AN INTEGRAL TRANSFORM WITH A WHITTAKER FUNCTION KERNEL

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Microfiche (MF) .65

ff 653 July 65

INTERIM REPORT
30 August 1968

Contract No. NAS9-7641

MRI Project No. 3162-P

For

NASA Manned Spacecraft Center
General Research Procurement Branch
Houston, Texas 77058

Attn: J. W. Carlson/BG731(48)

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NASA CR 92264

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by

Jet Wimp

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PREFACE

This report, written by Jet Wimp, Mathematics Branch, Midwest Research Institute, covers work performed from 20 June through 19 August 1968 on Contract No. NAS 9-7641.

Approved for:

MIDWEST RESEARCH INSTITUTE

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Sheldon L. Levy, Director
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10 September 1968

bc

I. INTRODUCTION

We investigate here an integral transform whose kernel contains a Whittaker function [1]. It is interesting that in both the transform and its inversion formula, integration is performed with respect to a parameter of the function, and thus our result is altogether different in form from previous transform pairs such as

$$\left.\begin{aligned} g(x) &= \frac{\Gamma(\mu+\frac{1}{2}+ix)\Gamma(\mu+\frac{1}{2}-ix)}{2\pi\Gamma(2\mu+1)^2}\, e^{-\frac{\pi i}{2}(2\mu+1)-x\pi} \int_0^\infty t^{-1} M_{ix,\mu}(it)f(t)dt \ , \\ f(t) &= \int_{-\infty}^{\infty} M_{ix,\mu}(it)g(x)dx \ , \end{aligned}\right\} \quad (1)$$

(Erdelyi, [2]);

$$\left.\begin{aligned} g(x) &= \Gamma(\tfrac{1}{2}-k-ix)\Gamma(\tfrac{1}{2}-k+ix) \int_0^\infty W_{k,ix}(t)f(t)dt \ , \\ f(t) &= \frac{1}{(t\pi)^2} \int_0^\infty x \sinh(2\pi x)W_{k,ix}(t)g(x)dx \ , \end{aligned}\right\} \quad (2)$$

(Wimp, [3]);

$$\left.\begin{aligned} g(x) &= \int_0^\infty e^{-\frac{xt}{2}} (xt)^{-k} W_{k,\mu}(xt)f(t)dt \ , \\ f(t) &= \frac{\Gamma\left(\frac{3}{2}-k+\mu\right)}{2\pi i\Gamma(1+2\mu)} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\frac{xt}{2}} (xt)^{k-1} M_{k-1,\mu}(xt)g(x)dx \ , \end{aligned}\right\} \quad (3)$$

(Meijer, [4]).

There is a similarity between our transform pair and several integrals due to Buchholz [5] which express various types of waves as integrals of the functions of the paraboloid of revolution, see [5,6] and the references given there.*

II. MAIN RESULT AND EXAMPLES

Theorem

Let i) $\left|\arg a\right| < \pi$, $a \neq 0$;

ii) $2\nu+1 \neq 0,-1,-2,\ldots$, $\mathrm{Re}(2\nu+1) < 0$;

iii) $g(x)\epsilon L^p(-\infty,\infty)$, $p = 1,2$.

If

$$g(x) = \int_{-\infty}^{\infty} \Gamma\left(\tfrac{1}{2}+\nu+i(x-t)\right)\Gamma\left(\tfrac{1}{2}+\nu-i(x-t)\right)M_{i(t-x),\nu}(a)f(t)dt \tag{4}$$

then $f(t)\epsilon L^2(-\infty,\infty)$ and almost everywhere

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* There are a number of errors in Chapter 15 of Ref. [6]. Not all of them are easily corrected. Formula (2) lacks the factor

$\cos^2 \frac{\phi}{2}\left(\tan \frac{\phi}{2}\right)^{-\alpha} e^{\frac{i\pi}{2}(\mu_1-\mu_2)}$ and the exponents of $(x-\xi')$, $(y-\eta')$ in the integrand should be diminished by 1. These errors make formula (3) wrong. Also formulae (5) and (6) of this chapter cannot hold since they violate the uniqueness property of the Mellin transform.

$$f(t) = \frac{(\nu+\frac{1}{2})\sin(2\nu\pi)}{2\pi^3} \int_{-\infty}^{\infty} \Gamma\left(-\nu-\tfrac{1}{2}+i(x-t)\right)\Gamma\left(-\nu-\tfrac{1}{2}-i(x-t)\right)$$

$$\times M_{i(t-x),-1-\nu}(a)\ g(x)dx \ . \qquad (5)$$

Proof: By Fourier transforms. We use the notation of Titchmarsh [7], throughout, e.g.,

$$G(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixu} g(x)dx \ . \qquad (6)$$

By [3, p. 75], $G(u)$ exists and $\epsilon\ L^2(-\infty,\infty)$. Now define

$$k(t) = M_{-it,\nu}(a)\Gamma\left(\frac{1}{2}+\nu+it\right)\Gamma\left(\frac{1}{2}+\nu-it\right) \qquad (7)$$

which by i), ii) is analytic in a and ν . By [8],* we have

$$K(u) = \sqrt{\pi}\,2^{-\frac{1}{2}-2\nu}\ \Gamma(1+2\nu)a^{\frac{1}{2}+\nu}\left(\cosh\frac{u}{2}\right)^{-1-2\nu} \exp\left\{-\frac{a}{2}\tanh\frac{u}{2}\right\} . \qquad (8)$$

Since $\nu + \frac{1}{2} < 0$, $G(u)/K(u)\ \epsilon\ L^2(-\infty,\infty)$.

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* The right hand side of the sixth transform pair on page 205 of this reference should read

$$(ac)^{-1}\frac{\Gamma\left(\frac{1+\nu+iya^{-1}}{2}\right)\Gamma\left(\frac{1+\nu-iya^{-1}}{2}\right)}{\Gamma(\nu+1)^2} M_{\frac{iy}{2a},\frac{\nu}{2}}\left[(b+\sqrt{b^2-c^2})^{\frac{1}{2}}\right] M_{\frac{iy}{2a},\frac{\nu}{2}}\left[(b-\sqrt{b^2-c^2})^{\frac{1}{2}}\right]$$

and the right-hand side of the last formula on page 206 requires the factor $\Gamma(2\nu+1)^2$.

The results [9] show that

$$\left|k(t)\right| = 0\left\{t^{c}\ e^{-\pi|t|+2|at|^{\frac{1}{2}}}\right\},\quad |t|\longrightarrow\infty\ , \tag{9}$$

so $k(t)\epsilon L(-\infty,\infty)$. We may apply the theorem [3, p. 315] to find, almost everywhere,

$$f(t) = \frac{\pi^{-\frac{3}{2}}\,2^{2\nu-\frac{1}{2}}}{\Gamma(1+2\nu)a^{\nu+\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-itu}\left(\cosh\frac{u}{2}\right)^{1+2\nu}\exp\left\{\frac{a}{2}\tanh\frac{u}{2}\right\}G(u)du\ . \tag{10}$$

Absolute convergence allows us to substitute (6) in (10) and interchange the order of integration. The inner integral is easily evaluated from (7)-(8) and the theorem results.

If $g(x)$ is replaced by $g^*(-x)$ and f by f^* , we get the form

$$g^*(x) = \int_{-\infty}^{\infty}\Gamma\left(\nu+\tfrac{1}{2}+i(x+t)\right)\Gamma\left(\nu+\tfrac{1}{2}-i(x+t)\right)M_{i(x+t),\nu}(a)f^*(t)dt\ , \tag{11}$$

$$f^*(t) = \frac{(\nu+\frac{1}{2})\sin(2\nu\pi)}{2\pi^3}\int_{-\infty}^{\infty}\Gamma\left(-\nu-\tfrac{1}{2}+i(x+t)\right)\Gamma\left(-\nu-\tfrac{1}{2}-i(x+t)\right)$$

$$\times M_{i(x+t),-1-\nu}(a)g^*(x)dx\ . \tag{12}$$

In the accompanying table, we give some examples of (11)-12). (The notation for all special functions is that of the Bateman volumes, [1].)

$f^*(t)$	$g^*(x)$
e^{-iyt}	$\pi 2^{-2\nu} a^{\nu+\frac{1}{2}} \Gamma(1+2\nu) e^{iyx} \left[\cosh \frac{y}{2}\right]^{-1-2\nu} \exp\left\{-\frac{a}{2}\tanh\frac{y}{2}\right\}$
$\Gamma(\frac{1}{2}+\sigma+\xi-it)\Gamma(\frac{1}{2}+\sigma-\xi+it)$ $\times M_{-it+\xi,\sigma}(b)$	$\frac{2\pi\Gamma(1+2\nu)\Gamma(1+2\sigma) a^{\nu+\frac{1}{2}} b^{\sigma+\frac{1}{2}}}{\Gamma(2+2\nu+2\sigma)(a+b)^{1+\nu+\sigma}} \Gamma(1+\nu+\sigma-\xi-ix)\Gamma(1+\nu+\sigma+\xi+ix)$ $\times M_{ix+\xi,\frac{1}{2}+\nu+\sigma}(a+b)$
$\Gamma(\frac{1}{2}+\sigma+it)\Gamma(\frac{1}{2}+\sigma-it)$ $\times M_{-it,\sigma}(b)\ M_{it,\sigma}(a+b)$	$\frac{\pi 2^{1-2\sigma}\left[b(a+b)\right]^{\frac{1}{2}+\sigma}}{\Gamma(2+2\nu+2\sigma)} a^{\nu+\frac{1}{2}} \Gamma(1+2\sigma)\Gamma(1+2\nu)\Gamma(1+\nu+\sigma-ix)\Gamma(1+\nu+\sigma+ix)$ $\times {}_2F_3\left(\begin{matrix}1+\nu+\sigma-ix, 1+\nu+\sigma+ix\\ 1+2\sigma, 1+\nu+\sigma, \frac{3}{2}+\nu+\sigma\end{matrix}\middle\vert \frac{b(a+b)}{16}\right)$
$\csc\pi(\lambda+it)$	$\frac{a^{-\frac{1}{2}}\Gamma\left(\frac{3}{2}+\nu-\lambda+ix\right)}{(\frac{1}{2}+\nu)} \Gamma(\frac{1}{2}+\nu+\lambda-ix)\ M_{\frac{1}{2}-\lambda+ix,\frac{1}{2}+\nu}(a)$
$\Gamma(\alpha-it)\Gamma(\beta+it)$	$\frac{2\pi\Gamma(1+2\nu)\Gamma(\alpha+\beta)}{\Gamma(1+2\nu+\alpha+\beta)} a^{-(\alpha+\beta)/2} \Gamma(\frac{1}{2}+\nu+\beta-ix)\ \Gamma(\frac{1}{2}+\nu+\alpha+ix)$ $\times M_{ix+\frac{\alpha-\beta}{2},\nu+\frac{\alpha+\beta}{2}}(a)$
$(1-b)^{it}\Gamma(\alpha+it)\Gamma(\delta-\alpha-it)$ $\times {}_2F_1\left(\begin{matrix}\alpha+it,\sigma\\ \delta\end{matrix}\middle\vert b\right)$	$\frac{2\pi\Gamma(\delta)\Gamma(1+2\nu) a^{\nu+\frac{1}{2}} e^{-\frac{a}{2}}}{\Gamma(1+2\nu+\delta)(1-b)^{\alpha}} \Gamma(\frac{1}{2}+\nu+\alpha-ix)\ \Gamma(\frac{1}{2}+\nu+\delta-\alpha+ix)$ $\times \Phi_1(\frac{1}{2}+\nu+\alpha-ix, \delta-\sigma, 1+2\nu+\delta, \frac{b}{b-1}, a)$
$\Gamma(\alpha+it)\Gamma(\gamma-\alpha-it)$ $\times \Phi_1(\alpha+it,\beta,\gamma;b,c)$	$\frac{2\pi\Gamma(1+2\nu) a^{\nu+\frac{1}{2}} e^{-\frac{a}{2}}}{\Gamma(1+2\nu+\gamma)} \Gamma(\gamma)\Gamma(\frac{1}{2}+\nu+\alpha-ix)\ \Gamma(\frac{1}{2}+\nu+\gamma-\alpha+ix)$ $\times \Phi_1(\frac{1}{2}+\nu+\alpha-ix,\beta,1+2\nu+\gamma;b,a+c)$
$\Gamma(\frac{1}{2}+\eta-\zeta+it) b^{it/2}$ $\times W_{\zeta-\frac{it}{2},\eta+\frac{it}{2}}(b)$	$2\pi\Gamma(1+2\nu) a^{\frac{1}{2}+\nu} b^{\eta+\frac{1}{2}} e^{-\frac{b}{2}}$ $\times\left\{\frac{\Gamma\left(\frac{1}{2}+\nu-2\eta+ix\right)\Gamma(1+\nu+\eta-\zeta-ix)}{\Gamma\left(\frac{3}{2}+2\nu-\eta-\zeta\right)} \Psi_2\left(1+\nu+\eta-\zeta-ix, \frac{1}{2}-\nu+2\eta-ix,\right.\right.$ $\left.\frac{3}{2}+2\nu-\eta-\zeta;b,a\right) + b^{\nu+\frac{1}{2}-2\eta+ix}\ \Gamma(-\frac{1}{2}-\nu+2\eta-ix)$ $\times \Psi_2\left(\frac{3}{2}+2\nu-\eta-\zeta, \frac{3}{2}+\nu-2\eta+ix, \frac{3}{2}+2\nu-\eta-\zeta;b,a\right)$
$\Gamma(2\zeta+2\eta+1-it)\Gamma(-2\eta+it) b^{\frac{it}{2}}$ $\times M_{\zeta-\frac{it}{2},\eta-\frac{it}{2}}(b)$	$\frac{e^{\frac{(b-a)}{2}}\ 2\pi\Gamma(1+2\nu)}{\Gamma(2\nu+2\zeta+2)} a^{\nu+\frac{1}{2}} b^{\eta+\frac{1}{2}} \Gamma\left(\nu+\frac{3}{2}+2\zeta+2\eta+ix\right)\Gamma(\nu+\frac{1}{2}-2\eta-ix)$ $\times H_4\left(\nu+\frac{1}{2}-2\eta-ix, \nu+\frac{3}{2}+2\zeta+2\eta, 2\nu+2\zeta+2, a, b\right)$
$\Gamma(\alpha-it)\Gamma(\beta+it)\ {}_2F_1\left(\begin{matrix}\beta+it,\gamma\\ 1+it-\alpha\end{matrix}\middle\vert b\right)$	$\frac{2\pi\Gamma(1+2\nu)\Gamma(\alpha+\beta) e^{-\frac{a}{2}}}{\Gamma(2\nu+\alpha+\beta+1)} a^{\nu+\frac{1}{2}} \Gamma(\nu+\beta+\frac{1}{2}-ix)\ \Gamma(\nu+\alpha+\frac{1}{2}+ix)$ $\times \Psi_1(\nu+\beta+\frac{1}{2}-ix, \gamma, \frac{1}{2}-\alpha-\nu-ix, 2\nu+\alpha+\beta+1; b, a)$

An interesting corollary of (11) - (12) follows if we replace $g^*(x)$ by $a^{\nu+\frac{1}{2}}g(x)$, $f^*(t)$ by $f(t)$, ν by $\xi - \frac{1}{2}$ and let $a \to 0$. Then, if

$$g(x) = \int_{-\infty}^{\infty} \Gamma\Big(\xi+i(x+t)\Big)\,\Gamma\Big(\xi-i(x+t)\Big)\,f(t)\,dt\ , \tag{13}$$

we have, almost everwhere,

$$f(t) = -\frac{\xi\ \sin(2\pi\xi)}{2\pi^3}\int_{-\infty}^{\infty} \Gamma\Big(-\xi+i(x+t)\Big)\,\Gamma\Big(-\xi-i(x+t)\Big)\,g(x)\,dx\ , \tag{14}$$

provided $g(x)\epsilon L^p(-\infty,\infty)$, $p = 1,2$, $2\xi \neq 0,-1,-2,\ldots$, $\mathrm{Re}\,\xi < 0$.

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9. Reference 1; V. I, Section (1.18), formula (6) and Section (6.13.2).