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NONLINEAR WAVES WITH A MAGNETIC FIELD IN THE COSMIC PLASMA

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SUMMARY

A new solution of two-fluid hydromagnetic equations is obtained in the form of large amplitude nonlinear ion waves. The magnetic field vector of such a wave rotates and changes its magnitude simultaneously. There exists experimental evidence of these nonlinear oblique waves propagating in cosmic space. A new mechanism of plasma penetration into the magnetosheath and into the magnetosphere is also suggested - average plasma flow due to the nonlinear character of these waves. Nonlinear ion waves of relativistic velocities are also considered giving a new mechanism for cosmic rays generation. As an example the acceleration of cosmic rays in a solar flare is computed.

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1. Nonlinear Hydromagnetic Waves. The Nonrelativistic Case.

The initial equations are two fluid electron-ions equations of cold plasma with a magnetic field. High amplitude large-scale periodic waves are sought for. Inasmuch as the ion waves are large-scale, two assumptions are made: 1) electron inertia is neglected considering that the motion velocity of wave is sufficiently low (of the order of Alfvén velocity), i.e.

$$E+\frac{1}{c}[V_e H]=0;$$

2) in the law of transverse momentum conservation, we have

$$nu(Mv_i+mv_e)=\frac{H_{\parallel}^0(\mathbf{H}-\mathbf{H}_{\perp 0})}{4\pi} \tag{1}$$

(where $H_{\parallel}{}^0$ is the longitudinal field and $H_{\perp}o,x, H_{\perp}o,y$ are arbitrary constants) we have neglected the electron pulse as compared with the ion pulse.

Expressing v_e and v_i of these relations by H & substituting in the Maxwell equation $\mathrm{rot}\, H = 4\pi enc^{-1}(v_i - v_e)$, we find the equations for the modulus and phase of the magnetic field H

$$(H = H_x + iH_y = H_{\perp 0}\lambda(z)\, e^{i\varphi(z)}; \quad H_{\perp 0}{}^2 = H^2{}_{\perp 0,x} + H^2{}_{\perp 0,y}: \qquad \dot{\lambda} = (\mu_{\|}{}^2 - Q_1)\sin\varphi + Q_1\cos\varphi \tag{2a}$$

$$\dot{\varphi} + (\mu_{\|}{}^2 - u) + \frac{1}{\lambda}\{(Q_1 - \mu_{\|}{}^2)\cos\varphi + Q_2\sin\varphi\} = 0. \tag{2b}$$

Here $u = 1 + \frac{\mu_\perp{}^2}{2}(1 - \lambda^2)$ -- is the dimensionless longitudinal velocity; $Q = Q_2 + iQ_2$ is the complex one & $\mu_{\|}{}^2$ and $\mu_\perp{}^2$ are the real integration constants.

$$\dot{\lambda} \equiv d\lambda / d\tau = \mu_{\|}\, u\, d\lambda / dz.$$

Let us denote $(Q_1 - \mu_{\|}{}^2)\cos\varphi + Q_2\sin\varphi$ by <u>s</u>, then from (2a) we obtain $d\varphi / d\tau = ds / d\lambda$.. Substituting this value into (2b) we shall obtain

$$\frac{ds}{d\lambda} + (\mu_{\|}{}^2 - u) + s/\lambda = 0. \tag{3}$$

Suppose that for $\phi = 0$ (i.e. at $H \parallel H_{\perp 0}$) the field modulus assumes a certain value $\lambda = \lambda_0$ (λ_0 being in this way one of wave constants). Then the solution of equation (3), which passes through point $\phi = 0, \lambda = \lambda_0$, is

$$s(\lambda) = \frac{\lambda_0}{\lambda}\left\{(Q_1 - \mu_{\|}{}^2) + \frac{1}{2\lambda_0}\left[\left(1 + \frac{\mu_\perp{}^2}{2} - \mu_{\|}{}^2\right)(\lambda^2 - \lambda_0{}^2) - \frac{\mu_\perp{}^2}{4}(\lambda^4 - \lambda_0{}^4)\right\}.$$

Substituting it in (2a) and taking into account that $\left(\frac{ds}{d\varphi}\right)^2 = |Q|^2 - s^2$, we obtain for the square of field modulus, i.e. for the magnetic field energy $g = \lambda^2$, the equation

$$g = 2\sqrt{-U}, \tag{4}$$

where

$$U = \left\{Q_1\lambda_0 + \frac{1}{2}\left[\left(1 + \frac{\mu_\perp{}^2}{2} - \mu_{\|}{}^2\right)(g - g_0) - \frac{\mu_\perp{}^2}{4}(g^2 - g_0{}^2)\right]\right\}^2 - |Q|^2 g,$$

$$g = \lambda^2.$$

Eq. (4) is easily integrated in elliptic functions. If, for instance, equation U = 0 has two real positive roots α and β ($\alpha > g > \beta$) and two other complex roots, the connection between τ and g is given by the elliptic integral ([1], formula 3.145.2)

$$\tau = \int_{\beta}^{g} \frac{dx}{[(\alpha - x)(x - \beta)[(x - m)^2 + n^2]^{1/2}} = [b_1 b_2]^{-1/2} F(\mu, k), \tag{5}$$

$$\mu = 2 \operatorname{arcctg} [b_2(\alpha - g)/(b_1 \times (g - \beta))], \quad k = \frac{1}{2}\sqrt{\frac{(\alpha - \beta)^2 - (b_1 - b_2)^2}{b_1 b_2}},$$

$$b_1^2 = n^2 + (m - \alpha)^2; \quad b_2^2 = n^2 + (m - \beta)^2.$$

Inverting this integral we obtain

$$g = \frac{\alpha + \beta B}{1 + B}, \quad \text{where} \quad B = \frac{b_1^2 sn^2(\tau\sqrt{b_1 b_2}, k)}{b_2^2[1 - cn^2(\sqrt{b_1 b_2}\tau, k)]^2} \dots \tag{6}$$

When the imaginary part n of the complex conjugate roots approaches zero, $k \to 1$, we have a nonlinear solitary pulse [2]. On the contrary, with the confluence of the real roots, $\alpha \to \beta$, and we have $k \to 0$. In this extreme case of low-amplitude waves ($\alpha \approx \beta$) the elliptic sine changes to an ordinary sinusoid; thus we obtain both branches of low-amplitude waves, namely the Alfvén branch and the magneto-acoustic branch.

Let us underscore some of the special features of the solution thus obtained.

1. Contrary to nonlinear solitary pulses, in periodic waves the plasma has a certain mean velocity $\bar{u} = \int u d\tau$ relative to the magnetic field, i.e. the plasma penetrates into the magnetic field. Postulating $\partial H / \partial t = -\operatorname{rot} E = 0$ and, therefore E = const, we have linked the coordinate system with the magnetic field of the wave. Consequently, the nonzero mean value for the period of velocity $\bar{u} = 1 + \mu_\perp^2/2 \cdot (1 - \lambda^2)$ implies that on the average the plasma moves relative to the field H. In linear approximation there is no such average plasma flow relative to the field. The average for the period from sine is zero. Similarly, the on-shore sea waves carry with them a specific momentum and flow of matter. Namely, at high amplitude of waves (in a storm) the nonlinear drift effect is greater also. A clear proof of the drift are the random objects, for instance pieces of wood, carried by the wave from the open sea to the shore.

Therefore, the nonlinear periodic waves constitute a form of plasma motion relative to the magnetic field. The compressed plasma penetrates into the magnetic field of the wave; this magnetic field is variable both in magnitude and direction. In this respect, nonlinear waves are essential in the turbulent transitional layer between the undisturbed solar wind and the magnetosphere: the magnetic field becomes variable and the plasma passes through it toward the magnetosphere.

Here the analogy with sea waves is not complete. The presence of the magnetic field changes considerably all the properties and the structure of a nonlinear wave. First of all, an oblique wave is always a wave with field rotation. It can be said that the plasma incident upon the magnetosphere does not break through the field but "twists its way" through it.

Secondly, the nonhomogenous field of a wave, especially of combination of nonlinear waves, retains the fast particles. Such a method of plasma retention by nonlinear waves can be used in plasma experiments. Let us note that in electron waves the field amplification is of the order $\sqrt{M/m} \geqslant 43$, so that there emerge very strongly retaining fields. Possible also, besides the magnetic retention is the electric retention, namely the trapping of particles in electric potential wells.

The polarization of nonlinear waves is essentially different from the linear case. Having written $\underline{s}$ as $\sin(\phi + \delta)\ |Q|$ we have the following complex relation between ϕ and λ:

$$\sin(\varphi+\delta) = \frac{1}{|Q|\lambda}\left\{Q_1\lambda_0 + \frac{1}{2}\left[\left(1+\frac{\mu_\perp}{2}-\mu_\pi^2\right)(\lambda^2-\lambda_0^2) - \right.\right.$$
$$\left.\left. - \frac{\mu_\perp^2}{4}(\lambda^4-\lambda_0^4)\right]\right\},$$

i.e. a more complicated polarization curve than the ellipse corresponding to linear waves.

Finally, let us underscore a general structural property of nonlinear waves: from the first integrals nu = const and $u = 1 + \frac{\mu_\perp^2}{2}(1-\lambda^2)$ it follows that plasma density is highest in the magnetic field maximum, and lowest in the magnetic field minimum.

The shape of the interplanetary field magnetograms measured on satellites, the curves of field vector rotation, the ratio between the constant and the variable field components (Explorer [3] Mariner-4 [4], OGO-1.3 [5]), all of them bear witness to the nonlinear nature of waves observed in the interplanetary medium and in the transition region [magneto sheath].

2. Relativistic Ion Waves and the Origin of Cosmic Rays. When Alfvén velocity approaches light velocity c, ions accelerate in nonlinear waves to energies of hundreds of Mev. The nonlinear nature of the wave is the prerequisite condition for acceleration, for only in nonlinear wave is the particle velocity of same order as the wave velocity. Particle velocities are low even in the fastest linear wave.

In the general relativistic case the two-fluid magnetic hydrodynamics equations can yield the following system of differential equations:

$$\dot{q} = B(p, q), \quad \dot{p} = -A(p, q), \tag{7}$$

where $q = H_x/H_{\perp 0}$; $p = \dfrac{H_y\sqrt{\cos^2\theta - \beta^2}}{\gamma H_{\perp 0}}$; $\beta = \dfrac{u_0}{c}$;

$$\gamma^2 = 1 - \beta^2;$$

$$A = q(2\operatorname{ctg}^2\theta - S + \gamma^2\lambda^2 + 2\beta^2 q) + \operatorname{Re} S_p + \beta^2(\lambda^2 - 2q) + S_1;$$

$$B = p\left(2\operatorname{ctg}^2\theta - 2\frac{\beta^2}{\gamma^2} - S + \gamma^2\lambda^2 + 2\beta^2 q\right) + \operatorname{Im} S_p;$$

and $S, \operatorname{Re} S_p, S_1, \operatorname{Im} S_p$ are arbitrary constants,

$$\dot{q} = \frac{dq}{d\tau} = G\frac{dq}{dz}, \text{ где } G = \frac{\gamma}{\beta}(\cos^2\theta - \beta^2)^{1/2}\frac{1}{4\pi e n_0 \sin\theta}; \quad \lambda^2 = p^2 + q^2.$$

Inasmuch as $\dfrac{\partial A}{\partial p} = \dfrac{\partial B}{\partial q}$, the system (7) is Hamiltonian and has for the first integral

$$\operatorname{Im} S_p\, p + \frac{\beta^4}{\gamma^2}q^2 + q(\beta^2\lambda^2 + S_1 + \operatorname{Re} S_p) + \lambda^2\left[\frac{1}{\gamma^2}\frac{\cos^2\theta - \beta^2}{\sin^2\theta} - \frac{S}{2}\right] + \frac{\gamma^2}{4}\lambda^4 = \text{const.} \tag{8}$$

For the field modulus we obtain the equation

$$q\operatorname{Im} S_p - p\left[2\frac{\beta^4}{\gamma^2}q + \beta^2\lambda^2 + S_1 + \operatorname{Re} S_p\right] = \frac{1}{2}\frac{d\lambda^2}{d\tau}\left(W + \frac{p_\pi}{\beta}\right)8\pi n_0\frac{\beta^2}{\gamma^2}\,.$$

$$\left(W = \frac{H_{\perp 0}^2}{8\pi n_0}\gamma^2[\lambda^2 - 2q + \text{const}] \text{ is the energy}\right). \tag{9}$$

In the particular case of ImSp = 0, integral (8) is reduced to a quadratic equation relative to q:

$$aq^2 + rq + d = 0.$$

Taking the quantity $\sigma = (r^2 - 4ad)^{1/2}$ for a new unknown function, we convert (9) to the following simple form:

$$p = (\sigma - K_1)\dot{\sigma} \tag{10}$$

(K_1 is a constant), where p^2 is expressed by a four degree polynomial relative to σ, and the solution of equation (10) is expressed by a linear combination of elliptic integrals.

Let us show how to determine the limits of particle acceleration in such a wave. We will illustrate it by the very simple example of a compression "soliton".

In the case of a nonlinear solitary pulse (soliton) (i.e. when the plasma is not perturbed at infinity), $\lambda = 1$; $|p| = p_{\parallel} = E = 0$ $W = Mc$ (p and $p_{\parallel}$ are the transverse and longitudinal momenta), the wave constants will respectively take the following values:

$$\operatorname{Re} S_p = -2\operatorname{ctg}^2\theta; \quad S_1 = \beta^2 + 2\frac{\beta^2}{\gamma^2}R_{\perp}^{-2}; \quad \operatorname{Im} S_p = 0;$$

$$S = 1 + \beta^2 + 2\frac{\beta^2}{\gamma^2}R_{\perp}^{-2}; \quad R_1 = R_{\perp}\sin\theta; \quad R = \frac{H_0}{\sqrt{4\pi n_0 M}}; \quad (c = 1).$$

And subsequently

$$p = (\sigma - \Delta)^2 \cdot P(\sigma), \text{ where } P(\sigma) = -\sigma^2 + 2\Delta\cdot\sigma\frac{1+\beta^2}{\gamma^2} + \frac{4}{\gamma^4}\left[\beta^2\frac{\Delta^2(1+\beta^2)^2}{4}\right]; \quad \Delta^2 = 1 + R_1^{-2} - \frac{\gamma^2}{\beta^2}\operatorname{ctg}^2\theta. \tag{11}$$

With substitution by variable $v = \frac{\gamma^2}{2\beta^2\Delta}\sigma - \Delta$ Eq. (10) is easily integrated in elementary functions.

The amplitude of the nonlinear solitary pulse compression $\sigma+$ and the minimal field value in a nonlinear solitary pulse refraction σ are determined as the roots of equation $P(\sigma) = 0$:

$$\sigma_{\pm} = \frac{\Delta(1+\beta^2)\pm 2\beta}{\gamma^2}, \quad \text{i.e. } \lambda_{\pm} = \frac{2\beta\Delta \pm (1+\beta^2)}{\gamma^2}.$$

Further relations for the extreme points of the wave coincide with those obtained in [6], where the author has investigated the particular case of nonlinear solitary pulses, limiting himself to the extreme points and without solving the differential wave equations.

First of all, let us find the maximum velocity for the nonlinear solitary pulse. At infinity $\cos\phi = 1$, i.e. $\phi = 0$. At this point for $\phi = 0$ $d^2(\cos\phi)/d\lambda^2 < 0$. By differentiating twice the expression for $\cos\phi$ as a function of λ, which follows from (8), we find $\beta < \beta_{max} = R/\sqrt{R^2+1}$. Substituting this maximum velocity in relation (12) we obtain

$$\lambda = 3 + 4R^2. \tag{13}$$

Now, let us consider that relativistic nonlinear solitary pulses propagate toward the initial field only at a sufficiently small angle. At an angle, larger than the limit angle, change over takes place and the motion becomes multiflow.

The condition $n_0/n = 1 + P_{\parallel}/\beta W > 0$ in the momentum maximum at $\cos\phi = -1$ yields

$$\frac{n_0}{n} = 1 + \frac{\gamma^2}{\beta^2}\frac{1-\lambda^2}{R_{\perp}^{-2} + \gamma^2(1+\lambda)^2} > 0, \tag{14}$$

wherefrom, using (13):

$$\sin^2\theta < \frac{1}{4(1+R^2)}. \tag{15}$$

The particle energy is

$$W = H^2\sin^2\theta / [\gamma^2(\lambda^2 - 2q + 1) + R_{\perp}^{-2}](8\pi n_0)^{-1}.$$

In an unperturbed plasma we have $W_{min} = H^2\sin^2\theta R_{\perp}^{-2}/8\pi n_0 = Mc^2$. The energy in the momentum maximum ($\cos\varphi = -1$; $\lambda^2 = 3 + 4R^2$) we have

$$W_{max} = \frac{H^2}{8\pi n_0}\frac{1}{4(1+R^2)}\left[\frac{16}{R^2+1}(1+R^2)^2 + R_{\perp}^{-2}\right],$$

i.e.

$$\Delta W = W_{max} - W_{min} = 4H_0^2/8\pi n_0. \tag{16}$$

As an example, let us investigate the conditions of a strong chromospheric flare in the Sun. We postulate that $H_0 = 500$ gauss, $n_0 = 10^8$ cm^{-3}. Then, according to formula (16), $\Delta W = 250$ Mev is the characteristic energy of solar cosmic rays.

Softer protons (for instance $\Delta W \sim 5$ Mev), are also emitted by a quiet Sun [7]. Let us estimate the field required for acceleration: $H^2 = 2\pi n_0 \Delta W = 10^4$ gauss^2 ($n_0 = 10^8$ cm^{-3}, $\Delta W = 5$ Mev).

Such fields of $\sim$ 100 gauss are actually characteristic for solar regions emitting corpuscles [7].

A characteristic peculiarity of the relativistic waves is the presence in them of the longitudinal electric field $E_{\parallel}$. This is a potential field (rot $E_{\parallel} = 0$), and a potential well is formed, in which ions (in the rarefaction non linear solitary pulse minimum) and electrons (in the condensation non linear solitary pulse maximum) effecting there a finite motion, are trapped. If only for the fact, that trapped particles move along together with the wave, they have also a considerable kinetic energy. Let us evaluate it for the simplest case of the condensation of nonlinear solitary pulse

$$\frac{W}{Mc^2} = \frac{1}{\sqrt{1-\beta^2}} = \sqrt{(Mc^2)^2 + \frac{H_0^2 Mc^2}{4\pi n_0}}.$$

The energy of a trapped particle is lower than the energy of a particle accomplishing an infinite motion, but is of the same order. When the wave is destroyed the trapped particles excape into the surrounding plasma, gnerating also cosmic rays.

All the results of this article can be generalized for the case of a hot plasma, taking into account the electron and ion pressure. Then, only the equations of particle motion along the axis OZ undergo a change. Pressure will introduce into them the additional term $1/n\partial p/\partial z$, and, accordingly

$$u = 1 + \frac{\mu_\perp^2}{2}(1-\lambda^2) + \gamma_1 \nu^2 (1 - u^{-\gamma_1})$$

where $\gamma_1 = c_p/c_v$ is the ratio of heat capacities and $\nu = \varkappa T_e / M u_0^2$ is the inverse Mach number for acoustic waves. The electronic pressure is part of this formula since the wave velocity of ions is lower than the thermal velocity of electrons and higher than that of ions.

The method that made it possible to find the first integral of Eqs. (2) can be also applied in the presence of pressure:

$$s\lambda = \frac{1}{2}\int (u - \mu_{\parallel}^2)\, d\lambda^2 = \frac{1}{\mu_\perp^2}\left\{ \frac{\nu^2\gamma_1^2}{1-\gamma_1} u^{1-\gamma_1} + \gamma_1 \mu_{\parallel}^2 \nu^2 u^{-\gamma_1} - \right.$$
$$\left. - \frac{u^2}{2} + \mu_{\parallel}^2 u \right\} + \text{const.}$$

Substituting this expression into equation $\frac{1}{2}\frac{d\lambda^2}{d\tau} = |Q|^2\lambda^2 - (s\lambda)^2$, we obtain a differential equation for u. With a special selection of constants the solution of this equation can be described by means of a double-humped curve.

As may be seen from the formula for u, the change over effect is not observed in waves with pressure, no matter what the parameter values.

In the isothermic case:

$$s\lambda = \frac{1}{\mu_\perp^2}\left(v^2 \ln u + \frac{\mu_\parallel^2 v^2}{u} + \mu_\parallel^2 u - \frac{u^2}{2}\right) + C.$$

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REFERENCES

1. N.M. RYZHIK, I.S. GRADSHTEYN: Tables of Integrals. Izd-vo 4-ye, Fizmatgiz, 1962.
2. A.P. KAZANTSEV: ZhETF, 44, 1283, 1963; 46, 620, 1964.
3. L. CAHILL, P. AMAZEEN: J. Geophys. Res. 68, 1835, 1963.
4. V. SISCOE and al: J. Geophys. Rev. 72, 19, 1967
5. V. HOLZER, R.M. McLEOD, E. SMITH: J. Geophys. Rev., 71, 1481, 1966.
6. M.A. GINTSBURG: Astron. Zh., 43, 550, 1966; J. Geophys. Res. 72, 2749, 1967.
7. J. WILCOX, N. NESS: J. Geophys. Res., 70, 5793, 1966.

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