bar{\ })$	vector cross product

Subscripts designate the body to which a given property belongs and superscripts designate the coordinate frame in which the property is represented. Bars over symbols denote vectors. A dot over a symbol denotes a time derivative with respect to the coordinate frame denoted by the superscript and $\frac{d(\ )}{dt}$ denotes a time derivative with respect to an inertial coordinate frame.

ANALYSIS

This method is not applicable to ideal locking joints (joints where the torque rates instantaneously change from finite to infinite values) during locking time. However, ideal locking joints may be approximated satisfactorily with highly damped, high-spring-rate joints during locking periods. After joint locking occurs, this method is applicable without approximation.

Mathematical Model

The dynamical system model under consideration is one which has gross changes in its physical geometry and one which can be constructed by adding a single rigid body with a single attachment point to form a new system from an old one. An example of such a dynamical system is a spacecraft with large folding panels. The motions of such a system can be analyzed by considering the motions of the bodies about the attachment points. When a dynamical system consists of only one body, that system's motion is described by the path of the center of its mass and its rotational motion. If a second body is added to this system at some common attachment point, the interaction of the second body with the first body must be found. There will be a force vector and a torque vector present which reflect physically the attachment of the two bodies. When a third body is added to the system, its interaction with one of the first two bodies must be found. Again there will be force and torque vectors that show how the third body is added to the older system to form a newer system. By this procedure a dynamical system of N rigid bodies can be constructed and studied.

Outline and Discussion of Method

Dynamical systems are governed by second-order differential equations where position coordinates and velocity coordinates must be specified before the accelerations can be calculated. Although the governing equations of motion may be nonlinear functions of position and velocity coordinates, these equations are always linear functions of the accelerations. In general, for free dynamical systems of N rigid bodies there are N vector angular acceleration and N vector translational acceleration equations. For the dynamical systems considered here, there are always $N - 1$ vector translational constraint equations which can be used to reduce the number of vector equations from $2N$ to $N + 1$ with $N + 1$ vector unknowns, these being N vector angular accelerations and one vector translational acceleration. Rotational constraints may also be present in the system, and thus reduce still further the number of equations to be solved simultaneously. When the number of equations has been reduced to the smallest number, the mathematical similarity between different systems is lost. The method of solution given here uses a different approach. The advantages of this method will depend upon the dynamical system involved. Here the emphasis is placed on the similarity between systems and the analysis is carried as far as possible without using detailed knowledge of the system contraints.

The method of solution defines the motion of an arbitrary body in the system in terms of that body's interaction with the system. The body's motion is described by its translational and angular acceleration, whereas its interaction with the system is described by the forces and torques impressed on the system by the body. Because all these vector equations are linear in accelerations, forces, and torques, these quantities can be found by classical matrix algebra. The matrix notation used here is introduced in appendix A.

The actual solution for a given system can be obtained by forming an independent set of vector equations equal in number to the total number of translational and angular accelerations as well as the interaction forces and torques, and then solving this set of vector equations for the unknowns. The angular accelerations are then integrated to find the angular coordinates and velocities at a new time. The translational accelerations need not be integrated because the coefficients in the governing set of equations are functions of angular variables.

Solution of One and Two Body Systems With No Rotational Constraints

The equations of motion which govern the motion of one and two body systems are developed in appendix B. Let C_i represent the coordinate frame fixed in body i, where body i is a single body system. The unit orthogonal base vectors for C_i are $\bar{e}_i(1)$, $\bar{e}_i(2)$, and $\bar{e}_i(3)$. From appendix B the translational momentum equation for body i is

$$m_i \left[a_i^i\right\rangle - \left[f_i^i\right\rangle = \left[0\right\rangle \tag{1}$$

where the subscripts designate the body to which a given property belongs and superscripts designate the coordinate frame in which the property is represented and m_i is the mass of body i, $\left[a\right\rangle$ is acceleration of the center of mass and $\left[f\right\rangle$ is the total external force on the body. The brackets $\left[\ \right\rangle$ denotes column vectors. The angular momentum equation for body i taken from appendix B is

$$\left[A_i^i\right]\left[\dot{\omega}_{iY}^i\right\rangle = \left[T_i^i\right\rangle - \left\lfloor\omega_{iY}^i\right\rceil\left[A_i^i\right]\left[\omega_{iY}^i\right\rangle \tag{2}$$

where $\left[\omega\right\rangle$ is the angular velocity and the double subscript indicates that $\left[\omega_{iY}\right\rangle$ is the rotation of C_i relative to C_Y where C_Y is the inertial coordinate frame. $\left[A\right]$ is the moment of inertia matrix about the body's center of mass, $\left\lfloor\omega\right\rceil$ is the antisymmetric matrix made up from $\left[\omega\right\rangle$ analogous to the $\bar{\omega}\times$ vector operator as shown in appendix A and $\left[T\right\rangle$ is the total external torque on the body about its center of mass. These are the equations of motion for a single rigid body system.

Consider a system composed of two rigid bodies which are attached at a common attachment point (reference point of body j) as shown in figure 1. Equations (1) and (2) are the equations of motion for body i. The translational momentum equation for body j taken from appendix B is

$$m_j\left(\left[q_j^j\right\rangle - \left\lfloor \alpha_j^j \right\rceil \left[\dot{\omega}_{jY}^j\right\rangle\right) - \left[f_j^j\right\rangle = -m_j \left\lfloor \omega_{jY}^j \right\rceil^2 \left[\alpha_j^j\right\rangle \tag{3}$$

where $\left[q\right\rangle$ is the inertial acceleration of the reference point of body j, $\left[\alpha_j\right\rangle$ is the vector from the reference point of body j to the center of mass of body j. The angular momentum equation for body j, measured about its reference point, taken from appendix B is

$$\left[B_j^j\right] \left[\dot{\omega}_{jY}^j\right\rangle + m_j \left\lfloor \alpha_j^j \right\rceil \left[q_j^j\right\rangle = \left[L_j^j\right\rangle - \left\lfloor \omega_{jY}^j \right\rceil \left[B_j^j\right] \left[\omega_{jY}^j\right\rangle \tag{4}$$

where $\left[B\right]$ is the moment-of-inertia matrix measured about the reference point of the body and $\left[L\right\rangle$ is the total torque on the body about its reference point. The translational constraint equation for the attachment of body j to body i taken from appendix B (and noticing that $\left[a_i^i\right\rangle$ equals $\left[q_i^i\right\rangle$) is

$$\left[q_j^j\right\rangle + \left[M_{ij}\right] \left\lfloor \beta_j^i \right\rceil \left[\dot{\omega}_{iY}^i\right\rangle - \left[M_{ij}\right] \left[a_i^i\right\rangle = \left[M_{ij}\right] \left\lfloor \omega_{iY}^i \right\rceil^2 \left[\beta_j^i\right\rangle \tag{5}$$

where $\left[M\right]$ is a transformation matrix and its double subscript indicates that it converts column vectors in C_i representation to C_j representation. $\left[\beta_j\right\rangle$ is a vector from the reference point of body i to the reference point of body j. In every case a reference point of a body is the same as its attachment point if it has an attachment point.

Next consider the interaction force and torque at the reference point of body j and designate as $\left[F(ij)\right\rangle$ and $\left[L(ij)\right\rangle$, respectively. The term $\left[L_j^j\right\rangle$ from equation (4) can be rewritten as

$$\left[L_j^j\right\rangle = \left[L_j^j(ij)\right\rangle + \left[N_j^j\right\rangle \tag{6}$$

and $\left[f_j^j\right\rangle$ from equation (3) can be rewritten as

$$\left[f_j^j\right\rangle = \left[F_j^j(ij)\right\rangle + \left[G_j^j\right\rangle \tag{7}$$

In equation (6), $\left[N_j\right\rangle$ is the external torque applied to body j about its reference as opposed to the interaction torque $\left[L_j(ij)\right\rangle$, and in equation (7), $\left[G_j\right\rangle$ is the external force applied to body i and $\left[F_j(ij)\right\rangle$ is the interaction force. The term $\left[T_i^i\right\rangle$ in equation (2) can likewise be rewritten as

$$\left[T_i^i\right\rangle = -\left[M_{ji}\right]\left[L_j^j(ij)\right\rangle - \left\lfloor\beta_j^i\right\rceil\left[M_{ji}\right]\left[F_j^j(ij)\right\rangle + \left[N_i^i\right\rangle \tag{8}$$

and $\left[f_i^i\right\rangle$ in equation (1) can be rewritten as

$$\left[f_i^i\right\rangle = -\left[M_{ji}\right]\left[F_j^j(ij)\right\rangle + \left[G_i^i\right\rangle \tag{9}$$

Equations (6) to (9) can be substituted into equations (1) to (5) to obtain the following set of equations A(i,j):

$$\text{Set } A(i,j) = \begin{cases} m_i\left[a_i^i\right\rangle + \left[M_{ji}\right]\left[F_j^j(ij)\right\rangle = \left[G_i^i\right\rangle & (10) \\ \left[A_i^i\right]\left[\dot{\omega}_{iY}^i\right\rangle + \left\lfloor\beta_j^i\right\rceil\left[M_{ji}\right]\left[F_j^j(ij)\right\rangle = -\left\lfloor\omega_{iY}^i\right\rceil\left[A_i^i\right]\left[\omega_{iY}^i\right\rangle - \left[M_{ji}\right]\left[L_j^j(ij)\right\rangle + \left[N_i^i\right\rangle & (11) \\ m_j\left[q_j^j\right\rangle - m_j\left\lfloor\alpha_j^j\right\rceil\left[\dot{\omega}_{jY}^j\right\rangle - \left[F_j^j(ij)\right\rangle = -m_j\left\lfloor\omega_{jY}^j\right\rceil^2\left[\alpha_j^j\right\rangle + \left[G_j^j\right\rangle & (12) \\ \left[B_j^j\right]\left[\dot{\omega}_{jY}^j\right\rangle + m_j\left\lfloor\alpha_j^j\right\rceil\left[q_j^j\right\rangle = \left[L_j^j(ij)\right\rangle - \left\lfloor\omega_{jY}^j\right\rceil\left[B_j^j\right]\left[\omega_{jY}^j\right\rangle + \left[N_j^j\right\rangle & (13) \\ \left[M_{ji}\right]\left[q_j^j\right\rangle + \left\lfloor\beta_j^i\right\rceil\left[\dot{\omega}_{iY}^i\right\rangle - \left[a_i^i\right\rangle = \left\lfloor\omega_{iY}^i\right\rceil^2\left[\beta_j^i\right\rangle & (14) \end{cases}$$

where equation (14) was obtained from equation (5) by multiplying equation (5) by $\left[M_{ji}\right]$. Equations (10) to (14) are the equations of motion for a dynamical system composed of two attached rigid bodies where no rotational constraints exist. The terms on the right-hand side of these equations must be known.

For the case of a single body system, $\left[F_j^j(ij)\right\rangle$ and $\left[L_j^j(ij)\right\rangle$ are zero and thus equations (10) and (11) remain as the describing equations where $\left[N\right\rangle$ and $\left[G\right\rangle$ are prescribed forcing functions. Then $\left[a_i^i\right\rangle$ and $\left[\dot{\omega}_{iY}^i\right\rangle$ can be calculated.

For the case of a two-body system, equations (10) to (14) are the five governing vector equations. The vector unknowns are $\left[a_i^i\right\rangle$, $\left[\dot{\omega}_{iY}^i\right\rangle$, $\left[q_j^j\right\rangle$, $\left[\dot{\omega}_{jY}^j\right\rangle$, and $\left[F_j^j(ij)\right\rangle$. It can be shown that this set of equations does have a unique solution for the unknowns.

Application of Rotational Constraints

Consider the case in which the angular velocity of body j relative to body i is constrained to be zero about one or two axes and $\left[L_j^j(ij)\right\rangle$ can no longer be specified as a function of position and angular velocity. The equation set A(i,j) defined by equations (10) to (14) can no longer give the required solution because $\left[L_j^j(ij)\right\rangle$ has now become an unknown. Let $\left[La_j^j(ij)\right\rangle$ be the known interaction torque specified by position and angular velocity; also let $\left[Lc_j^j(ij)\right\rangle$ be the unknown constraint torque of body j on body i. Then

$$\left[L_j^j(ij)\right\rangle = \left[La_j^j(ij)\right\rangle + \left[Lc_j^j(ij)\right\rangle \tag{15}$$

Two cases are considered. In the first case, body j is allowed to rotate relative to body i about two axes; in the second case, body j is allowed to rotate relative to body i about one axis.

<u>Relative rotation about two axes</u>.- In the first case consider three unit vectors $\left[\eta_j^j(1)\right\rangle$, $\left[\eta_j^j(2)\right\rangle$, and $\left[\eta_j^j(3)\right\rangle$ such that the first two form the instantaneous axes (not necessarily orthogonal) of relative rotation of body j to body i and the last one is

$$\left[\eta_j^j(3)\right\rangle = \frac{\left\lfloor\eta_j^j(2)\right\rceil \left[\eta_j^j(1)\right\rangle}{\left|\left\lfloor\eta_j^j(2)\right\rceil \left[\eta_j^j(1)\right\rangle\right|} \tag{16}$$

The component $\left[L_j^j(ij)\right\rangle$ is known along $\left[\eta_j^j(1)\right\rangle$ and $\left[\eta_j^j(2)\right\rangle$. The unknown component $\left[Lc_j^j(ij)\right\rangle$ must therefore lie along $\left[\eta_j^j(3)\right\rangle$ and must be perpendicular to $\left[\eta_j^j(1)\right\rangle$ and $\left[\eta_j^j(2)\right\rangle$. Consider also the vectors $\left[\zeta_j^j(1)\right\rangle$, $\left[\zeta_j^j(2)\right\rangle$, and $\left[\zeta_j^j(3)\right\rangle$ which are reciprocal to $\left[\eta_j^j(1)\right\rangle$, $\left[\eta_j^j(2)\right\rangle$, and $\left[\eta_j^j(3)\right\rangle$ as defined in appendix C. Let $\left[Pa_j^j\right]$ and $\left[Pc_j^j\right]$ be defined by

$$\left[Pa_j^j\right] = \left[\eta_j^j(1)\right\rangle \left\langle\zeta_j^j(1)\right] + \left[\eta_j^j(2)\right\rangle \left\langle\zeta_j^j(2)\right] \tag{17}$$

and

$$\left[Pc_j^j\right] = \left[\eta_j^j(3)\right\rangle \left\langle\zeta_j^j(3)\right] \tag{18}$$

The matrix $\left[Pa_j^j\right]$ is the projection matrix for the angular velocity of body j relative to body i, $\left[\omega_{ji}\right\rangle$, and for $\left[La_j^j(ij)\right\rangle$. Projection matrices and reciprocal vectors are defined in appendix C. The matrix $\left[Pc_j^j\right]$ is the projection matrix for $\left[Lc_j^j(ij)\right\rangle$. The vector $\left[\omega_{ji}\right\rangle$ in c_j representation is given by

$$\left[\omega_{ji}^j\right\rangle = \left[\omega_{jY}^j\right\rangle - \left[M_{ij}\right]\left[\omega_{iY}^i\right\rangle \tag{19}$$

It follows from equations (17) and (18) that

$$\left[Pa_j^j\right]\left[L_j^j(ij)\right\rangle = \left[La_j^j(ij)\right\rangle \tag{20}$$

and

$$\left[Pc_j^j\right]\left[\omega_{ji}^j\right\rangle = \left[0\right\rangle \tag{21}$$

Equations (20) and (21) are the additional equations needed to solve the problem where the rotational motion of body j relative to body i is constrained to lie along two axes.

Relative rotation about one axis.- Let the axis of relative rotation be $\left[\eta_j^j(1)\right\rangle$. In this case, $\left[L_j^j(ij)\right\rangle$ along $\left[\eta_j^j(1)\right\rangle$ is known so that $\left[Lc_j^j(ij)\right\rangle$ is perpendicular to $\left[\eta_j^j(1)\right\rangle$. Because $\left[Lc_j^j(ij)\right\rangle$ is a linear combination of $\left[\eta_j^j(2)\right\rangle$ and $\left[\eta_j^j(3)\right\rangle$, $\left[\eta_j^j(2)\right\rangle$ and $\left[\eta_j^j(3)\right\rangle$ are both perpendicular to $\left[\eta_j^j(1)\right\rangle$. Since $\left[\zeta_j^j(1)\right\rangle$ is also perpendicular to $\left[\eta_j^j(2)\right\rangle$ and $\left[\eta_j^j(3)\right\rangle$, $\left[\eta_j^j(1)\right\rangle$ and $\left[\zeta_j^j(1)\right\rangle$ must be parallel and equal. Thus, $\left[Pa_j^j\right]$ is defined by

$$\left[Pa_j^j\right] = \left[\eta_j^j(1)\right\rangle \left\langle\zeta_j^j(1)\right] \tag{22}$$

and since the sum of $\left[P_{a_j}^{j}\right]$ and $\left[P_{c_j}^{j}\right]$ is equal to the identity matrix $\left[I\right]$, $\left[P_{c_j}^{j}\right]$ is always defined by

$$\left[P_{c_j}^{j}\right] = \left[I\right] - \left[P_{a_j}^{j}\right] \tag{23}$$

and $\left[P_{a_j}^{j}\right]$ and $\left[P_{c_j}^{j}\right]$ are again known. Equations (20) and (21) can now be used to impose the desired constraints on rotational motion of body j relative to body i.

Return to equation set A(i,j) and recall that when $\left[L_{c_j}^{j}(ij)\right\rangle$ is different from $\left[0\right\rangle$, equation set A(i,j) has six vector unknowns and only five vector equations. One more vector equation or three scalar equations are required to obtain values for the unknowns. If body j is allowed to rotate relative to body i about one (or two) axes, equation (20) provides one (or two) new independent scalar equations. Also the time derivative of equation (21) provides two (or one) new independent scalar relations between $\left[\dot{\omega}_{iY}^{i}\right\rangle$ and $\left[\dot{\omega}_{jY}^{j}\right\rangle$. Therefore equation (20) plus the time derivative of equation (21) provide the three necessary new scalar equations necessary to solve equation set A(i,j) for the unknowns. These three independent scalar relations may be put into vector form simply by adding the time derivative of equation (21) $\left(\text{premultiplied by } \left[B_j^{j}\right] \text{ to obtain the correct order of magnitude}\right)$ directly to equation (20) to get

$$\begin{aligned}\left[B_j^{j}\right]\left[P_{c_j}^{j}\right]\left[\dot{\omega}_{jY}^{j}\right\rangle = {} & \left[B_j^{j}\right]\left[P_{c_j}^{j}\right]\left[M_{ij}\right]\left[\dot{\omega}_{iY}^{i}\right\rangle - \left[B_j^{j}\right]\left[P_{c_j}^{j}\right]\left[\omega_{ji}^{j}\right]\left[M_{ij}\right]\left[\omega_{iY}^{i}\right\rangle \\ & - \left[B_j^{j}\right]\left[\dot{P}_{c_j}^{j}\right]\left[\omega_{ji}^{j}\right\rangle - \left[P_{a_j}^{j}\right]\left[L_j^{j}(ij)\right\rangle + \left[L_{a_j}^{j}(ij)\right\rangle\end{aligned} \tag{24}$$

Equation set A(i,j) plus equation (24) can now be solved directly by matrix inversion to get unique values for the unknowns. However, if one desires to find the unknowns by algebraic substitution (this will generally be the case because direct matrix inversion will require more computer storage and longer computation time than solution by algebraic substitution), a useful alternate to equation (24) can be found by premultiplying equation (13) by $\left[P_{a_j}^{j}\right]$ and adding the result to equation (24) then

$$\left(\left[Pa_j^j\right]\left[B_j^j\right] + \left[B_j^j\right]\left[Pc_j^j\right]\right)\left[\dot{\omega}_{jY}^j\right\rangle + m_j\left[Pa_j^j\right]\left\lfloor \alpha_j^j \right\rceil\left[q_j^j\right\rangle - \left[B_j^j\right]\left[Pc_j^j\right]\left\lceil M_{ij}\right]\left[\dot{\omega}_{iY}^i\right\rangle$$

$$= \left[La_j^j(ij)\right\rangle - \left[Pa_j^j\right]\left\lfloor \omega_{jY}^j \right\rceil\left[B_j^j\right]\left[\omega_{jY}^j\right\rangle - \left[B_j^j\right]\left[Pc_j^j\right]\left\lfloor \omega_{ji}^j \right\rceil\left\lceil M_{ij}\right]\left[\omega_{iY}^i\right\rangle$$

$$- \left[B_j^j\right]\left[\dot{P}c_j^j\right]\left[\omega_{ji}^j\right\rangle + \left[Pa_j^j\right]\left[N_j^j\right\rangle \tag{25}$$

Let the matrix $\left[Bm_j^j\right]$ be defined by

$$\left[Bm_j^j\right] = \left(\left[Pa_j^j\right]\left[B_j^j\right] + \left[B_j^j\right]\left[Pc_j^j\right]\right) \tag{26}$$

which can be shown to be nonsingular. When equation (26) is substituted into equation (25), there follows

$$\left[Bm_j^j\right]\left[\dot{\omega}_{jY}^j\right\rangle + m_j\left[Pa_j^j\right]\left\lfloor \alpha_j^j \right\rceil\left[q_j^j\right\rangle - \left[B_j^j\right]\left[Pc_j^j\right]\left\lceil M_{ij}\right]\left[\dot{\omega}_{iY}^i\right\rangle$$

$$= \left[La_j^j(ij)\right\rangle - \left[Pa_j^j\right]\left\lfloor \omega_{jY}^j \right\rceil\left[B_j^j\right]\left[\omega_{jY}^j\right\rangle$$

$$- \left[B_j^j\right]\left[Pc_j^j\right]\left\lfloor \omega_{ji}^j \right\rceil\left\lceil M_{ij}\right]\left[\omega_{iY}^i\right\rangle - \left[B_j^j\right]\left[\dot{P}c_j^j\right]\left[\omega_{ji}^j\right\rangle$$

$$+ \left[Pa_j^j\right]\left[N_j^j\right\rangle \tag{27}$$

Since $\left[Bm_j^j\right]$ is nonsingular, equation (27) can be combined algebraically with the members of set A(i,j) to get unique solutions for $\left[a_i^i\right\rangle$, $\left[\dot{\omega}_{iY}^i\right\rangle$, $\left[q_j^j\right\rangle$, $\left[\dot{\omega}_{jY}^j\right\rangle$, $\left[F_j^j(ij)\right\rangle$, and $\left[L_j^i(ij)\right\rangle$.

Extension to multibody systems.- Consider now an N body dynamical system. From these bodies, arbitrarily select a reference body, body i, and let its reference point be its center of mass. The motion of this body is governed by equation set D(i) defined as equations (10) and (11). Any other body, body j, which is attached to the reference body has its attachment point as its reference point and its motion is governed by equation set E(i,j) defined as equations (12) to (14) plus equation (27). Any other body,

body k, which is attached to a body j has its attachment point as its reference point and its motion is governed by equation set E(j,k) and so on. Thus the reference body, body i, has its motion governed by D(i) and every other body, body m, has its motion governed by $E(l,m)$ where body l is the body to which body m is attached and the reference point of body m is the point at which it is attached to body l. The set of equations which governs the motion of the entire system S(1,2,...,N) is D(i) plus all the $E(l,m)$ for every other body m in the system. When the physical properties of the system along with initial coordinates and coordinate velocities have been specified, the system's unknowns can be found by classical matrix inversion or by algebraic substitution between the members of S(1,2,...,N) according to the desires of the investigator. Any desired number of unknowns may be calculated without calculating the remaining unknowns. If one is to generate a time history of the motion of this type of dynamical system, at least all the angular accelerations must be calculated to provide angular positions and angular velocities which in turn determine the coefficients in the equation set S(1,2,...,N). A five-body system is illustrated in the example problem.

One word of caution is in order. When the system's motions are solved from S(1,2,...,N), the correct accelerations will automatically be found. However, when these accelerations are integrated, one must be exceedingly careful to insure that the integration scheme conforms to the motion constraints applied to the system. This precaution is necessary because all numerical integrations schemes have some inherent error and over long time intervals, motion constraints might otherwise become grossly violated.

APPLICATION

Translation of a Spacecraft's Physical Properties into Matrix Variables

The analysis is illustrated by studying the deployment motions of a proposed NASA micrometeorite detection satellite shown in figure 2. This satellite would essentially be composed of five rigid bodies connected at four attachment points. The body which would house the satellite's communication electronics (the hub) is centrally located between four other bodies (the wings) which are the meteorite detection sensor arrays. Each wing is made up from three smaller rigid members, a main panel plus two auxiliary panels. In the launch configuration the auxiliary panels are folded down to the main panel and all four main panels are folded about the ends adjacent to the hub into a square cylinder, the hub forming a cap for the cylinder, as shown in figure 2(a).

The initial phase of deployment consists of the four wings folding out of the cylindrical configuration (with the auxiliary panels still locked to the main panels) into a planar configuration as shown in figure 2(b). Then the auxiliary panels are unfolded into the position shown in figure 2(c).

The motion studied here encompasses the time from which the wings begin to deploy from the cylindrical configuration until they are in the planar configuration. Since the auxiliary panels (during this period) are folded against the main panels and locked in that position, the wings are assumed to be rigid bodies, and the whole system is to be treated as the ideal system shown in figure 3. Each wing is assumed to rotate relative to the hub about two axes. (See figs. 3 and 4.) The first axis of rotation, the hinge line, is the axis of wing deployment. The second axis of rotation is perpendicular to the hinge line and passes through the center of mass of the individual wing. This degree of freedom is intended to provide a pseudo-simulation of any twist which develops in the wings during deployment. The coordinate frame for the hub is shown in figure 4 along with the hinge line for body 2 and the line through the center of mass of body 2 about which body 2 can rotate relative to body 1.

Figure 5 gives the modified Euler angles which relate C_1 (coordinate frame fixed in body 1) to C_2 (coordinate frame fixed in body 2). Consider an intermediate coordinate frame C which initially is alined with C_1. First, rotate about $\bar{e}(3)$ counterclockwise through angle ψ_2. Second, rotate about the new $\bar{e}(1)$ counterclockwise through angle θ_2. Finally, rotate about the new $\bar{e}(2)$ counterclockwise through angle γ_2. The intermediate coordinate frame is now alined with C_2. It follows that a vector in C_1 representation $\left[V^1\right\rangle$ is related to the same vector in C_2 representation $\left[V^2\right\rangle$ by

$$\left[V^1\right\rangle = \left[R_1\right]\left[R_2\right]\left[R_3\right]\left[V^2\right\rangle = \left[M_{21}\right]\left[V^2\right\rangle \tag{28}$$

where

$$\left[R_1\right] = \begin{bmatrix} \cos\psi_2 & -\sin\psi_2 & 0 \\ \sin\psi_2 & \cos\psi_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29}$$

$$
\left[R_2\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix} \tag{30}
$$

$$
\left[R_3\right] = \begin{bmatrix} \cos\gamma_2 & 0 & \sin\gamma_2 \\ 0 & 1 & 0 \\ -\sin\gamma_2 & 0 & \cos\gamma_2 \end{bmatrix} \tag{31}
$$

Rotation may occur about two axes, $\bar{\eta}_2(1)$ and $\bar{\eta}_2(2)$. Figure 5 shows that $\bar{\eta}_2(1)$ (the xis of θ_2 rotation), $\bar{\eta}_2(2)$ (the axis of γ_2 rotation) and $\bar{\eta}_2(3)$ (the axis along which he constraint torque lies) form a set of mutually orthogonal unit vectors. In all such ases, the vectors $\bar{\zeta}_2(i)$ (which are reciprocal to the vectors $\bar{\eta}_2(i)$) are equal to the ectors $\bar{\eta}_2(i)$. From figure (5) it can be seen that

$$
\left.\begin{aligned}
\bar{\eta}_2(1) &= \bar{e}_2(1)\cos\gamma_2 + \bar{e}_2(3)\sin\gamma_2 \\
\bar{\eta}_2(2) &= \bar{e}_2(2) \\
\bar{\eta}_2(3) &= -\bar{e}_2(1)\sin\gamma_2 + \bar{e}_2(3)\cos\gamma_2
\end{aligned}\right\} \tag{32}
$$

t now follows directly from equation (17) that

$$
\begin{aligned}
\left[Pa_2^2\right] &= \left[\begin{matrix} \cos\gamma_2 \\ 0 \\ \sin\gamma_2 \end{matrix}\right\rangle \left\langle \cos\gamma_2, \quad 0, \quad \sin\gamma_2 \right] + \left[\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right\rangle \left\langle 0, \quad 1, \quad 0 \right] \\
&= \begin{bmatrix} (\cos\gamma_2)^2 & 0 & \cos\gamma_2 \sin\gamma_2 \\ 0 & 1 & 0 \\ \cos\gamma_2 \sin\gamma_2 & 0 & (\sin\gamma_2)^2 \end{bmatrix}
\end{aligned} \tag{33}
$$

also since

$$\left[\eta_2^2(3)\right\rangle = \left[\begin{matrix} -\sin\gamma_2 \\ 0 \\ \cos\gamma_2 \end{matrix}\right\rangle$$

Then from equation (18)

$$\left[Pc_2^2\right] = \left[\begin{matrix} -\sin\gamma_2 \\ 0 \\ \cos\gamma_2 \end{matrix}\right\rangle \left\langle -\sin\gamma_2, \quad 0, \quad \cos\gamma_2\right]$$

$$= \begin{bmatrix} (\sin\gamma_2)^2 & 0 & -\sin\gamma_2 \cos\gamma_2 \\ 0 & 0 & 0 \\ -\cos\gamma_2 \sin\gamma_2 & 0 & (\cos\gamma_2)^2 \end{bmatrix} \tag{34}$$

The three other wings (bodies 3, 4, and 5) are physically connected to the hub in the same manner as body 2; thus, the matrices $\left[Pa_j^j\right]$, $\left[Pc_j^j\right]$, and $\left[M_{ji}\right]$ for each wing can be found for each wing by replacing $\left(\gamma_2, \theta_2, \psi_2\right)$ by $\left(\gamma_j, \theta_j, \psi_j\right)$ in equations (29), (30), (31), (33), and (34). The angles ψ_j give the spacing of the jth wing around the hub, whereas θ_j and γ_j represent the degrees of freedom of each wing relative to the hub. The torques applied to each wing were assumed to be applied at the point of attachment for the attachment of the wing to the hub. One torque component was assumed proportional to both θ_j and $\dot{\theta}_j$ and parallel to the axis of rotation for θ_j. A second torque component was assumed to be proportional to γ_j and parallel to the axis of rotation of γ_j. The physical properties of the example spacecraft are given in table I. The vector, $\left[\beta_j^1\right\rangle$, in C_1 is measured from the center of mass of the hub to the attachment point of the jth wing. The vector $\left[\alpha_j^j\right\rangle$ is measured in C_j coordinates from the attachment point of the hub

TABLE I.- PHYSICAL PROPERTIES OF EXAMPLE SPACECRAFT

Radius vector from attachment to center of mass for wings 2, 3, 4, and 5, dimension, m:

$$\left[\beta_2^1\right\rangle = \begin{pmatrix} 0 \\ 2.23 \\ 0 \end{pmatrix} \qquad \left[\beta_3^1\right\rangle = \begin{pmatrix} -2.23 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\beta_4^1\right\rangle = \begin{pmatrix} 0 \\ -2.23 \\ 0 \end{pmatrix} \qquad \left[\beta_5^1\right\rangle = \begin{pmatrix} 2.23 \\ 0 \\ 0 \end{pmatrix}$$

Radius vector from reference point of hub to attachment point, dimension, m:

$$\left[\alpha_2^2\right\rangle = \left[\alpha_3^3\right\rangle = \left[\alpha_4^4\right\rangle = \left[\alpha_5^5\right\rangle = \begin{pmatrix} 0 \\ 4.80 \\ 0 \end{pmatrix}$$

Moment-of-inertia matrix for hub about its center of mass, dimension, kg-m^2:

$$\left[A_1^1\right] = \begin{bmatrix} 877 & 0 & 0 \\ 0 & 877 & 0 \\ 0 & 0 & 1755 \end{bmatrix}$$

Moment-of-inertia matrix for jth wing about its attachment point to hub, dimension, kg-m^2

$$\left[B_j^j\right] = \begin{bmatrix} 83000 & 0 & 0 \\ 0 & 2610 & 0 \\ 0 & 0 & 85700 \end{bmatrix}$$

to the wing to the center of mass of the wing. The mass of the hub and wings were each 2620 kg. The moment of inertia matrix for the hub about its center of mass $\left[A_1^1\right]$ is given in table I. The moment of inertia matrices for all wings about their attachment points $\left[B_j^j\right]$ were identical to each other. All four wings are identical and their respective coordinate frames are each oriented relative to each body in identical arrangements and this arrangement is given in figure 6. The spacing of the wings around the hub is given by ψ_2, ψ_3, ψ_4, and ψ_5 equal to 0, $\pi/2$, π, and $3\pi/2$ radians, respectively. The torques applied to the wings about their point of attachment are governed by the torque coefficients Z_j, W_j, and Y_j for each wing: Z_j is the torque rate (N-m-rad^{-1}) about the axis of θ_j rotation, W_j is the torque damping rate (N-m-sec-rad^{-1}) about the axis of θ_j rotation, and Y_j is the torque rate (N-m-rad^{-1}) about the axis of γ_j rotation. Thus, $\left[La_j^j(1j)\right\rangle$ (the torque applied to body j at its point of attachment) is given (in C_j coordinates) by

$$\left[La_j^j(1j)\right\rangle = \left[\begin{matrix} -Z_j\left(\cos\gamma_j\right)\theta_j - W_j\left(\cos\gamma_j\right)\dot{\theta}_j \\ -Y_j\ \gamma_j \\ -Z_j\left(\sin\gamma_j\right)\theta_j - W_j\left(\sin\gamma_j\right)\dot{\theta}_j \end{matrix}\right\rangle \tag{35}$$

When the initial spin rates $\left[\omega_{jY}^j\right\rangle$ of all the bodies in the system are specified along with ψ_j, θ_j, and γ_j for each of the four wings, equation (28) is used to calculate $\left[M_{j1}\right]$ for all four wings, after which equation (19) is used to calculate $\left[\omega_{j1}^j\right\rangle$ for each wing. From figure 5 it is seen that

$$\left[\omega_{j1}^j\right\rangle = \left[\begin{matrix} \dot{\theta}_j\cos\gamma_j \\ \dot{\gamma}_j \\ \dot{\theta}_j\sin\gamma_j \end{matrix}\right\rangle \tag{36}$$

Since $\left[\omega_{j1}^j\right\rangle$ is known for each wing, $\dot{\theta}_j$ and $\dot{\gamma}_j$ can be found for each wing and $\left[La_j^j(1j)\right\rangle$ can be calculated by using equation (35). Forces and torques are applied to the wings and hub only at the attachment points so that since the external forces and torques are all zero, $\left[G_1^1\right\rangle$, $\left[N_1^1\right\rangle$, $\left[G_j^j\right\rangle$, and $\left[N_j^j\right\rangle$ in equations (10) to (13) are all zero. Thus the equation set (D(1)) governing the motion of the hub is

$$
\text{Set D(1)} = \begin{cases} M_1 \left[a_1^1\right\rangle + \sum\limits_{j=2}^{j=5} \left[M_{j1}\right] \left[F_j^j(1j)\right\rangle = 0 & (37) \\ \\ \left[A_1^1\right] \left[\dot{\omega}_{1Y}^1\right\rangle + \sum\limits_{j=2}^{j=5} \left(\left\lfloor \beta_j^1 \right\rceil \left[M_{j1}\right] \left[F_j^j(1j)\right\rangle + \left[M_{j1}\right] \left[L_j^j(1j)\right\rangle \right) = - \left\lfloor \omega_{1Y}^1 \right\rceil \left[A_1^1\right] \left[\omega_{1Y}^1\right\rangle & (38) \end{cases}
$$

and the equation set E(1,j) is

$$
\text{Set E(1,j)} = \begin{cases} m_j \left[q_j^j\right\rangle - m_j \left\lfloor \alpha_j^j \right\rceil \left[\dot{\omega}_{jY}^j\right\rangle - \left[F_j^j(1j)\right\rangle = -m_j \left\lfloor \omega_{jY}^j \right\rceil^2 \left[\alpha_j^j\right\rangle & (39) \\ \\ \left[B_j^j\right] \left[\dot{\omega}_{jY}^j\right\rangle + m_j \left\lfloor \alpha_j^j \right\rceil \left[q_j^j\right\rangle - \left[L_j^j(1j)\right\rangle = - \left\lfloor \omega_{jY}^j \right\rceil \left[B_j^j\right] \left[\omega_{jY}^j\right\rangle & (40) \\ \\ \left[M_{j1}\right] \left[q_j^j\right\rangle + \left\lfloor \beta_j^1 \right\rceil \left[\dot{\omega}_{1Y}^1\right\rangle - \left[a_1^1\right\rangle = \left\lfloor \omega_{1Y}^1 \right\rceil^2 \left[\beta_j^1\right\rangle & (41) \\ \\ \left[Bm_j^j\right] \left[\dot{\omega}_{jY}^j\right\rangle + m_j \left[Pa_j^j\right] \left\lfloor \alpha_j^j \right\rceil \left[q_j^j\right\rangle - \left[B_j^j\right] \left[Pc_j^j\right] \left[M_{1j}\right] \left[\dot{\omega}_{1Y}^1\right\rangle = \left[La_j^j(1j)\right\rangle & \\ - \left[Pa_j^j\right] \left\lfloor \omega_{jY}^j \right\rceil \left[B_j^j\right] \left[\omega_{jY}^j\right\rangle - \left[B_j^j\right] \left[Pc_j^j\right] \left\lfloor \omega_{j1}^j \right\rceil \left[M_{1j}\right] \left[\omega_{1Y}^1\right\rangle - \left[B_j^j\right] \left[\dot{Pc}_j^j\right] \left[\omega_{j1}^j\right\rangle & (42) \end{cases}
$$

The equation set S(1,2,3,4,5) which governs this dynamical system is given by

$$
S(1,2,3,4,5) = D(1) + E(1,2) + E(1,3) + E(1,4) + E(1,5) \qquad (43)
$$

This set of simultaneous equations can be solved by classical matrix inversion or, since all the equations in the set are three dimensional, they may be solved by algebraic substitution within the set.

The system of equations defined by S(1,2,3,4,5) in equation (43) was programed for a digital computer. This linear system of equations was solved by algebraic substitution to get unique values for $\left[a_1^1\right\rangle$, $\left[\dot{\omega}_{1Y}^1\right\rangle$, $\left[F_j^j(1j)\right\rangle$, $\left[L_j^j(1j)\right\rangle$, $\left[q_j^j\right\rangle$, and $\left[\dot{\omega}_{jY}^j\right\rangle$. Then $\left[\dot{\omega}_{1Y}^1\right\rangle$ and $\left[\dot{\omega}_{jY}^j\right\rangle$ were used to determine $\left[\omega_{1Y}^1\right\rangle$ and $\left[\omega_{jY}^j\right\rangle$ at a new time, and equation (36) was used to update θ_j and γ_j so that $\left[M_{j1}\right]$ could be updated. Equation (A4) was used to update $\left[M_{1Y}\right]$ in the form of iteration by

$$\left[M_{1Y}\right]_{t+\Delta t} = \left[M_{1Y}\right]_t + \frac{1}{2}\left[\left(\left[M_{1Y}\right]\left\lfloor\omega_{1Y}^1\right\rceil\right)_t + \left(\left[M_{1Y}\right]\left\lfloor\omega_{1Y}^1\right\rceil\right)_{t+\Delta t}\right]\Delta t \tag{44}$$

and it was orthogonalized at each time increment by averaging $\left[M_{1Y}\right]$ with $\left(\left[M_{1Y}\right]^{-1}\right)^T$. This cycle was repeated until the desired time history of the system was produced. A flow chart for the computation sequence is given in table II.

It should be noticed that $\left[a_1^1\right\rangle$ appears as a quantity which connects the hub back to its initial coordinates in inertial space and it need not be retained if one desires only the time histories of all $\left[M_{j1}\right]$.

The results of three computer runs are presented here. These runs correspond to two different methods of deploying the spacecraft's wings from the launch configuration.

Motions From Nonsymmetrical Deployment With No Spin

One method of deploying the wings is initially to have the spacecraft nonrotating in space and use torsion bars with viscous damping to fold the wings into the planar configuration. This method of deployment was studied for two damping rates to establish the effects of damping on nonsymmetrical deployment motions.

Case 1 is the simulation of this type of deployment with small damping in the deployment system. This case shown in figure 7 has the initial conditions $\left[\omega_{jY}^j\right\rangle$ equal $\left[0\right\rangle$ for all four wings and hub, and θ_j equal $\pi/2$. The torque rate about the axis of θ_j rotation Z_j for each wing was 482 N-m-rad^{-1}, the torque damping rate about the axis of θ_j rotation W_j for each wing was 560 N-m-sec-rad^{-1}, and γ_j was constrained to be zero. Wing 2 was released 2 seconds before wings 3, 4, and 5. The time integration was continued until wing 2 rotated from θ_2 equal $\pi/2$ to θ_2 equal zero. The integration was stopped at that time to avoid the event of panel locking. Figure 7 (time histories of θ_2, θ_3, θ_4, and θ_5) shows that the 2-second delay caused an angular lag for the wings released late.

TABLE II.- FLOW CHART OF COMPUTATION SEQUENCE

1. Input physical properties:

$$\left[A_1^1\right];\quad \left[B_j^j\right],\quad \left[\alpha_j^j\right\rangle,\quad \left[\beta_j^i\right\rangle \qquad (j = 2,3,4,5;\;\; i = 1)$$

$$\left[\eta_j(i)\right\rangle \qquad (j = 2,3,4,5;\;\; i = 1,2,3)$$

$$m_1;\quad m_j,\quad Z_j,\quad W_j,\quad Y_j \qquad (j = 2,3,4,5)$$

2. Input initial conditions:

$$\left[M_{1Y}\right];\quad \left[\omega_{iY}^i\right\rangle \qquad (i = 1,2,3,4,5)$$

$$\theta_j,\quad \gamma_j,\quad \psi_j \qquad (j = 2,3,4,5)$$

3. Calculate all coefficients and constants in the linear set of equations S(1,2,3,4,5) at the initial time.

4. Calculate all the unknowns in the set of linear equations S(1,2,3,4,5) by algebraic substitution at the initial time.

5. On first iteration cycle, set the acceleration at the new time equal to the accelerations at the initial time.

6. Integrate accelerations to get the position coordinates and velocity coordinates at the new time by the formulas:

$$Q_{t+\Delta t} = Q_t + \dot{Q}_t\,\Delta t + \frac{1}{2}\left(\frac{\ddot{Q}_t}{2} + \frac{\ddot{Q}_{t+\Delta t}}{2}\right)\Delta t^2$$

$$\dot{Q}_{t+\Delta t} = \dot{Q}_t + \left(\ddot{Q}_t + \ddot{Q}_{t+\Delta t}\right)\frac{\Delta t}{2}$$

7. Calculate all coefficients and constants in the linear set of equations S(1,2,3,4,5) at the new time.

8. Calculate all the unknowns in the set of linear equations S(1,2,3,4,5) by algebraic substitution for the new time.

9. Cycle N times between points 6 and 8.

10. Output desired quantities; replace initial conditions with new time position coordinates and velocity coordinates; go to point 5.

Case 2 is the simulation of this type of deployment with large damping in the system. For this case the initial conditions and constants were the same as those of case 1 except that the damping torque rate about the axis of θ_j rotation W_j for each wing was 5530 N-m-sec-rad^{-1}. Figure 8 (time histories of θ_2, θ_3, θ_4, and θ_5) shows again that there is an angular lag for the wings released late and the deployment time is increased about 50 percent.

Figure 9 gives a comparison of the pitching motion of the hub induced by the nonsymmetrical deployment of the wings for cases 1 and 2. The angle time histories plotted correspond to the angle between the vector $\bar{e}_1(3)$ at time zero and the vector $\bar{e}_1(3)$ for times greater than zero, where $\bar{e}_1(3)$ is the third axis of the coordinate frame fixed in body 1. Figure 9 shows that increased damping (in the deployment system) decreased the pitching motion of the hub when a nonsymmetrical deployment occurs. This result probably would not hold true for dynamical systems different from the one studied here.

In order to establish a check on the digital computer program used to calculate time histories of the example problems, an independent exact solution was calculated for a symmetrical deployment in which no damping was present and the results are plotted in figure 10. Figure 10 is in good agreement with figure 7. It should be remembered that both runs had no spin but the solution given in figure 7 had a small amount of damping.

Motions From Symmetrical Deployment With Spin

Another method of deploying the wings is to use a combination of damped torsion bars with vehicle spin about $\bar{e}_1(3)$ so that centrifugal force on the wings speeds up the deployment process. Case 3 shown in figures 11, 12, and 13 gives such a deployment time history. This case has the initial conditions $\left[\omega_{j1}^{j}\right\rangle$ equal $\left[0\right\rangle$ for all four wings and $\left[\omega_{1Y}^{1}\right\rangle$ equals 10 revolutions per minute about $\bar{e}_1(3)$. Initially, the deployment angle θ_j was $\pi/2$ and the twist angle γ_j was zero for all four wings. For this case, γ_j was not constrained to be zero; the elastic torque rates Y_j were each 4330 N-m-rad^{-1}; the elastic torque rates Z_j were each 482 n-m-rad^{-1}; and the damping rates W_j were all zero. Figure 11 gives the time history of the deployment angle θ_j from which it can be seen that the deployment time has been appreciably shortened relative to the no spin case of figure 7. Figure 12 gives the time history of the wing twisting which develops as the wings deploy. The size of the twist angle suggests that a stiffer wing is needed or else a smaller initial value of $\left[\omega_{1Y}^{1}\right\rangle$ should be used. Figure 13 gives the third component of $\left[\omega_{1Y}^{1}\right\rangle$ and demonstrates the manner in which the spacecraft spin slows down as the wings deploy.

This case also serves as a numerical check on the correctness of the overall approach because it is easy to verify that the angular momentum of the system about $\bar{e}_1(3)$ has been conserved during the deployment of the wings as it should. In all cases, ψ_j for all four wings was constrained to remain constant.

CONCLUDING REMARKS

The motions of dynamic systems built up of interconnected rigid bodies, which undergo large relative angular displacements, can be calculated with matrix calculus. The method given in this report makes no use of such quantities as "generalized coordinates" and therefore applies (as is) to any dynamical system of interconnected rigid bodies.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., February 20, 1968,
124-09-14-04-23.

APPENDIX A

PROPERTIES OF SPECIAL MATRICES

Consider an orthogonal coordinate frame, designated C, and use subscripts to refer to specific coordinate frames. If a vector's components $\left[V^i\right\rangle$ are given in C_i, the same vector's components in C_j are given by reference 7 as

$$\left[V^j\right\rangle = \left[M_{ij}\right]\left[V^i\right\rangle \tag{A1}$$

For $\left[M_{ij}\right]$ the double subscript shows the direction of the change in the coordinate frames; in this case the subscript ij shows that the C_i representation is being changed to the C_j representation.

Next consider two vectors $\bar{A}$ and $\bar{B}$ and their cross product $\bar{C}$ so that

$$\bar{A} \times \bar{B} = \bar{C}$$

If

$$\left[A\right\rangle = \left[\begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix}\right\rangle$$

and if an antisymmetric square matrix $\lfloor A \rceil$ is formed from the components of $\left[A\right\rangle$, so that

$$\lfloor A \rceil = \begin{bmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{bmatrix}$$

it is simple to verify that

$$\lfloor A \rceil \left[B\right\rangle = \left[C\right\rangle \tag{A2}$$

The scalar product of $\left[A\right\rangle$ and $\left[B\right\rangle$ is given by

$$\left\langle A\right]\left[B\right\rangle = \left\langle B\right]\left[A\right\rangle \tag{A3}$$

where $\left\langle A\right]$ is the transpose of $\left[A\right\rangle$. Finally consider again coordinate systems C_i and C_j and the orthogonal transformation matrix $\left[M_{ij}\right]$. The time derivative of $\left[M_{ij}\right]$ is given by (see ref. 7)

$$\left[\dot{M}_{ij}\right] = \left[M_{ij}\right]\left\lfloor\omega_{ij}^{i}\right\rceil \tag{A4}$$

where $\left\lfloor\omega_{ij}^{i}\right\rceil$ is the antisymmetric matrix made from $\left[\omega_{ij}^{i}\right\rangle$ as in equation (A2) and $\left[\omega_{ij}^{i}\right\rangle$ is the angular velocity vector of C_i relative to C_j and represented in C_i. For orthogonal matrices, the inverse is equal to the transpose; thus,

$$\left[M_{ij}\right]^{-1} = \left[M_{ij}\right]^{T} \tag{A5}$$

The matrix notation used here is similar to Dirac notation (see ref. 6) and it has the property that the outermost brackets in the product of two matrices always indicates the matrix character of the product.

APPENDIX B

THE TRANSLATION OF VECTOR EQUATIONS OF MOTION OF RIGID BODIES INTO MATRIX NOTATION

The equations of motion for a rotating rigid body are

$$m\bar{a} = \bar{f} \tag{B1}$$

and

$$\frac{d(\bar{H})}{dt} = \bar{L} \tag{B2}$$

where m is the body mass, $\bar{a}$ is acceleration of the center of mass, $\bar{f}$ is total force applied to the body, $\bar{H}$ is total angular momentum about a reference point, $\bar{L}$ is the total torque applied at the same reference point, bars over quantities denote vector character and $\frac{d(\ )}{dt}$ denotes time differentiation of the parenthetical quantity in the inertial coordinate frame.

Consider body i in figure 1 and let its reference point be its center of mass. Equation (1) then becomes

$$m_i\bar{a}_i^i = \bar{f}_i^i \tag{B3}$$

where the subscripts denote that a property belongs to body i and superscripts denote the coordinate frame in which the nonscalar quantities are measured. Equation (B2) becomes

$$\dot{\bar{H}}_i^i + \bar{\omega}_{iY}^i \times \bar{H}_i^i = \bar{L}_i^i \tag{B4}$$

where $\dot{\bar{H}}$ denotes time differentiation of $\bar{H}$ in a body-fixed coordinate frame (denoted by the superscript) and $\bar{\omega}$ is angular velocity and its double subscript indicates that it measures the spin of coordinate frame i relative to the inertial coordinate frame C_Y.

Next consider body j in figure 1 and let its reference point be point P (the point of attachment of body j to body i). In this case $\bar{a}_j^j$ (the acceleration of the center of mass of body j) can be written as

$$\bar{a}_j^j = \bar{q}_j^j - \frac{d}{dt}\left(\bar{\alpha}_j^j \times \bar{\omega}_{jY}^j\right) \tag{B5}$$

where $\bar{q}_j$ is the acceleration of the reference point of body j and $\bar{\alpha}_j$ is the position vector measured from the reference point of body j (fixed in body j) to the center of mass of body j. When equation (B5) is expanded,

$$\bar{a}^j_j = \bar{q}^j_j - \bar{\omega}^j_{jY} \times \left(\bar{\alpha}^j_j \times \bar{\omega}^j_{jY}\right) - \bar{\alpha}^j_j \times \dot{\bar{\omega}}^j_j \tag{B6}$$

When equation (B1) is written for body j in terms of its reference-point acceleration, the result is

$$m_j\left[\bar{q}^j_j - \bar{\alpha}^j_j \times \dot{\bar{\omega}}^j_{jY} - \bar{\omega}^j_{jY} \times \left(\bar{\alpha}^j_j \times \bar{\omega}^j_{jY}\right)\right] = \bar{f}^j_j \tag{B7}$$

The angular momentum equation for "body j" about its reference point is

$$\frac{d}{dt}\left(\bar{H}^j_j\right) = \bar{L}^j_j \tag{B8}$$

Equations (B3), (B4), and (B7) can be put in matrix notation by replacing all vectors and vector operations by matrices as in appendix A. In equation (B4) let $\bar{H}^i_i$ be replaced by $\left[H^i_i\right\rangle$ and then let $\left[H^i_i\right\rangle$ be replaced by $\left[A^i_i\right]\left[\omega^i_{iY}\right\rangle$ where $\left[A^i_i\right]$ is the inertia matrix of body i about its center of mass. The results are

$$m_i\left[a^i_i\right\rangle = \left[f^i_i\right\rangle \tag{B9}$$

$$\left[A^i_i\right]\left[\dot{\omega}^i_{iY}\right\rangle + \left\lfloor\omega^i_{iY}\right\rceil\left[A^i_i\right]\left[\omega^i_{iY}\right\rangle = \left[T^i_i\right\rangle \tag{B10}$$

and for equation (B7)

$$m_j\left(\left[q^j_j\right\rangle - \left\lfloor\alpha^j_j\right\rceil\left[\dot{\omega}^j_{jY}\right\rangle + \left\lfloor\omega^j_{jY}\right\rceil^2\left[\alpha^j_j\right\rangle\right) = \left[f^j_j\right\rangle \tag{B11}$$

Equation (B8) can be transformed into matrix notation after noticing that

$$\left[H^j_j\right\rangle = \left[B^j_j\right]\left[\omega^j_{jY}\right\rangle + m_j\left\lfloor\Omega^j_j\right\rceil\frac{d}{dt}\left[\Omega^j_j\right\rangle - m_j\left\lfloor\alpha^j_j\right\rceil\frac{d}{dt}\left[\alpha^j_j\right\rangle \tag{B12}$$

where $\left[B^j_j\right]$ is the inertia matrix of body j about its reference point and $\left[\Omega^j_j\right\rangle$ is a position vector measured from a point in inertial space (coinciding with its reference point) to the center of mass of body j. It must be noted that although $\bar{\Omega}_j$ and $\bar{\alpha}_j$ are equal at the time instant under consideration, their derivatives are not equal and thus a distinction must be made whenever differentiation is performed. When equation (B12) is substituted into the matrix equivalent of equation (B8) and after performing the indicated differentiation, let $\left\lfloor \alpha_j^j \right\rceil$ equal $\left\lfloor \Omega_j^j \right\rceil$ and let $\left[q_j^j \right\rangle$ equal $\frac{d^2}{dt^2}\left[\Omega_j^j \right\rangle - \frac{d^2}{dt^2}\left[\alpha_j^j \right\rangle$ to get

$$\left[B_j^j \right] \left[\dot{\omega}_{jY}^j \right\rangle + \left\lfloor \omega_{jY}^j \right\rceil \left[B_j^j \right] \left[\omega_{jY}^j \right\rangle + m_j \left\lfloor \alpha_j^j \right\rceil \left[q_j^j \right\rangle = \left[L_j^j \right\rangle \tag{B13}$$

Finally consider the constraint caused by body j being attached to body i. The resulting equation in terms of accelerations is

$$\bar{q}_j^i = \bar{q}_i^i + \frac{d}{dt}\left(\bar{\omega}_{iY}^i \times \bar{\beta}_j^i \right) \tag{B14}$$

where $\bar{\beta}_j$ is the vector from the reference point of body i to the reference point of body j as shown in figure 1. When the indicated differentiation in equation (B14) is performed, there follows

$$\bar{q}_j^i = \bar{q}_i^i + \bar{\omega}_{iY}^i \times \left(\bar{\omega}_{iY}^i \times \bar{\beta}_j^i \right) + \dot{\bar{\omega}}_{iY}^i \times \bar{\beta}_j^i \tag{B15}$$

which, when translated into matrix notation, gives

$$\left[M_{ji} \right] \left[q_j^j \right\rangle = \left[q_i^i \right\rangle - \left\lfloor \beta_j^i \right\rceil \left[\dot{\omega}_{iY}^i \right\rangle + \left\lfloor \omega_{iY}^i \right\rceil^2 \left[\beta_j^i \right\rangle \tag{B16}$$

where $\left[M_{ji} \right]$ is the square matrix that changes column vectors represented in the jth coordinate system to the same column vector represented in the ith coordinate system as shown in appendix A.

APPENDIX C

PROPERTIES OF PROJECTION MATRICES

Projection matrices are a special class of matrices, closely related to the identity matrix, which may be singular. These matrices are useful when dealing with nonorthogonal sets of vectors.

Consider three linearly independent, nonorthogonal, unit column vectors $\left(\left[\eta(i)\right\rangle,\ i = 1,2,3\right)$ in a physical three space. Consider also three other column vectors $\left(\left[\zeta(i)\right\rangle\right.$ where $\left. i = 1,2,3\right)$ which are related to the first three vectors by the relationship

$$\left\langle\zeta(i)\right]\left[\eta(j)\right\rangle = \delta_{ij} \tag{C1}$$

where $\langle\ ]$ denotes transpose of $[\ \rangle$ and δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = 0 \qquad (i \neq j)$$

$$\delta_{ij} = 1 \qquad (i = j)$$

Next consider an arbitrary column vector $\left[V\right\rangle$ such that

$$\left[V\right\rangle = a_1\left[\eta(1)\right\rangle + a_2\left[\eta(2)\right\rangle + a_3\left[\eta(3)\right\rangle \tag{C2}$$

If equation (C2) is multiplied on the left by $\left\langle\zeta(i)\right]$, there follows from equation (C1)

$$a_i = \left\langle\zeta(i)\right]\left[V\right\rangle \tag{C3}$$

Equation (C3) gives a practical use for the $\left[\zeta(i)\right\rangle$ values; also the $\left[\eta(i)\right\rangle$ and $\left[\zeta(i)\right\rangle$ terms are said to be mutually reciprocal. Next when $\left[V\right\rangle$ is written out in its expanded form, it is seen that

$$\left[V\right\rangle = \left(\left[\eta(1)\right\rangle\left\langle\zeta(1)\right] + \left[\eta(2)\right\rangle\left\langle\zeta(2)\right] + \left[\eta(3)\right\rangle\left\langle\zeta(3)\right]\right)\left[V\right\rangle \tag{C4}$$

which shows that the identity matrix is defined by

$$\left[I\right] = \left[\eta(1)\right\rangle\left\langle\zeta(1)\right] + \left[\eta(2)\right\rangle\left\langle\zeta(2)\right] + \left[\eta(3)\right\rangle\left\langle\zeta(3)\right] \tag{C5}$$

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Let the term $\left[\eta(i)\right\rangle\left\langle\zeta(i)\right]$ be dropped from $\left[I\right]$ and let $\left[V\right\rangle$ be premultiplied by the result. It is seen from equations (C1) and (C2) that $\left[V\right\rangle$ remains unchanged except that the corresponding term $a_i\left[\eta(i)\right\rangle$ in equation (C2) is dropped. Thus when terms are dropped from $\left[I\right]$, the resulting matrix is said to be the projection matrix for the directions corresponding to the terms retained in $\left[I\right]$.

The vectors $\left[\zeta(i)\right\rangle$ may be found from the vectors $\left[\eta(i)\right\rangle$ by forming a matrix $\left[X\right]$ (where the ith column of $\left[X\right]$ corresponds to the components of $\left[\eta(i)\right\rangle$) and then forming the inverse of $\left[X\right]$. Then the components of the jth row of $\left[X\right]^{-1}$ are the components of $\left[\zeta(j)\right\rangle$. Since $\left[\eta(i)\right\rangle$ terms are linearly independent, $\left[X\right]^{-1}$ exists.

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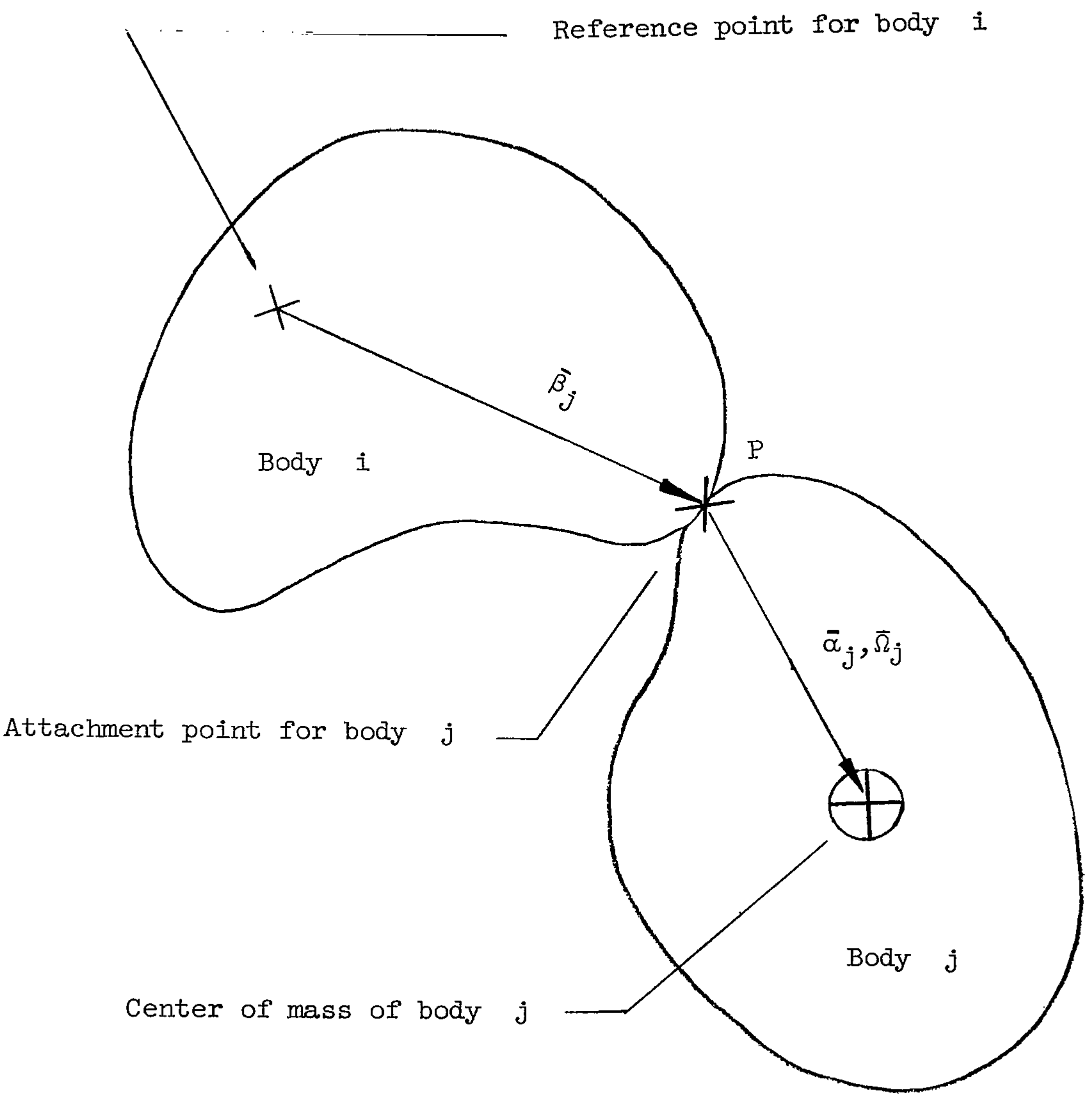

Figure 1.- Sketch showing body j rotating relative to body i about two axes.

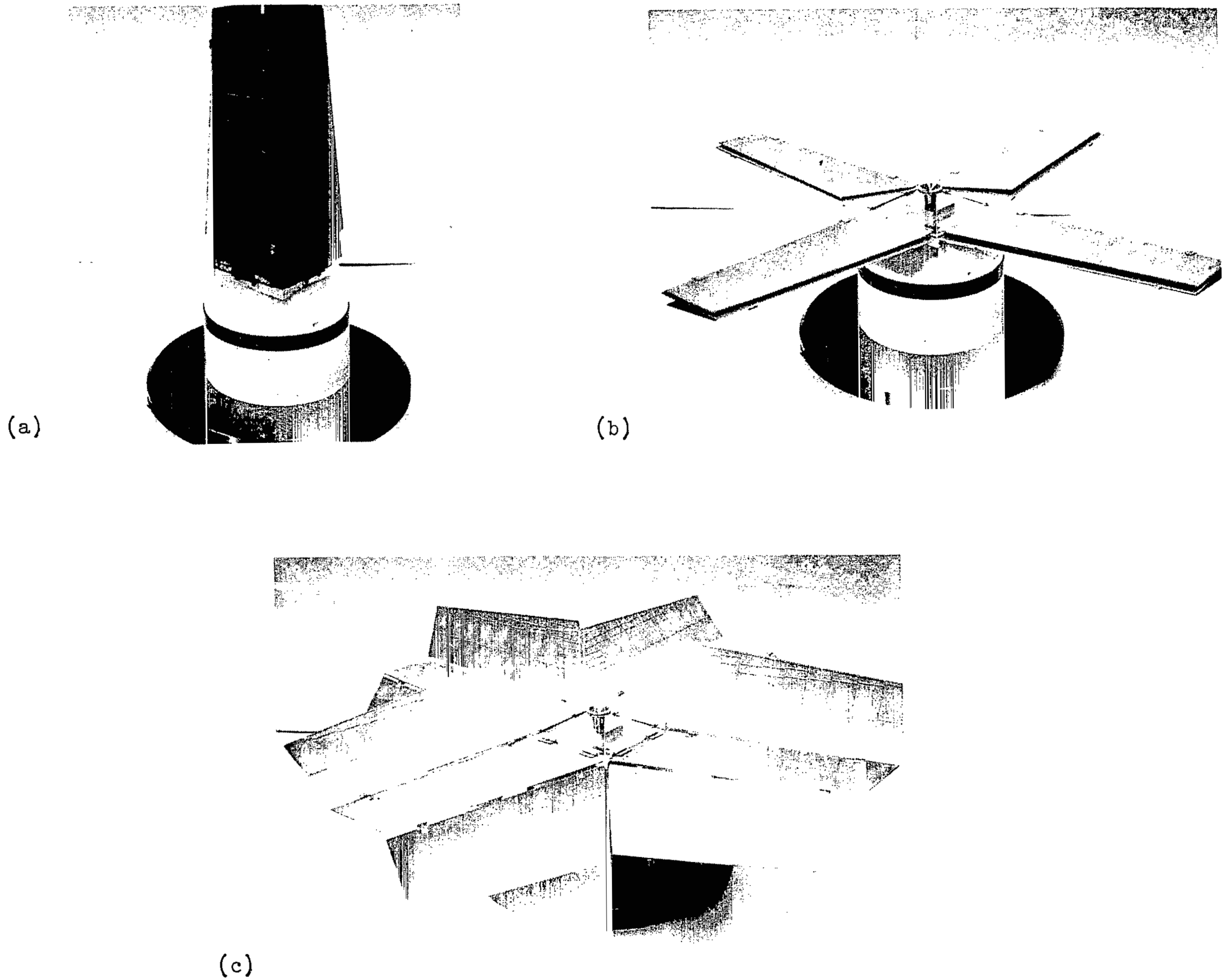

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Figure 2.- Model of the proposed NASA micrometeorite detection satellite showing its deployment sequence. (Deployment motions studied are from models shown in figures 2(a) and 2(b).)

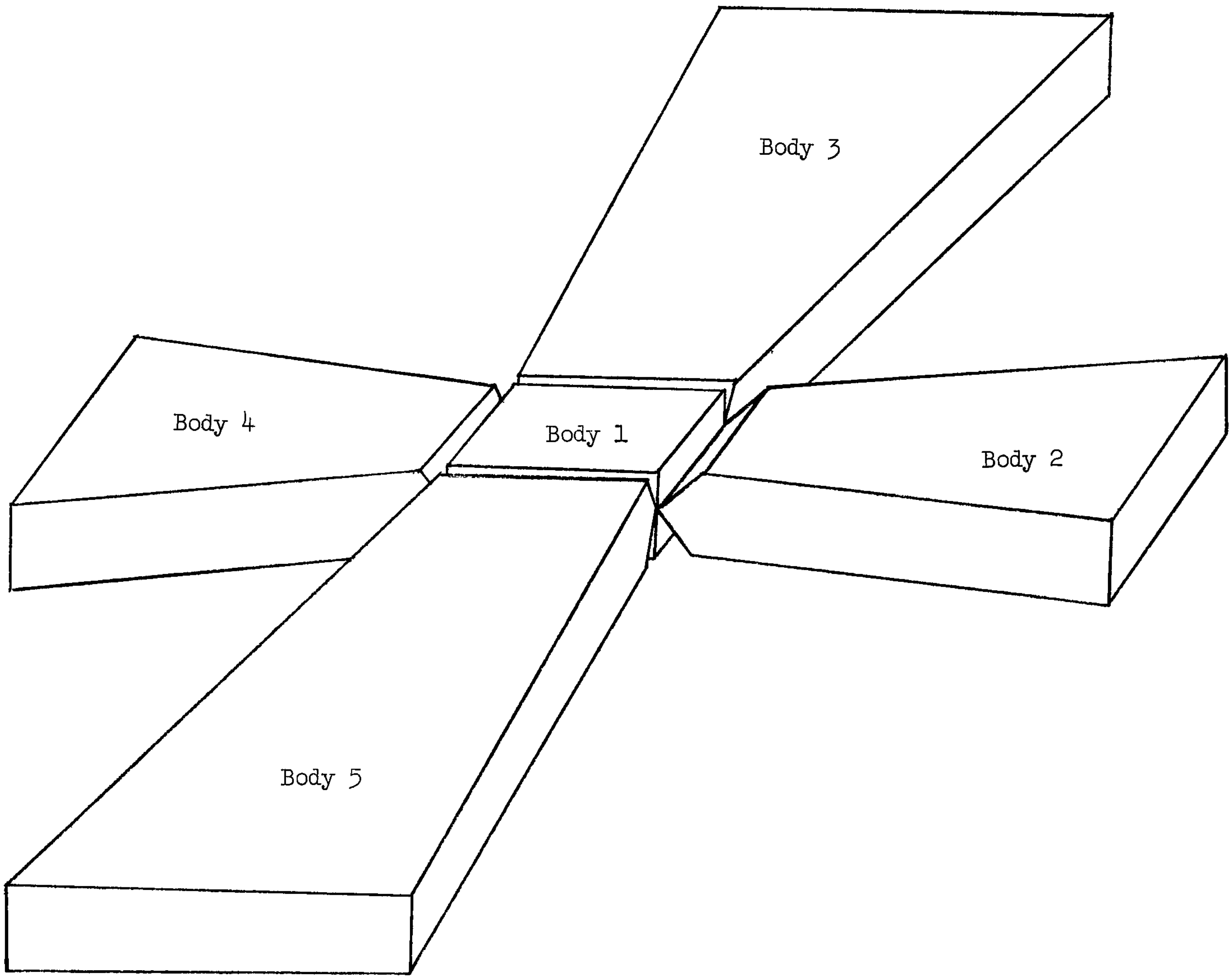

Figure 3.- Idealized dynamical system with a central hub and four wings.

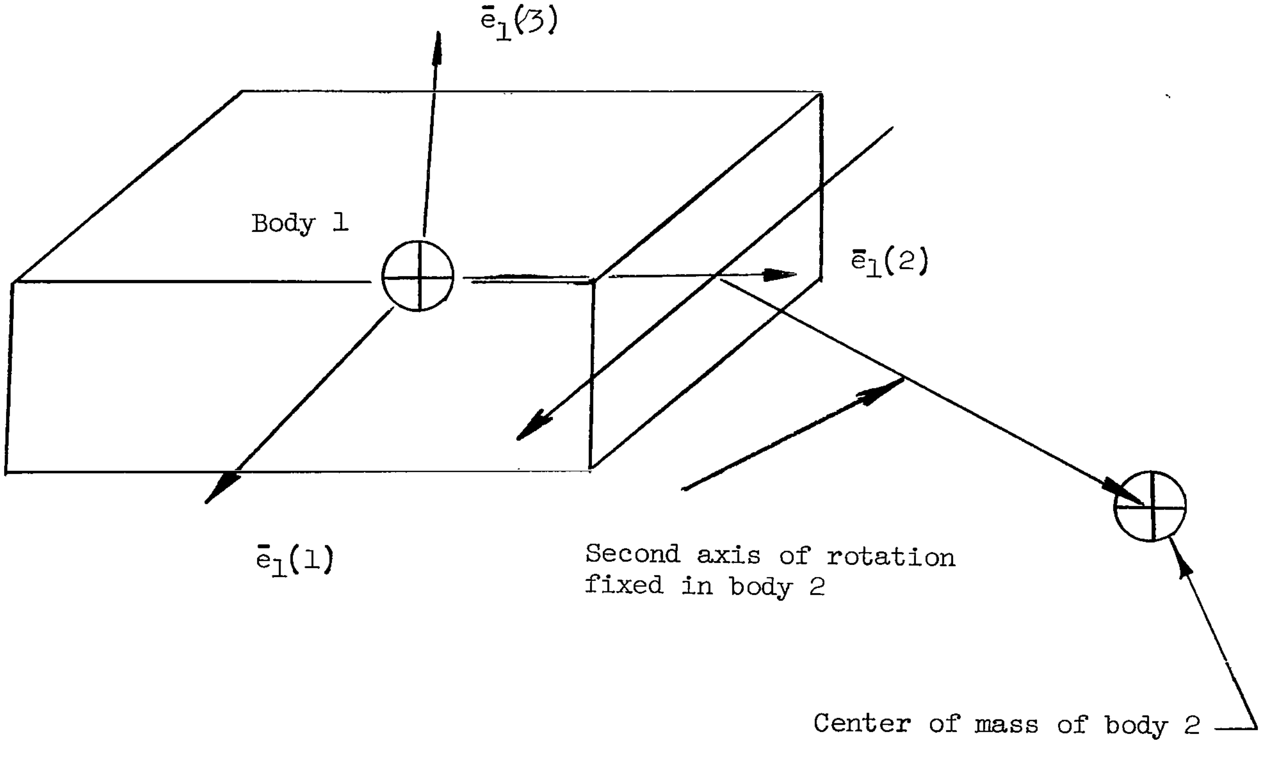

Figure 4.- Sketch showing coordinate system for body 1 fixed in body 1 along with axes of rotation for body 2 relative to body 1.

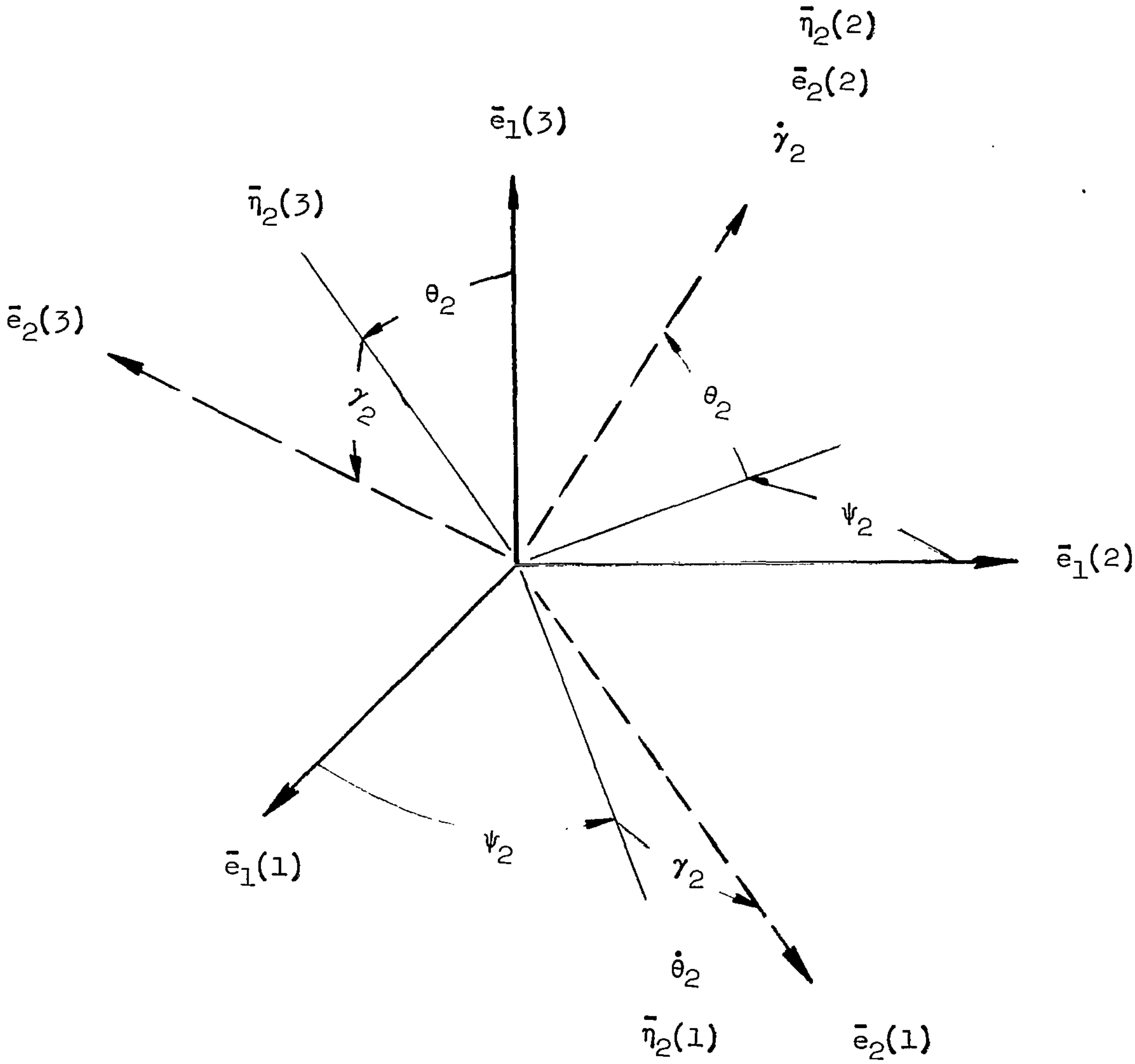

Figure 5.- Sequence of rotations relating C_1 (coordinate frame fixed in body 1) to C_2 (coordinate frame fixed in body 2).

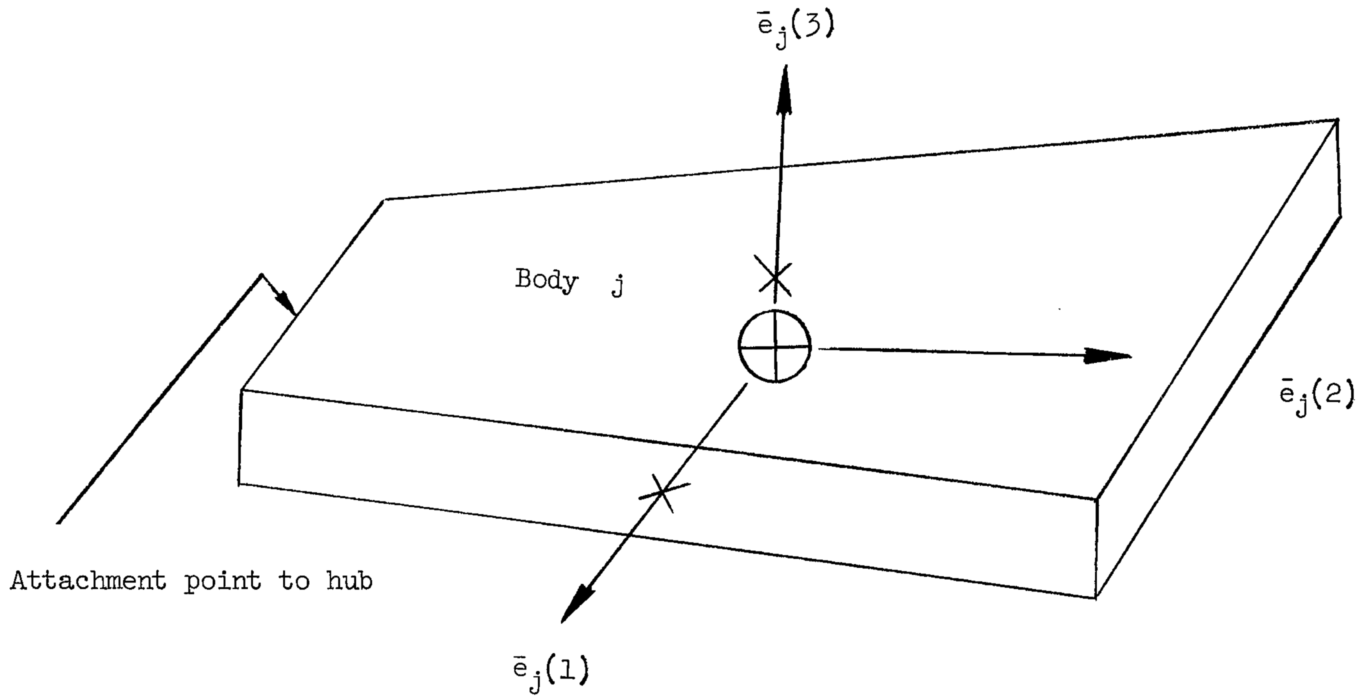

Figure 6.- Arrangement of body j coordinate frame relative to body j.

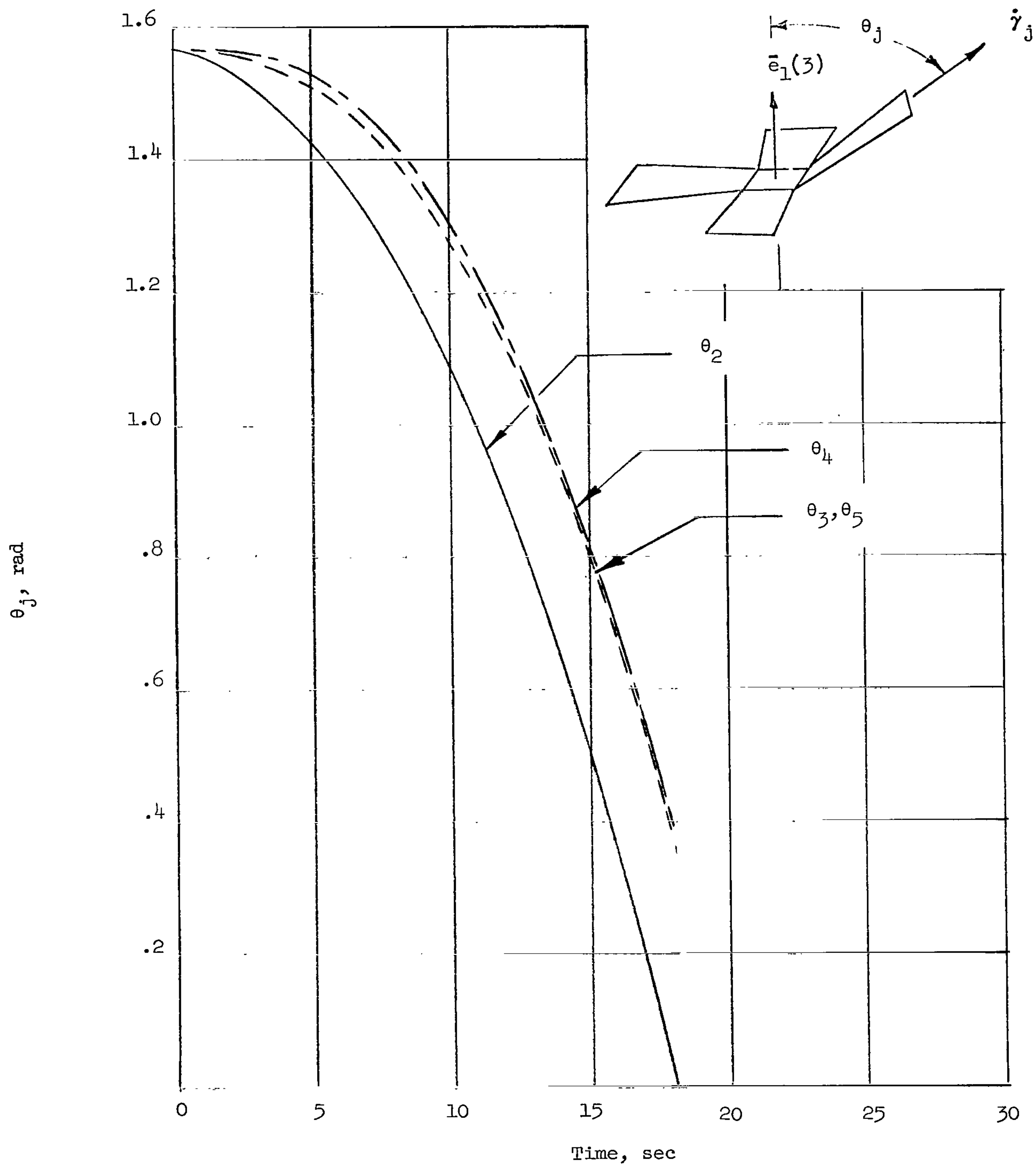

Figure 7.- Deployment-angle time history of wings when spacecraft has no initial spin. $W_j = 560$ m-N-rad^{-1}.

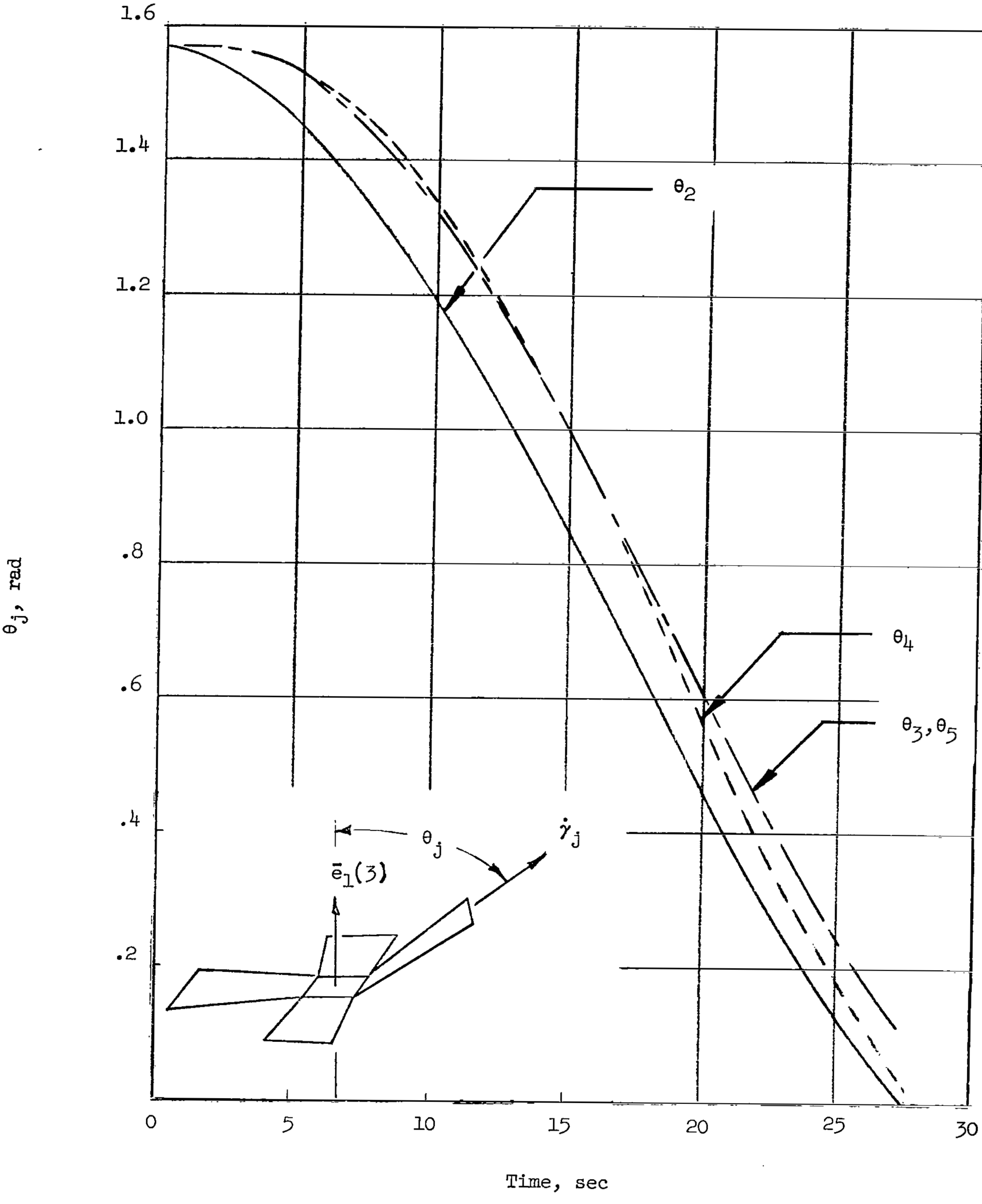

Figure 8.- Deployment-angle time history of wings when spacecraft has no initial spin. $W_j = 5530$ m-N-sec-rad^{-1}.

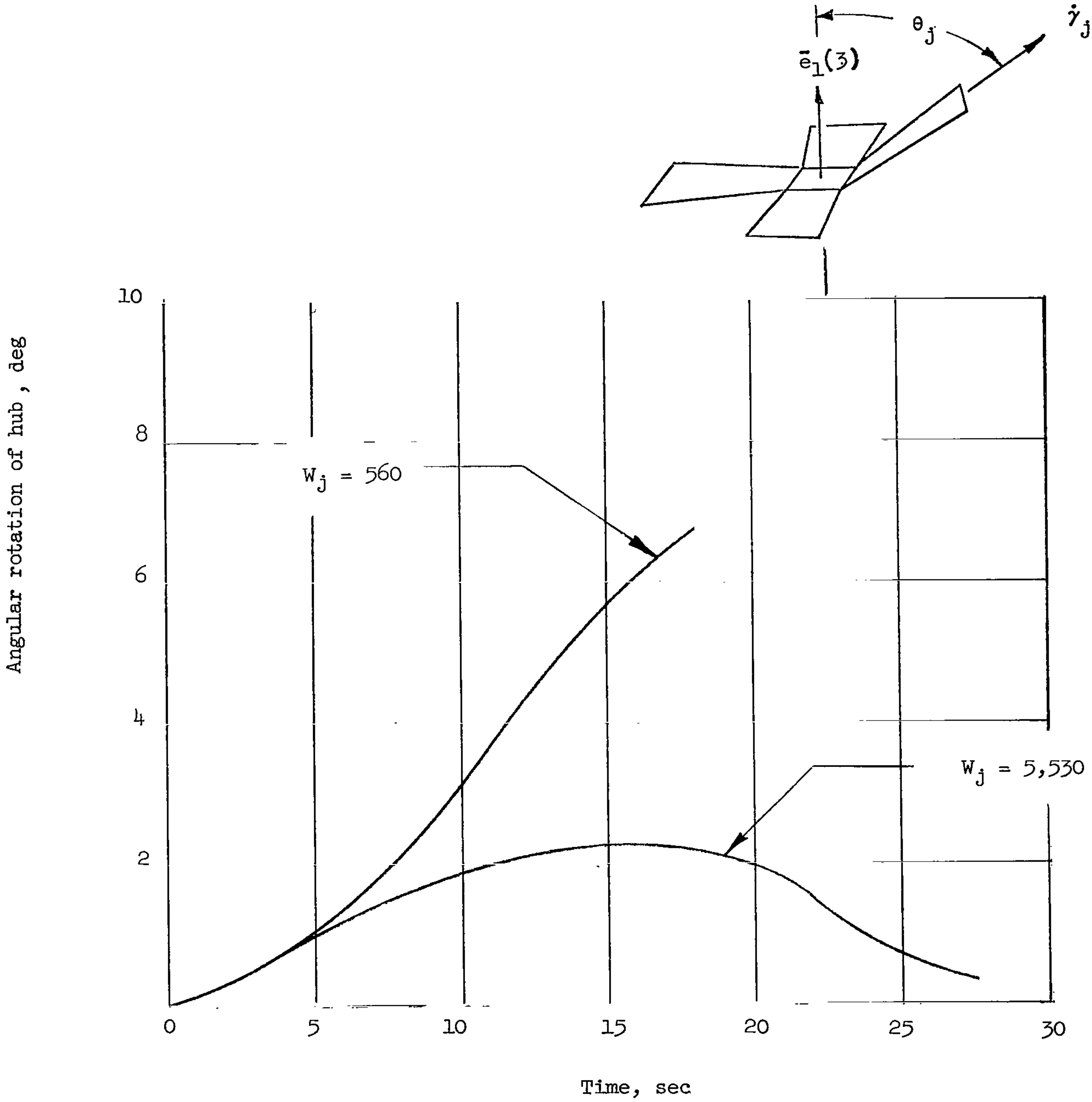

Figure 9.- Pitching motion of hub due to one wing being deployed early when the spacecraft has no initial spin.

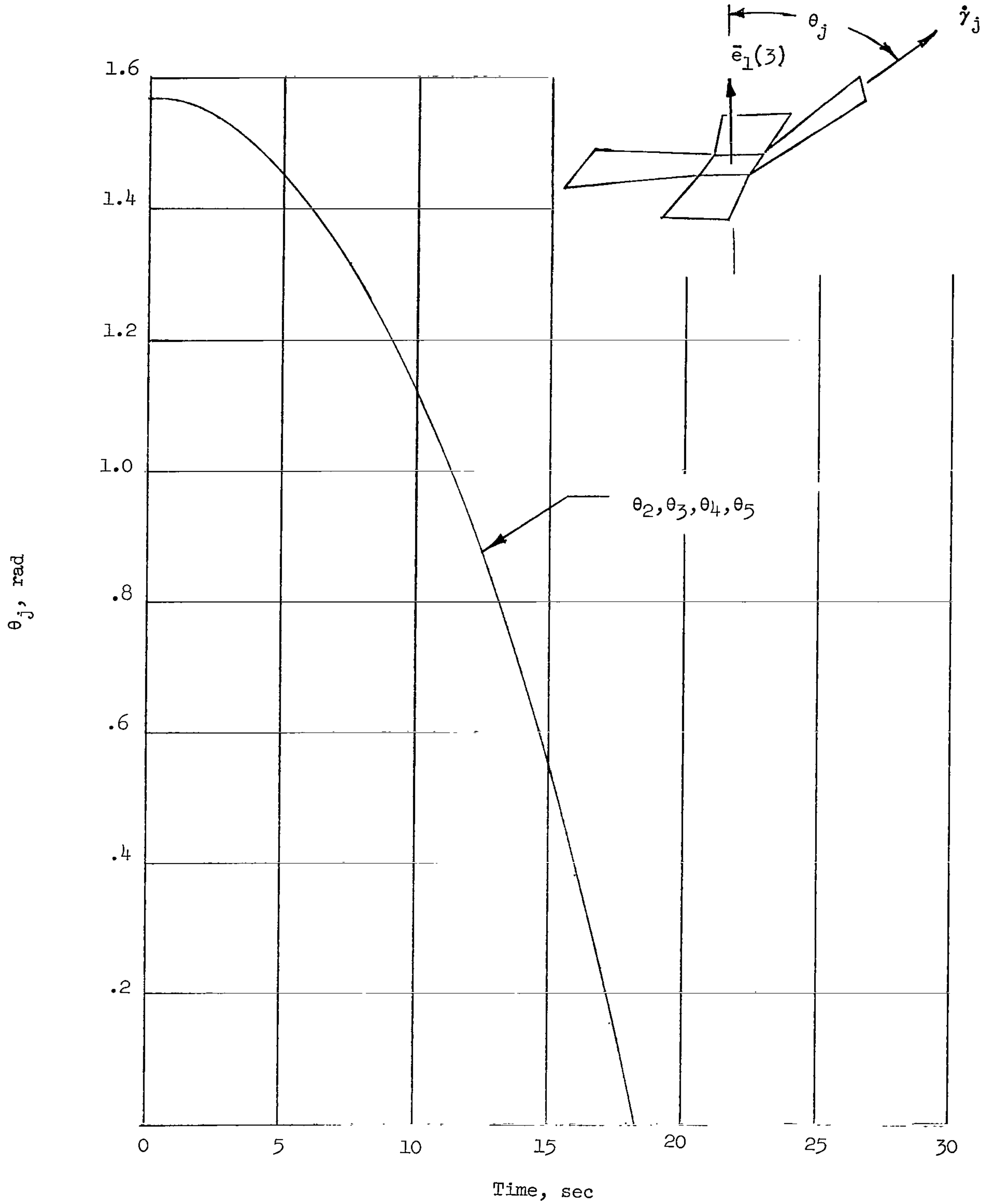

Figure 10.- Exact solution of deployment-angle time history of wings with no initial spin and no damping.

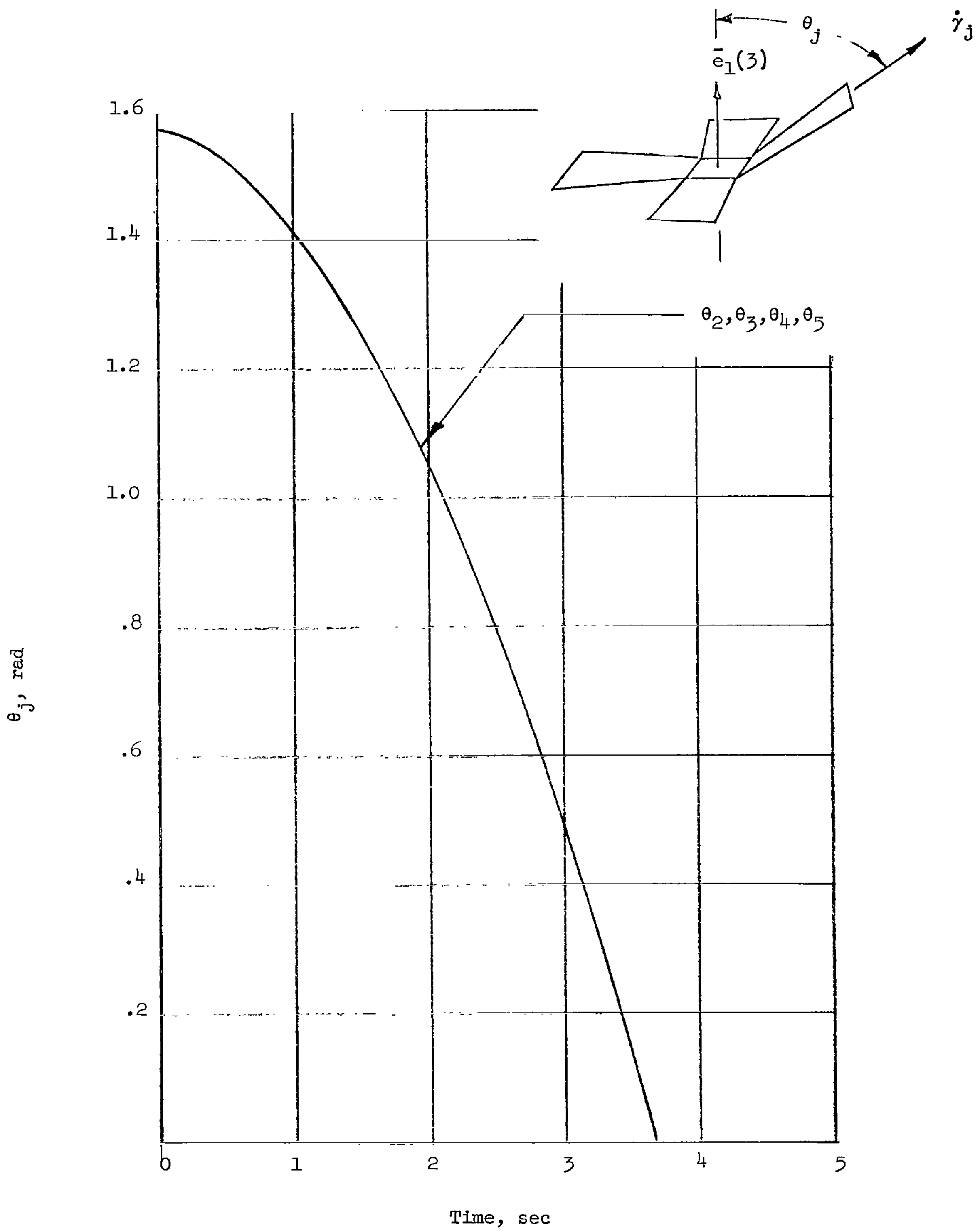

Figure 11.- Time history of wing deployment angle θ_j for deployment with spin.

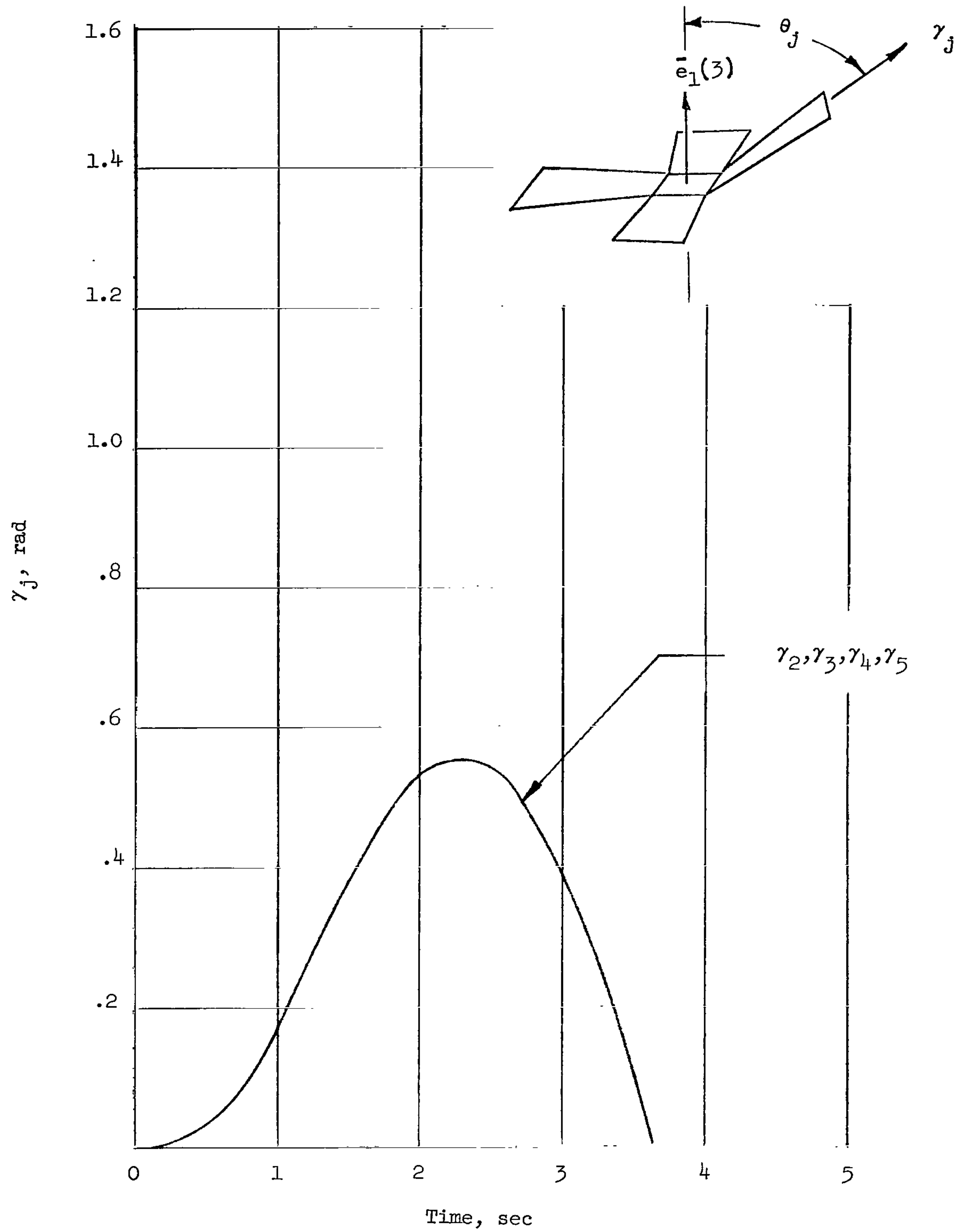

Figure 12.- Time history of twist angle γ_j for deployment with spin.

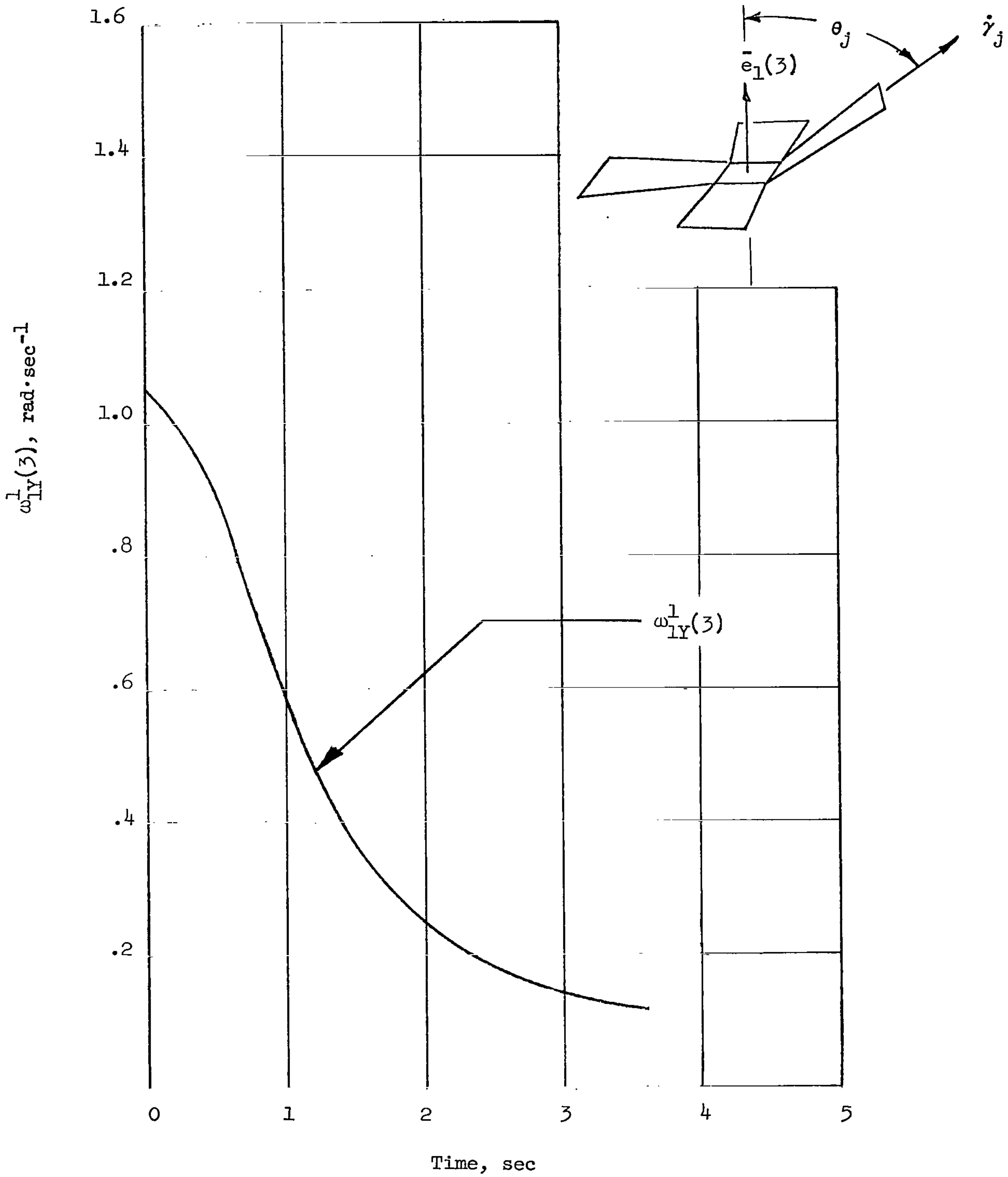

Figure 13.- Time history of hub spin rate for deployment with spin.

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