A REPRESENTATION FORMULA FOR THE SOLUTIONS OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATION*

Submitted

by

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Let p and q be continuous functions on a bounded closed interval I containing x_o. It is the purpose of this paper to give a simple existence and uniqueness proof for the initial value problem

$$(1) \qquad y'' - py' - qy = 0, \qquad y(x_o) = y_o, \qquad y'(x_o) = y_o',$$

y_o and y_o' arbitrary numbers. A product of the proof is a representation formula for the solution of (1). That the representation formula is indeed a solution under the hypothesis stated can be verified by direct substitution into (1).

A function ϕ on I is a solution of (1) only if ϕ' and ϕ'' exist on I and

$$\phi''(x) - p(x)\phi'(x) - q(x)\phi(x) = 0, \qquad x \in I.$$

From this then,

$$D_x\phi'(x)\exp(-\int_{x_o}^{x} p(v)dv) = q(x)\phi(x)\exp(-\int_{x_o}^{x} p(v)dv),$$

so that, since ϕ is continuous on I,

$$\phi'(x)\exp(-\int_{x_o}^{x} p(v)dv) = y_o' + \int_{x_o}^{x} q(s)\phi(s)\exp(-\int_{x_o}^{s} p(v)dv)ds.$$

Thus, upon setting

$$E(x) = \exp \int_{x_0}^{x} p(v)\,dv$$

and $E^{-1}(x) = 1/E(x)$, one has

$$\phi'(x) = E(x)[y_0' + \int_{x_0}^{x} E^{-1}(s)q(s)\phi(s)ds], \tag{2}$$

which implies

$$\phi(x) = y_0 + y_0'\int_{x_0}^{x} E(t)dt + \int_{x_0}^{x}\int_{x_0}^{t} E(t)E^{-1}(s)q(s)\phi(s)dsdt \tag{3}$$

for $x \in I$. Conversely now, suppose ϕ is continuous on I with (3) holding for $x \in I$. Then it follows from (3) that ϕ' exists and in fact that ϕ' is given on I by (2). Whence, from (2), ϕ'' exists and, by simple inspection, $\phi'' = p\phi' + q\phi$; moreover, from (3), $\phi(x_0) = y_0$, and from (2), $\phi'(x_0) = y_0'$. Thus, ϕ is a solution of (1), completing the proof of the following.

<u>Theorem 1</u>. A continuous function ϕ on I is a solution of (1) if and only if (3) holds for $x \in I$.

Thus the problem of solving (1) is equivalently replaced by the problem of finding a function ϕ continuous on I for which (3) holds for $x \in I$. The form of (3), an integral equation, is considerably simplified with the introduction of the operator S defined for continuous functions f on I by

$$Sf(x) = \int_{x_0}^{x}\int_{x_0}^{t} E(t)E^{-1}(s)q(s)f(s)dsdt, \qquad x \in I.$$

Indeed, (3) is a special case of $\emptyset(x) = f(x) + S\emptyset(x)$, $x \in I$, or equivalently,

(4) $$\emptyset = f + S\emptyset,$$

where f is restricted to the set of continuous functions on I. The operator S, and therefore the corresponding integral in (3), plays a singularly important role in generating a function $\emptyset$ satisfying (4), hence in solving (1).

Note that Sf is a continuous function on I if f is, in which case so is SSf. With this in mind, operators S^n, iterants of S, are defined inductively for continuous functions f on I in the following way. Set

$$S^o f = f, \qquad S^1 f = Sf,$$

and then

$$S^n f = SS^{n-1} f, \qquad n = 1, 2, \ldots \quad .$$

Pertinent properties of S^n are given below in the form of three lemmas. As their proofs are straightforward, they are omitted.

Lemma 1. S^n is a linear operator. That is, if $f_1, \ldots, f_k$ are continuous functions on I, then

$$S^n(c_1 f_1 + \ldots + c_k f_k) = c_1 S^n f_1 + \ldots + c_k S^n f_k$$

for any set $c_1, \ldots, c_k$ of numbers.

Lemma 2. Let f be continuous on I, hence bounded on I by say M. Then $S^n f$ is continuous on I; moreover, there exists K such that $|S^n f(x)| \leq MK^n/n!$ for $n = 0, 1, 2, \ldots$ and $x \in I$.

Lemma 3. If f is continuous on I, then the series $\sum_{n=0}^{\infty} S^n f$ converges uniformly to a continuous function on I.

Having disposed of the preliminaries, the main theorem concerning (4) follows.

Theorem 2. Let f be continuous on I. Then there exists one and only one continuous function ϕ on I such that $\phi = f + S\phi$; indeed, $\phi = \sum_{n=0}^{\infty} S^n f$.

Proof. Define

$$\phi_0 = f$$

and then

$$\phi_n = f + S\phi_{n-1}, \qquad n = 1, 2, \ldots .$$

In view of lemma 1,

$$\phi_n = \sum_{k=0}^{n} S^k f.$$

But the sequence $\{\phi_n\}$ is the sequence of partial sums defining the series $\sum_{n=0}^{\infty} S^n f$, which by lemma 3 converges uniformly to a continuous function ϕ on I. That

$$\phi = \sum_{n=0}^{\infty} S^n f$$

satisfies (4) follows from

$$f + S\phi = f + S \sum_{n=0}^{\infty} S^n f$$

$$= f + \sum_{n=0}^{\infty} S^{n+1} f$$

$$= S^o f + \sum_{n=1}^{\infty} S^n f \quad = \quad \phi,$$

the second equality holding because of the uniform convergence on I of $\sum S^n f$. To show that it is the only such continuous function, suppose there are two, say ϕ and Ψ. Then

$$\phi - \Psi = S(\phi - \Psi).$$

But this implies by induction that

$$\phi - \Psi = S^n(\phi - \Psi), \qquad n = 1, 2, \ldots$$

This in turn implies by lemma 2 that, for some M and K,

$$|\phi(x) - \Psi(x)| \leq \frac{MK^n}{n!}, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad x \in I.$$

Hence, for $x \in I$,

$$|\phi(x) - \Psi(x)| \leq \lim_{n\to\infty} \frac{MK^n}{n!} = 0,$$

implying $\phi = \Psi$.

Corollary. A function ϕ is a solution of (1) if and only if

$$\phi(x) = \sum_{n=0}^{\infty} S^n\left(y_0 + y_0' \int_{x_0}^{x} E(r)dr\right), \quad x \in I. \tag{5}$$

A pair of linearly independent solutions of (1) obtained from (5) is the pair

$$y_1(x) = 1 + \int_{x_0}^{x}\int_{x_0}^{t_1} E(t_1)E^{-1}(s_1)q(s_1)ds_1dt_1$$

$$+ \int_{x_0}^{x}\int_{x_0}^{t_2} E(t_2)E^{-1}(s_2)q(s_2)\int_{x_0}^{s_2}\int_{x_0}^{t_1} E(t_1)E^{-1}(s_1)q(s_1)ds_1dt_1ds_2dt_2$$

$$+ \ldots,$$

$$y_2(x) = \int_{x_0}^{x} E(r)dr + \int_{x_0}^{x}\int_{x_0}^{t_1} E(t_1)E^{-1}(s_1)q(s_1)\int_{x_0}^{s_1} E(r)drds_1dt_1$$

$$+ \int_{x_0}^{x}\int_{x_0}^{t_2} E(t_2)E^{-1}(s_2)q(s_2)\int_{x_0}^{s_2}\int_{x_0}^{t_1} E(t_1)E^{-1}(s_1)q(s_1)\int_{x_0}^{s_1} E(r)drds_1dt_1ds_2dt_2$$

$$+ \ldots$$

It is not at all difficult to compute ϕ' directly from (5), formally or by reference to convergence theorems, to obtain (2) and then to conclude as in the last part of the proof of theorem 1 that ϕ as given by (5) is a solution of (1), by direct verification.

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