CONTROLABILITY OF HIGHER ORDER LINEAR SYSTEMS

by

H. O. FATTORINI*

Consejo Nacional de Investigaciones Cientificas y Tecnicas and University of Buenos Aires, Buenos Aires, Argentina

Now at Brown University, Division of Applied Mathematics, Providence, R. I.

Presented in the Conference on the Mathematical Theory of Control, January 30 to February 1, 1967, University of Southern California, Los Angeles, California

*This research was supported in part by the National Aeronautics and Space Administration under Grant No. NGR-40-002-015 and in part by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AF-AFOSR Grant No. AF-AFOSR-693-66.

[redacted]

Introduction We consider in this paper a dynamical system whose evolution in time is described by a second-order linear differential equation in a complex Banach space $E = \quad u , v, \ldots$

$$u''(t) = Au(t) + Bf(t) \tag{1}$$

Here A is a linear, possibly unbounded operator with domain $D(A) \subseteq E$ and range in E, B a bounded operator from another Banach space F to E. We shall assume $\rho(A)$, the resolvent set of A to be non-void, i.e. there exists a λ such that $R(\lambda ; A) = (\lambda I - A)^{-1}$ exists and is bounded. The state of the system at time t is given by the pair $(u(t), u'(t))$ of elements of E; the F-valued function $f(\cdot)$ is the <u>input</u> or <u>control</u> by means of which we govern the system.

The problem of <u>complete controllability</u> consists, roughly speaking, in selecting a control f in a given class $\mathcal{L}$ in such a way that the systems evolutions from a given initial state to the vicinity of a given final state. If the initial state is taken to be $(0, 0)$ then the problem is that of <u>null controllability</u>. We introduce in Section 1 some results on the theory of the equation (1) and apply them in Section 2 to show that the problem of complete controllability of the system (1) can be reduced to the corresponding one for the first-order system $u' = Au + Bf$ if A satisfies a certain condition (Condition (2.6). Finally, we examine in Section 3 the relation between null and complete controllability for first and second-order systems. An appendix refers a systems described by higher order equations $u^{(n)} = Au + Bf$, $n \geq 3$. We shall use without proofs some results on controllability of first order systems $u' = Au + Bf$; we refer to (6) for proofs and further details.

§1. Let $g(\cdot)$ be a E - valued, strongly continuous function defined for $t \geqslant 0$. We shall understand by a <u>solution</u> of

$$u''(t) = Au(t) + g(t) \tag{1.1}$$

an E - valued function $u(\cdot)$ defined and with two continuous derivatives in $t \geq 0$, such that $u(t) \in D(A)$ and Eq. (1.1) is satisfied for all $t \geqslant 0$. [redacted]

We shall assume that the Cauchy problem for the homogenous equation

$$u''(t) = Au(t) \tag{1.2}$$

is <u>uniformly well posed</u> in $t \geq 0$, i.e. we shall suppose that

(a) There exists a dense subspace D of E such that if $u_0, u_1 \in D$ there exists a solution $u(\cdot)$ of (1.2) with $u(0) = u_0$, $u'(0) = u_1$.

(b) For each $t > 0$ there exists a constant $K_t < \infty$ such that

$$|u(s)| \leq K_t (|u(0)| + |u'(0)|), \quad 0 \leq s \leq t$$

for any solution $u(\cdot)$ of (1.2)

Let $u \in E$, $u(\cdot)$ (resp. $v(\cdot)$) be a solution of Eq (1.2) with $u(0) = u$, $u'(0) = 0$ (resp. $v(0) = 0$, $v'(0) = u$). Define

$$S(t)u = u(t) \quad (\text{resp. } T(t)u = v(t))$$

By virtue of (a) and (b) $S(t)$, $T(t)$ are well defined and bounded for all $t \geq 0$, at least in D. Thus they can be extended to bounded operators in E that we shall denote with the same symbols. It follows from a simple approximation argument that $S(\cdot)$, $T(\cdot)$ are strongly continuous functions of t. We shall call S, T the <u>solution operators</u> of Eq. (1.2)

We take from (<u>4</u>), Section 4 the following properties of A, $R(\lambda; A)$, $S(\cdot)$, $T(\cdot)$.

(c) For each $u \in E$, $t \geq 0$

$$T(t)u = \int_0^t S(s)u \, ds \tag{1.3}$$

(d) There exist constants $w < \infty$, $K < \infty$ such that

$$|S(t)| \leq K e^{wt}, \quad |T(t)| \leq K e^{wt}, \quad t \geq 0$$

(e) $\sigma(A)$, the spectrum of A is contained in the region $\{\lambda; \ \operatorname{Re}\lambda \leq w^2 - (\operatorname{Im}\lambda)^2/4w^2\}$ and

$$R(\lambda^2; A)u = \lambda^{-1} \int_0^\infty e^{-\lambda t} S(t)u \, dt =$$

$$= \int_0^\infty e^{-\lambda t} T(t)u \, dt \tag{1.4}$$

for $\operatorname{Re}\lambda > w$.

With the help of $T(\cdot)$ we can construct solutions of the inhomogenous equation (1.1) . In fact, we have

<u>1.1 LEMMA</u> <u>Let</u> $g(\cdot)$ <u>be continuously differentiable . Then</u>

$$u(t) = \int_0^t T(t - s)g(s)ds \tag{1.5}$$

<u>is a solution of Eq.</u> (1.1) <u>with</u> $u(0) = u'(0) = 0$

Proof: Let $0 \leqslant t < t'$. We have

$$\frac{u(t') - u(t)}{t' - t} = \frac{1}{t' - t}\int_t^{t'} T(t' - s)g(s)ds +$$

$$+ \int_0^t \left[\frac{T(t' - s) - T(t - s)}{t - t'}\right] g(s)ds = I_1 + I_2$$

Using now Eq. (1.3) and the fact that $|S(\cdot)|$ is bounded on compacts of $[0, \infty)$ (a consequence of the principle of uniform boundedness, (1), Chapter I) we see that $|T(r)| = O(r)$ as $r \to 0$ and thus $I_1 \to 0$ as $t' - t \to 0$. Making use again of (1.3) the integrand in I_2 is seen to converge to $S(t - s)g(s)$ as $t' - t \to 0$; by the theorem of the mean of differential calculus is bounded in norm by

$$\sup_{0 \leqslant r \leqslant t'} |S(r)g(s)|$$

Thus u is continuously differentiable in $t \geqslant 0$ and

$$u'(t) = \int_0^t S(t - s)g(s)ds \tag{1.6}$$

Interchanging now s by $t - s$ in Eq. (1.6) and proceeding similarly as before we see that $u'(\cdot)$ is continuously differentiable in $t \geqslant 0$ and

$$u''(t) = S(t)g(0) + \int_0^t S(s)g'(t - s)ds \tag{1.7}$$

Let us now compute $Au(t)$. Integrating the expression (1.5) by parts we get

$$u(t) = \int_0^t T(s)g(0)ds + \int_0^t \left(\int_0^s T(r)dr\right) g'(t - s)ds$$

Using now the fact that

$$v_a = \int_0^a T(s)u\,ds$$

is an element of $D(A)$ for any $u \in E$ and $Av_a = S(a)u - u$ ((4) , Section 5) we see that $u(t) \in D(A)$ for all $t \geq 0$ and $Au(t) = u''(t) - g(t)$ as desired.

We close this section with another result on the equation (1.2) . Let $E^* = \{ u^* , v^* , \ldots \}$ be the dual space of E ; denote $\langle u^* , u \rangle$ or $\langle u , u^* \rangle$ the value of the functional u^* at the point $u \in E$.

1.2 LEMMA Let E be reflexive. Then the Cauchy problem for the equation

$$(u^*)''(t) = A^*u^*(t) \tag{1.8}$$

is uniformly well posed. If $S^*(\cdot)$, $T^*(\cdot)$ are the solution operators of Eq. (1.8) we have $S^*(t) = (S(t))^*$, $T^*(t) = (T(t))^*$, where S , T are the solution operators of (1.2)

The proof is a consequence of the characterization of operators A for which the Cauchy problem for Eq. (1.2) is well posed ((4), Theorem 5.9) . We shall assume throughout the rest of this paper that E is reflexive so that Lemma 1.2 applies.

§2. Let $E^2 = E \times E$ be the space of all pairs (u_0 , u_1) of elements of E endowed with pointwise operations and any of its natural norms, for instance $|(u_0 , u_1)| = |u_0| + |u_1|$. The dual space $(E^2)^*$ can be identified algebraically and topologically with the space $(E^*)^2$, application of the functional $u^* = (u_0^* , u_1^*)$ to the element $u = (u_0 , u_1)$ being given by

$$\langle u^* , u \rangle = \langle u_0^* , u_0 \rangle + \langle u_1^* , u_1 \rangle$$

Let the linear control system

$$u''(t) = Au(t) + Bf(t) \tag{2.1}$$

(we we shall denote by L) be given . We shall assume the class $\mathcal{L}$ of controls to consist of all F - valued infinitely differentiable functions defined in $[0, \infty)$. Call $K_t(L)$, $t \geq 0$ the subspace of E^2 consisting of all the pairs (u_0 , u_1) ,

$$u_0 = \int_0^t T(t - s)Bf(s)ds , \quad u_1 = \int_0^t S(t - s)Bf(s)ds \tag{2.2}$$

In wiew of Lemma 1, $K_t(L)$ can be described as the subspace of all pairs $(u(t), u'(t))$, $u(\cdot)$ a solution of Eq. (2.1) with $u(0) = u'(0) = 0$, $f \in \mathcal{L}$ or simply as the subspace of all possible states of the system at time t - the initial state being $(0,0)$ for $t = 0$. We also define $K(L) = \bigcup_{t>0} K_t(L)$. We shall say that the system L is <u>null controllable</u> if $\mathrm{Cl}\, K(L) = E^2$, <u>null controllable at time</u> t_o if $\mathrm{Cl}\, K_{t_o}(L) = E^2$. It is a consequence of the Hahn-Banach theorem that $\mathrm{Cl}\, K(L) = E^2$ if and only if $K(L)^{\perp} = \{(u_o^*, u_1^*) \in (E^*)^2 \mid \langle (u_o^*, u_1^*), (u_o, u_1)\rangle = 0$ for all $(u_o, u_1) \in K(L)\} = \{0\}$, $\mathrm{Cl}\, K_t(L) = E$ if $K_t(L)^{\perp} = 0$, $K_t(L)^{\perp}$ similarly defined.

Our first results are analogous to Proposition 2.1 and Corollary 2.2 of (<u>6</u>)

<u>2.1 LEMMA</u> $(u_o^*, u_1^*) \in K(L)^{\perp}$ $(K_t(L)^{\perp})$ <u>if and only if</u>

$$B^*(T^*(s)u_o^* + S^*(s)u_1^*) = 0, \quad 0 \leq s \quad (0 \leq s \leq t) \quad (2.3)$$

<u>Proof</u> Assume $(u_o^*, u_1^*) \in K(L)^{\perp}$. Then, for any $f \in \mathcal{L}$

$$0 = \left\langle u_o^*, \int_0^t T(t-s)Bf(s)ds \right\rangle + \left\langle u_1^*, \int_0^t S(t-s)Bf(s)ds \right\rangle =$$

$$\int_0^t \left\langle B^*(T^*(t-s)u_o^* + S^*(t-s)u_1^*), f(s) \right\rangle ds$$

Taking now $f(s) = y(s)u$, u any element of E, $y(\cdot)$ any scalar-valued function we easily see that (2.3) holds. The reverse implication is clear. The proof is similar for $K_t(L)$.

Let us denote $\rho_o(A)$ the connected component of $\rho(A)$ that contains the half-plane $\mathrm{Re}\,\lambda > w^2$ (w the constant in (d), Section 1)

<u>2.2 COROLLARY</u> $(u_o^*, u_1^*) \in K(L)$ <u>if and only if</u>

$$B^*R(\lambda; A^*)(u_o^* + \lambda^{\frac{1}{2}}u_1^*) = 0 \quad \text{for } \lambda \in \rho_o(A) \qquad (2.4)$$

($\arg \lambda^{\frac{1}{2}} = \frac{1}{2}\arg\lambda$, $-\pi < \arg\lambda < \pi$.

<u>Proof:</u> We obtain (2.4) for λ real, $\lambda^{\frac{1}{2}} > w$ integrating $\exp(-\lambda^{\frac{1}{2}}t)B^*(T^*(s)u_o^* + S^*(s)u_1^*)$ in $(0, \infty)$ and applying (2.3) and Lemma (2.1). For $\rho_o(A)$ the result follows from an analytic continuation argument. The reverse implication is, as in (<u>6</u>), Corollary 2.2 a consequence of uniqueness of Laplace transforms.

We shall also consider in what follows the first order system M,

$$u'(t) = Au(t) + Bf(t) \qquad (2.5)$$

Now $K_t(M)$ is defined as the subspace of E consisting of all values (at time t) of solutions of (2.5) such that $u(0) = 0$, $f \in \mathcal{L}$, $K(M)$, $K(M)^{\perp}$, $K_t(M)^{\perp}$ are defined in a way similar to that for second-order systems (see (6) for more details)

2.3 THEOREM Assume A satisfies the condition

there exists a simple closed curve C entirely contained in $\rho_0(A)$ and such that the origin is contained in the interior of C (2.6)

Then $K(L) = \{ (u_0, u_1) \; ; \; u_0, u_1 \in K(M) \}$

Proof Obviously we only have to prove $K(L)^{\perp} = \{ (u_0^*, u_1^*) \; ; \; u_0^*, u_1^* \in K(M)^{\perp} \}$. We shall use the following characterization of the elements of $K(M)^{\perp}$ (see (6), Corollary 2.2); $u^* \in K(M)^{\perp}$ if and only if $B^*R(\lambda; A^*)u^* = 0$, $\lambda \in \rho_0(A)$. This makes clear that if $u_0^*, u_1^* \in K(M)^{\perp}$ then $(u_0^*, u_1^*) \in K(L)^{\perp}$. Conversely, assume $(u_0^*, u_1^*) \in K(L)^{\perp}$. Consider (2.4) for a given $\lambda \in C$. As λ turns once around the origin and returns to its original value, $\lambda^{\frac{1}{2}}$ changes sign. Adding up the two versions of (2.4) so obtained we get $B^*R(\lambda; A^*)u_0^* = 0$, $B^*R(\lambda; A^*)u_1^* = 0$ for $\lambda \in C$; by analytic continuation this holds as well for all $\lambda \in \rho_0(A)$, which ends the proof.

2.4 COROLLARY Assume A satisfies Condition (2.6). Then the control system L is null controllable if and only if M is null controllable

2.5 REMARK If Condition (2.6) is not satisfied then Theorem 2.3 may fail to hold. We construct in what follows an example of this situation.

Let $E = L^2 = L^2_y(-\infty, \infty) = \{ u(y), v(y), \dots \}$ Recall that the space H^2 of the upper half-plane consists of all those functions in L^2 that are boundary values of functions $u(y + i\mathfrak{f})$, holomorphic in the upper half-plane and such that

$$\sup_{\mathfrak{f} > 0} \int |u(y + i\mathfrak{f})|^2 \, dy < \infty$$

(all integrals hereafter shall be taken on $(-\infty, \infty)$) By the Paley-Wiener theorem ((2), Chapter 8) H^2 consists of all those functions on L^2 whose Fourier-Plancherel transform vanishes for $t \geq 0$, i.e. of those $u(\cdot)$ in L^2 such that

$$\hat{u}(t) = (2\pi)^{-\frac{1}{2}} \int u(y)\, e^{iyt} dy = 0 \text{ for } t \geq 0$$

We shall make use of the following

<u>2.6 LEMMA</u> <u>Let</u> $m \in L^2$, $m \geq 0$, $m \neq 0$. <u>There exists</u> $u \in H^2$ <u>such that</u> $|u(y)| = m(y)$ <u>if and only if</u>

$$\int |\log m(y)|\, (1 + y^2)^{-1} dy < \infty$$

For a proof for H^2 of the unit circle see (<u>3</u>), Theorem 7.33; it can be adapted to the case of the half-plane by using the results in (<u>2</u>), Chapter 8 .

<u>2.7 COROLLARY</u> <u>Let</u> $\{a_{ij}(y)\}$, $i,j = 1,2$ <u>be</u> a 2 x 2 matrix of functions in L^2 . Assume

$$\int |\log |\det\{a_{ij}(y)\}||\, (1 + y^2) dy < \infty \qquad (2.8)$$

Then there exist v_1 , v_2 , both different from zero almost everywhere and such that

$$v_1 a_{11} + v_2 a_{12} = v_1 a_{21} + v_2 a_{22} \in H^2 \quad (2.9)$$

<u>Proof</u> Let $w_1(y) = b(y)(a_{22}(y) - a_{12}(y))$, $w_2(y) =$ $= b(y)(a_{11} - a_{21}(y))$ where $b(y)^{-1} = \operatorname{sgn} \det\{a_{ij}(y)\}$.
We have

$$w_1 a_{11} + w_2 a_{12} = w_1 a_{21} + w_2 a_{22} = \det\{a_{ij}\}$$

In view of Lemma 2.6 there exists $u \in H^2$ such that $|\det\{a_{ij}(y)\}| = |u(y)|$ Thus if we set $v_i = w_i \operatorname{sgn} u$, $i = 1.2$ v_1 , v_2 satisfy (2.9)

Let us now pass to the example proper. Let $E = L^2 =$ $= L_x^2(\infty, \infty) = \{u(x), v(x), \dots\}$, A_r the (self adjoint) operator defined by

$$(A_r u)(x) = u''(x) + ru(x) , \qquad (2.10)$$

$D(A_r) = \{u \in L^2 ; u'' \in L^2\}$ (u'' understood in the sense of distributions) , $F = C^2 = \{(y_1,y_2), \dots\}$ two dimensional unitary space, $f(t) = (f_1(t), f_2(t))$, $B(y_1,y_2)(x) =$ $y_1 g_1(x) + y_2 g_2(x)$, g_1 , g_2 elements of L^2 to be determined later.

The Fourier-Plancherel transform $u(x) \longleftrightarrow \hat{u}(s)$ defines an isometric isomorphism of L^2 onto itself under which the operator A_r transforms into the multiplication operator

$$(A_r u)(s) = (-s^2 + r)u(s) , \tag{2.11}$$

$D(A_r) = \{ u \in L^2 \mid s^2 u(s) \in L^2 \}$. Thus we may consider $E = L^2 = L^2_s(-\infty, \infty) = \{ u(s) , v(s) , \dots \}$ A_r defined by (2.11) , $B(y_1,y_2)(s) = y_1 h_1(s) + y_2 h_2(s)$, $h_1 = \hat{g}_1$, $h_2 = \hat{g}_2$. The adjoint of B is given by $B^*u = (y_1(u) , y_2(u))$,

$$y_i(s) = \int u(s) k_i(s) ds , \; i = 1,2$$

(where we have set $k_i(s) = \bar{h}_i(s)$) . It is not difficult to see that the Cauchy problem for $u'' = A_r u$ is uniformly well posed for any r , the propagators being given by

$$S_r(t)u(s) = a(r,s,t)u(s) , \; T_r(t)u(s) = b(r,s,t)u(s) ,$$

$$a(r,s,t) = \begin{cases} \cosh (r - s^2)^{\frac{1}{2}} t & \text{if } r \geq 0 \text{ and } |s| \leq r^{\frac{1}{2}} , \\ \cos (s^2 - r)^{\frac{1}{2}} t & \text{if } r < 0 \text{ or if } r \geq 0 \text{ and } |s| \geq r^{\frac{1}{2}} \end{cases}$$

$$b(r,s,t) = \begin{cases} (r - s^2)^{-\frac{1}{2}} \sinh (r - s^2)^{\frac{1}{2}} t & \text{if } r \geq 0 \text{ and } |s| \leq r^{\frac{1}{2}} , \\ (s^2 - r)^{-\frac{1}{2}} \sin (s^2 - r)^{\frac{1}{2}} t & \text{if } r < 0 \text{ or if } r \geq 0 \text{ and } |s| \geq r^{\frac{1}{2}} . \end{cases}$$

The spectrum of A_r consists of the half-line $(-\infty , r]$; thus if $r < 0$ condition (2.6) is satisfied. By Theorem 2.3 the system

$$u''(t) = A_r u(t) + Bf(t) \tag{2.12}$$

is null controllable if and only if

$$u'(t) = A_r u(t) + Bf(t) \tag{2.13}$$

is null controllable. For $r = 0$ the system (2.13) has been considered in (5) , Section 4; it is null controllable if and only if

$$h_1(s)h_2(-s) - h_1(-s)h_2(s) \neq 0 \;\; \text{a.e.} \tag{2.14}$$

It is easy to see that the same result holds for any r (null controllability is "translation-invariant" for first-order systems) . Condition (2.14) holds for instance when $g_1(x) =$

$= \exp(-|x|)$, $g_2 = \exp(-|x+1|)$; then $h_1(s) = 2(1+s^2)^{-1}$, $h_2(s) = 2\exp(-is)(1+s^2)^{-1}$ and the expression on the left-hand side of (2.14) reduces to $8i(1+s^2)^{-2}\sin s$.

Let us now examine (2.12) for $r \geqslant 0$, with the same choice of g_1 , g_2 . Let u_o , $u_1 \in L^2_s$. It is plain that Eq. (2.3) will hold for them if and only if

$$\int (a(r,s,t)u_1(s) + b(r,s,t)u_o(s))k_i(s)ds = 0 \ , \ t \geqslant 0 \qquad (2.15)$$

It is easy to see by means of simple changes of variable that the part of the integral in the left-hand side of (2.15) extending over $|s| \geqslant r^{\frac{1}{2}}$ can be written

$$\int (\cos yt \, (K_r u_1)(y) \ + \ y^{-1}\sin yt \, (K_r u_o)(y))(K_r k_i)(y)dy \qquad (2.16)$$

where K_r is the isometric isomorphism of $L^2(|s| \geqslant r^{\frac{1}{2}})$ onto L^2_y given by

$$(K_r u)(y) = |y|^{\frac{1}{2}}(y^2 + r)^{-\frac{1}{4}} u((y^2+r)^{\frac{1}{2}} \operatorname{sgn} y)$$

If $u_o(y)/y$ is summable at the origin we can write (2.16) as follows:

$$\tfrac{1}{2}\int e^{iyt}(v_1(y)\tilde{k}_i(y) + v_2(y)\tilde{k}_i(-y))dy \ , \qquad (2.17)$$

$v_1(y) = \tilde{u}_1(y) - i\tilde{u}_o(y)/y$, $v_2(y) = \tilde{u}_1(-y) - i\tilde{u}_o(-y)/y$ (here we have written $\tilde{u}_i = K_r u_i$, $\tilde{k}_j = K_r k_j$) . Call now $a_{ij}(y) = k_i((-1)^j y)$, $i.j = 1.2$. The matrix $\{a_{ij}\}$ so defined satisfies the assumptions in Corollary 2.7 and thus there exist v_1 , v_2 in L^2 , $v_i \neq 0$ such that

$$v_1 a_{21} + v_2 a_{22} \in H^2 \ , \ i = 1,2 \qquad (2.18)$$

It is plain that (2.18) holds as well for $w_i(y) = v_i(y)(i+y)^{-1}$, $i = 1,2$. If we now define $\tilde{u}_1(y) = \frac{1}{2}(w_1(y) + w_2(-y))$, $\tilde{u}_2(y) = \frac{1}{2}iy(w_1(y) - w_2(-y))$ then u_1 , $u_2 \in L^2$ and (2.17) - a fortiori (2.16) - vanishes for $t \geqslant 0$. Taking now $u_i = K_r^{-1}\tilde{u}_i$ which are defined for $|s| \geqslant r^{\frac{1}{2}}$ and extending them to the entire real line by setting $u_i = 0$ in $|s| < r^{\frac{1}{2}}$ we obtain two non-vanishing elements of L^2_s such that (2.15) holds, which shows that the system (2.12) is not completely controllable for $r \geqslant 0$.

2.6 REMARK Our results on density of $K(L)$ generalize to other topologies in $E \times E$. We show briefly in what follows how this can be done.

Let E_o, E_1 be Banach spaces with norms $|\cdot|_o$, $|\cdot|_1$ such that $E_o \subseteq E$, $E_1 \subseteq E$ (no relation between the topologies of E, E_1, E_2 is postulated) Let $m > 0$; introduce in $D(A^m)$, the domain of A^m the topology given by the norm $|u|_{D(A^m)} = |u|_E + |A^m u|_E$ or the equivalent one $|u|'_{D(A^m)} = |(\lambda - A)^m u|_E$, λ any element of $\rho(A)$. We shall assume that:

$$D(A^m) \subseteq E_o \quad , \; D(A^{m-1}) \subseteq E_1$$

both inclusions being continuous (we shall always consider $D(A^m)$ endowed with the topology given before, $D(A^{m-1})$ with the similar topology obtained replacing m by m-1); moreover we suppose

$$B(F) \subseteq D(A^{m-1}) \; , \; S(t)E_i \subseteq E_i \; , \; T(t)E_i \subseteq E_i \; , \; t \geq 0 \; , \; i = 0,1$$

$S(\cdot)$, $T(\cdot)$ are strongly continuous functions in the topologies of E_1, E_2. Under all this conditions it is easy to show that if $u(\cdot)$ is a solution of (2.1) with $u(0) = u'(0) = 0$, $u(t) \in D(A^m)$, $u'(t) \in D(A^{m-1})$ for all $t \geq 0$. It is then natural to ask when $K(L)$ will be dense in $E_o \times E_1$, i.e. when the system (2.1) will be null controllable <u>in the topology of</u> $E_o \times E_1$.

Let E_o^*, E_1^* be the dual spaces of E_o, E_1, application of a functional $u_i^* \in E_i^*$ to an element $u_i \in E_i$ being indicated

$$\langle u_i^* , u_i \rangle_i \quad , \; i = 0,1$$

Assume (2.1) is not null controllable in $E_o \times E_1$ and let $(u_o^*, u_1^*) \in K(L)^{\perp}$. Then we have, in view of (1.5)

$$\langle u_o^* , \int_0^t T(s)Bf(s)ds \rangle_o + \langle u_1^* , \int_0^t S(s)Bf(s)ds \rangle_1 = 0 \quad (2.19)$$

for all $t \geq 0$, $f \in \mathcal{L}$. Setting $f(s) = \exp(-\lambda^{\frac{1}{2}} s)u$, $u \in F$, $\lambda^{\frac{1}{2}} > w$ (w the constant in (d), Section 1) and letting $t \to \infty$ we get from (2.19) that

$$\langle u_o^* , R(\lambda ;A)Bu \rangle_o + \langle u_1^* , \lambda^{\frac{1}{2}} R(\lambda ;A)Bu \rangle_1 = 0$$

for $\lambda > w^2$ and a fortiori for all $\lambda \in \rho_o(A)$. Assume now A satisfies Condition (2.6). Then, by using the same trick in the proof of Theorem (2.3) we can show that

$$\langle u_o^*, R(\lambda;A)Bu\rangle_o = \langle u_1^*, R(\lambda;A)Bu\rangle_1 = 0 \qquad (2.19')$$

for all $\lambda \in \rho_o(A)$. Assume now the first-order system M is null controllable. Then, if $B^*R(\lambda;A^*)u^* = 0$ for some $u^* \in E^*$ and all $\lambda \in \rho_o(A)$, $u^* = 0$ or, what amounts to the same thing, the subspace of E generated by all elements of the form

$$R(\lambda;A)Bu \qquad (2.20)$$

$u \in F$, $\lambda \in \rho_o(A)$ is dense in E. Let us see that the same thing happens with the subspace of E generated by the elements

$$(\mu - A)^m R(\lambda;A)Bu \qquad (2.21)$$

μ a fixed element of $\rho(A)$, $u \in F$, $\lambda \in \rho_o(A)$. In fact, assume this is not true. Then there exists $u^* \in E^*$ such that

$$\langle u^*, (\mu - A)^m R(\lambda;A)Bu\rangle = 0 \qquad (2.22)$$

for all $u \in F$, $\lambda \in \rho_o(A)$. Adding up (2.22) for two different elements λ_o, λ_1 of $\rho_o(A)$ and using the first resolvent equation we get

$$\langle u^*, (\mu - A)^m R(\lambda_o;A)R(\lambda_1;A)Bu\rangle = 0 \qquad (2.23)$$

Differentiating (2.23) with respect to λ_1 m-1 times we get

$$\langle u^*, (\mu - A)^m R(\lambda_o;A)R(\lambda_1;A)^m Bu\rangle = 0$$

for all $u \in F$, $\lambda_o \in \rho_o(A)$. Then

$$B^*R(\lambda_o;A^*)(\mu - A^*)^m R(\lambda_1;A^*)^m u^* = 0$$

which, in view that M is completely controllable, implies

$$(\lambda - A^*)^m R(\lambda_1;A^*) = 0 ,$$

a fortiori, $u^* = 0$.

Let us observe next that to assert that the subspace generated by all elements of the form (2.21) is dense in E is equivalent to assert that the subspace generated by all elements of the form

(2.20) is dense in $D(A^m)$. But then it will also be dense in E_o ; thus, in wiew of (2.19') , $u_o^* = 0$. The second term in (2.19') can be treated in the same way than the first. Collecting all our observations we have

2.7 THEOREM Let E_o , E_1 be Banach spaces satisfying all the conditions in Remark 2.6. Assume the first-order control system (2.5) is null controllable, and assume A satisfies Condition (2.6) . Then the system (2.1) is null controllable in the topology of $E_o \times E_1$

§3. Let us call the system (2.1) completely controllable if, given u_o , $u_1 \in D$, v_o , $v_1 \in E$, $\epsilon > 0$ there exists $f \in \mathcal{L}$ such that the solution of Eq. (2.1) with $u(0) = u_o$, $u'(0) = u_1$ satisfies

$$|u(t) - v_o| \leq \epsilon \ , \ |u'(t) - v_1| \leq \epsilon$$

for some $t > 0$. It is plain that complete controllability of L implies null controllability. The reverse implication is also true; this follows from the fact that the solutions of Eq. (2.1) can be translated and inverted in time, i.e. if $u(\cdot)$ is a solution of (2.1) for some $f(\cdot) \in \mathcal{L}$ then $v(t) = u(a - t)$ is also a solution of Eq. (2.1) for $g(t) = f(a - t)$. Thus to steer the system from (u_o , u_1) to the vicinity of (v_o , v_1) we only have to steer first to the vicinity of the origin (using null controllability and the inversion property just mentioned) and then from the origin to the vicinity of (v_o , v_1) .

The situation is diferent for first-order systems; in fact a first-order system may be null controllable without being completely controllable. There are, however, two important particular cases where the equivalence holds; these are (a) the case where A generates an analytic semigroup and (b) the case where A generates a group and $\rho_o(A) = \rho_o(-A)$, this last condition meaning that we can unite the points $+\infty$ and $-\infty$ of the real axis by means of a curve that does not meet the spectrum of A.

§4. Problems similar to the ones we have considered for the systems $u^{(n)} = Au + Bf$, $n = 1 , 2$ may also be considered for $n \geq 3$. However, the interest of these generalizations is limited by the fact that the assumption of well posedness of the homogenous problem $u^{(n)} = Au$ implies the boundedness of A ((4), Section 3), thus precluding applications to partial differential equations. The results are as follows: if L is the n-th order system

$$u^{(n)}(t) = Au(t) + Bf(t) \tag{2.24}$$

and M, as usual, is the first-order system

$$u'(t) = Au(t) + Bf(t) \tag{2.25}$$

then the four notions, null controllability, null controllability at time t_o, complete controllability, complete controllability at time t_o are equivalent for the system L and equivalent to the corresponding notions for the system M. The proof is a consequence of the fact that the solution operators of Eq. (2.24) - and also of equation (2.25) - are analytic when A is bounded.

References

1. E. HILLE - R. S. PHILLIPS, Functional Analysis and Semi-groups, American Mathematical Society Colloquium Publications, vol. XXXI , Providence, Rhode Island, 1957

2. K. HOFFMAN, Banach space of analytic functions, Prentice-Hall, Englewood Cliffs, New Jersey, 1962

3. A. ZYGMUND, Trigonometric series, second edition, vol. I, Cambridge University Press, Cambridge, 1959.

4. H. O. FATTORINI, Ordinary differential equations in linear topological spaces, submitted for publication to Pacific J. Math. Announcement of results in: Notices of the Amer. Math. Soc. Abstract 66T-474 (vol. 13) 734, Abstract 67T-41 (vol. 14), 140.

5. ________. On complete controllability of linear systems, to appear in Journal of Differential Equations 3 (1967)

6. ________. Some remarks on complete controllability, J. Soc. Ind. Appl. Math., ser. A: On Control, 4 (1966) 686-694