AN ASCENT EXISTENCE AND UNIQUENESS PROOF FOR LINEAR ORDINARY DIFFERENTIAL EQUATIONS[1]

Submitted

by

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1. Introduction. The typical existence and uniqueness theorem for initial value problems for linear ordinary differential equations is traditionally obtained as a corollary to an initial value problem for a linear or non-linear vector differential equation. This paper gives a method of ascent proof which, starting with the solutions of the first order linear differential equation, describes inductively the solutions of the n-th order equation in terms of those of the n-1st order. The proof provides a representation formula which can easily be verified by direct substitution to be a solution indeed. Particularly interesting is the manner in which the solutions of the n-1st order equation appear in the formula, especially in connection with their role in the generation of linearly independent solutions. A second form, a Volterra form, of the solution is included. Merit of the first is seen in its simplicity, in both derivation and form, and in its adaptability to verification. The method itself is a counterpart of the method of reduction of order.

2. <u>The Inductive Assumption</u>. Let $p_1, \ldots, p_n$ be continuous functions on the bounded closed interval I containing the point $x = 0$. Define the k-th order linear differential operator L_k for $k = 1, \ldots, n$ symbolically by

$$L_k y = y^{(k)} - p_1 y^{(k-1)} - \ldots - p_k y.$$

In statements of initial value problems $y_o, y_o', \ldots, y_o^{(n-1)}$ denote as is the custom arbitrary numbers.

In order to implement the inductive argument presented, the following assumption concerning the solutions of $L_{n-1} y = 0$ is made.

<u>Assumption</u>. The initial value problem $L_{n-1} y = 0$, $y(0) = y_o, \ldots, y^{(n-2)}(0) = y_o^{(n-2)}$ has exactly one solution.

The assumption is trivially valid for $n = 2$. In fact, a function ϕ is a solution of $L_1 y = 0$, $y(0) = y_o$ if and only if $\phi(x) = y_o \exp\int_o^x p_1(s)ds$ for $x \in I$.

To further implement the argument, the following theorem, an easy consequence of the assumption, is stated without proof. The theorem and its proof presumes only the most elementary knowledge of the Wronskian as required by the simple and effective method of variation of parameters. The somewhat more sophisticated Green's function method could be used if one prefers.

Theorem 1. Let $u_1, \ldots, u_{n-1}$ be solutions of $L_{n-1}y = 0$ satisfying $u_j^{(i-1)}(0) = \delta_{ij}$, $i = 1, \ldots, n-1$. Then, a function ϕ is a solution of the initial value problem

$$L_{n-1}y = p_n, \quad y(0) = y_o, \ldots, y_o^{(n-2)}(0) = y_o^{(n-2)}$$

if and only if

$$\phi(x) = y_o u_1(x) + \ldots + y_o^{(n-2)} u_{n-1}(x) + \int_o^x G(x,s) p_n(s) ds, \qquad x \in I,$$

where

$$G(x,s) = [u_1(x)W_1(s) + \ldots + u_{n-1}(x)W_{n-1}(s)]/W(s)$$

in which W is the Wronskian of $u_1, \ldots, u_{n-1}$ and W_j is the cofactor of $u_j^{(n-2)}$ in W, with the agreement that $W_1(s) \equiv 1$ when $n = 2$.

3. An Equivalent Integral Equation. Assume that ϕ is a solution of the initial value problem

(1) $$L_n y = 0, \quad y(0) = y_o, \ldots, y^{(n-1)}(0) = y_o^{(n-1)}.$$

Then, for $x \in I$,

(2) $$L_{n-1}\phi'(x) = p_n(x)\phi(x),$$

$$\phi'(0) = y_o', \ldots, \phi^{(n-1)}(0) = y_o^{(n-1)}$$

from which, in the presence of theorem 1, it follows that ϕ' is uniquely described by

(3) $$\phi'(x) = y_o'u_1(x) + \ldots + y_o^{(n-1)}u_{n-1}(x) + \int_o^x G(x,s)p_n(s)\phi(s)ds, \quad x \in I,$$

so that

(4) $$\phi(x) = y_o + \int_o^x [y_o'u_1(t) + \ldots + y_o^{(n-1)}u_{n-1}(t)]dt + \int_o^x\int_o^t G(t,s)p_n(s)\phi(s)dsdt, \quad x \in I.$$

Conversely now, if ϕ is a continuous function on I for which (4) holds with $u_1, \ldots, u_{n-1}$ and G as per theorem 1, then ϕ' exists on I and is in fact given by (3). From this then it is easily argued with the aid of knowledge of the method of variation parameters (or the properties of Green's function) that $\phi'', \ldots, \phi^{(n)}$ also exist on I and moreover that ϕ' satisfies (2) for $x \in I$. Thus, ϕ is a solution of (1) since, from (4), $\phi(0) = y_o$. This completes the proof of the following equivalence theorem.

Theorem 2. A continuous function ϕ on I is a solution of (1) if and only if ϕ satisfies (4).

4. The Solution. The problem of finding a solution of (1) is equivalently replaced by the problem of finding a continuous function ϕ on I for which (4) holds. The task is notationally simplified with the introduction of the linear double integral operator S defined for continuous functions f on I by

$$Sf(x) = \int_0^x \int_0^t G(t,s)p_n(s)f(s)\,ds\,dt, \quad x \in I.$$

Clearly, if f is continuous on I, so is Sf, hence so is SSf. For continuous functions f on I then, define $S^0 f = f$, $S^1 f = Sf$, and then $S^k f = SS^{k-1}f$ for $k = 1, 2, \ldots$.

Concerning S^k, the following two lemmas prove useful in the statement and proof of theorem 3 below. As their proofs are straightforward, they are omitted.

Lemma 1. Let f be a continuous function on I, hence bounded on I by say M. Then $S^k f$ is continuous on I; moreover, there exists H such that $|S^k f(x)| \leq MH^k/k!$ for $k = 0, 1, 2, \ldots$ and $x \in I$.

Lemma 2. If f is continuous on I, then $\sum_{k=0}^{\infty} S^k f$ converges uniformly to a continuous function on I.

Theorem 3. Let f be a continuous function on I. Then there exists one and only one continuous function $\emptyset$ on I for which $\emptyset = f + S\emptyset$; indeed, $\emptyset = \sum_{k=0}^{\infty} S^k f$.

Proof. That $\emptyset = f + S\emptyset$ for $\emptyset$ so given is immediate in the presence of lemma 2. The uniqueness of $\emptyset$ is a consequence of lemma 1. For if Ψ is continuous on I and $\Psi = f + S\Psi$, then $\emptyset - \Psi = S(\emptyset - \Psi)$ implies $\emptyset - \Psi = S^k(\emptyset - \Psi)$, $k = 1, 2, \ldots,$ and hence in turn that $\emptyset = \Psi$ via lemma 1.

Corollary. A function $\emptyset$ is a solution of (3) if and only if

(5) $$\emptyset(x) = \sum_{k=0}^{\infty} S^k\left(y_o + \int_o^x [y_o' u_1(t) + \ldots + y_o^{(n-1)} u_{n-1}(t)]dt\right), \qquad x \in I.$$

Agreeing that $S^k f = S^k$ when $f(x) \equiv 1$, a set of linearly independent solutions $y_1, \ldots, y_n$ such that $y_j^{(i-1)}(0) = \delta_{ij}$, $i = 1, \ldots, n$, generated by (5) is given by

$$y_1(x) = \sum_{k=0}^{\infty} S^k(x),$$

$$y_j(x) = \sum_{k=0}^{\infty} S^k \int_o^x u_{j-1}(s)ds, \qquad j = 2, \ldots, n.$$

5. The Case $n = 2$. The second order linear equation, interesting in itself, deserves individual attention. For this case the integral equation (4) becomes

$$\emptyset(x) = y_o + y_o'\int_o^x E(t)dt$$

$$+ \int_o^x\int_o^t E(t)E^{-1}(s)p_2(s)\emptyset(s)dsdt, \qquad x \in I,$$

where $E(x) = \exp\int_o^x p_1(s)ds$ and $E^{-1}(x) = 1/E(x)$. The function $\emptyset$ defined by

$$\emptyset(x) = \sum_{k=0}^{\infty} S^k(y_o + y_o'\int_o^x E(s)ds), \qquad x \in I.$$

is the unique solution of $L_2y = 0$, $y(0) = y_o$, $y'(0) = y_o'$. A pair of linearly independent solutions of $L_2y = 0$ is the pair

$$y_1(x) = 1 + \int_o^x \int_o^{t_1} E(t_1)E^{-1}(s_1)p_2(s_1)ds_1dt_1$$

$$+ \int_o^x \int_o^{t_2} E(t_2)E^{-1}(s_2)p_2(s_2) \int_o^{s_2} \int_o^{t_1} E(t_1)E^{-1}(s_1)p_2(s_1)ds_1dt_1ds_2dt_2$$

$$+ \dots,$$

$$y_2(x) = \int_0^x E(t)dt + \int_0^x \int_0^{t_1} E(t_1)E^{-1}(s_1)p_2(s_1) \int_0^{s_1} E(t)dtds_1dt_1$$

$$+ \int_0^x \int_0^{t_2} E(t_2)E^{-1}(s_2)p_2(s_2) \int_0^{s_2} \int_0^{t_1} E(t_1)E^{-1}(s_1)p_2(s_1) \int_0^{s_1} E(t)dtds_1dt_1ds_2dt_2$$

$$+ \dots \quad .$$

A comparison of the solution

$$y(x) = y_o E(x) + \int_0^x E(x)E^{-1}(s)p_2(s)ds$$

of $L_1 y = p_2$, $\dot{y}(0) = y_o$ with the integral equation and the pair y_1, y_2 is worthy of note.

6. <u>A Simple Reduction of the Integral Equation to a Volterra Equation</u>. With the restriction that f be continuous on I and $x \in I$, a change in the order of integration in Sf results in

$$Sf(x) = \int_0^x g(x,s)f(s)ds,$$

where

$$g(x,s) = \int_s^x G(t,s)p_n(s)dt, \qquad x,s \in I.$$

With this then, (4) assumes the form

(6) $$\emptyset(x) = f(x) + \int_0^x g(x,s)\emptyset(s)ds,$$

a Volterra integral equation. Borrowing from the theory of such equations, the unique solution of (6) for f continuous on I is

(7) $$\emptyset(x) = f(x) + \int_0^x K(x,s)f(s)ds,$$

where K is the resolvent of g defined for $x,s \in I$ by

$$K(x,s) = \sum_{k=0}^{\infty} K_k(x,s)$$

with $K_o = g$ and

$$K_k(x,s) = \int_s^x g(x,t)K_{k-1}(t,s)dt, \qquad k = 1, 2, \ldots .$$

That $\sum K_k$ converges uniformly in (x,s) to K, continuous in (x,s) for $x,s \in I$, is immediate because every K_k is continuous in (x,s) for $x,s \in I$ and because, moreover, there exists M and H such that $|K_k(x,s)| \leq MH^k/k!$, $k = 0, 1, 2, \ldots$ and $x,s \in I$.

The solution of (6), hence of (4), hence of (1), in the form of (7) can also be obtained by observing that

$$S^k f(x) = \int_0^x g(x,s) S^{k-1} f(s)\,ds, \qquad k = 1, 2, \ldots ,$$

so that, after appropriate changes in the order of integration,

$$S^k f(x) = \int_0^x K_{k-1}(x,s) f(s)\,ds, \qquad k = 1, 2, \ldots ,$$

and hence that

$$\sum_{k=0}^{\infty} S^k f(x) = f(x) + \int_0^x K(x,s) f(s)\,ds,$$

in view of uniform convergence.

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FOOTNOTE

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