MARTINGALE CONVERGENCE AND THE RADON-NIKODYM THEOREM IN BANACH SPACES

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Martingale Convergence and the Radon-Nikodym Theorem in Banach Spaces

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§ 1 . Introduction:

In recent years, several authors have considered various extensions of the martingale convergence theorems of Doob [8] to the case where the random variables take values in a Banach space (B-space) e.g. Chatterji [4 (a), (b)] Scalora [17], A.I. and C.I. Tulcea [18 (a)] and Metivier [12]; the last named author has even considered the general case of locally convex topological vector spaces. Whereas certain types of convergence theorems were shown to be valid [4 (a),(b)] for arbitrary B-spaces, a counter-example in Chatterji [4 (a)] shows that without some condition on the B-space concerned, some of the most important convergence theorems of the scalar-valued case are invalid. The main purpose of this paper is to elucidate this latter situation, by demonstrating that the validity of almost any general theorem for martingales taking values in a B-space is equivalent to the fact that the Radon-Nikodym theorem is valid for set-functions taking values in such spaces. At the same time, this paper offers self-contained proofs of almost everywhere (a.e.) convergence theorems for B-space-valued martingales, theorems which are more general than those to be found in [17,18 (a)]. The method of proof yields, as a by-product, several known Radon-Nikodym theorems for B-spaces, including one due to Phillips [13].

§ 2 . Notation and preliminary remarks:

For the sake of clarity of exposition, I shall consider only the case where the underlying measure space is a probability space S, with σ-algebra Σ of measurable subsets and P a σ-additive positive measure on Σ with $P(S) = 1$. Suitable generali-

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[redacted]

zations to the case of an arbitrary measure space will be obvious to the interested reader. X will be used to denote a B-space with norm $|\cdot|$ and all random variables f with values in X will be assumed to be strongly (or Bochner) measurable functions on S with values in X. The integral of such a function, denoted by E(f) or $\int f(s)P(ds)$ or simply $\int f$ will always be considered in the Bochner-sense. These and other measure-theoretic concepts and notations are to be found in Dunford and Schwartz [9] Hille and Phillips [11].

Given a sub-σ- algebra Σ_i of Σ, there exists a well-defined linear operator of norm one, the conditional expectation operator E_i mapping $L^1(\Sigma,X) \longrightarrow L^1(\Sigma_i,X)$ and satisfying

$$\int_A f = \int_A E_i f \qquad A \in \Sigma_i \quad .$$

Here $L^1(\Sigma,X) = \{f \mid f \text{ is } \Sigma\text{-measurable}, \; \|f\|_1 = \int |f| < \infty\}$. If $f = \sum_{k=1}^{n} a_k C_{A_k}(s)$, $a_k \in X$, $A_k \in \Sigma$ ($C_A(s) = 1$ if $s \in A$ and 0 if $s \notin A$) then $E_i f = \sum_{k=1}^{n} P_i(C_{A_k}) a_k$ where P_i stands for conditional probability given Σ_i as in Doob [8]. For a general f, $E_i f$ can be easily shown to exist by a standard approximation argument. This procedure is necessary since given a X-valued σ-additive set-funktion μ on Σ such that $\mu(A) = 0$ wheneven $P(A) = 0$, μ is not necessarily an indefinite integral of a function with respect to P, even though the total variation $V_\mu(A) = \sup\{\sum_{k=1}^{n} |\mu(A_k)| \mid A_k \in \Sigma, A_k \subset A, A_k \text{ disjoint}\}$ which is always a non-negative measure on Σ is totally-finite. Thus the standard argument for the existence of the conditional expectation operator E_i is not applicable. It is convenient to introduce at this point the following definition.

Definition 1: The B-space X has the RN-property with respect to (S,Σ,P) if every X-valued σ-additive set-funktion μ of bounded variation (i.e. $V_\mu(S) < \infty$) which is absolutely continuous with respect to P (i.e. $P(A) = 0 \Rightarrow \mu(A) = 0$ or equivalently $V_\mu \ll P$) has an integral representation i.e. $\exists f \in L^1(\Sigma,X)$ such that $\mu(A) = \int_A f(s)P(ds)$, $\forall A \in \Sigma$. X will be said to have property (D) if it has the RN-property with respect to Lebesgue measure on the Borel sets of the unit interval.

Bochner and Taylor [2] had defined property (D) for a B-space X as being the property that a function of strong bounded variation on the unit interval is differentiable (strongly) almost everywhere. It can be easily seen from the methods of the present paper that their definition of property (D) is equivalent to mine.

It will follow from the work in the next section that if P is not purely atomic, X has the RN-property with respect to (S,Σ,P) if and only if X has property (D). So for all practical purposes in this connection property (D) is what really matters. If P is purely atomic, then any B-space X has the RN-property with respect to (S,Σ,P), as can be immediately verified.

Definition 2: Given a directed set $(N, \leq)$ and a family of σ-algebras $\Sigma_i \subset \Sigma$, $i \in N$, the system $\{f_i,\Sigma_i,\ i \in N\}$ forms a X-valued martingale if $f_i \in L^1(\Sigma_i,X)$, $i \leq j \Rightarrow \Sigma_i \subset \Sigma_j$ and $E_i f_j = f_i$. The following two special examples of X-valued martingales will play special roles:

Example (i). Let Σ_i, N be as above and let $f \in L^1(\Sigma,X)$. If $f_i = E_i f$ then $\{f_i,\Sigma_i,\ i \in N\}$ is a X-valued martingale.

Example (ii). Let μ be a X-valued σ-additive set-function and let I be the directed set of all partitions $\pi = \{A_1,A_2,\dots,A_n\}$ of S where $n \geq 1$, $A_i \in \Sigma$, $P(A_i) > 0$, $\bigcup_{i=1}^{n} A_i = S$, A_i's disjoint. $\pi_1 \leq \pi_2$ if every set in the partition π_2 is contained (P almost surely) in a set of the partition π_1. Define

$$f_\pi(s) = \frac{\mu(A_i)}{P(A_i)} \quad \text{if } s \in A_i .$$

Then $\{f_\pi,\Sigma_\pi,\ \pi \in I\}$ is a X-valued martingale where Σ_π = σ-algebra generated by sets in the partition π. For this latter fact actually the additivity of μ is all that is necessary. These f_π martingales have been used often in measure theory. See e.g. Dunford and Schwarz [9] pp. 297.

As an illustration of the connection between the convergence of martingales and RN-property, I shall state the following result which is of an elementary nature.

Theorem 1:

(a) Let $f \in L^p(\Sigma,X)$ i.e. f is Σ-measurable and $\|f\|_p^p = \int |f|^p < \infty$, $1 \leq p < \infty$. Then for any directed set N and σ-algebras Σ_i the martingale $\{f_i, \Sigma_i, i \in N\}$ of example (i) has the property that

$$\lim_i \| f_i - f_\infty \|_p = 0$$

where

$$f_\infty = E_\infty f = \text{conditional expectation of f given the } \sigma\text{-algebra } \Sigma_\infty \text{ generated by } \bigcup_{i \in N} \Sigma_i$$

(b) In example (ii) if $\mu(A) = \int_A f(s)P(ds)$, $f \in L^p(\Sigma,X)$

then

$$\lim_\pi \| f_\pi - f \|_p = 0 \text{ .}$$

(c) If in exaple (ii) $\lim_{\pi_1, \pi_2} \| f_{\pi_1} - f_{\pi_2} \|_p = 0$ i.e. f_π is a L^p-Cauchy sequence then $\mu(A) = \int_A f(s)P(ds)$ for some $f \in L^p(\Sigma,X)$.

Remarks: Theorem 1(a) is a generalization to directed sets of a corresponding theorem in [4(b)] where N = positive integers. Since the method of proof is exactly the same and in any case of utter simplicity, only a bare sketch will be provided. Parts (b) and (c) were proved by Rønnow [16] for the case p = 1 slightly differently. Here (b) is an immediate corollary of (a) since $f_\pi = E_\pi f$ and clearly $\Sigma_\infty = \Sigma$ in this case. As regards (c), it will be noticed that when $1 < p < \infty$ and X = complex numbers, the much weaker condition that $\sup_\pi \| f_\pi \|_p < \infty$ is sufficient (and clearly always necessary) for the conclusion. This is indeed a classical theorem of F. Riesz where the condition is expressed as

$$\sup_\pi \sum_{i=1}^{n} \frac{|\mu(A_i)|^p}{[P(A_i)]^{p-1}} < \infty \text{ .}$$

This latter assertion (not valid even in the classical case for p = 1) will follow from the main theorem of this paper for a wide class of spaces X; in fact, it would show, in some sense, exactly which class of spaces X allow such a theorem.

Proof: (a) Assume first that f is Σ_∞-measurable. If f is measurable with respect to the algebra $\bigcup_{i\in N}\Sigma_i$ then $E_i f = f$ for $i \geq i_0$. Hence for this case the conclusion follows. A general f which is Σ_∞-measurable can be approximated arbitrarily closely in L^p-norm by functions measurable $\bigcup_{i\in N}\Sigma_i$. So the conclusion holds for such f. Finally for any $f \in L^p(\Sigma,X)$ $f_i = E_i f = E_i E_\infty f = E_i f_\infty$. As pointed out above (b) follows immediately.

(c) From the completeness of $L^p(\Sigma,X)$ it follows that $\exists f \in L^p(\Sigma,X)$ such that $\lim_\pi \| f_\pi - f\|_p = 0$.

I shall now show that $f_\pi = E_\pi f$. Assertion (a) then will justify the conslusion of (c). Now given $\varepsilon > 0$ $\exists$ π_ε such that $\| f_{\pi'} - f\|_p < \varepsilon$ if $\pi' \geq \pi_\varepsilon$. To any π, since the set I of partitions is directed, there is a partition π_1 which is finer than both π and π_ε i.e. $\pi_1 \geq \pi$, $\pi_1 \geq \pi_\varepsilon$. It has already been remarked that $\{f_{\pi'}, \Sigma_{\pi'}, \pi' \in I\}$ is a martingale and hence for any set $A \in \pi$

$$\int_A f_\pi = \int_A f_{\pi_1} .$$

Now

$$\left|\int_A f_\pi - \int_A f\right| = \left|\int_A f_{\pi_1} - \int_A f\right| \leq \| f_{\pi_1} - f\|_1 \leq \|f_{\pi_1} - f_\pi\|_p < \varepsilon .$$

Since ε is arbitrary and f_π is Σ_π-measurable, $E_\pi f = f_\pi$. This concludes the proof.

An interesting corollary, noted by Rønnow [16] in the case p = 1, will be stated here for later application.

Corollary: In order that an additive X-valued set-function μ is the integral of a function $f \in L^p(\Sigma,X)$ either of the following two conditions is necessary and sufficient:

(1) For every monotone sequence π_n of partitions (i.e. $\pi_n \leq \pi_{n+1}$) the functions f_{π_n} $n \geq 1$ as defined in example (ii) above should be Cauchy convergent in L^p.

(2) The restriction of μ to every separable σ-subalgebra of Σ (i.e. one generated by a denumerable number of sets) has an integral representation

by means of a function from $L^P(\Sigma,X)$.

§.3. Discussion of the RN-property:

If P is purely atomic i.e. there exists a sequence of disjoint sets $E_n \in \Sigma$, $P(E_n) > 0$, $P(\bigcup_{n=1}^{\infty} E_n) = 1$ such that E_n's are P-atoms in Σ i.e. $F \in \Sigma$, $F \subset E_n$ implies $P(F) = 0$ or $P(E_n)$, then every B-space X has the RN-property with respect to (S,Σ,P). Indeed given any σ-additive, P-absolutely continuous, X-valued set function μ of bounded variation, the function $f(s) = \sum_{n=1}^{\infty} a_n C_{E_n}(s)$ with $a_n = \mu(E_n)/P(E_n)$ is easily seen to be an integrable function such that $\mu(E) = \int_E f$ for all $E \in \Sigma$. Now an arbitrary probability measure can be written down, essentially uniquely, as a convex combination $dP_1 + (1-d)P_2$, $0 \leq d \leq 1$, of two probability measures P_1, P_2 where P_1 is purely atomic and P_2 is purely nonatomic. It follows, therefore, that X will have the RN-property with respect to (S,Σ,P) if and only if it possesses RN-property with respect to (S,Σ,P_2). I shall assume now that P is purely nonatomic on Σ. By virtue of the corollary of the last section, X will possess the RN-property with respect to (S,Σ,P) if and only if this happens with respect to (S,Σ_0,P) for every separable σ-subalgebra Σ_0. Clearly, Σ_0 can be so chosen that P restricted to Σ_0 is also purely non-atomic. For instance, Σ_0 can be defined to be the σ-algebra generated by a sequence π_n of successively finer partitions such that $\pi_n = \{A_{n1}, A_{n2}, \ldots, A_{n2^n}\}$. and $P(A_{nk}) = 2^{-n}$ for $n \geq 1$. This is possible since P is nonatomic. Now if A is any set belonging to Σ_0 with $P(A) > 0$ then there exist indices n and k such that $0 < P(AA_{nk}) < P(A)$ thus proving the non-existence of atoms in Σ_0. By a theorem of Halmos and Von Neumann [10, pp.173] the measure algebra $(\tilde{\Sigma}_0, P)$ is isomorphic to the measure algebra $(\tilde{\mathcal{B}}, m)$ of the unit interval with Lebesgue measure "m" on the Borel sets. It is easy to see that the measure algebra isomorphism T between $\tilde{\Sigma}_0$ and $\tilde{\mathcal{B}}$ can be extended to an isometry between the whole of $L^1(\Sigma_0,X)$ and $L^1(\mathcal{B},X)$ (considered as equivalence classes of functions) in such a way that $\int_A f dP = \int_{TA} Tf \, dm$ holds. It is to be noted that T is to be thought of as working on equivalence classes of X-valued functions and that no assumption is made concerning the possibility of inducing the measure-algebra isomorphism T through a 1-1 point-transformation between S and the unit interval. This latter which may be impossible if S is "pathological" is not necessary in the present discussion. Since any X-valued σ-additive P-absolutely continuous (m-absolutely continuous) set function μ can be lifted to the respective measure algebras $\tilde{\Sigma}_0(\tilde{\mathfrak{B}})$, it is clear from the above that X has property (D) if and only if X has the RN property with respect to (S, Σ_0, P).

I shall now summarize the conclusions of the above discussion in the form of a theorem:

Theorem 2:

(a) If $(S, \Sigma . P)$ is purely atomic then every B-space has the RN-property with respect to it.

(b) If P is not purely atomic then a B-space has the property (D) if and only if it has the RN-property with respect to (S, Σ, P).

Thus we see that the RN-property is really independent of the underlying probability space and can be considered entirely in relation to the unit interval.

§ 4. Preliminary a.e. convergence theorems:

The purpose of this section is to prove a convergence theorem which ensures a.e. convergence of the martingales of Theorem 2(a) above in case the directed set $N = \{1, 2, 3, \dots\}$ under the natural ordering. No assumptions are necessary on the space X for this theorem. In this generality, the theorem was first proved by using a deep theorem of Banach, in Chatterji [4(b)] and also by A.I. and C.I. Tulcea [18(a)] later. The proof presented here is totally elementary and depends on the following lemma which is stated in the present form for later use.

Lemma 1: Let $\{f_n, \Sigma_n, n \geqq 1\}$ be a X-valued martingale and let $A \in \Sigma_N$. Then for any $\varepsilon > 0$

$$P\{s \in A, \sup_{n \geqq N} |f_n(s)| \geqq \varepsilon\} \leqq \frac{1}{\varepsilon} \sup_{n \geqq N} \int_A |f_n| .$$

The lemma is an easy consequence of the fact that $|f_n|$ is a positive submartingale and is, in this sense well-known. See Doob [8] pp. 314 .

Theorem 3: Let $f \in L^1(\Sigma,X)$ and let $f_n = E_n f$ = conditional expectation with respect to Σ_n. Let $\Sigma_n \subset \Sigma_{n+1}$, $n = 1,2,\ldots$. Then

$$\lim_{n\to\infty} f_n = f_\infty$$

exists (strongly) a.e. and $f_\infty = E_\infty f$ = conditional expectation of f given Σ_∞, the σ-algebra generated by the algebra $\bigcup_{n=1}^{\infty} \Sigma_n$.

Proof: Since the proof is exactly the same as one of the proofs for the scalar-valued case (see Billingsley [1] or Dunford and Schwartz[9] pp. 208) it will be presented only briefly here. If f is measurable with respect to $\bigcup_{n=1}^{\infty} \Sigma_n$ then $f_n = f$ from some point on and hence the conclusion above is immediate. If f is measurable Σ_∞ then, given $\varepsilon > 0$, $\delta > 0$, a g can be found, measurable $\bigcup_{n=1}^{\infty} \Sigma_n$, and such that $\| f - g\|_1 < \frac{\varepsilon\delta}{2}$. By the linearity of the operators E_n, one has

$$|f_n-f_m| \leq |E_n g-E_m g| + |E_n(f-g) - E_m(f-g)|$$

$$\leq |E_n g-E_m g| + 2 \sup_{n\geq 1} E_n|f-g| .$$

Hence $\limsup_{m,n\to\infty} |f_n - f_m| \leq h = 2 \sup_{n\geq 1} E_n|f-g|$ so that

$$P\{\limsup_{m,n\to\infty} |f_n-f_m| \geq \varepsilon\} \leq P\{h \geq \varepsilon\}$$

$$\leq \frac{2}{\varepsilon} \| f-g\| < \delta$$

by an application of lemma 1 to the real-valued martingale $E_n|f-g|$. δ being arbitrary, $P\{\limsup_{m,n\to\infty} |f_n-f_m| \geq \varepsilon\} = 0$ whence ε being arbitrary, the existence of $\lim_{n\to\infty} f_n$ is demonstrated. For a general $f \in L^1(\Sigma,X)$, since $f_n = E_n f = E_n f_\infty$ and f_∞ is Σ_∞-measurable, the existence of $\lim_{n\to\infty} f_n$ is assured. The identification of the limit as being f_∞ follows immediately from Theorem 2(a) above.

For general reference, I shall state a theorem here for the case $N = \{0,-1,-2,\ldots\}$ which was proved in [4(b)], again by the afore-mentioned theorem of Banach and can now be proved by the method given above, without any use of scalar-valued martingale theory.

Theorem 4: Let $\{f_n,\Sigma_n, n \geq 0\}$ be a X-valued martingale then

$$\lim_{n\to-\infty} f_n = f_{-\infty}$$

exists strongly a.e. and also in $L^1(\Sigma,X)$ where $f_{-\infty} = E_{-\infty}f_0$ = conditional expectation of f_0 given $\Sigma_{-\infty} = \bigcap_{n\leq 0} \Sigma_n$.

It may be appropriate to add here that generalizations of theorems 3 and 4 to arbitrary index sets N are not possible, even in the scalar-valued case, without some further assumptions on the structure of the σ-algebras Σ_n. The first counter-example was given by Diendonne [7] . A much simpler counter-example has recently been given by Chow [5] . I should like to point out here a more obvious way of looking at Chow's example. Let $\{g_n, n \geq 1\}$ be a sequence of independent r.v.'s with $E(g_n) = 0$, taking values in an arbitrary B-space X. Let $f = \Sigma g_n$ exist a.e. but suppose that the series is almost surely not unconditionally convergent. Let further $f \in L^1(\Sigma,X)$. Define $f_\pi = \sum_{n\in\pi} g_n$ where π is a finite set of positive integers. Let the π's be ordered by inclusion. If Σ_π is the smallest σ-subalgebra with respect to which $\{g_n, n \in \pi\}$ are measurable, then clearly $f_\pi = E_\pi f$. Further, $\lim_\pi f_\pi$ cannot exist almost surely since this is equivalent to the unconditional convergence of Σg_n almost surely. Note, however, that theorem 2(a) implies that $\| f_\pi - f\|_1 \longrightarrow 0$ all the same. A convenient way of choosing g_n is to take $g_n = \frac{\epsilon_n}{n} a$ where $0 \neq a \in X$ and $\epsilon_n = \pm 1$ with probability 1/2 and are independent. In this case, $f \in L^2(\Sigma,X)$ even, since $E|f|^2 = |a|\Sigma\, 1/n^2 < \infty$ and hence by Theorem 2(a) f_π even converges to f in $L^2(\Sigma,X)$. This choice was made by Chow in [5] pp. 1490 but the point made here is that no calculation is necessary to show that $\lim f_\pi$ does not exist since the series Σg_n is blatantly unconditionally convergent. This latter in the real-valued case automatically implies that $\lim\sup f_\pi = +\infty$ and $\lim\inf f_\pi = -\infty$.

A counter-example to theorem 4 i.e. the "decreasing" index case is also possible. Consider "Riemann sums" $f_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(x+k/n)$ where $f \in L^1(0,1)$ with respect to Lebesgue measure and + is addition module 1. Then, it is easy to verify that $f_n = E_n f$ = conditional expectation of f given Σ_n, the σ-algebra of Borel-sets of the unit interval with period 1/n. If $n_1 | n_2$ then $\Sigma_{n_1} \supset \Sigma_{n_2}$. Define $n_2 \ll n_1$ if $n_1 | n_2$. Then $\{f_n, \Sigma_n\}$ is a martingale which need not converge a.e. as shown by the counter-example in Rudin [15] even though $f \in L^\infty(0,1)$. The analogue of theorem 2(a) however, shows that in all cases however $f_n \longrightarrow a$ in $L^1(0,1)$ where $a = \int_0^1 f$.

§ 5. A decomposition theorem for X-valued set-functions:

In order to avoid interrupting the continuity of the proof of the main theorem in the next section, I shall present here a theorem concerning finitely additive X-valued set-functions. As proto-type of this theorem, in the scalar-valued case, can be considered a theorem of Hewitt and Yosida which states that every finitely additive (scalar) set-function on an algebra can be uniquely decomposed into the sum of a σ-additive and a purely finitely additive set-function. A convenient reference is [9] pp. 163-64. The present theorem for X-valued set-functions is not as sharp as the above theorem but is enough for my purposes.

Theorem 5: Let P be a probability measure on (S,Σ) where Σ is assumed only to be an algebra of sets and let μ be a X-valued finitely additive set-function on Σ of bounded total variation. Then $\mu = \sigma + \eta$ where σ is a σ-additive set-function whose total variation V_σ is finite and P-absolutely continuous and η is a finitely additive set-function whose total variation V_η is finite and P-singular i.e. given $\varepsilon, \delta > 0$ $\exists$ $A \in \Sigma$ such that $P(A) < \varepsilon$ and $V_\eta(A') < \delta$, A' = complement of A.

Proof: The method to be used is fairly standard and is incorporated in pp.311-13 of Dunford and Schwartz [9]. Given the space (S,Σ), there is a space S_1 which is a compact Hausdorff space which has the following properties: (1) S_1 is totally disconnected i.e. the algebra Σ_1 of simultaneously closed and open (clopen) sets form a basis for the topology of S_1 and (2) there is an isometric isomorphism H between $B(S,\Sigma)$ the space of bounded scalar-valued Σ-measurable functions on S and $C(S_1)$, the space of scalar-valued continuous functions on S_1, both spaces being considered under the uniform norm. Let the correspondence $H(C_A(s)) = C_{A_1}(s_1)$ (C's standing for characteristic functions) induce the set-algebra isomorphism τ between Σ and Σ_1 i.e. define $\tau(A) = A_1$. This correspondence is such that $\tau(\Sigma) = \Sigma_1$. Now given an additive or σ-additive (X-valued or scalar-valued) set-function Q on Σ, the formula $Q_1(A_1) = Q(\tau^{-1}(A_1))$ always defines a σ-additive set-function on Σ_1, whether or not Q was σ-additive to start with. The reason for this is that the σ-additivity equation for Q_1 viz. $Q_1(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} Q_1(A_n)$ if $A_n \in \Sigma_1$,

A_n's disjoint and $\bigcup_{n=1}^{\infty} A_n \in \Sigma_1$ is trivially satisfied since the compactness of S_1 precludes the possibility of the existence of an infinite sequence of non-empty disjoint A_n's $\in \Sigma_1$ such that $\bigcup_{n=1}^{\infty} A_n \in \Sigma_1$ also. Clearly if Q is of finite total variation, so is Q_1 on Σ_1. If this is so, then Q_1 can be extended to the σ-algebra Σ_2 generated by Σ_1. If Q is scalar-valued, this is possible by a classical theorem of Caratheodory. If Q_1 is X-valued then also this fact has been known for a long time. For convenient reference, see [18(a)]pp. 119 and foot-note (6). Now let P_1, μ_1 be these transpositions of P, μ of the theorem to the space (S_1, Σ_1). Let P_1, μ_1 stand also for the extended set-functions on (S_1, Σ_2). μ_1 is further of bounded total variation on Σ_2 also.

According to a theorem of Rickart [14] which generalizes the classical Lebesgue decomposition theorem for scalar-valued set-functions, $\mu_1 = \sigma_1 + \eta_1$ on the σ-algebra Σ_2 where σ_1, η_1 are of bounded variation if μ_1 is so (as in this case) and σ_1 is P_1-absolutely continuous and η_1 is P_1-singular. Let σ, η be the inverse images of the restrictions of σ_1, η_1 to Σ_1. Then on the given space (S, Σ), $\mu = \sigma + \eta$ where V_σ is P-absolutely continuous and V_η is P-singular. The σ-additivity of σ follows trivially from the fact that V_σ is absolutely continuous with respect to a σ-additive function P. Thus the decomposition theorem is completely established.

It seems likely that η should be further decomposable into a sum of two set-functions, one σ-additive and P-singular and the other purely finitely additive by which is meant that its total variation is singular to all σ-additive set-functions on Σ. I have not been able to prove this yet.

§ 6. The Main Theorem of this paper will bow be stated as follows:

Theorem 6. For a B-space X and a probability space (S, Σ, P) the following statements are equivalent:

Every X-valued martingale $\{f_n, \Sigma_n\}$ $n \geq 1$, with the property that

(1) $\sup_{n \geq 1} \|f_n\|_1 < +\infty$ is such that $f_\infty = \lim_{n \to \infty} f_n$ exists strongly a.e.

(2) $\sup_{n \geq 1} \|f_n\|_1 < +\infty$ is such that $f_\infty = \lim_{n \to \infty} f_n$ exists weakly a.e.

in the sense that $\exists\ f_\infty$ strongly measurable such that $\forall y^* \in X^*$,

$\lim_{n \to \infty} < f_n(s), y^* > = < f_\infty(s), y^* >$ for $s \notin N_{y^*}$, $P(N_{y^*}) = 0$. It is enough to know that f_∞ is a.e. separable-valued to deduce a version of it which is strongly measurable. See proof of Theorem 7 later for an elucidation of this condition.

(3) for some $C > 0$ $\sup_{n \geq 1} |f_n(s)| < C$ a.e. is such that $f_\infty = \lim_{n \to \infty} f_n$ exists strongly a.e.

(4) for some $C > 0$, $\sup_{n \geq 1} |f_n(s)| < C$ a.e. is such that $f_\infty = \lim_{n \to \infty} f_n$ exists weakly a.e. in the sense of statement (2)

(5) f_n's are uniformly integrable (i.e. $\lim_{N \to \infty} \int |f_n| C_{\{|f_n| > N\}} = 0$ uniformly in $n \geq 1$) is such that $\exists\ f_\infty \in L^1(\Sigma, X)$ with $\lim_{n \to \infty} \| f_n - f_\infty \|_1 = 0$

(6) $\sup_{n \geq 1} \| f_n \|_p < \infty$, $1 < p < \infty$, is such that $\exists\ f_\infty \in L^p(\Sigma, X)$ with $\lim_{n \to \infty} \| f_n - f_\infty \|_p = 0$

(7) X has the RN-property with respect to (S, Σ, P).

Remark: The reader is reminded that in view of the discussion of the RN-property given above, the convergence properties of X-valued martingales are rather independent of the underlying probability space. If P is purely atomic, then all the 7 statements above hold for all B-spaces X. If P is not purely atomic and if X has one of the above 7 properties then X has all of them with respect to any other probability space and in particular X has property (D). I should like to remark that the equivalence of (5) and (7) have also been pointed out by Rønnow [16]. Some of the equivalences above (e.g. (2) <=> (5)) can be deduced very easily, independently and are listed for their possible utility and for completeness.

Proof. The major part of the proof consists in showing that (7) => (1). All the other implications then follow by fairly routine arguments. So I begin with proving that

(7) $\Rightarrow$ (1): Given the martingale $\{f_n, \Sigma_n, n \geq 1\}$ with the property that $\sup_{n \geq 1} E|f_n| < \infty$, let the X-valued set-function μ_n be defined on Σ_n by the formula $\mu_n(A) = \int_A f_n(s)P(ds)$. Clearly, the martingale property of the f_n's is equivalent to the statement that μ_{n+1} is an extension of μ_n to $\Sigma_{n+1} \supset \Sigma_n$. Hence the formula $\mu(A) = \lim_{n \to \infty} \mu_n(A)$ defines an X-valued set-function on the algebra $\Sigma_\omega = \bigcup_{n=1}^{\infty} \Sigma_n$ which is clearly finitely additive. Let $V_\mu(A) = \sup\{\sum_{i=1}^{k} |\mu(B_i)| \mid B_i \in \Sigma_\omega, B_i \subset A, B_i$ disjoint, $1 \leq k < \infty\}$ be the total variation of μ for a set $A \in \Sigma_\omega$. It is easy to see that $V_\mu(A) = \lim_{n \to \infty} \int_A |f_n| < +\infty$. In other words, μ is a finitely additive set-function of bounded total variation on the algebra Σ_ω. One of the difficulties in proving (1) is that μ may not be σ-additive, a difficulty which may arise even in the scalar-valued case. I shall obviate this difficulty by using Theorem 5 of the preceeding section. According to that theorem $\mu = \sigma + \eta$ where σ is σ-additive and whose variation is P-absolutely continuous. By the RN-property (i.e. (7)), $\sigma(A) = \int_A g$, $A \in \Sigma_\omega$ and $g \in L^1(\Sigma_\infty, X)$ $\Sigma_\infty = \sigma$-algebra generated by Σ_ω. If σ_n is the restriction of σ to Σ_n then clearly $\sigma_n(A) = \int_A g_n$ $A \in \Sigma_n$, where $g_n = E_n g$. Since, by assumption, the restriction μ_n of μ to Σ_n is also an integral, the restriction η_n of η to Σ_n must be of the form $\int_A h_n$. Indeed $f_n = g_n + h_n$, and $\{g_n, \Sigma_n\}$, $\{h_n, \Sigma_n\}$ are X-valued martingales. Moreover, since $g_n = E_n g$, by Theorem 3, $\lim_{n \to \infty} g_n = g$ exists strongly a.e. I shall now show that $\lim_{n \to \infty} h_n = 0$ strongly a.e. Because of the P-singularity of V_η, given $0 < \varepsilon, \delta < 1$, I can find $A \in \Sigma_\omega$ (and hence $A \in \Sigma_N$ for some N) such that

$$P(A') + V_\eta(A) < \frac{\varepsilon\delta}{2} .$$

Now

$$P\{\sup_{n \geq N} |h_n| > \varepsilon\} = P\{A'; \sup_{n \geq N} |h_n| > \varepsilon\} + P\{A; \sup_{n \geq N} |h_n| > \varepsilon\}$$

$$< \frac{\varepsilon\delta}{2} + \frac{1}{\varepsilon} \sup_{n \geq N} \int_A |h_n| P(ds) \quad \text{(by lemma 1)}$$

$$= \frac{\varepsilon\delta}{2} + \frac{1}{\varepsilon} V_\eta(A) < \frac{\varepsilon\delta}{2} + \frac{\delta}{2} < \delta .$$

Hence

$$P\{\limsup_{n \to \infty} |h_n| > \varepsilon\} \leq P\{\sup_{n \geq N} |h_n| > \varepsilon\} < \delta .$$

ε,δ being arbitrary, it follows that $\lim_{n\to\infty} |h_n| = 0$ a.e. This proves that $\lim_{n\to\infty} f_n$ exists strongly a.e. and to some extent characterizes the limit function.

Proof of

(1) => (5):

Suppose f_n's are uniformly integrable. Then $\sup_{n\geq 1} \|f_n\|_1 < \infty$ and hence by (1) the limit $\lim_{n\to\infty} f_n = f_\infty$ exists strongly a.e. Clearly $f_\infty \in L^1(\Sigma_\infty, X)$ since by Faton's lemma $E|f_\infty| \leq \lim_{n\to\infty} \|f_n\|_1$. Hence $|f_n(s) - f_\infty(s)|$ as a sequence of real-valued functions is uniformly integrable and tends to 0 a.e. Therefore

$$\lim_{n\to\infty} \|f_n - f_\infty\|_1 = \lim_{n\to\infty} E|f_n - f_\infty| = 0.$$

Proof of

(5) => (7):

By the Corollary to Theorem 1, given a P-absolutely continuous X-valued σ-additive function μ of bounded total variation on Σ, to prove that μ is a P-integral, it is enough to verify that for every sequence π_n of finer and finer partitions, the sequence of X-valued r.v.'s f_n (denoted there by f_{π_n}) which forms a martingale $\{f_n, \Sigma_n\}$ (Σ_n = σ-algebra formed by π_n), is such that f_n's converge in $L^1(\Sigma, X)$. If I can show that f_n's are uniformly integrable then by virtue of (5), this latter will follow and (7) will be deduced. Because of the inequality

$$P(|f_n| \geq N) \leq \frac{1}{N}\|f_n\|_1 \leq \frac{1}{N} V_\mu(S)$$

given $\varepsilon > 0$, one can choose N so large that

$$P(|f_n| \geq N) < \varepsilon \quad \text{for all } n \geq 1.$$

Because V_μ is P-absolutely continuous, given $\delta > 0$, there exists $\varepsilon > 0$ such that

$$P(A) < \varepsilon \text{ implies } V_\mu(A) < \delta \text{ for } A \in \Sigma.$$

Hence for any $\delta > 0$,

$$\int_{\{|f_n| \geq N\}} |f_n| \leq V_\mu\{|f_n| \geq N\} < \delta \qquad n = 1,2,\ldots$$

if first ε and then N are chosen as indicated above.

This proves uniform integrability of f_n and proves (7).

Proof of (2) => (5):

Let $\{f_n, \Sigma_n\}$ be an uniformly integrable X-valued martingale. Clearly $\sup_{n \geq 1} \|f_n\|_1 < \infty$; hence by (2) there exists f_∞, which can be easily seen to be in $L^1(\Sigma_\infty, X)$, such that $\lim_{n \to \infty} < f_n(s), y^* > = < f_\infty(s), y^* >$ a.e. for any $y^* \in X^*$. Since the uniform integrability of f_n's, clearly implies the same for $< f_n(s), y^* >$, it follows that for every $y^* \in X^*$, $\{ < f_n, y^* >, \Sigma_n, 1 \leq n \leq \infty\}$ is a scalar-valued martingale i.e. in particular, for $A \in \Sigma_n$, the relation

$$< \int_A f_n, y^* > = \int_A < f_n, y^* > = \int_A < f_\infty, y^* = < \int_A f_\infty, y^* >$$

is valid for every $y^* \in X^*$. Hence $\int_A f_n = \int_A f_\infty$ for all $A \in \Sigma_n$. In other words, $f_n = E_n f_\infty$. Theorem 1 then implies that $\| f_n - f_\infty \|_1 \to 0$.

The implication (1) =>(2) being trivial, the above arguments show that (1), (2), (5), and (7) are equivalent.

Proof of (3) => (7):

If condition (3) holds for some $C > 0$ then clearly it holds for all $0 < C < \infty$. Suppose first that the X-valued set-function μ is such that $\|\frac{dV_\mu}{dP}\|_\infty \leq N$ i.e. $P\{\frac{dV_\mu}{dP} \leq N\} = 1$ which means that $V_\mu(A) \leq NP(A)$ for all $A \in \Sigma$ and some integer $N \geq 1$. Because of the corollary to Theorem 1, as in the proof of (5) => (7), it suffices to prove that for every sequence of increasily finer partitions π_n, the associated martingale $\{f_n, \Sigma_n\}$ is such that f_n's converge in $L^1(\Sigma, X)$. Since $V_\mu(A) \leq NP(A)$, it follows that $\sup_{n \geq 1} |f_n(s)| \leq N$ and by (3) f_n's converge strongly a.e. to a function f_∞ which is then automatically in $L^1(\Sigma, X)$. By the dominated convergence theorem, since $|f_n(s) - f_\infty(s)| \leq 2N$ a.e. $\| f_n - f_\infty \|_1 \to 0$. Thus every X-valued set-function μ under consideration, with the above-mentioned extra property is representable as an integral. For a general μ, the proof now proceeds by a standard argument, which has nothing to do with martingale theory, as follows. Let $A_N = \{s \mid \frac{dV_\mu}{dP} \leq N\}$. Clearly $A_N \subset A_{N+1}$ and $\Omega = \bigcup_{N=1}^{\infty} A_N$. Let $\mu_N(B) = \mu(BA_N)$ for $B \in \Sigma$. Then

$V_{\mu_N}(B) = V_\mu(BA_N)$ and so $V_{\mu_N}(B) \leq NP(BA_N) \leq NP(B)$. By what has already been proved, it follows that $\mu_N(B) = \int_B f_N$ for some $f_N \in L^1(\Sigma,X)$. It is easily seen that $f_N = 0$ a.e. on A'_N and that for $N > M$ $(A_N \supset A_M)$ $f_N = f_M$ a.e. on A_M. Hence for $N > M$,

$$\int|f_N - f_M| = \int_{A'_M}|f_N| = V_{\mu_N}(A'_M) = V_\mu(A_N A'_M) \leq V_\mu(A'_M)$$

so that $\| f_N - f_M\|_1 \to 0$ as $M,N \to \infty$. Hence there exists an $f \in L^1(\Sigma,X)$ such that $\| f_N - f\|_1 \to 0$ as $N \to \infty$. Since

$$\mu(B) = \lim_{N\to\infty} \mu(BA_N) = \lim_{N\to\infty} \int_B f_N = \int_B f(s)P(ds),$$

holds, (7) is proved.

The argument of (2) => (5), shows that (4) => (3) since the condition in (3) implies uniform integrability and once it has been shown that there exists f_∞ such that $\| f_n - f_\infty\|_1 \to 0$, it would follow that $f_n = E_n f_\infty$ whence Theorem 3 would lead to the conclusion of (3).

Since the implications (3) ==> (4) and (1) => (3) are immediate, it follows that (1), (2), (3), (4), (5), and (7) are equivalent.

As regards (6), notice first that (6) ==> (3) by an argument used already. For if $\sup_n|f_n(s)| < C$ a.e. then $\| f_n\|_p < C$ for $n \geq 1$. Therefore by (6) there exists $f_\infty \in L^p(\Sigma,X)$ such that $\| f_n - f_\infty\|_p \to 0$ $(1 < p < \infty)$. It follows then that $f_n = E_n f_\infty$ and Theorem 3 does the rest.

On the other hand (5) => (6), because given a martingale $\{f_n,\Sigma_n\}$ with $\sup_{n\geq 1} \|f_n\|_p < \infty$, $1 < p < \infty$, it follows immediately that f_n's are uniformly integrable and hence by (5), there exists f_∞ such that $\| f_n - f_\infty\|_1 \to 0$. This implies as before that $f_n = E_n f_\infty$. Further $f_\infty \in L^p(\Sigma_\infty,X)$ since by Fatou's lemma $\int|f_\infty|^p \leq \lim_{n\to\infty} \int|f_n|^p < \infty$ by the assumption of (6). Theorem 1 now implies that $\| f_n - f_\infty\|_p \to 0$.

Thus the equivalence of (1) - (7) is established.

Applications:

In this section the main theorem will be used to deduce some well-known Radon-Nikodym theorems for X-valued set-functions. To emphasize the simplicity of these deductions, I should like to point out that what is needed is not the whole strength of the main theorem but rather the following elementary version of it.

Let μ be a X-valued σ-additive set-function of bounded total variation on the probability space (S,Σ,P) and let $\mu(A) = 0$ whenever $P(A) = 0$. Then for any sequence of partitions π_n, $n \geq 1$, which become increasingly finer, the functions $f_{\pi_n}(s)$ of Example (ii) of section (2), are uniformly integrable. μ has the integral representation $\int_A f(s)P(ds)$ if and only if for every sequence π_n of increasingly finer partitions the corresponding sequence f_{π_n} converges weakly a.e. (P) to a strongly measurable function $f_\infty(s)$ in the sense that for all $y^* \in X^*$, there is a set of P-measure zero N_{y^*}, possibly depending on y^*, such that if $s \notin N_{y^*}$

then $\lim_{n \to \infty} < f_n(s), y^* > = < f_\infty(s), y^* >$.

It is left to the interested reader to verify that the "non-elementary" argument (7) => (1) of the main theorem is nowhere needed in a proof of the above statement.

Using it, I shall now derive a theorem originally due to Phillips [13]. A variety of other theorems of this sort e.g. the Dunford-Pettis theorem, the Dunford-Pettis-Phillips theorem (see Bourbaki [3]) follow effortlessly in a similar manner, without any separability assumption on the space X as was originally made and later removed by the use of "lifting" arguments by A.I. and C.I. Tulcea [18(b)] . These and some more recent theorems of Mr. M.A. Rieffel (to be published) and representations by means of integrals other than Bochner-integrals will be deferred to a more systematic treatment in a later publication.

Theorem 7 (see Phippips [13]).

Let μ be a X-valued σ-additive set-function of totally bounded variation on a probability space (S,Σ,P) such that $\mu(A) = 0$ whenever $P(A) = 0$. If for every integer $N \geq 1$, the set $K_N = \{ \frac{|\mu(A)|}{P(A)} \mid \frac{|\mu(A)|}{P(A)} \leq N,\ P(A) > 0\}$ is relatively weakly compact then $\mu(A) = \int_A f(s)P(ds)$ where $f \in L^1(\Sigma,P)$.

Proof: - I shall suppose first that for some integer $N \geq 1$, $|\mu(A)| \leq NP(A)$ for all $A \in \Sigma$. The general statement can be derived from this special case exactly by means of the method sketched in the proof, (3) => (7), of the main theorem.

By virtue of the remarks made at the beginning of this section, it suffices to show that if π_n is an increasingly finer sequence of partitions of S, then the corresponding functions f_n converge weakly a.e. to a strongly measurable function f_∞ in the sense described before. Actually, it is enough to know that f_∞ is separable-valued a.e. to deduce its strong measurability since the limit relation $\lim_{n \to \infty} < f_n(s), y^* > = < f_\infty(s), y^* >$ a.e. (even if the null-set depends on $y^* \in X^*$), implies that for each $y^* \in X^*$ the function $< f_\infty(s), y^* >$ is measurable with respect to the σ-algebra Σ^*, the completion of Σ under the probability measure P. By a known theorem, (see Hille-Phillips [11]), f_∞ is then strongly measurable with respect to Σ^*. Clearly f_∞ can then be changed on a set of P-measure zero, so that the new version is Σ-strongly measurable and such that the weak-convergence of f_n to f_∞ in the above sense remains unaltered.

From the definition of the f_n's it is to be seen that these finitely-valued r.v.'s , take their values in the set defined in the statement of the theorem. Let X_0 be the closed separable linear manifold spanned by the values of $f_n(s)$, $s \in S$, $n \geq 1$. Two things about X_0 are to be noticed: (i) X_0 is automatically weakly closed also by a general theorem (see [9] pp. 422, Theorem 13) and that because of the hypothesis of Theorem 7, (ii) the subset of X_0 consisting of the values of $f_n(s)$ is relatively weakly compact. For any point $s \in S$, let a subsequence n_k be chosen so that $f_{n_k}(s)$ converges weakly to $f_\infty(s)$, an element of X_0. This is possible because of (i) and (ii) above. (An application of the axiom of choice is involved in this procedure). Now for any $y^* \in X^*$ the sequence $< f_n(s), y^* >$, being a scalar-valued martingale, converges a.e. Hence :

$\lim_{n \to \infty} < f_n(s), y^* > = < f_\infty(s), y^* >$ a.e. Since $f_\infty(s)$ is separable-valued, the remarks made before show that it may be chosen to be strongly Σ-measurable. Hence the criterion given at the beginning of the section ensures that μ has an integral representation by means of a function from $L^1(\Sigma, X)$.

Corollary: The following classes of B-spaces X have property (D) and hence the RN property with respect to any probability space (S,Σ,P)

(i) the reflexive spaces

(ii) separable duals of Banach spaces i.e. X is separable and there is a B-space Y such that $Y^* = X$.

(iii) weakly complete spaces with separable duals, i.e. X is weakly complete and X^* is separable.

That the reflexive spaces have the property (D) follows immediately from Theorem 7. For the other two classes, the property (D) can be derived similarly. The details are omitted. From the counter-example of the next section, will be seen that neither separability nor weak completeness can be left out in the description of the classes (ii) and (iii). The classes (i)-(iii) have been known to possess property (D) for some time. I hope to discuss property (D) in greater detail in a later publication.

A counter-example:

Several examples are known of X-valued set-functions which are σ-additive, P-absolutely continuous, totally bounded variation but not integrals. E.g. if S = the unit interval (with P = Lebesgue measure on Σ= Borel sets) and $X = L^1$ over this space, then $\mu(A) = C_A(x) \in L^1$ is an old example of this nature. In Chatterji [4(a)] a martingale is constructed from this in the obvious way, which converges almost nowhere in any sense. As [18(a)] points out, this shows in particular that L^1 is not the dual of any space, by virtue of (ii) of the Corollary above, a fact pointed out by Dieudonné first. An example of a non-convergent martingale has been recently given by Rønnow [16]. I should like to present it here in a different and very simple form and in a way which illustrates various new features of the theory of X-valued r.v.'s. The underlying probability space is again that of the unit interval and let the B-space involved be c_0 = the space of real or complex sequences which converge to zero with $|x| = \sup_{j \geq 1} |x_j|$, $x = (x_1, x_2, \dots)$. Let $\gamma_n(s)$ be the sequence of Rademacher functions on the unit interval. These are known to be stochastically independent under Lebesgue measure. (Definition of $\gamma_n(s)$: let $s = \sum_{n=1}^{\infty} a_n(s) 2^{-n}$ be the binary expansion of $0 \leq s \leq 1$; then $\gamma_n(s) = 1 - 2a_n(s) = \pm 1$ with probability 1/2).

Let $e_n = (0,0,\ldots 1,0,\ldots) \in c_0$(1 at the nth place); $|e_n| = 1$, $n \geq 1$. Define $f_n(s) = \sum_{k=1}^{n} \gamma_k(s)e_k = (\gamma_1(s),\gamma_2(s),\ldots,\gamma_n(s),0,\ldots\ldots)$. It is immediate that $\{f_n,\Sigma_n\}_{n\geq 1}$ is a martingale, where Σ_n = σ-algebra generated by intervals of the type $(\frac{k}{2^n},\frac{k+1}{2^n})$, $0 \leq k \leq 2^n-1$. Actually f_n is the sum of n independent c_0-valued r.v.s, each of which takes two values and each of which has expected value 0. Clearly $|f_n(s)| \equiv 1$ and $E|f_n| = \|f_n\|_1 = 1$. But $f_n(s)$ does not converge strongly in c_0 or even in the bigger space 1^∞ at any irrational point s. On the other hand, since $(c_0)^* = 1^1$, and since the sequence $< f_n(s), y >$ converges for every s, for any $y \in 1^1 = (c_0)^*$, the sequence $f_n(s)$ converges weakly but not to any element of c_0. Further, since $1^\infty = (1^1)^*$, it follows that a martingale f_n taking values in a space $X = (Y)^*$, may be convergent to f_∞ in the weak*-topologi of X (i.e. the Y topology of X) without being strongly or weakly convergent. The last remark is verified by noting that $f_\infty(s) = (\gamma_1(s),\ldots,\gamma_n(s),\ldots\ldots)$ has a non-separable range in 1^∞.

It is to be noted however that for any sequence a_n, tending to 0, however slowly the series of c_0-valued independent r.v.'s $\Sigma\, a_n\gamma_n(s)e_n$ converges everywhere unconditionally but not absolutely if $\Sigma|a_n| = +\infty$. But $E|a_n\gamma_n(s)e_n|^2 = |a_n|^2$ so that the variance series may be chosen to diverge. Thus one may have a c_0-valued sequence of independent r.v.'s Y_n which are uniformly bounded and of 0 expectation and such that ΣY_n converges a.e. (even unconditionally) without the convergence of the variance series, in contradiction to a known theorem in the scalar-valued case. I hope to pursue this matter further in other publications. The example above may also be looked at as the martingale version of a counter-example of Clarkson [6] pp. 414 of a 1^∞-valued function of bounded variation which is not differentiable anywhere, although it satisfies a Lipschitz condition.

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