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ALMOST SURE STABILITY OF LONG CYLINDRICAL SHELLS WITH RANDOM IMPERFECTIONS

by Rena Scher Fersht

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CALIFORNIA INSTITUTE OF TECHNOLOGY
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TABLE OF CONTENTS

		Page
	Abstract	1
	Introduction	1
1.	Preliminaries	3
2.	Basic Equations	6
3.	Method of Solution	7
4.	Derivation of Stability Condition	12
5.	Numerical Example	15
6.	Concluding Remarks	16

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ABSTRACT

In this paper a Lyapunov method is used to obtain sufficient conditions for the buckling stability of cylindrical shells with axisymmetric random imperfections. A perturbed system of equations in the neighborhood of the prebuckling solution is investigated. By reducing the problem to a system of integral equations, it is observed that the stability boundary value problem of a long shell is similar to that of a dynamical system with random parametric excitations.

The present method for obtaining stability conditions has been tested numerically for the particular case of axisymmetric random imperfections of a cylindrical shell under axial compression. Initial imperfections were assumed to have Gaussian distribution and an exponential cosine correlation function. The critical load was obtained as a function of the root mean square of the imperfections. Results obtained are qualitatively similar to those of Koiter for a periodic imperfection (Ref. 1).

INTRODUCTION

In the last three decades it has been recognized that small geometrical imperfections are the major cause for the reduction in the buckling strength of cylindrical shells, subjected to axial loads. Particular analytical studies of the problem, using approximate techniques and considering simple periodic modes of imperfections, have been carried out by Koiter (Refs. 1,2), Donnell and Wan(Ref. 3), Hutchinson (Ref. 4), Budiansky and Hutchinson (Ref. 5), Babcock and Sechler (Ref. 6) and others. Few attempts have been made to study problems associated with local imperfections, almost periodic, and stationary random imperfections. In other words, the studies that have been carried out so far are related to ideal cases and give qualitative insight to the problem.

In the search for a more realistic description of the geometry of imperfections, it was suggested by Bolotin (Ref. 7) that the imperfection function should be considered as a random variable. By using statistical techniques based on probability distributions and their transformations one could evaluate the probabilities for buckling failure. This outlined procedure is perhaps too general and becomes impractical as the number of random variables increases.

Considering the problem of long cylindrical shells, a particular class of random imperfections, which is of practical significance, is the stationary state of imperfections with respect to the axial variable. By expanding the imperfection function in Fourier series in the circumferential direction, one can set up the problem considering the Fourier coefficients as the random variables. These coefficients are assumed to be stationary with respect to the axial independent variable and may be cross correlated. In addition it is assumed that the joint probability distribution for these coefficients is known. Further simplification is obtained by assuming that the random variables satisfy the ergodic property.

In the present paper the case of axisymmetric random imperfections is treated. The stability analysis is based upon Lyapunov's direct method, which has recently been used by Caughey and Gray (Ref. 8) in dynamical systems with stationary random parametric excitations. By considering the perturbation equations of the prebuckling solution it is possible to obtain a linear system of ordinary differential equations with constant and random parametric coefficients. By disregarding the terms with parametric coefficients the system is reduced to a stable one as long as the load is below the classical buckling load.

When the parametric coefficients are included by reducing the problem into a set of integral equations it was observed that, by proper modifications, the stability analysis is similar to that of a dynamic system where the axial variable replaces the time variable. As soon as this part of the analysis is established, the application of the Lyapunov technique becomes straightforward.

Lyapunov's method yields sufficient conditions for stability, but it often occurs that this technique leads to extremely conservative conditions. One of the major problems with Lyapunov's method is that of determining the proper matrix inequalities in order to derive sharper stability conditions. This part of the problem has been handled with particular care, yet it is felt that this part is still open, as in dynamical systems, to improvement.

The present method of stability has been tested numerically for the particular case of axisymmetric random imperfections. By considering a Gaussian distribution and an exponential cosine correlation function, the critical load was obtained as a function of the root mean square of the imperfections. The curves obtained are similar to those of Koiter for the cases where the peak of the power spectrum function coincides with the frequency of the critical linear buckling mode.

Finally one should point out that the present study is perhaps only the first step in this direction. By using the same technique, sufficient conditions for stability of cylindrical shells, subjected to other types of loads, as well as deterministic, almost periodic states of imperfections, can be obtained.

I. PRELIMINARIES

The present study treats the stability of a boundary value problem. In general Lyapunov's second method treats asymptotic stability of dynamic systems, in other words it is related to initial value problems. In order to relate the boundary value problem to an equivalent dynamic system in a steady state response or a stationary response in a statistical sense let us investigate the following system of equations.

$$\frac{d^2X}{d\xi^2} = BX + F(\xi)X \qquad -\infty < \xi < \infty \tag{1.1}$$

where X is an N-column vector with the components x_i, $i = 1, 2, \ldots, N$, B is a constant N x N matrix and $F(\xi)$ is an N x N matrix whose nonzero elements are stochastic processes:

$$F(\xi) = [f_{ij}(\xi)] \tag{1.2}$$

It is assumed that the matrix B has at least one square root A,

$$A \cdot A = A^2 = B$$

the eigenvalues of which are distinct and have negative real parts.

Now consider the system of equations

$$\frac{d^2X}{d\xi^2} = A^2X \qquad -\infty < \xi < \infty \tag{1.3}$$

with the conditions at infinity

$$X \to 0 \quad \text{as} \quad \xi \to \pm\infty \tag{1.4}$$

The solution of (1.3) as $\xi \to +\infty$ can be obtained from the equation

$$\frac{dX}{d\xi} = AX \tag{1.5}$$

Furthermore, as $\xi \to -\infty$ the solution can be obtained from

$$\frac{dX}{d\xi} = -AX \tag{1.6}$$

Equations (1.5) and (1.6) can be combined to one equation as follows

$$\frac{dX}{d|\xi|} = AX \qquad \text{for } |\xi| \to \infty \tag{1.7}$$

The stability is defined in the sense that as $\xi \to \pm\infty$ the lateral deflection of the shell tends to zero. This is known as asymptotic stability, and the term almost sure stability is associated with it. One can therefore state that conditions (1.4) can be met if and only if there is a matrix A, the eigenvalues of which have negative real parts. This last condition together with (1.4) assures stable solutions as $|\xi| \to \infty$, or by considering (1.7) and selecting the proper A it assures that (1.7) is asymptotically stable.

Turning now to (1.1) and assuming that for $F(\xi) = 0$ this system is stable, let it also be assumed that the elements of $F(\xi)$, $f_{ij}(\xi)$, satisfy the following properties,

a. The processes are continuous in $-\infty < \xi < \infty$

b. The processes are strictly stationary.

c. The processes satisfy an ergodic property, guaranteeing the equality of the averages with respect to ξ and the ensemble averages.

On the basis of the assumptions with respect to A and the boundary conditions at $\xi = \pm\infty$, one can construct a Green's function matrix associated with (1.3) or (1.7)

$$G(\xi,\eta) = G(|\xi - \eta|) \tag{1.8}$$

Equations (1.1) can therefore be converted into a system of integral equations of the form,

$$X(\xi) = \tfrac{1}{2}A^{-1} \int_{-\infty}^{\infty} G(|\xi-\eta|)\, F(\eta)\, X(\eta)\, d\eta \tag{1.9}$$

By observation one realizes that equations (1.9) can be obtained from the system of equations

$$\frac{dX}{d|\xi|} = AX \pm \tfrac{1}{2}A^{-1}\, F(\xi)\, X \tag{1.10}$$

where the positive sign in the second term, on the right hand side, is taken as ξ increases and the negative sign as it decreases. Due to symmetry it will be sufficient to investigate the asymptotic stability of (1.10) only for $\xi \to \infty$. Hence equations (1.10) can be reduced to the form

$$\frac{dX}{d\xi} = AX + \tfrac{1}{2}A^{-1}\, F(\xi)X \tag{1.11}$$

where a proper condition at $X(0) = X_o$ can be selected.

From this point, the analysis will follow Caughey and Gray (Ref. 8). If A is a stability matrix, there exists a Hermitian positive definite matrix V, such that (Ref. 9)

$$A^{*}V + VA = -I \tag{1.12}$$

where $A^{*} = \overline{A}^{T}$.

A Hermitian matrix $Q(\xi)$ can be formed as follows

$$Q(\xi) = \tfrac{1}{2}\{V^{-\frac{1}{2}}[A^{-1}F(\xi)]^{*}V^{\frac{1}{2}} + V^{\frac{1}{2}}[A^{-1}F(\xi)]\ V^{-\frac{1}{2}}\} \tag{1.13}$$

where $V^{\frac{1}{2}}$ and $V^{-\frac{1}{2}}$ are positive definite Hermitian matrices obtained as follows: Since V is positive definite Hermitian matrix there exists an orthogonal transformation Θ such that

$$\Theta^{*}\Theta = I, \quad \Theta^{*}V\Theta = \begin{bmatrix} \diagdown \mu \diagdown \end{bmatrix}$$

V possesses a unique square root $V^{\frac{1}{2}}$

$$V^{\frac{1}{2}} = \Theta \begin{bmatrix} \diagdown \mu^{\frac{1}{2}} \diagdown \end{bmatrix} \Theta^{*}$$

also

$$V^{-\frac{1}{2}} = \Theta \begin{bmatrix} \diagdown \mu^{-\frac{1}{2}} \diagdown \end{bmatrix} \Theta^{*}$$

Now, let $\|Q(\xi)\|$ be the norm of $Q(\xi)$, if $E\{\|Q(\xi)\|\}$ exists and is less than $1/\mu_{max}$, the system of equation (1.11) is almost surely stable in the large.

In the particular case that $F(\xi)$ may be written in the form

$$F(\xi) = \sum_{i=1}^{M} G_i f_i(\xi) \tag{1.14}$$

where G_i are constant matrices and $f_i(\xi)$ are scalar functions of ξ and $M < N^2$, it is possible to have a sharper condition of stability. If $\sum_{i=1}^{M} |\eta^{(i)}|_{max}\ E\{f_i(\xi)\}$ exists and is less than $1/\mu_{max}$ then equation (1.11) is almost surely stable in the large, where $|\eta^{(i)}|_{max}$ is the numerically largest eigenvalue of the matrix

$$B_i = \tfrac{1}{2}\,[V^{-\frac{1}{2}}(A^{-1}G_i)^{*}V^{\frac{1}{2}} + V^{\frac{1}{2}}(A^{-1}G_i)V^{-\frac{1}{2}}] \tag{1.15}$$

2. BASIC EQUATIONS

Let a point on the cylindrical surface of radius R be specified by its axial and circumferential coordinates x and y. Due to the presence of imperfections each point is radially displaced from the cylindrical surface by $\overline{w}(x)$. It is assumed that

$$|\overline{w}(x)| << R \qquad |\overline{w}_{,x}| << 1$$

In the absence of surface loads, the equations expressing equilibrium in the x and y direction for a shallow shell involve only the membrane stress resultants N_x, N_y and N_{xy}. These equations are satisfied by introducing the stress function $F(x,y)$,

$$N_x = F,_{yy} \qquad N_y = F,_{xx} \qquad N_{xy} = -F,_{xy}$$

Let $w(x,y)$ (positive inwards) be the radial displacement of the shell. In the case of axisymmetric imperfections the functions $F(x,y)$ and $w(x,y)$ satisfy the following two nonlinear equations,

$$\frac{1}{Eh}\nabla^4 F + \frac{1}{R} w,_{xx} = -w,_{xx} w,_{yy} - \overline{w},_{xx} w,_{yy} + (w,_{xy})^2 \qquad (2.1)$$

$$D\nabla^4 w - \frac{1}{R} F,_{xx} = \overline{w},_{xy} F,_{yy} + w,_{xx} F,_{yy} + w,_{yy} F,_{xx} - 2w,_{xy} F,_{xy} \qquad (2.2)$$

where E is Young's modulus, ν Poisson's ratio, h is the shell thickness, Eh = membrane rigidity,

$$D = \frac{Eh^3}{12(1-\nu^2)} = \text{bending rigidity}$$

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$$

Equation (2.1) is the compatibility equation in membrane strains and (2.2) is the radial equilibrium equation.

3. METHOD OF SOLUTION

Equations (2.1), (2.2) admit, for an axisymmetric imperfect cylindrical shell under axial compression, an axisymmetric prebuckling solution which may be written as

$$\begin{aligned} F(x,y) &= -\tfrac{1}{2}\sigma h y^2 + \phi^o(x) \\ w(x,y) &= \frac{\nu\sigma}{E} R + w^o(x) \end{aligned} \qquad (3.1)$$

where σ is the axial compressive stress. Substituting (3.1) into (2.1) and (2.2) yields

$$\frac{1}{Eh}\,\phi^{o},_{xxxx} + \frac{1}{R}\,w^{o},_{xx} = 0 \tag{3.2}$$

$$D\,w^{o},_{xxxx} + \sigma h\,(\overline{w},_{xx} + w^{o},_{xx}) - \frac{1}{R}\,\phi^{o},_{xx} = 0 \tag{3.3}$$

These equations can be simplified by reducing them to a nondimensional form. Let

$$x = \sqrt{\frac{Rh}{2\gamma}}\,\xi \qquad y = \sqrt{\frac{Rh}{2\gamma}}\,\eta$$

$$w^{o} = hW^{o} \qquad \overline{w} = h\overline{W} \qquad \phi^{o} = \frac{\sigma_{c\ell}h^{2}R}{2}\,\psi^{o} \tag{3.4}$$

$$\gamma = \sqrt{3(1-\nu^{2})}\,, \qquad \beta = \frac{\sigma}{\sigma_{c\ell}} \qquad \sigma_{c\ell} = \frac{Eh}{R\gamma}$$

Introducing these relations into (3.2) and (3.3) yields

$$\begin{aligned} &\psi^{o},_{\xi\xi\xi\xi} + W^{o},_{\xi\xi} = 0 \\ &W^{o},_{\xi\xi\xi\xi} - \psi^{o},_{\xi\xi} + 2\beta W^{o},_{\xi\xi} = -2\beta\,\overline{W},_{\xi\xi} \end{aligned} \tag{3.5}$$

The solution of (3.5) can be written in the form

$$\begin{aligned} W^{o}(\xi) &= \int_{-\infty}^{\infty} G^{o}(\xi-\eta)\,\overline{W}(\eta)\,d\eta \\ \psi^{o}(\xi) &= \int_{-\infty}^{\infty} H^{o}(\xi-\eta)\,\overline{W}(\eta)\,d\eta \end{aligned} \tag{3.6}$$

where $G^{o}(\xi-\eta)$ and $H^{o}(\xi-\eta)$ are the Green's functions associated with the homogeneous part of (3.5). These functions are

$$\begin{aligned} G^{o}(\xi) &= \frac{\beta}{2}\,e^{-a|\xi|}\left[\frac{1}{a}\cos b\xi - \frac{1}{b}\sin b\,|\xi|\right] \\ H^{o}(\xi) &= \frac{\beta}{\sqrt{1-\beta}}\,e^{-a|\xi|}\left[-b\cos b\xi + a\sin b\,|\xi|\right] \end{aligned} \tag{3.7}$$

where

$$a = \sqrt{(1-\beta)/2} \qquad b = \sqrt{(1+\beta)/2}$$

This solution remains finite as long as $\beta < 1$.

Considering the case of stationary and ergodic random imperfections with zero mean, the autocorrelation functions are defined as follows

$$R_{\overline{W}}(\xi) = E\{\overline{W}(\eta+\xi)\,\overline{W}(\eta)\} \tag{3.8}$$

$$R_{W^o}(\xi) = E\{W^o(\eta+\xi)\,W^o(\eta)\} \tag{3.9}$$

where $E\{f(\xi)\}$ is the expectation of the function f. Now

$$E\{W^o(\xi+\eta)\,W^o(\eta)\} = E\left\{\iint_{-\infty}^{\infty} G^o(\xi_1)G^o(\xi_2)\overline{W}(\eta+\xi-\xi_1)\overline{W}(\eta-\xi_2)d\xi_1 d\xi_2\right\}$$

Introducing the expectation operator into the double integral, yields

$$R_{W^o}(\xi) = \iint_{-\infty}^{\infty} G^o(\xi_1)G^o(\xi_2)\,R_{\overline{W}}(\xi-\xi_1+\xi_2)d\xi_1 d\xi_2 \tag{3.10}$$

This is the desired relation for the linear part of the solution.

Following Koiter (Ref. 1), the nonlinear equations (2.1), (2.2) may admit an asymmetric solution adjacent to the symmetric one which is specified by $w'(x,y)$ and $\phi'(x,y)$. Hence considering

$$F(x,y) = -\tfrac{1}{2}\sigma h y^2 + \phi^o(x) + \phi'(x,y) \tag{3.11}$$

$$w(x,y) = \nu\frac{\sigma}{E}R + w^o(x) + w'(x,y) \tag{3.12}$$

and taking into account that the deviation from the axisymmetric configuration is infinitesimal one may linearize the equations with respect to $w'(x,y)$ and $\phi'(x,y)$. The compatibility condition and the equilibrium equation therefore are

$$\frac{1}{Eh}\nabla^4\phi' + \frac{1}{R}w'_{,xx} = -(w^o_{,xx} + \overline{w}_{,xx})\,w'_{,yy} \tag{3.13}$$

$$D\nabla^4 w' + \sigma h w'_{,xx} - \frac{1}{R}\phi'_{,xx} = (\overline{w}_{,xx} + w^o_{,xx})\phi'_{,yy} + \phi^o_{,xx}\,w'_{,yy} \tag{3.14}$$

Now, let

$$\phi'(x,y) = \sum \phi_n'(x) \cos \frac{ny}{R}$$
$$w'(x,y) = \sum w_n'(x) \cos \frac{ny}{R} \quad . \tag{3.15}$$

Introducing the last expressions into (3.13) and (3.14) yields

$$\frac{1}{Eh} \left(\frac{d^2}{dx^2} - \frac{n^2}{R^2}\right)^2 \phi_n' + \frac{1}{R} w'_{n,xx} = \frac{n^2}{R^2} (w^o_{,xx} + \overline{w}_{,xx}) w_n' \tag{3.16}$$

$$D\left(\frac{d^2}{dx^2} - \frac{n^2}{R^2}\right)^2 w_n' + \sigma h w'_{n,xx} - \frac{1}{R} \phi'_{n,xx} =$$
$$= - \frac{n^2}{R^2} \left[(\overline{w}_{,xx} + w^o_{,xx}) \phi_n' + \phi^o_{,xx} w_n' \right] \tag{3.17}$$

As before, these equations can be reduced to a nondimensional form using (3.4) and the relations,

$$w_n' = hW \qquad \phi_n' = \tfrac{1}{2} \sigma_{c\ell} h^2 R \psi \qquad \alpha^2 = \frac{h}{R} n^2$$

which is

$$\left(\frac{d^2}{d\xi^2} - \alpha^2\right)^2 \psi + W_{,\xi\xi} = F_1(\xi) W \tag{3.18}$$

$$\left(\frac{d^2}{d\xi^2} - \alpha^2\right)^2 W - \psi_{,\xi\xi} + 2\beta W_{,\xi\xi} = -[F_1(\xi)\psi + F_2(\xi)W] \tag{3.19}$$

where

$$F_1(\xi) = \alpha^2 (W^o + \overline{W})_{,\xi\xi} \ ; \quad F_2(\xi) = \alpha^2 \psi^o_{,\xi\xi}$$

Now, let

$$\psi = x_1 \ ; \qquad W = x_2$$

$$\left(\frac{d^2}{d\xi^2} - \alpha^2\right) x_1 = x_3 \ ; \qquad \left(\frac{d^2}{d\xi^2} - \alpha^2\right) x_2 = x_4$$

equations (3.18) and (3.14) yield

$$x_1'' = \alpha^2 x_1 + x_3$$
$$x_2'' = \alpha^2 x_2 + x_4 \tag{3.20}$$
$$x_3'' = \alpha^2 x_3 - (\alpha^2 x_2 + x_4) + F_1(\xi) x_2$$

$$x_4'' = \alpha^2 x_4 - 2\beta(\alpha^2 x_2 + x_4) + \alpha^2 x_1 + x_3 - F_1(\xi)x_1 + F_2(\xi)x_2$$

where prime denotes differentiation with respect to ξ.
In matrix notation (3.20) may be written as

$$X'' = A^2 X + \overline{F}(\xi)X \tag{3.21}$$

where

$$A^2 = \begin{bmatrix} \alpha^2 & 0 & 1 & 0 \\ 0 & \alpha^2 & 0 & 1 \\ 0 & -\alpha^2 & \alpha^2 & -1 \\ \alpha^2 & -2\beta\alpha^2 & 1 & \alpha^2-2\beta \end{bmatrix} \tag{3.22}$$

$$\overline{F}(\xi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} F_1(\xi) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} F_2(\xi) \tag{3.23}$$

and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Constructing a transformation matrix P such that

$$P^{-1} A^2 P = \Lambda^2 = \begin{bmatrix} \ddots & & \\ & \lambda^2 & \\ & & \ddots \end{bmatrix}$$

where Λ^2 is a diagonal matrix containing the eigenvalues of A^2, one can find the matrix A

$$A = P \Lambda P^{-1} \tag{3.24}$$

where

$$\Lambda = \begin{bmatrix} \ddots & & \\ & \lambda & \\ & & \ddots \end{bmatrix} \quad \text{and} \quad \mathrm{Re}\{\lambda_i\} < 0$$

4. DERIVATION OF STABILITY CONDITION

For the following analysis, the autocorrelation function $R_{\overline{W}}(\xi)$ will be assumed as an exponential cosine function

$$R_{\overline{W}}(\xi) = \kappa^2 e^{-\varepsilon|\xi|} \cos\theta\xi \tag{4.1}$$

and $\overline{W}(\xi)$ will be assumed to have a Guassian distribution. Obviously κ represents the root mean square of the imperfections.

For a function f(x) with Gaussian distribution

$$E\{|f(x)|\} = \sqrt{\frac{2}{\pi}} \left[R_f(0)\right]^{\frac{1}{2}} \tag{4.2}$$

In the case of the cylindrical shell,

$$E\{|F_1(\xi)|\} = \sqrt{\frac{2}{\pi}}\ \alpha^2 \left[R_{(W^o+\overline{W}),_{\xi\xi}}(0)\right]^{\frac{1}{2}} \tag{4.3}$$

and

$$E\{|F_2(\xi)|\} = \sqrt{\frac{2}{\pi}}\ \alpha^2 \left[R_{W^o}(0)\right]^{\frac{1}{2}} \tag{4.4}$$

where

$$\begin{aligned} R_{(W^o+\overline{W}),_{\xi\xi}}(0) &= \iint_{-\infty}^{\infty} Q(\xi_1)Q(\xi_2)R_{\overline{W}}(\xi_2-\xi_1)d\xi_1 d\xi_2 \\ Q(\xi) &= \frac{e^{-a|\xi|}}{\sqrt{2}} \left[\frac{\beta(1-2\beta)}{\sqrt{1-\beta}}\cos b\xi + \frac{\beta(1+2\beta)}{\sqrt{1+\beta}}\sin b|\xi|\right] \end{aligned} \tag{4.5}$$

and

$$R_{W^o}(0) = \iint_{-\infty}^{\infty} G^o(\xi_1)G^o(\xi_2)R_{\overline{W}}(\xi_2-\xi_1)d\xi_1 d\xi_2$$

After rather cumbersome integrations

$$R_{W^o}(0) = \frac{1}{4}\beta^2\kappa^2 \Bigg\{ \Big\{ \frac{1}{\sqrt{1-\beta}} (a+\varepsilon)(K+N) - \frac{1}{\sqrt{1+\beta}} [(b+\theta)K+(b-\theta)N] \Big\} \cdot$$

$$\Big\{ \frac{1}{\sqrt{1-\beta}} [(a-\varepsilon)(M+L) + (a+\varepsilon)(K+N)] - \frac{1}{\sqrt{1+\beta}} [(b+\theta)(M+K)+(b-\theta)(L+N)] \Big\}$$

$$+ \Big\{ \frac{1}{\sqrt{1-\beta}} [(b+\theta)K-(b-\theta)N] - \frac{1}{\sqrt{1+\beta}} (a+\varepsilon)(N-K) \Big\} \cdot$$

$$\cdot \Big\{ \frac{1}{\sqrt{1-\beta}} [(b+\theta)(M-K)-(b-\theta)(L-N)] - \frac{1}{\sqrt{1+\beta}} [(a-\varepsilon)(L-M)+(a+\varepsilon)(K-N)] \Big\}$$

$$+ \tfrac{1}{2} \Big\{ \frac{1}{\sqrt{1-\beta}} \frac{2a^2+b^2}{a(a^2+b^2)} - \frac{1}{\sqrt{1+\beta}} \frac{b}{a^2+b^2} \Big\} \cdot$$

$$\cdot \Big\{ \frac{1}{\sqrt{1-\beta}} [-(a-\varepsilon)(M+L)+(a+\varepsilon)(K+N)] + \frac{1}{\sqrt{1+\beta}} [(b+\theta)(M-K)+(b-\theta)(L-N)] \Big\}$$

$$+ \tfrac{1}{2} \Big\{ \frac{1}{\sqrt{1-\beta}} \frac{b}{a^2+b^2} - \frac{1}{\sqrt{1+\beta}} \frac{b^2}{a(a^2+b^2)} \Big\} \cdot$$

$$\cdot \Big\{ \frac{1}{\sqrt{1-\beta}} [(b+\theta)(M-K)+(b-\theta)(L-N)] - \frac{1}{\sqrt{1+\beta}} [(\varepsilon-a)(M+L)+(a+\varepsilon)(K+N)] \Big\} \Bigg\} \tag{4.6}$$

and

$$R_{(W^o+\overline{W}),\xi\xi}(0) = \frac{1}{4}\beta^2\kappa^2 \Bigg\{ \Big\{ \frac{1-2\beta}{\sqrt{1-\beta}} (a+\varepsilon)(K+N) + \frac{1+2\beta}{\sqrt{1+\beta}} [(b+\theta)K+(b-\theta)N] \Big\} \cdot$$

$$\cdot \Big\{ \frac{1-2\beta}{\sqrt{1-\beta}} [(a-\varepsilon)(M+L)+(a+\varepsilon)(K+N)] + \frac{1+2\beta}{\sqrt{1+\beta}} [(b+\theta)(M+K)+(b-\theta)(L+N)] \Big\}$$

$$+ \Big\{ \frac{1-2\beta}{\sqrt{1-\beta}} [(b+\theta)K-(b-\theta)N] + \frac{1+2\beta}{\sqrt{1+\beta}} (a+\varepsilon)(N-K) \Big\} \cdot$$

$$\cdot\left\{\frac{1-2\beta}{\sqrt{1-\beta}}\left[(b+\theta)(M-K)-(b-\theta)(L-N)\right]+\frac{1+2\beta}{\sqrt{1+\beta}}\left[(a-\varepsilon)(L-M)+(a+\varepsilon)(K-N)\right]\right\}$$

$$+\tfrac{1}{2}\left\{\frac{1-2\beta}{\sqrt{1-\beta}}\ \frac{2a^2+b^2}{a(a^2+b^2)}+\frac{1+2\beta}{\sqrt{1+\beta}}\ \frac{b}{a^2+b^2}\right\}\cdot$$

$$\cdot\left\{\frac{1-2\beta}{\sqrt{1-\beta}}\left[-(a-\varepsilon)(M+L)+(a+\varepsilon)(K+N)\right]-\frac{1+2\beta}{\sqrt{1+\beta}}\left[(b+\theta)(M-K)+(b-\theta)(L-N)\right]\right\}$$

$$+\tfrac{1}{2}\left\{\frac{1-2\beta}{\sqrt{1-\beta}}\ \frac{b}{a^2+b^2}+\frac{1+2\beta}{\sqrt{1+\beta}}\ \frac{b^2}{a(a^2+b^2)}\right\}\cdot$$

$$\cdot\left\{\frac{1-2\beta}{\sqrt{1-\beta}}\left[(b+\theta)(M-K)+(b-\theta)(L-N)\right]+\frac{1+2\beta}{\sqrt{1+\beta}}\left[(\varepsilon-a)(M+L)+(a+\varepsilon)(K+N)\right]\right\}\Bigg\} \tag{4.7}$$

where

$$K=\frac{1}{(a+\varepsilon)^2+(b+\theta)^2} \qquad M=\frac{1}{(a-\varepsilon)^2+(b+\theta)^2}$$

$$N=\frac{1}{(a+\varepsilon)^2+(b-\theta)^2} \qquad L=\frac{1}{(a-\varepsilon)^2+(b-\theta)^2}$$

To complete the stability analysis established in part I, one has to find the Hermitian matrix V such that

$$A^*V+VA=-I$$

where A is formulated as shown in (3.24). Then one has to find the transformation matrix Θ such that

$$\Theta^*\Theta=I \qquad \Theta^*V\Theta=\begin{bmatrix}\diagdown & & \\ & \mu & \\ & & \diagdown\end{bmatrix}$$

where μ_i are the eigenvalues of V. Let G_1 and G_2 be the following matrices

$$G_1=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\-1&0&0&0\end{bmatrix} \qquad G_2=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&-1&0&0\end{bmatrix} \tag{4.8}$$

Then,

$$B_1 = \tfrac{1}{2}\,[\,V^{-\frac{1}{2}}(A^{-1}G_1)^*\,V^{\frac{1}{2}} + V^{\frac{1}{2}}(A^{-1}G_1)V^{-\frac{1}{2}}] \tag{4.9}$$

and

$$B_2 = \tfrac{1}{2}\,[\,V^{-\frac{1}{2}}\,(A^{-1}G_2)^*V^{\frac{1}{2}} + V^{\frac{1}{2}}\,(A^{-1}G_2)\,V^{\frac{1}{2}}] \tag{4.10}$$

Let $\eta_i^{(1)}$ be the eigenvalues of B_1 and $\eta_i^{(2)}$ the eigenvalues of B_2. The stability condition for the cylindrical shell will then be

$$|\eta^{(1)}|_{max}\,E\{\,|F_1(\xi)|\,\} + |\eta^{(2)}|_{max}E\{\,|F_2(\xi)|\} < \frac{1}{\mu_{max}} \tag{4.11}$$

Now from (4.3), (4.4), (4.6) and (4.7), it is easily seen that

$$E\{|F_1(\xi)|\} = C_1\kappa$$

and

$$E\{\,|F_2(\xi)|\} = C_2\kappa$$

where C_1 and C_2 are constants.
Introducing these relations into (4.11), yields

$$\kappa < \frac{1}{|\eta^{(1)}|_{max}C_1 + |\eta^{(2)}|_{max}\,C_2}\;\;\frac{1}{\mu_{max}} \tag{4.12}$$

This is the desired stability condition.

5. NUMERICAL EXAMPLE

In order to evaluate the stability boundary determined in equation (4.12) a specific numerical example has been carried out. The following parameters were used in the calculation.

$$R/h = 800$$
$$\nu = 0.3$$
$$\epsilon = 0.2$$
$$\theta = 1.0$$

The data (ϵ, θ) for the correlation function of the initial imperfections were selected so that the peak of the power spectrum would be in the neighborhood of the peak of the response kernel for $W^o(\xi)$. This will assure consideration of the most critical situation. The numerical evaluation determines the following relation.

$$\zeta(\beta) = \frac{1}{|\eta^{(1)}|_{max} C_1 + |\eta^{(2)}| C_2} \frac{1}{\mu_{max}}$$

The shell will remain stable as long as $\kappa(\beta) < \zeta(\beta)$.

The calculation was carried out varying the wave number α in the vicinity of $\alpha = \frac{1}{2}$. The stability boundary obtained is shown in Figure 1. This result is similar qualitatively to the deterministic cases associated with sinusoidal imperfections.

It should be pointed out that the imperfections at particular points might be higher than κ by a factor of 10 or more.

6. CONCLUDING REMARKS

The stability condition is only a sufficient criteria for the stability of the shell. The buckling problem is still open for sharper conditions, nevertheless the present condition does not require any further assumptions with respect to β or the power spectrum functions of the initial imperfections.

It should be pointed out that for certain particular cases a sharper stability condition can be obtained by means of other techniques. For example where the load is close to the linear critical one and the power spectrum function of the imperfections varies slowly in the vicinity of the axisymmetric response function, the case can be solved in a simplified manner, considering a narrow band filter technique. The stability condition obtained will be sharper than that obtained by Lyapunov's method.

After completion of the present report the author's attention was drawn to a study carried out at Harvard (15) on the same subject using different techniques. At the present time no comparison of results has been made.

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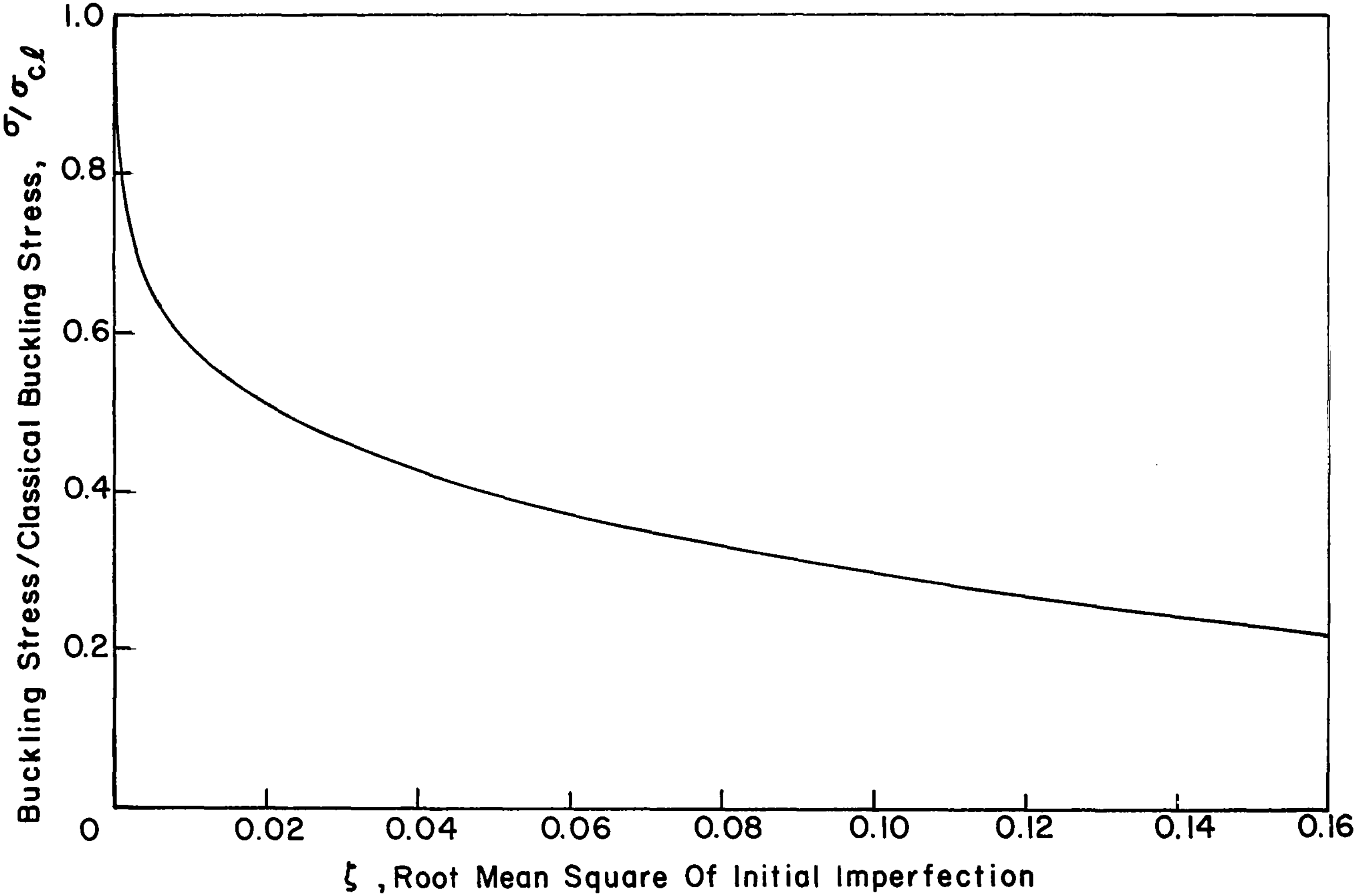

FIG. I STABILITY BOUNDARY FOR A CYLINDRICAL SHELL WITH AXISYMMETRIC IMPERFECTIONS

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