On a Periodic Solution of the Central Differential Equation in the Relativity Theory of Gravitation

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Summary. Assuming the earth's gravitational field is spherically symmetric and the effect of the gravitational field of the sun on an artificial earth satellite is negligible, then the motion of an artificial earth satellite is governed by the ordinary geodesics of the static Schwarzschild metric, which in simplified form is

(1) $\left(\frac{du}{d\phi}\right)^2 = 2mu^3 - u^2 + \frac{2mc^2}{h^2}u - \gamma$

For a complete discussion of the classical equation (1) and its notations, refer to G. McVittie (General Relativity and Cosmology, The University of Illinois Press, Urbana, 1965). Differentiating equation (1) with respect to the true anomaly ϕ and setting

$\frac{mc^2}{h^2} = \frac{1}{p}$, we obtain

(2) $$\frac{d^2u}{d\phi^2} + u = \frac{1}{p} + 3mu^2,$$

where the semi-latus rectum p is related to the semi-major axis a and the eccentricity e of the Keplerian orbit by $p = a(1-e^2)$.
An approximate solution of (2) has been given by P. Bergmann (Introduction to the Theory Relativity, Prentice-Hall, Inc., 1960),

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and an approximate periodic solution of (2) starting from the perigee, that is to say, satisfying the initial conditions

$$u(0)=\frac{1}{p}\,,\quad \left(\frac{du}{d\phi}\right)_0=0 \tag{3}$$

is obtained by applying the Lindstedt perturbation method to equation (2). To apply Lindstedt method we change the independent variable ϕ to θ through the relation

$$\phi=\theta\left(1+c_1\varepsilon+c_2\varepsilon^2\cdots\cdots\right), \tag{4}$$

where c_1, c_2, c_3, . . . are the unknown coefficients and ε is an arbitrary small positive parameter. The smallness of the gravitational radius m (m = 0.443 cm.) enables us to suppose $\varepsilon = 3m$, and then equation (2) becomes

$$\frac{d^2u}{d\theta^2}+u-\frac{1}{p}+\varepsilon\left(-\frac{2c_1}{p}+2c_1u-u^2\right)+O(\varepsilon^2)=0. \tag{5}$$

Let us write the solution of (5) in a power series with respect to the small parameter ε, (6) $\quad u=\sum_{n=0}^{\infty}\varepsilon^n u_n(\theta)$

and limit ourselves to the first order approximation such that

$$u=u_0+\varepsilon u_1+O(\varepsilon^2) \tag{7}$$

Computing $d^2u/d\theta^2$ and u^2 from (6) and substituting them into equation (5) and comparing the terms which are constant, those which are multiplied by ε one finds that the leading term u_0 is a solution of the unperturbed equation

(8) $\quad \dfrac{d^2u_0}{d\theta^2}+u_0=\dfrac{1}{p}$ $\qquad$ and u_1 satisfies

$$\frac{d^2u_1}{d\theta^2}+u_1=\frac{2c_1}{p}-2c_1u_0+u_0^2=\frac{1+e^2}{p^2}+\frac{2e(1-c_1p)}{p^2}\cos\theta+\frac{e^2}{2p^2}\cos 2\theta \tag{9}$$

The unknown coefficient c_1 is to be determined such that no secular term appears in the solution. Hence we choose c_1 so that

(10) $c_1 = 1/p$, and therefore equation (9) assumes the form

(11) $$\frac{d^2 u_1}{d\theta^2} + u_1 = \frac{2+e^2}{2p^2} + \frac{e^2}{2p^2} \cos 2\theta .$$

The solution of (11) satisfying the initial conditions

(12) $$u_1(0) = 0 , \quad \left(\frac{du_1}{d\theta}\right)_0 = 0$$

is (13) $$u_1 = \frac{2+e^2}{2p^2}(1 - \cos 2\theta).$$

Therefore, the first order approximate solution of (3) is

(14) $$u = u_0 + \varepsilon u_1 + O(\varepsilon^2) = \frac{1+e\cos\theta}{p} + \frac{\varepsilon(2+e^2)}{2p^2}(1 - \cos 2\theta) + O(\varepsilon^2),$$

where (15) $$\theta = \frac{\phi}{1+c_1\varepsilon+c_2\varepsilon^2 \dots} \sim \frac{\phi}{1+c_1\varepsilon} = \frac{\phi}{1+\frac{\varepsilon}{p}} = \phi\left(1-\frac{\varepsilon}{p}\right).$$

Formulae (14) and (15) show that the change in the frequencies is dependent upon the amplitude a and the eccentricity e of the Keplerian orbit and also of the parameter ε , property which belongs to the periodic solutions of all nonlinear autonomous differential equations.

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Abstract. This investigation is directed toward the application of perturbation techniques to the central differential equation in the relativity theory of gravitation.

Assuming the earth's gravitational field is spherically symmetric and the effect of the gravitational field of the sun on an artificial earth satellite is negligible, then the motion of an artificial earth satellite is governed by the ordinary geodesics of the Schwarzschild metric.

An approximate periodic solution of the satellite orbit is obtained and it is shown that the frequency of the approximate solution is a function of the amplitude and eccentricity of the corresponding Keplerian orbit.

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