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by

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Vestsi Akademii Navuk Belaruskay SSR,
Seryya Fizika Matematychnykh Navuk,
No. 4, pp. 110-120, 1965.

Translated from the Russian

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NUMERICAL SOLUTION OF HYPERBOLIC EQUATIONS AND SYSTEMS WITH TWO INDEPENDENT VARIABLES BY A METHOD OF THE RUNGE-KUTTA TYPE. I

by

Nguyen Kong Tuy

Discussed is the application of two-iteration Runge-Kutta algorithms in the solution of the Cauchy problem for hyperbolic equations and systems with two independent variables.

The question of the application of Runge-Kutta type methods to the numerical solution of partial differential equations has been considered by several authors [1-3].

In the present article, use is being made of two-iteration Runge-Kutta algorithms in the solution of the Cauchy problem for hyperbolic equations and systems with two independent variables. The results for one equation $u_{xy} = f(x, y, u, u_x, u_y)$ with Cauchy conditions along the line segment $x - y = \text{const}$ are generalized for the case of a system of such equations and are being used for the solution of a nonlinear Monge-Ampere equation. An analogous problem with Cauchy conditions along a curve segment will be discussed later.

1. Statement of the Problem

Let us consider the equation:

$$u_{xy} = f(x, y, u, p, q) \quad , \tag{1}$$

(u is an unknown function of x and y; $p = u_x$; $q = u_y$) with initial conditions given along the segment AB of the straight line $x + y = \text{const}$, in the form:

$$u^0 = u^0(x) \quad , \qquad p^0 = p^0(x) \quad , \qquad q^0 = q^0(x) \quad . \tag{2}$$

u^0, p^0, q^0, f are considered to be continuous and differentiable a sufficient number of times.

Let us construct in the region ABC of definition of the solution a uniform rectangular grid with step h (Figure 1). We will distribute nodes of the grid in layers parallel to the initial line segment AB. In making calculations on the k^{th} layer, for simplicity, we will denote the earlier found values u, p, q, in the nodes of the $(k - 1)^{st}$ layer, as well as the initial data in the problem (1), (2), as u^0, p^0, q^0.

We will introduce the following designations: $f(x,y,u,p,q) = F(x,y)$, $F(x_1,y_1) = F(M)$.

For the elementary triangle MNP of the grid with vertices $M(x_1,y_1)$, $N(x_2 = x_1 + h,\ y_2 = y_1 - h)$ on the $(k - 1)^{st}$ layer and vertex $P(x_2,y_1)$ on the k^{th} layer (Figure 2) the following relationships are known:

$$u(P) = u^0(x_1) + \int_{x_1}^{x_1 + h} p^0(x)\,dx + \iint F(x,y)\,dxdy \ , \qquad (3.a)$$

$$p(P) = p^0(x_2) + \int_{y_1 - h}^{y_1} F(x_2,y)\,dy \ , \qquad (3.b)$$

$$q(P) = q^0(x_1) + \int_{x_1}^{x_1 + h} F(x,y_1)\,dx \ , \qquad (3.c)$$

where the multiple integral in (3.a) is being taken over the region of MNP.

Let us take as an initial point the point M, corresponding to $h = 0$. Then, using (3) we can find an expansion in powers of h at the point M for the increments $\Delta^* u = u(P) - u(M)$, $\Delta^* p = p(P) - p(M)$, $\Delta^* q = q(P) - q(M)$ in the form

$$\Delta^* u = hp^0(x_1) + \frac{h^2}{2!}\frac{dp^0(x_1)}{dx} + \frac{h^3}{3!}\frac{d^2p^0(x_1)}{dx^2} + \frac{h^2}{2!}F(M)$$

$$+ \frac{h^3}{3!}\left[2\frac{\partial F(M)}{\partial x} - \frac{\partial F(M)}{\partial y}\right] + O(h^4) \ , \qquad (4.a)$$

$$\Delta^* p = h \frac{dp^0(x_1)}{dx} + \frac{h^2}{2!} \frac{d^2p^0(x_1)}{dx^2} + hF(M)$$

$$+ \frac{h^2}{2!} \left[2 \frac{\partial F(M)}{\partial x} - \frac{\partial F(M)}{\partial y} \right] + O(h^3) \ , \qquad (4.b)$$

$$\Delta^* q = hF(M) + \frac{h^2}{2!} \frac{\partial F(M)}{\partial x} + O(H^3) \ . \qquad (4.c)$$

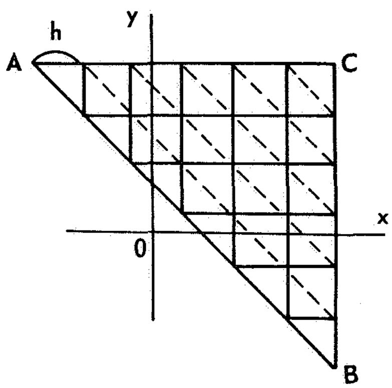


FIGURE 1

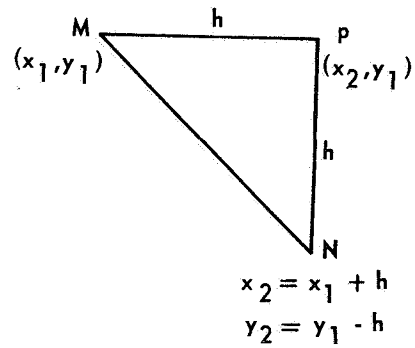


FIGURE 2

In the present article the problem is being set forth of constructing Runge-Kutta type algorithms, which give approximate values of the increments Δu, Δp, Δq.

2. Formal Algorithms of the Runge-Kutta Type

Values of the function $F(x,y)$ used in the i^{th} iteration $(i = 1,2)$ for the calculation of Δu, Δp, Δq, will hereafter be denoted as F_i, $F_{\hat{i}}$, $F_{\bar{i}}$, respectively. Let the values u, p, q at the point p, obtained after the i^{th} iteration, be u^i, p^i, q^i, and $\int_{x_1}^{x_1 + h} p^0(x)\,dx = I(h)$ [for the calculation of I(h) see Paragraph 4].

At the first iteration we assume that:

$$u^1 = u^0(x_1) + I(h) + \frac{1}{2} h^2 F_1 \ , \qquad (5.a)$$

$$F_1 = \omega_1 F(M) + \tau_1 F(N) \quad ; \qquad \omega_1, \tau_1 \geq 0 \; , \qquad \omega_1 + \tau_1 = 1 \; , \tag{5.b}$$

$$p^1 = p^0(x_2) + hF_{\hat{1}} \quad ; \qquad F_{\hat{1}} = F(N) \; , \tag{5.c}$$

$$q^1 = q^0(x_1) + hF_{\bar{1}} \quad ; \qquad F_{\bar{1}} = F(M) \; .$$

We form the elementary increments:

$$\left.\begin{aligned} \Delta^1 u &= u^1 - u^0(x_1) \\ \Delta^1 p &= p^1 - p^0(x_1) \\ \Delta^1 q &= q^1 - q^0(x_1) \end{aligned}\right\} \; , \quad (5.\text{d}) \qquad \left.\begin{aligned} \delta^1 u &= u^1 - u^0(x_2) \\ \delta^1 p &= p^1 - u^0(x_2) \\ \delta^1 q &= q^1 - u^0(x_2) \end{aligned}\right\} \; . \tag{5.e}$$

Increments (5.d) and (7.d) below relate to the point M, and increments (5.e) — to the point N.

Then, we introduce the intermediate points $M_2^j (j = u, q)$, $N_2^j (j = u, p)$:

$$M_2^j = \Big[x_1 + \rho^j h ; \; y_1 + \sigma^j h ; \; u^0(x_1) + \alpha^j \Delta^1 u \; ;$$

$$p^0(x_1) + \beta^j \Delta^1 p \quad ; \qquad q^0(x_1) + \gamma^j \Delta^1 \bar{q} \Big] \; , \tag{6.a}$$

$$N_2^j = \Big[x_2 + \bar{\rho}^{\,j} j \quad ; \qquad y_2 + \bar{\sigma}^{\,j} h \quad ; \qquad u^0(x_2) + \bar{\alpha}^{\,j} \delta^1 u \quad ;$$

$$p^0(x_2) + \bar{\beta}^{\,j} \delta^1 p \quad ; \qquad q^0(x_2) + \bar{\gamma}^{\,j} \delta^1 q) \quad , \tag{6.b}$$

ρ^j, $\bar{\rho}^{\,j}$, σ^j, $\bar{\sigma}^{\,j}$, $\ldots$, γ^j, $\bar{\gamma}^{\,j}$ are numerical parameters.

At the second iteration we assume that:

$$u^2 = u^0(x_1) + I(h) + \frac{1}{2} h^2 F_2 \; , \tag{7.a}$$

$$F_2 = \omega_2 F\left(M_2^u\right) + \tau_2 F\left(N_2^u\right) \; ; \qquad \omega_2, \tau_2 \geq 0 \; ; \qquad \omega_2 + \tau_2 = 1 \; ,$$

$$p^2 = p^0(x_2) + hF_{\hat{2}} \quad ; \qquad F_{\hat{2}} = F\left(N_2^p\right) \; , \tag{7.b}$$

$$q^2 = q^0(x_1) + hF_{\bar{2}} \; ; \qquad F_{\bar{2}} = F\left(M_2^q\right) \tag{7.c}$$

$\left[\text{by } F\left(M_2^j\right)\right]$ is understood $f\left[x_1 + \rho^j h;\ y_1 + \sigma^j h;\ u^0(x_1) + \alpha^j \Delta^1 u;\ p^0(x_1) + \beta^j \Delta^1 p;\ q^0(x_1) + \gamma^j \Delta^1 q\right]$; analogously with $\left[F\left(N_2^j\right)\right]$,

$$\left.\begin{aligned} \Delta^2 u &= u^2 - u^0(x_1) \\ \Delta^2 p &= p^2 - p^0(x_1) \\ \Delta^2 q &= q^2 - q^0(x_1) \end{aligned}\right\} \quad . \tag{7.d}$$

Finally, as Δu, Δp, Δq we take linear combinations:

$$\left.\begin{aligned} \Delta u &= \lambda_1 \Delta^1 u + \lambda_2 \Delta^2 u \\ \Delta p &= \mu_1 \Delta^1 p + \mu_2 \Delta^2 p \\ \Delta q &= \nu_1 \Delta^1 q + \nu_2 \Delta^2 q \end{aligned}\right\} \quad , \tag{8}$$

λ_i, μ_i, ν_i are numerical parameters.

3. Numerical Determination of Runge-Kutta Parameters

For $\overset{i}{\Delta} u$, $\overset{i}{\Delta} p$, $\overset{i}{\Delta} q$ the following expansions in powers of h are valid:

$$\overset{i}{\Delta} u = hp^0(x_1) + \frac{h^2}{2!}\frac{dp^0(x_1)}{dx} + \frac{h^3}{3!}\frac{d^2p^0(x_1)}{dx^2} + \frac{h^2}{2!} F(M) + \frac{h^3}{3!}\, 3 \left(\frac{dF_i}{dh}\right)_M + O(h^4) \quad , \tag{9.a}$$

$$\overset{i}{\Delta} p = h\frac{dp^0(x_1)}{dx} + \frac{h^2}{2!}\frac{d^2p^0(x_1)}{dx^2} + hF(M) + \frac{h^2}{2!}\, 2 \left(\frac{d\hat{F}_i}{dh}\right)_M + O(h^3) \quad , \tag{9.b}$$

$$\overset{i}{\Delta} q = hF(M) + \frac{h^2}{2!}\, 2 \left(\frac{d\bar{F}_i}{dh}\right)_M + O(h^3) \quad . \tag{9.c}$$

$(d/dh)_M$ in (9) and below means that the derivative with respect to h is being taken at the point M.

Taking into account (8) we obtain expansions in powers of h for Δu, Δp, Δq:

$$\Delta u = (\lambda_1 + \lambda_2)\left[hp^0(x_1) + \frac{h^2}{2!}\frac{dp^0(x_1)}{dx} + \frac{h^3}{3!}\frac{d^2p^0(x_1)}{dx^2}\right]$$

$$+ (\lambda_1 + \lambda_2)\frac{h^2}{2!}F(M) + \frac{h^3}{3!}\, 3\left(\lambda_1\frac{dF_1}{dh} + \lambda_2\frac{dF_2}{dh}\right)_M$$

$$+ O(h^4) \quad , \tag{10.a}$$

$$\Delta p = (\mu_1 + \mu_2)\left. h\frac{dp^0(x_1)}{dx} + \frac{h^2}{2!}\frac{d^2p^0(x_1)}{dx^2}\right] + (\mu_1 + \mu_2)hF(M)$$

$$+ \frac{h^2}{2!}\, 2\left(\mu_1\frac{d\hat{F}_1}{dh} + \mu_2\frac{dF_2}{dh}\right)_M + O(h^3) \quad , \tag{10.b}$$

$$\Delta q = (\nu_1 + \nu_2)hF(M) + \frac{h^2}{2!}\, 2\left(\nu_1\frac{d\bar{F}_1}{dh} + \nu_2\frac{d\bar{F}_2}{dh}\right)_M + O(h^3) \quad . \tag{10.c}$$

The selection of parameters is effected so that the expansions (10) and (4) coincide at the arbitrary function $F(x,y)$ and arbitrary step h. As a result of comparison of (10) and (4) we obtain the following conditions:

$$\lambda_1 + \lambda_2 = \mu_1 + \mu_2 = \nu_1 + \nu_2 = 1 \quad , \tag{11}$$

$$3\left(\lambda_1\frac{dF_1}{dh} + \lambda_2\frac{dF_2}{dh}\right)_M = 2\,\frac{\partial F(M)}{\partial x} - \frac{\partial F(M)}{\partial y} \quad , \tag{12}$$

$$2\left(\mu_1\frac{d\hat{F}_1}{dh} + \mu_2\frac{d\hat{F}_2}{dh}\right)_M = 2\,\frac{\partial F(M)}{\partial x} - \frac{\partial F(M)}{\partial y} \quad , \tag{13}$$

$$2\left(\nu_1\frac{d\bar{F}_1}{dh} + \nu_2\frac{d\bar{F}_2}{dh}\right)_M = \frac{\partial F(M)}{\partial x} \quad . \tag{14}$$

For the right sides of (12) through (14) we have:

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} ,$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} ,$$

where

$$\frac{\partial u}{\partial x} = p \ ; \qquad \frac{\partial u}{\partial y} = q \ ; \qquad \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} = f \ ;$$

$$\frac{\partial p}{\partial x} = \lim_{h \to 0} \frac{\Delta^* p}{h} = \frac{dp^0(x)}{dx} + f \ ;$$

$$\frac{\partial q}{\partial y} = \lim_{h \to 0} \frac{q^0(x_1) + \Delta^* q - q^0(x_2)}{h} = -\frac{dq^0(x)}{dx} + f \ . \tag{15}$$

From (15) it follows that:

$$2\frac{\partial F(M)}{\partial x} - \frac{\partial F(M)}{\partial y} = \left[2\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}(2p^0 - q^0) + \frac{\partial f}{\partial p}\left(2\frac{dp^0}{dx} + f\right) + \frac{\partial f}{\partial q}\left(\frac{dq^0}{dx} + f\right)\right]_M . \tag{16}$$

For the calculation of the left sides of (12) through (14) we will find $(d/dh)_M$ for $F(N)$, $F\left(M_2^j\right)$, $F\left(N_2^j\right)$:

$$\frac{dF(N)}{dh} = \frac{dF(x_1 + h,\ y_1 - h)}{dh} = \frac{\partial F(N)}{\partial x} - \frac{\partial F(N)}{\partial y} ,$$

so that on the basis of (15) we have

$$\left[\frac{d}{dh}F(N)\right]_M = \left[\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}(p^0 - q^0) + \frac{\partial f}{\partial p}\frac{dp^0}{dx} + \frac{\partial f}{\partial q}\frac{dq^0}{dx}\right]_M . \tag{17}$$

Furthermore, from (9) it follows that:

$$\left(\frac{d\Delta^1 u}{dh}\right)_M = p^0(x_1) , \qquad \left(\frac{d\Delta^1 p}{dh}\right)_M = \frac{dp^0(x_1)}{dx} + f(M) , \qquad \left(\frac{d\Delta^1 q}{dh}\right)_M = f(M) , \tag{18}$$

and from comparison of $\delta^1 u$, $\delta^1 p$, $\delta^1 q$ with $\Delta^1 u$, $\Delta^1 p$, $\Delta^1 q$ we obtain:

$$\left(\frac{d\delta^1 u}{dh}\right)_M = q^0(x_1) \ , \quad \left(\frac{d\delta^1 p}{dh}\right)_M = f(M) \ , \quad \left(\frac{d\delta^1 q}{dh}\right)_M = -\frac{dq^0(x_1)}{dx} + F(M) \ . \tag{19}$$

From (6.a), by using (18), we have

$$\left[\frac{d}{dh} F\left(M_2^j\right)\right]_M = \left[\rho^j \frac{\partial f}{\partial x} + \sigma^j \frac{\partial f}{\partial y} + \alpha^j p^0 \frac{\partial f}{\partial u} + \beta^j \left(\frac{dp^0}{dx} + f\right) \frac{\partial f}{\partial p} + \gamma^j f \frac{\partial f}{\partial q}\right]_M . \tag{20}$$

Analogously, from (6.b), where $x_2 = x_1 + h$, $y_2 = y_1 - h$, by using (19), we have

$$\left[\frac{d}{dh} F\left(N_2^j\right)\right]_M = \left[\left(1 + \bar{\rho}^j\right) \frac{\partial f}{\partial x} + \left(\bar{\sigma}^j - 1\right) \frac{\partial f}{\partial y} + \left(p^0 - q^0 + \bar{\alpha}^j q^0\right) \frac{\partial f}{\partial u} + \left(\frac{dp^0}{dx} + \beta^j f\right) \frac{\partial f}{\partial p} + \frac{dq^0}{dx} - \gamma^j \frac{dq^0}{dx} + \bar{\gamma}^j f \Bigg) \frac{\partial f}{\partial q}\right]_M . \tag{21}$$

Let us return to the expression (12). The right side of this equality is given in (16).

By using (5.a), (7.a), (17), (20), and (21), its left side can be expressed in the following form:

$$3 \frac{\partial f}{\partial x} (\lambda_1 \tau_1 + \lambda_2 \tau_2 + \lambda_2 \omega_2 \rho + \lambda_2 \tau_2 \bar{\rho}) + 3 \frac{\partial f}{\partial y} (-\lambda_1 \tau_1 - \lambda_2 \tau_2 + \lambda_2 \omega_2 \sigma + \lambda_2 \tau_2 \bar{\sigma})$$
$$+ 3 \frac{\partial f}{\partial u} [(\lambda_1 \tau_1 + \lambda_2 \tau_2 + \lambda_2 \omega_2 \alpha) p^0 - (\lambda_1 \tau_1 + \lambda_2 \tau_2 - \lambda_2 \tau_2 \bar{\alpha}) q^0]$$
$$+ 3 \frac{\partial f}{\partial p} \left[(\lambda_1 \tau_1 + \lambda_2 \tau_2 + \lambda_2 \omega_2 \beta) \frac{dp^0}{dx} + (\lambda_2 \omega_2 \beta + \lambda_2 \tau_2 \bar{\beta}) f\right]$$
$$+ 3 \frac{\partial f}{\partial q} \left[(\lambda_1 \tau_1 + \lambda_2 \tau_2 - \lambda_2 \tau_2 \bar{\gamma}) \frac{dq^0}{dx} + (\lambda_2 \omega_2 \gamma + \lambda_2 \tau_2 \bar{\gamma}) f\right] . \tag{22}$$

Here all values are given at the point M and, for simplicity, all indices j = u are omitted. Assume:

$$\bar{\alpha} = \alpha \ , \quad \bar{\beta} = \beta \ , \quad \bar{\gamma} = \gamma \ , \quad \bar{\rho} = \rho - 1 \ , \quad \bar{\sigma} = \sigma + 1 \ . \tag{23}$$

(Such a choice of $\bar{\sigma}$, $\bar{\rho}$ is dictated by the form of the region MNP; with this choice the corresponding projections of points M_2^u and N_2^u in the plane xy coincide.)

Taking into account (11), (23) and equating (22) to the right side of (16) we obtain the following system of equations for the parameters at Δu:

$$\lambda_1\tau_1 + \lambda_2\rho = \frac{2}{3} \ , \qquad \lambda_1\tau_1 - \lambda_2\sigma = \frac{1}{3} \ ,$$

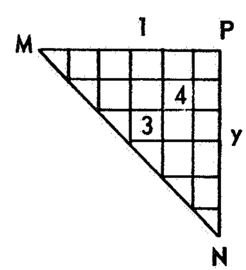
$$\lambda_1\tau_1 + \lambda_2\tau_2 + \lambda_2\omega_2\alpha = \lambda_1\tau_1 + \lambda_2\tau_2 + \lambda_2\omega_2\beta = \frac{2}{3} \ ,$$

$$\lambda_1\tau_1 + \lambda_2\tau_2 - \lambda_2\tau_2\alpha = \lambda_1\tau_1 + \lambda_2\tau_2 - \lambda_2\tau_2\gamma = \frac{1}{3} \ ,$$

$$\lambda_2\beta = \lambda_2\gamma = \frac{1}{3} \ . \tag{24}$$

For example, the parameters from Table I will constitute solutions of the system (24).

TABLE I

	ω_1	τ_1	ω_2	τ_2	λ_1	λ_2	ρ	σ	$\bar{\rho}$	$\bar{\sigma}$	$\alpha, \bar{\alpha}$	$\beta, \bar{\beta}$	$\gamma, \bar{\gamma}$	Location of Projections of M_2^u and N_2^u in the Plane xy
1)	0	1	1	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	0	-	-	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	
2)	1	0	0	1	$\frac{1}{3}$	$\frac{2}{3}$	-	-	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	
3)	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{2}{3}$	$-\left(\frac{1}{3}\right)$	$-\left(\frac{1}{3}\right)$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	
4)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{6}$	$-\left(\frac{1}{6}\right)$	$-\left(\frac{1}{6}\right)$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	

Analogously, at Δp and Δq we obtain for the parameters:

$$\mu_2\bar{\rho}^p = 0 , \quad \mu_2\bar{\sigma}^p = \frac{1}{2} , \quad \mu_2\bar{\alpha}^p = \mu_2\bar{\beta}^p = \mu_2\bar{\gamma}^p = \frac{1}{2} , \tag{25}$$

$$\nu_2\rho^q = \frac{1}{2} , \quad \sigma^q = 0 , \quad \nu_2\alpha^q = \nu_2\beta^q = \nu_2\gamma^q = \frac{1}{2} . \tag{26}$$

Equations (25) and (26) can be satisfied upon selection of parameters, for example, from variants in Table II.

TABLE II

μ_1, ν_1	μ_2, ν_2	$\bar{\rho}^p, \sigma^q$	$\bar{\sigma}^p, \rho^q$	$\bar{\alpha}^p, \alpha^q$	$\bar{\beta}^p, \beta^q$	$\bar{\gamma}^p, \gamma^q$
0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	1

The factual scheme of calculations according to proposed algorithms has the form:

$$\left.\begin{aligned} u(P) &= u^0(x_1) + \lambda_1\Delta^1 u + \lambda_2\Delta^2 u , \\ p(P) &= p^0(x_2) + \mu_1\delta^1 p + \mu_2\delta^2 p , \\ q(P) &= q^0(x_1) + \nu_1\Delta^1 q + \nu_2\Delta^2 q , \end{aligned}\right\} . \tag{27}$$

where

$$\left.\begin{aligned} \Delta^1 u &= I(h) + \frac{1}{2} h^2 F_1 , \\ \delta^1 p &= hF_{\hat{1}} , \\ \Delta^1 q &= hF_{\bar{1}} ; \end{aligned}\right\} \tag{28.a}$$

$$\left.\begin{aligned} \delta^1 u &= \Delta^1 u + u^0(x_1) - u^0(x_2) , \\ \Delta^1 p &= \delta^1 p + p^0(x_2) - p^0(x_1) , \\ \delta^1 q &= \Delta^1 q + q^0(x_1) - q^0(x_2) ; \end{aligned}\right\} \tag{28.b}$$

$$\left.\begin{aligned} \Delta^2 u &= I(h) + \frac{1}{2} h^2 F_2 , \\ \delta^2 p &= h\hat{F_2} , \\ \Delta^2 q &= h\bar{F_2} . \end{aligned}\right\} \qquad (28.c)$$

If we adopt the first variants of Tables I and II, then at each step we will have to calculate only four values of f: $f(M)$, $f(N)$, $f\left(M_2^u\right) = f\left(M_2^q\right)$, $f\left(N_2^p\right)$.

4. Investigation of the Stability of the Method

We find $\int_{x_1}^{x_1+h} p^0(x)dx = I(h)$ according to Simpson's formula

$$I(h) \approx \frac{1}{6} h[p^0(M) + 4p^0(Q) + p^0(N)] ,$$

where Q is the midpoint of the segment MN. We calculate the unknown $p^0(Q)$ with the aid of Bessel's interpolation polynomial or, if need be, with Newton's polynomial. Whereupon, if the values of $p^0(x)$ in the nodes of the given layer are found with an error η, and if we neglect the remainder of order $O(h^5)$ in Simpson's formula, then the irremovable error in the calculation of $I(h)$ will be approximately $h\eta$.

Relationships (27) on the k^{th} layer will have the following form:

$$u_k = u_{k-1} + \lambda_1\Delta^1 u_{k-1} + \lambda_2\Delta^2 u_{k-1} , \qquad (29.a)$$

$$p_k = p_{k-1} + \mu_1\delta^1 p_{k-1} + \mu_2\delta^2 p_{k-1} , \qquad (29.b)$$

$$q_k = q_{k-1} + \nu_1\Delta^1 q_{k-1} + \nu_2\Delta^2 q_{k-1} , \qquad (29.c)$$

where the indices k relate to the point P of the triangle MNP, indices k - 1 in (29.a), (29.c) relate to the point M, and in (29.b) relate to the point N.

Along with (29) we write:

$$u_k^* = u_{k-1}^* + \lambda_1\Delta^1 u_{k-1}^* + \lambda_2\Delta^2 u_{k-1}^* + r_1 , \qquad (30.a)$$

$$p_k^* = p_{k-1}^* + \mu_1\delta^1 p_{k-1}^* + \mu_2\delta^2 p_{k-1}^* + r_2 , \tag{30.b}$$

$$q_k^* = q_{k-1}^* + \nu_1\Delta^1 q_{k-1}^* + \nu_2\Delta^2 q_{k-1}^* + r_3 . \tag{30.c}$$

Here, $\Delta^1 u_{k-1}^*$, $\Delta^2 u_{k-1}^*$, $\delta^1 p_{k-1}^*$, ..., $\Delta^2 q_{k-1}^*$ are elementary increments calculated with respect to (28.a) through (28.c), in which the approximate values u, p, q are replaced by the exact values u^*, p^*, q^*; r_1, r_2, r_3 are the local errors of the method which in our case are of orders $O(h^4)$, $O(h^3)$, $O(h^3)$ [equation (10)].

We denote the errors as follows:

$$|u_k^* - u_k| = \epsilon_k , \quad |p_k^* - p_k| = \eta_k, \quad |q_k^* - \mu_k| = \Theta_k . \tag{31}$$

Subtracting (29) from (30) and using designations (31) we obtain:

$$\left.\begin{aligned}
\epsilon_k &\le \epsilon_{k-1} + \lambda_1|\Delta^1 u_{k-1}^* - \Delta^1 u_{k-1}| + \lambda_2|\Delta^2 u_{k-1}^* - \Delta^2 u_{k-1}| + r_1 ,\\
\eta_k &\le \eta_{k-1} + \mu_1|\delta^1 p_{k-1}^* - \delta^1 p_{k-1}| + \mu_2|\delta^2 p_{k-1}^* - \delta^2 p_{k-1}| + r_2 ,\\
\Theta_k &\le \Theta_{k-1} + \nu_1|\Delta^1 q_{k-1}^* - \Delta^1 q_{k-1}| + \nu_2|\Delta^2 q_{k-1}^* - \Delta^2 q_{k-1}| + r_3 ;
\end{aligned}\right\} \tag{32}$$

Let $f(x,y,u,p,q)$ satisfy a Lipschitz condition with respect to u, p, q, with constant K, and let

$$U = K\left(\epsilon_{k-1} + \eta_{k-1} + \Theta_{k-1}\right) .$$

On the strength of (28.a) we have

$$\left.\begin{aligned}
&|\Delta^1 u_{k-1}^* - \Delta^1 u_{k-1}| \le h\eta_{k-1} + \frac{1}{2} h^2 U ,\\
&|\delta^1 p_{k-1}^* - \delta^1 p_{k-1}| \le hU , \quad |\Delta^1 q_{k-1}^* - \Delta^1 q_{k-1}| \le hU .
\end{aligned}\right\} \tag{33}$$

Furthermore, from (33) and (28.b) it follows that:

$$|\delta^1 u^*_{k-1} - \delta^1 u_{k-1}| \le 2\epsilon_{k-1} + h\eta_{k-1} + \frac{1}{2} h^2 U ,$$

$$|\Delta^1 p^*_{k-1} - \Delta^1 p_{k-1}| \le 2\eta_{k-1} + hU ,$$

$$|\delta^1 q^*_{k-1} - \delta^1 q_{k-1}| \le 2\Theta_{k-1} + hU .$$

Taking into account (6.a) and (6.b) we have:

$$|f\left(M_2^{j*}\right) - f\left(M_2^{j}\right)| \le V = K \left[\left(\epsilon_{k-1} + h\eta_{k-1} + \frac{1}{2} h^2 U\right) + \left(3\eta_{k-1} + hU\right) + \left(\Theta_{k-1} + hU\right)\right] ;$$

$$|f\left(N_2^{j*}\right) - f\left(N_2^{j}\right)| \le W = K \left[\left(3\epsilon_{k-1} + h\eta_{k-1} + \frac{1}{2} h^2 U\right) + \left(\eta_{k-1} + hU\right) + \left(3\Theta_{k-1} + hU\right)\right] .$$

From the preceding and from the equality (28.c) we obtain:

$$\left.\begin{aligned} &|\Delta^2 u^*_{k-1} - \Delta^2 u_{k-1}| \le h\eta_{k-1} + \frac{1}{2} h^2 (V + W) , \\ &|\delta^2 p^*_{k-1} - \delta^2 p_{k-1}| \le hW , \quad |\Delta^2 q^*_{k-1} - \Delta^2 q_{k-1}| \le hV . \end{aligned}\right\} \tag{34}$$

Using (33) and (34) we write (32) in the form:

$$\begin{aligned} \epsilon_k &\le \epsilon_{k-1} + h\eta_{k-1} + \frac{1}{2} h^2 [\lambda_1 U + \lambda_2 (V + W)] + r_1 , \\ \eta_k &\le \eta_{k-1} + h(\mu_1 U + \mu_2 W) + r_2 , \\ \Theta_k &\le \Theta_{k-1} + h(\nu_1 U + \nu_2 V) + r_3 . \end{aligned} \tag{35}$$

We denote vectors with components ϵ_k, η_k, Θ_k and r_1, r_2, r_3 correspondingly by X_k and r ($r \sim \{O(h^4), O(h^3), O(h^3)\}$), and we assume that they are accurately given on the zero layer of u^0, p^0, q^0. On the basis of (35) we have

$$X_k \leq AX_{k-1} + r \ , \quad k = 1, 2, \ldots, \left[\frac{AC}{h}\right] .$$

Diagonal elements of matrices of A are

$$1 + c_1 h^2 + \ldots, \quad 1 + c_2 h + \ldots, \quad 1 + c_3 h + \ldots,$$

where c_1, c_2, c_3 are constants not depending on h; the dots designate certain addends of higher powers in h; non-diagonal elements of A are values of the order not higher than O(h).

Thus, at usual norms of vectors and matrices the distribution coefficient of errors at each step does not exceed 1 + O(h); hence ensues the stability of the method in the sense of [4].

5. Some Generalizations of the Method

Let there be a system of n equations

$$u_{xy} = f(x, y, u, p, q) \tag{36}$$

with initial conditions:

$$u^0 = u^0(x) \ , \qquad p^0 = p^0(x) \ , \qquad q^0 = q^0(x) \ , \tag{37}$$

given along the segment AB of the straight line x + y = const, where u is n-dimensional vector function of x, y; $p = u_x$; $q = u_y$. All that was stated in Paragraphs 2 and 3 remains valid for the problem (36) and (37), with the difference that all functions and increments in algorithms (27), (28) are vectorial-values depending on the scalar step h; λ_i, μ_i, ν_i, $\alpha^i, \beta^i, \gamma^i, \bar{\alpha}^i, \bar{\beta}^i, \bar{\gamma}^i, \rho^j$, $\bar{\rho}^j, \sigma^j, \bar{\sigma}^j$ are scalar coefficients. The proof is analogous to that carried out in Paragraphs 2 and 3; there remain in force conditions (11), (24 through (26) and the results of Tables I and II.

Let us apply the stated method to the solution of the Cauchy problem for nonlinear equations of the hyperbolic type with two independent variables by reducing such equations to the system (36) with initial conditions (37). Let us consider, for example, the Cauchy problem for the Monge-Ampere equation

$$ar + 2bs + ct + g(rt - s^2) + f = 0 \ , \qquad (g \neq 0) \ , \tag{38}$$

where $r = u_{xx}$; $s = u_{xy}$; $t = u_{yy}$; u is an unknown function of x, y; a, b, c, g, f are given functions of x, y, u, p, q, while $p = u_x$ and $q = u_y$. Assume that initial conditions are given along the segment A'B' of a noncharacteristic curve Γ in the form of $u^0 = u^0(x)$, $p^0 = p^0(x)$, $q^0 = q^0(x)$.

It is known [5] that in the hyperbolic case the equation (38) is equivalent to the system

$$\left.\begin{aligned} & c\frac{\partial x}{\partial \alpha} + \varphi_1\frac{\partial y}{\partial \alpha} + g\frac{\partial p}{\partial \alpha} = 0 \ , \\ & \varphi_2\frac{\partial x}{\partial \alpha} + \alpha\frac{\partial y}{\partial \alpha} + g\frac{\partial q}{\partial \alpha} = 0 \ , \\ & p\frac{\partial x}{\partial \alpha} + q\frac{\partial y}{\partial \alpha} - \frac{\partial u}{\partial \alpha} = 0 \ , \\ & c\frac{\partial x}{\partial \beta} + \varphi_2\frac{\partial y}{\partial \beta} + g\frac{\partial p}{\partial \beta} = 0 \ , \\ & \varphi_1\frac{\partial x}{\partial \beta} + \alpha\frac{\partial y}{\partial \beta} + g\frac{\partial q}{\partial \beta} = 0 \ , \end{aligned}\right\} \tag{39}$$

where α, β are parameters of characteristic families; φ_1, φ_2 are roots of the characteristic equation $\varphi^2 - 2b\varphi + ac - gf = 0$, whereas φ_1 corresponds to a family α, and φ_2 to a family β.

We will differentiate the first three equations of the system (39) in accordance with Levi and Friedrichs, with respect to β, and the remainder of them — with respect to α, and we will solve the system thus obtained in terms of the second derivatives. As a result we will obtain a system composed of five equations:

$$x_{\alpha\beta} = f_1 \ , \qquad y_{\alpha\beta} = f_2 \ , \qquad u_{\alpha\beta} = f_3 \ , \qquad p_{\alpha\beta} = f_4 \ , \qquad q_{\alpha\beta} = f_5 \ , \tag{40}$$

where f_j ($j = 1 \div 5$) are known functions of x, y, u, p, q, x_α, y_α, u_α, p_α, q_α, x_β, y_β, u_β, p_β, q_β.

We will select the transformation $(x, y) \to (\alpha, \beta)$ so that the segment A'B' of the initial curve is transformed into the segment AB of the straight line $\alpha + \beta = \text{const}$. Along AB we know x^0, y^0, u^0, p^0, q^0 and we can uniquely determine x^0_α, y^0_α, u^0_α, p^0_α, q^0_α, x^0_β, y^0_β, ..., q^0_β. (Details of the stated facts have been given previously [6, 7]). Thus, the Cauchy problem for (38) was reduced to the problem (36), (37); hence follows its solvability by means of the proposed algorithms.

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13. ABSTRACT

Discussed is the application of two-iteration Runge-Kutta algorithms in the solution of the Cauchy problem for hyperbolic equations and systems with two independent variables.

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