NASA TECHNICAL NOTE

NASA TN D-4774

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NASA TN D-4774

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • OCTOBER 1968

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SUMMARY

Variational expressions of the admittance of a rectangular aperture covered with homogeneous material are derived. The electric field inside the waveguide is assumed to be a dominant mode plus the first higher-order symmetrical mode. Admittance expressions are also given for semi-infinite media.

Admittance calculations for polystyrene slabs are given. These calculations are shown to agree closely with the results given in the literature. Also given are calculations for lossy plasma slabs with electron densities both above and below the critical density. For plasma slabs approximately equal to or greater than $\frac{1}{3}\lambda_0$, the medium can be considered semi-infinite, particularly for overdense plasmas.

For changing slab thickness, the field of the higher-order mode has a greater effect on the admittance in the dielectric slab than in the plasma slab. However, this effect is small; hence, only the dominant-mode field is needed for computing aperture admittance.

INTRODUCTION

The input admittance of a waveguide-fed rectangular aperture, opening onto a dielectric or plasma-coated ground plane, has been investigated by many authors (refs. 1 to 8). The variational expressions derived by most of these authors have assumed the TE_{01} mode only as a trial field at the aperture. In papers by Galejs (refs. 1, 2, and 5), a trial field in the aperture was assumed to be a superposition of a sine wave and a shifted cosine wave. The expressions for the aperture admittance obtained from this trial field were also variational; however, the infinite ground-plane structure was approximated by a large waveguide. The symmetry of the problem suggests that a variational solution which assumes the dominant TE mode plus higher-order odd modes may adequately represent the field in the aperture. The number of these higher-order TE modes (TE_{03}, TE_{05}, and so forth) needed to describe the trial field for accurate admittance calculations is not presently available.

The investigation herein concerns the input admittance of a rectangular aperture terminated in a flat ground plane covered with a slab of homogeneous material and assumes the dominant TE_{01} mode plus a higher-order evanescent mode at the aperture. The variational technique used to determine the admittance expressions is similar to the one appearing in reference 6; that is, the fields in the slab and free-space regions are derived from both the magnetic and electric vector potentials. The admittance expressions are then divided into two components; namely, TE admittance and TM admittance. These expressions are integrated numerically for a nonlossy dielectric and for lossy plasmas in which the electron densities assume values both above and below the critical density.

The problem of how thin a layer of plasma can be approximated by a semi-infinite layer is also of interest because the admittance expressions for semi-infinite conditions are simpler. Such considerations have been limited to nonlossy, underdense plasmas (ref. 3).

SYMBOLS

$A_{lm}(\alpha,\beta_n)$ general representation of equations (21)

a short dimension of waveguide

b long dimension of waveguide

C_0,C_1,C_3 functions defined in equations (6)

E electric field intensity

E_o amplitude of incident wave

$f(\beta,z),g(\beta,z)$ normalized Fourier transforms of vector potential

$G = \dfrac{R}{E_o(1 + \Gamma)}$

g_s surface-wave conductance

$g_{s,n}$ surface-wave conductance where n refers to specific poles

H magnetic field intensity

I reaction integral

$j = \sqrt{-1}$

$k^2 = \left(N^2 - 1\right)k_0^2 Z_o^2$

k_1^2, k_2^2 values of k^2 for specific values of Z_o

k_0 wave number in free space, $\omega\sqrt{\epsilon_0 \mu_0}$

k_x, k_y Cartesian components of wave number

$k_{z,01}, k_{z,03}$ wave numbers in region (defined in eqs. (2))

$\overline{k_z}^{II}$ wave number in region II

$\overline{k_z}^{III}$ wave number in region III

N index of refraction, $\sqrt{\epsilon_1/\epsilon_0}$

P,Q defined by equations (A14)

R amplitude of TE_{03} mode

t time

u real part of relative permittivity in plasma

v imaginary part of relative permittivity in plasma

x,y,z Cartesian coordinates

Y_0 characteristic admittance of free space

Y_{01}, Y_{03} characteristic admittance of the TE_{01} and TE_{03} modes, respectively, in region I (defined in eqs. (2))

Y_{ap} aperture admittance

Y_{lm} term defined by equation (16)

y_{03} normalized value of Y_{03}, $\frac{Y_{03}}{Y_{01}}$

y_{ap} normalized aperture admittance

y_{lm} normalized expression for Y_{lm}, $\frac{Y_{lm}}{Y_{01}}$

$Z = k_0 Z_o \sqrt{N^2 - \beta^2}$

Z_o thickness of slab

α,β polar component for $\frac{k_x}{k_0}$ and $\frac{k_y}{k_0}$, respectively

β_n surface-wave pole

Γ reflection coefficient

ϵ_0 permittivity of free space

ϵ_1 permittivity of region II

ϵ_I imaginary part of permittivity in dielectric

ϵ_R real part of permittivity in dielectric

λ_0 free-space wavelength

μ_0 permeability of free space

ν angular collision frequency

ω angular operating frequency

ω_p angular plasma frequency

Superscripts:

I waveguide region

II material slab region

TE transverse electric

TM transverse magnetic

Subscripts:

lm integer number combination (11, 13, or 33)

x,y,z direction components of Cartesian coordinates

A double bar over a symbol indicates a double Fourier transform. A prime denotes a derivative with respect to one of the Cartesian coordinates.

THEORY

A rectangular waveguide is terminated in a flat ground plane of infinite extent in both the x and the y direction. A slab of homogeneous dielectric material of thickness Z_o is assumed to cover the ground plane as well as the open-end waveguide. The geometry of the problem which is divided into three regions is shown in figure 1.

In region I, which is the region inside the waveguide, a TE_{01} mode is assumed to be incident upon the aperture from the left. The discontinuity at $z = 0$ excites reflected modes both propagating and nonpropagating. However, since the TE_{01} mode is assumed to be incident upon the aperture, only a reflected TE_{01} propagating mode is excited. Higher-order nonpropagating modes (evanescent modes) are excited but because of the symmetry only odd modes exist. For this problem only the TE_{03} evanescent mode is assumed to be present. From the foregoing assumptions and with $e^{j\omega t}$ time dependence assumed, the fields in region I (waveguide) are written

$$\left.\begin{aligned}
E_x{}^I(y,z) &= E_o\left(e^{-jk_{z,01}z} + \Gamma e^{jk_{z,01}z}\right)\cos\frac{\pi y}{b} + Re^{jk_{z,03}z}\cos\frac{3\pi y}{b} \\
E_y{}^I(y,z) &= 0 \\
H_x{}^I(y,z) &= 0 \\
H_y{}^I(y,z) &= Y_{01}E_o\left(e^{-jk_{z,01}z} - \Gamma e^{jk_{z,01}z}\right)\cos\frac{\pi y}{b} - Y_{03}Re^{jk_{z,03}z}\cos\frac{3\pi y}{b}
\end{aligned}\right\} \tag{1}$$

where

$$\left.\begin{aligned}
k_{z,01} &= k_0\sqrt{1 - \left(\frac{\pi}{k_0 b}\right)^2} \\
k_{z,03} &= -jk_0\sqrt{\left(\frac{3\pi}{k_0 b}\right)^2 - 1} \\
Y_{01} &= Y_0\sqrt{1 - \left(\frac{\pi}{k_0 b}\right)^2} \\
Y_{03} &= -jY_0\sqrt{\left(\frac{3\pi}{k_0 b}\right)^2 - 1}
\end{aligned}\right\} \tag{2}$$

The tangential components of E and H are continuous across the boundary $z = 0$; that is,

$$\left.\begin{aligned}
E_x^{I}(0) &= E_x^{II}(0) \\
E_y^{I}(0) &= E_y^{II}(0) \\
H_x^{I}(0) &= H_x^{II}(0) \\
H_y^{I}(0) &= H_y^{II}(0)
\end{aligned}\right\} \tag{3}$$

From Swift (ref. 6) the electric fields of equations (3) are rewritten as

$$\left.\begin{aligned}
E_x^{I}(0) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\bar{E}}_x^{II}\left(k_x,k_y,0\right)e^{-jk_x x}e^{-jk_y y}\,dk_x\,dk_y \\
E_y^{I}(0) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\bar{E}}_y^{II}\left(k_x,k_y,0\right)e^{-jk_x x}e^{-jk_y y}\,dk_x\,dk_y
\end{aligned}\right\} \tag{4}$$

Hence

$$\left.\begin{aligned}
\bar{\bar{E}}_x^{II}\left(k_x,k_y,0\right) &= \bar{\bar{E}}_x^{I}\left(k_x,k_y,0\right) = E_0(1 + \Gamma)C_0\left(k_x\right)\left[C_1\left(k_y\right) + GC_3\left(k_y\right)\right] \\
\bar{\bar{E}}_y^{II}\left(k_x,k_y,0\right) &= \bar{\bar{E}}_y^{I}\left(k_x,k_y,0\right) = 0
\end{aligned}\right\} \tag{5}$$

where

$$
\left.
\begin{aligned}
G &= \frac{R}{E_o(1+\Gamma)} \\
C_0(k_x) &= \int_{-\frac{a}{2}}^{a/2} e^{jk_x x}\,dx = \frac{a \sin \frac{k_x a}{2}}{\frac{k_x a}{2}} \\
C_1(k_y) &= \int_{-\frac{b}{2}}^{b/2} e^{jk_y y} \cos \frac{\pi y}{b}\,dy = \frac{2\pi b \cos \frac{k_y b}{2}}{\pi^2 - (k_y b)^2} \\
C_3(k_y) &= \int_{-\frac{b}{2}}^{b/2} e^{jk_y y} \cos \frac{3\pi y}{b}\,dy = \frac{-6\pi b \cos \frac{k_y b}{2}}{(3\pi)^2 - (k_y b)^2}
\end{aligned}
\right\} \qquad (6)
$$

From reference 6, the double Fourier transform of the magnetic field $\overline{\overline{H}}_y^{\,II}$ is written in terms of $\overline{\overline{E}}_x^{\,II}$ as

$$
\overline{\overline{H}}_y^{\,II}(k_x,k_y,0) = -j\,\frac{k_0^{\,2} Y_0 \overline{\overline{E}}_x^{\,II}(k_x,k_y,0)}{k_x^{\,2} + k_y^{\,2}} \left[-\left(\frac{k_y}{k_0}\right)^2 \left(\frac{g'(0)}{k_0 g(0)}\right) + \left(\frac{k_x}{k_0}\right)^2 \frac{\epsilon_1}{\epsilon_0} \left(\frac{k_0 f(0)}{f'(0)}\right) \right] \qquad (7)
$$

where, for homogeneous media, the boundary conditions at $z = Z_o$ and $z = 0$ give:

$$
\left.
\begin{aligned}
\frac{g'(0)}{g(0)} &= \frac{\overline{k_z}^{II} \sin \overline{k_z}^{II} Z_o - j\overline{k_z}^{III} \cos \overline{k_z}^{II} Z_o}{\cos \overline{k_z}^{II} Z_o + j\,\dfrac{\overline{k_z}^{III}}{\overline{k_z}^{II}} \sin \overline{k_z}^{II} Z_o} \\
\frac{f(0)}{f'(0)} &= \frac{\cos \overline{k_z}^{II} Z_o + j\,\dfrac{\epsilon_1}{\epsilon_0}\dfrac{\overline{k_z}^{III}}{\overline{k_z}^{II}} \sin \overline{k_z}^{II} Z_o}{\overline{k_z}^{II} \sin \overline{k_z}^{II} Z_o - j\,\dfrac{\epsilon_1}{\epsilon_0}\,\overline{k_z}^{III} \cos \overline{k_z}^{II} Z_o}
\end{aligned}
\right\} \qquad (8a)
$$

where

$$
\overline{k_z}^{II} = \pm\sqrt{\omega^2 \mu_0 \epsilon_1 - \left(k_x^{\,2} + k_y^{\,2}\right)} \qquad \left.\left(k_x^{\,2} + k_y^{\,2} < \omega^2 \mu_0 \epsilon_1\right)\right\}
$$

(Equations continued on next page)

$$\left.\begin{aligned}
\overline{k_z}^{II} &= \pm j\sqrt{k_x^2 + k_y^2 - \omega^2\mu_0\epsilon_1} && \left(k_x^2 + k_y^2 > \omega^2\mu_0\epsilon_1\right) \\
\overline{k_z}^{III} &= \sqrt{k_0^2 - \left(k_x^2 + k_y^2\right)} && \left(k_x^2 + k_y^2 < k_0^2\right) \\
\overline{k_z}^{III} &= -j\sqrt{k_x^2 + k_y^2 - k_0^2} && \left(k_x^2 + k_y^2 > k_0^2\right)
\end{aligned}\right\} \tag{8b}$$

Since tangential E and H are continuous across the boundary $z = 0$, the reaction integral (ref. 9) is also continuous; that is,

$$I = \int_{-\frac{a}{2}}^{a/2} \int_{-\frac{b}{2}}^{b/2} E_x^{I}(0) H_y^{I}(0) dx\, dy = \int_{-\frac{a}{2}}^{a/2} \int_{-\frac{b}{2}}^{b/2} E_x^{II}(0) H_y^{II}(0) dx\, dy \tag{9}$$

The reaction integral due to the fields inside the waveguide is determined by substituting equations (1) and (2) into equation (9) and performing the integration. The reaction integral becomes

$$I = Y_{01} E_o^2 (1 + \Gamma)(1 - \Gamma)\frac{ab}{2} - Y_{03} R^2 \frac{ab}{2} \tag{10}$$

By defining (ref. 9) Y_{ap} as

$$Y_{ap} = Y_{01} \frac{1 - \Gamma}{1 + \Gamma} \tag{11}$$

equation (10) is written as

$$Y_{ap} = \frac{2}{ab} \frac{1}{E_o^2 (1 + \Gamma)^2} I + Y_{03} G^2 \tag{12}$$

By applying Parseval's theorem, the reaction integral due to the fields inside region II at $z = 0$ becomes

$$I = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\overline{E}}_x^{II}(0) \overline{\overline{H}}_y^{II}(0) dk_x\, dk_y \tag{13}$$

where $\overline{\overline{E}}_x^{II}$ and $\overline{\overline{H}}_y^{II}$ are the double Fourier transforms given by equations (5) and (7), respectively.

The aperture admittance found by substituting equations (5), (7), and (13) into equation (12) is

$$Y_{ap} = -j \frac{2Y_0 k_0{}^2}{ab(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{C_0(k_x)C_0(k_x)}{k_x{}^2 + k_y{}^2} \left[C_1(k_y)C_1(k_y) + 2GC_1(k_y)C_3(k_y) + G^2C_3(k_y)C_3(k_y)\right]$$

$$\times \left[-\left(\frac{k_y}{k_0}\right)^2 \left(\frac{g'(0)}{k_0 g(0)}\right) + \left(\frac{k_x}{k_0}\right)^2 \frac{\epsilon_1}{\epsilon_0}\left(\frac{k_0 f(0)}{f'(0)}\right)\right] + Y_{03}G^2 \tag{14}$$

Equation (14) may be written as

$$Y_{ap} = Y_{11} + 2GY_{13} + \left(Y_{33} + Y_{03}\right)G^2$$

or

$$y_{ap} = \frac{Y_{ap}}{Y_{01}} = \frac{Y_{11}}{Y_{01}} + 2G\frac{Y_{13}}{Y_{01}} + \left(\frac{Y_{33}}{Y_{01}} + \frac{Y_{03}}{Y_{01}}\right)G^2 \tag{15}$$

where

$$Y_{lm} = -j \frac{2Y_0 k_0{}^2}{ab(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_0(k_x)C_0(k_x)C_l(k_y)C_m(k_y)\left[-\left(\frac{k_y}{k_0}\right)^2 \left(\frac{g'(0)}{k_0 g(0)}\right) + \left(\frac{k_x}{k_0}\right)^2 \frac{\epsilon_1}{\epsilon_0}\left(\frac{k_0 f(0)}{f'(0)}\right)\right]$$

$$(lm = 11,\ 13,\ \text{or}\ 33) \tag{16}$$

The normalized aperture admittance is stationary (see ref. 8) and, therefore,

$$\frac{\partial y_{ap}}{\partial G} = \frac{2Y_{13}}{Y_{01}} + 2\left(\frac{Y_{33}}{Y_{01}} + \frac{Y_{03}}{Y_{01}}\right)G = 0$$

and

$$G = -\frac{\dfrac{Y_{13}}{Y_{01}}}{\dfrac{Y_{33}}{Y_{01}} + \dfrac{Y_{03}}{Y_{01}}} \tag{17}$$

Substituting equation (17) into (15) yields

$$y_{ap} = \frac{Y_{11}}{Y_{01}} - \frac{\left(\dfrac{Y_{13}}{Y_{01}}\right)^2}{\dfrac{Y_{33}}{Y_{01}} + \dfrac{Y_{03}}{Y_{01}}} = y_{11} - \frac{y_{13}{}^2}{y_{33} + y_{03}} \tag{18}$$

By making the change of variable $k_x = k_0\beta \cos \alpha$ and $k_y = k_0\beta \sin \alpha$, the normalized admittances are written explicitly as

$$
\left.
\begin{aligned}
y_{11} &= -j\,\frac{2k_0^2ab}{\frac{Y_{01}}{Y_0}}\int_{\beta=0}^{\infty}\int_{\alpha=0}^{2\pi}\left[\frac{\sin\left(\frac{k_0\beta a\cos\alpha}{2}\right)}{\frac{k_0\beta a\cos\alpha}{2}}\right]^2\left[\frac{\cos\left(\frac{k_0\beta b\sin\alpha}{2}\right)}{\pi^2-\left(k_0\beta b\sin\alpha\right)^2}\right]^2\left[-\sin\alpha^2\left(\frac{g'(\beta,0)}{k_0g(\beta,0)}\right)\right.\\
&\quad\left.+\frac{\epsilon_1}{\epsilon_0}\cos^2\alpha\left(\frac{k_0f(\beta,0)}{f'(\beta,0)}\right)\right]\beta d\beta\,d\alpha\\
y_{13} &= -j\,\frac{2k_0^2ab}{\frac{Y_{01}}{Y_0}}\int_{\beta=0}^{\infty}\int_{\alpha=0}^{2\pi}\left[\frac{\sin\left(\frac{k_0\beta a\cos\alpha}{2}\right)}{\frac{k_0\beta a\cos\alpha}{2}}\right]^2\left\{\frac{-3\cos^2\left(\frac{k_0\beta b\sin\alpha}{2}\right)}{\left[\pi^2-\left(k_0\beta b\sin\alpha\right)^2\right]\left[(3\pi)^2-\left(k_0\beta b\sin\alpha\right)^2\right]}\right\}\\
&\quad\times\left[-\sin^2\alpha\left(\frac{g'(\beta,0)}{k_0g(\beta,0)}\right)+\frac{\epsilon_1}{\epsilon_0}\cos^2\alpha\left(\frac{k_0f(\beta,0)}{f'(\beta,0)}\right)\right]\beta d\beta\,d\alpha\\
y_{33} &= -j\,\frac{2k_0^2ab}{\frac{Y_{01}}{Y_0}}\int_{\beta=0}^{\infty}\int_{\alpha=0}^{2\pi}\left[\frac{\sin\left(\frac{k_0\beta a\cos\alpha}{2}\right)}{\frac{k_0\beta a\cos\alpha}{2}}\right]^2\left[\frac{3\cos\left(\frac{k_0\beta b\sin\alpha}{2}\right)}{(3\pi)^2-\left(k_0\beta b\sin\alpha\right)^2}\right]^2\\
&\quad\times\left[-\sin^2\alpha\left(\frac{g'(\beta,0)}{k_0g(\beta,0)}\right)+\frac{\epsilon_1}{\epsilon_0}\cos^2\alpha\left(\frac{k_0f(\beta,0)}{f'(\beta,0)}\right)\right]\beta d\beta\,d\alpha\\
y_{03} &= -j\,\frac{Y_0}{Y_{01}}\sqrt{\left(\frac{3\pi}{k_0b}\right)^2-1}
\end{aligned}
\right\}\quad(19)
$$

The expressions y_{11}, y_{13}, and y_{33} represent the following sums of TE and TM admittances

$$
\left.y_{11}^{TE}=-\int_0^{\infty}\int_0^{2\pi}A_{11}(\alpha,\beta)\sin^2\alpha\left(\frac{g'(\beta,0)}{k_0g(\beta,0)}\right)\beta d\beta\,d\alpha\right\}
$$

(Equations continued on next page)

$$\left.\begin{aligned} y_{13}^{TE} &= -\int_0^{\infty}\int_0^{2\pi} A_{13}(\alpha,\beta)\sin^2\alpha\left(\frac{g'(\beta,0)}{k_0 g(\beta,0)}\right)\beta d\beta\, d\alpha \\ y_{33}^{TE} &= -\int_0^{\infty}\int_0^{2\pi} A_{33}(\alpha,\beta)\sin^2\alpha\left(\frac{g'(\beta,0)}{k_0 g(\beta,0)}\right)\beta d\beta\, d\alpha \end{aligned}\right\} \tag{20a}$$

$$\left.\begin{aligned} y_{11}^{TM} &= \int_0^{\infty}\int_0^{2\pi} A_{11}(\alpha,\beta)\cos^2\alpha\, N^2\left(\frac{k_0 f(\beta,0)}{f'(\beta,0)}\right)\beta d\beta\, d\alpha \\ y_{13}^{TM} &= \int_0^{\infty}\int_0^{2\pi} A_{13}(\alpha,\beta)\cos^2\alpha\, N^2\left(\frac{k_0 f(\beta,0)}{f'(\beta,0)}\right)\beta d\beta\, d\alpha \\ y_{33}^{TM} &= \int_0^{\infty}\int_0^{2\pi} A_{33}(\alpha,\beta)\cos^2\alpha\, N^2\left(\frac{k_0 f(\beta,0)}{f'(\beta,0)}\right)\beta d\beta\, d\alpha \end{aligned}\right\} \tag{20b}$$

where

$$N^2 = \frac{\epsilon_1}{\epsilon_0}$$

$$\left.\begin{aligned} A_{11}(\alpha,\beta) &= -j\,\frac{2k_0^2ab}{\frac{Y_{01}}{Y_0}}\left[\frac{\sin\left(\frac{k_0\beta a\cos\alpha}{2}\right)}{\frac{k_0\beta a\cos\alpha}{2}}\right]^2\left[\frac{\cos\left(\frac{k_0\beta b\sin\alpha}{2}\right)}{\pi^2-\left(k_0\beta b\sin\alpha\right)^2}\right]^2 \\ A_{13}(\alpha,\beta) &= -j\,\frac{2k_0^2ab}{\frac{Y_{01}}{Y_0}}\left[\frac{\sin\left(\frac{k_0\beta a\cos\alpha}{2}\right)}{\frac{k_0\beta a\cos\alpha}{2}}\right]^2\left\{\frac{-3\cos\left(\frac{k_0\beta b\sin\alpha}{2}\right)}{\left[(3\pi)^2-\left(k_0\beta b\sin\alpha\right)^2\right]\left[\pi^2-\left(k_0\beta b\sin\alpha\right)^2\right]}\right\} \\ A_{33}(\alpha,\beta) &= -j\,\frac{2k_0^2ab}{\frac{Y_{01}}{Y_0}}\left[\frac{\sin\left(\frac{k_0\beta a\cos\alpha}{2}\right)}{\frac{k_0\beta a\cos\alpha}{2}}\right]^2\left[\frac{3\cos\left(\frac{k_0\beta b\sin\alpha}{2}\right)}{(3\pi)^2-\left(k_0\beta b\sin\alpha\right)^2}\right]^2 \end{aligned}\right\} \tag{21}$$

The integrands of equations (20) must be examined carefully for singularities in the range of integration before evaluation of the integrals can be performed. No singularities occur over the range of α, but over the range of β two types can occur; namely, branch points and poles. The singularities are contained in the ratios $\frac{g'(\beta,0)}{k_0 g(\beta,0)}$ and $\frac{k_0 f(\beta,0)}{f'(\beta,0)}$ (ref. 6).

In the region on the real axis where those poles exist, the numerical integration is performed symmetrically about each pole so that the integrals of the integrand on either side of the poles cancel each other; that is, the integrand is antisymmetrical about each pole (ref. 7).

For lossy materials $\left(N^2 \text{ complex}\right)$, the integration of equations (20) presents no difficulties except at the branch point $\beta = 1$ where a proper root change of $\sqrt{1 - \beta^2}$ must be taken into account. However, for nonlossy materials the index of refraction N is real, and thereby poles exist on the real β axis. A complete discussion of the integrating problems is given in the appendix.

In Swift's paper (ref. 6), where only the dominant TE was considered, two integrals had to be evaluated; one for the TE admittance and one for the TM admittance. The evaluation of the admittance with the higher-order mode assumed requires six integrals; three TE integrals and three TM integrals.

In tables I and II, a summary of admittance expressions for both lossy and nonlossy conditions of plasmas and real dielectrics is given. Similar tables for a single-mode assumption in the aperture are given in reference 6.

RESULTS

Equations (18) and (19) were numerically integrated for a given set of parameters for both lossy plasma and lossless dielectric slabs. In each case the self admittance (dominant-mode admittance) and aperture admittance (admittance with a higher-order mode assumed) were computed by varying the thickness of material for a given set of parameters; X-band waveguide dimensions ($a = 1.016$ cm and $b = 2.286$ cm) were used.

For the lossy plasma case, the complex index of refraction given in equation (A15) is used in equations (19). This index of refraction is dependent upon the collision frequency, the plasma frequency, and the operating frequency. To investigate the manner in which the admittance varies as the losses increase or decrease, three collision-frequency ratios, $\nu/\omega = 0.004$, 0.04, and 0.4, were chosen. For each of these ratios, the self admittance was calculated and plotted as a function of material thickness for three plasma ratios, $\left(\frac{\omega_p}{\omega}\right)^2 = 0.6$, 1.2, and 4.0, in figures 2, 3, and 4, respectively. All admittance calculations for the plasma cases were made by assuming an operating frequency of 10.0 GHz. The semi-infinite admittance values are also shown in these figures. The differences in the aperture admittance and self admittance are very small; and therefore, difficult to indicate graphically. Hence, a sample of these admittances is tabulated in table III.

For the lossless dielectric case, equations (18) and (19) were again integrated numerically but with the index of refraction equal to a real number, independent of frequency. The relative dielectric constant N^2 of the material covering the rectangular aperture was chosen to be 2.55. The operating frequency for these calculations was 10.5 GHz. Plots of the dominant-mode and the higher-order-mode aperture admittances as a function of material (polystyrene) thickness are shown in figure 5 which also includes data obtained from reference 5.

The normalized electric field distribution across the aperture is given as

$$\left|\frac{E_x(y)}{E_x(0)}\right| = \left|\frac{\cos\frac{\pi y}{b} - \frac{y_{13}}{y_{33} + y_{03}}\cos\frac{3\pi y}{b}}{1 - \frac{y_{13}}{y_{33} + y_{03}}}\right| \tag{22}$$

where y_{13} and y_{33} are determined from equations (19). In figure 6, equation (22) is plotted for several slab thicknesses of polystyrene. The electric field distribution with only the dominant mode assumed is also included in this figure. The electric field distributions for varying plasma densities are shown in figures 7 and 8.

The aperture-admittance equations for a semi-infinite medium are obtained from equations (19) by allowing the thickness Z_o to approach infinity or by using the method described by Deschamps (ref. 10). These equations are written explicitly as

$$\left.\begin{aligned}
y_{11} &= -j\frac{2k_0^2ab}{\frac{Y_{01}}{Y_0}}\int_{\beta=0}^{\infty}\int_{\alpha=0}^{2\pi}\left[\frac{\sin\left(\frac{k_0\beta a\cos\alpha}{2}\right)}{\frac{k_0\beta a\cos\alpha}{2}}\right]^2\left[\frac{\cos\left(\frac{k_0\beta b\sin\alpha}{2}\right)}{\pi^2 - (k_0\beta b\sin\alpha)^2}\right]^2 \\
&\quad\times\left[-\sin^2\alpha\left(-j\sqrt{N^2 - \beta^2}\right) + \cos^2\alpha\left(\frac{jN^2}{\sqrt{N^2 - \beta^2}}\right)\right]\beta d\beta\, d\alpha \\
y_{13} &= -j\frac{2k_0^2ab}{\frac{Y_{01}}{Y_0}}\int_{\beta=0}^{\infty}\int_{\alpha=0}^{2\pi}\left[\frac{\sin\left(\frac{k_0\beta a\cos\alpha}{2}\right)}{\frac{k_0\beta a\cos\alpha}{2}}\right]^2\left\{\frac{-3\cos^2\left(\frac{k_0\beta b\sin\alpha}{2}\right)}{\left[\pi^2 - (k_0\beta b\sin\alpha)^2\right]\left[(3\pi)^2 - (k_0\beta b\sin\alpha)^2\right]}\right\} \\
&\quad\times\left[-\sin^2\alpha\left(-j\sqrt{N^2 - \beta^2}\right) + \cos^2\alpha\left(\frac{jN^2}{\sqrt{N^2 - \beta^2}}\right)\right]\beta d\beta\, d\alpha
\end{aligned}\right\}$$

(Equations continued on next page)

$$
\left.\begin{aligned}
y_{33} = -j\,\frac{2k_0^{\,2}ab}{\dfrac{Y_{01}}{Y_0}}\int_{\beta=0}^{\infty}\int_{\alpha=0}^{2\pi}&\left[\frac{\sin\left(\dfrac{k_0\beta a\cos\alpha}{2}\right)}{\dfrac{k_0\beta a\cos\alpha}{2}}\right]^2\left[\frac{3\cos\left(\dfrac{k_0\beta b\sin\alpha}{2}\right)}{(3\pi)^2-\left(k_0\beta b\sin\alpha\right)^2}\right]^2\\
&\times\left[-\sin^2\alpha\left(-j\sqrt{N^2-\beta^2}\right)+\cos^2\alpha\left(\frac{jN^2}{\sqrt{N^2-\beta^2}}\right)\right]\beta d\beta\, d\alpha
\end{aligned}\right\}\quad(23)
$$

where the semi-infinite medium can be either a dielectric or plasma with a real or complex relative dielectric constant N^2. Since a branch point now occurs at $\beta = N$, a proper root change must be made in accordance with equation (16). The resulting integrations can then be substituted into equation (18) to obtain the total aperture admittance.

For plasma thicknesses greater than 1 centimeter $\left(\frac{1}{3}\lambda\right)$, equations (23) can be used to compute the admittance, particularly for the overdense plasmas. This approximation can be observed in figures 2, 3, and 4.

DISCUSSION

The previously derived expressions of the admittance of a rectangular aperture covered with a slab of homogeneous material are variational. Therefore, because of the stationary character of these expressions the variational procedure was used to determine the coefficient of the higher-order mode TE_{03}. This procedure only assures that for the chosen trial field the solution will best approximate the exact field if it were known.

The admittance values obtained for an aperture covered with varying thicknesses of polystyrene material were compared to the results obtained by Galejs (ref. 5) for the same material. For the most part, the calculations for the assumed trial fields agree closely with one another. The major disagreements occur for a thickness of approximately 0.5 cm $\left(0.175\lambda_0\right)$ for conductance and 0.8 cm $\left(0.279\lambda_0\right)$ for susceptance. The discrepancies in the two sets of calculations can possibly be attributed to the assumptions made on the termination of the rectangular aperture; that is, Galejs assumed a small guide radiating into a larger one filled with polystyrene material, whereas this paper assumes a waveguide radiating into a laterally unbounded slab of polystyrene material. Other sources of error could possibly be caused by the actual numerical evaluation of the equations on the form of the trial aperture field. For increasing slab thickness, the conductance and susceptance approach their semi-infinite values in an oscillatory manner; that is, the admittance oscillates in a decaying manner about its semi-infinite value.

For plasma slabs the admittance becomes quite insensitive to increases in slab thickness, particularly for overdense plasmas. However, for thin plasma layers, appreciable changes in admittance values were observed. These observations are in agreement with Villeneuve (ref. 3). For overdense plasmas a slab thickness of approximately 1.0 cm $\left(\frac{1}{3}\lambda_0\right)$ can be used to approximate semi-infinite conditions. However, for underdense plasmas, the approximated slab thickness is not so well defined. The susceptance for this case may be inductive or capacitive depending on the thickness; therefore, in choosing the thickness to approximate semi-infinite conditions the magnitude as well as the phase of the semi-infinite admittance must be considered.

Two expressions representing the admittance of a rectangular aperture covered with slabs of material were derived by Galejs (ref. 5). The expression for thin layers assumed a larger waveguide termination filled with material and the expression for thick layers assumed laterally unbounded material. In the thick-layer formulation the admittance calculations are valid for slab thicknesses greater than a wavelength. For the thin-layer solution the conductance curves vary smoothly for slab thicknesses less than a wavelength, but, for thicknesses greater than a wavelength, the curves oscillate. The average of this oscillatory plasma conductance was shown to be approximately equal to the values obtained by using the thick-slab solution. Therefore, by using these average values, the admittance computations for this case can be used for all thicknesses.

The expressions derived in this paper give a compact representation of the aperture admittance for dielectric or plasma-coated rectangular apertures. No restrictions in regard to thickness are placed on these coatings; hence, the determination of the aperture admittances for all thicknesses can be obtained by using the laterally unbounded model.

In reference 3, a qualitative discussion was presented in regard to the different effects that plasma and dielectric slabs may have on the aperture admittance. The essence of this discussion was that dielectric slabs would have a greater effect than the plasma slabs. The calculations obtained with the higher-order-mode assumption substantiate this argument. Although, a slab-thickness change had more effect on the admittance in the dielectric slab than in the plasma slab, the effect was small; therefore, the computations for plasma and dielectric slabs can be simplified by assuming only the dominant mode in the aperture.

For changing slab thickness in the dielectric case a significant change in the electric field distribution was observed. However, in the plasma case the distribution across the aperture varies only slightly for varying slab thicknesses. A greater variation in the distribution for the dielectric case is expected since the dielectric slab has a greater effect on the aperture admittance than does the plasma slab. For increasing electron density, the electric field distribution for the higher-order-mode assumption is shown to approach the distribution obtained when only the dominant-mode field was assumed. This agrees with the general observation made by R. L. Fante in reference 11.

CONCLUDING REMARKS

Variational expressions of the admittance of a rectangular aperture covered with homogeneous material are derived. The electric field inside the waveguide is assumed to be a dominant mode TE_{01} plus the first higher-order symmetrical mode TE_{03}. Admittance expressions are also given for semi-infinite media.

Admittance calculations for polystyrene slabs are given. These calculations are shown to agree closely with the results given in the literature. Also given are calculations for lossy plasma slabs with electron densities both above and below the critical density. For plasma slabs approximately equal to or greater than $\frac{1}{3}\lambda_0$, the medium can be considered semi-infinite, particularly for overdense plasmas.

For changing slab thickness, the field of the higher-order mode has a greater effect on the admittance more in the dielectric slab than in the plasma slab. However, this effect is small; hence, only the dominant-mode field is needed for computing aperture admittance.

For changing slab thickness in the dielectric slab, a significant change in the electric field distribution was observed. In the plasma slab the distribution across the aperture varies only slightly for changing slab thickness. For increasing electron density, the electric field distribution for the higher-order-mode assumption is shown to approach the distribution obtained when only the dominant-mode field is assumed.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., June 18, 1968,
125-22-02-02-23.

APPENDIX

PROBLEMS IN INTEGRATION

The expressions appearing in equations (20) are computed numerically by breaking up the integral over β as suggested in tables I and II. Contour plots showing path and surface-wave poles for $N^2 > 1$ and $N^2 < -1$ are given in figure 9. For $N^2 > 1$ (real dielectrics) the contour integral along the real β axis for the TE and TM admittances $\left(y_{11}, y_{13}, \text{ and } y_{33}\right)$ is shown in figure 9(a). Because this time dependence is assumed to be $e^{j\omega t}$, the branch cut at $\beta = 1$ must be chosen as shown to meet the requirements for the proper root of $\overline{k_z}^{III}$; that is,

$$\left.\begin{aligned} \overline{k_z}^{III} &= k_0\sqrt{1 - \beta^2} && \left(\beta^2 < 1\right) \\ \overline{k_z}^{III} &= -jk_0\sqrt{\beta^2 - 1} && \left(\beta^2 > 1\right) \end{aligned}\right\} \qquad \text{(A1)}$$

In the interval between $\beta = 1$ and $\beta = \infty$ for $N^2 > 1$, the integration contributes only to the susceptance in the admittance expressions for both TE and TM modes. In the range from $\beta = 0$ to $\beta = 1$ the integration contributes both to the conductance and susceptance. The only other contribution to the conductance is due to the residues of the simple poles in the interval $1 < \beta < N$. The simple poles are termed "surface-wave poles." The conductance as a result of the surface-wave poles is expressed as

$$\left.\begin{aligned} g_{s,n_{lm}}^{TE} &= -\pi j \operatorname{Res}\left(y_{lm}^{TE}\right)_{N^2>1} = j\pi \sum_n \frac{\left(N^2 - \beta_n^2\right)\int_0^{2\pi} A_{lm}\left(\alpha,\beta_n\right)\sin^2\alpha \, d\alpha}{k_0 Z_o\left[1 - \frac{\left(N^2 - 1\right)}{\left(\beta_n^2 - 1\right)} \frac{\sin 2k_0 Z_o\sqrt{N^2 - \beta_n^2}}{2k_0 Z_o\sqrt{N^2 - \beta_n^2}}\right]} \\ g_{s,n_{lm}}^{TM} &= -\pi j \operatorname{Res}\left(y_{lm}^{TM}\right)_{N^2>1} = j\pi \sum_n \frac{N^2\int_0^{2\pi} A_{lm}\left(\alpha,\beta_n\right)\cos^2\alpha \, d\alpha}{k_0 Z_o\left[1 + \frac{\left(N^2 - 1\right)}{\left(\beta_n^2 - 1\right)} \frac{\sin 2k_0 Z_o\sqrt{N^2 - \beta_n^2}}{2k_0 Z_o\sqrt{N^2 - \beta_n^2}}\right]} \end{aligned}\right\} \qquad \text{(A2)}$$

When l and m equal 1, equations (A2) are identical to the surface-wave conductance appearing in Swift's paper (ref. 6); that is,

$$
\left.
\begin{aligned}
g_{s,n}^{TE} &= \frac{4\pi k_0{}^2 ab}{\frac{Y_{01}}{Y_0}} \sum_n \frac{\left(N^2 - \beta_n{}^2\right) \displaystyle\int_0^{\pi} \left[\frac{\sin\left(\frac{k_0\beta_n a \cos\alpha}{2}\right)}{\frac{k_0\beta_n a \cos\alpha}{2}}\right]^2 \left[\frac{\cos\left(\frac{k_0\beta_n b \sin\alpha}{2}\right)}{\pi^2 - \left(k_0\beta_n b \sin\alpha\right)^2}\right]^2 \sin^2\alpha \, d\alpha}{k_0 Z_o\left[1 - \frac{\left(N^2 - 1\right)}{\left(\beta_n{}^2 - 1\right)} \frac{\sin 2k_0 Z_o\sqrt{N^2 - \beta_n{}^2}}{2k_0 Z_o\sqrt{N^2 - \beta_n{}^2}}\right]} \\
g_{s,n}^{TM} &= \frac{4\pi k_0{}^2 ab}{\frac{Y_{01}}{Y_0}} \sum_n \frac{N^2 \displaystyle\int_0^{\pi} \left[\frac{\sin\left(\frac{k_0\beta_n a \cos\alpha}{2}\right)}{\frac{k_0\beta_n a \cos\alpha}{2}}\right]^2 \left[\frac{\cos\left(\frac{k_0\beta_n b \sin\alpha}{2}\right)}{\pi^2 - \left(k_0\beta_n b \sin\alpha\right)^2}\right]^2 \cos^2\alpha \, d\alpha}{k_0 Z_o\left[1 + \frac{\left(N^2 - 1\right)}{\left(\beta_n{}^2 - 1\right)} \frac{\sin 2k_0 Z_o\sqrt{N^2 - \beta_n{}^2}}{2k_0 Z_o\sqrt{N^2 - \beta_n{}^2}}\right]}
\end{aligned}
\right\} \quad (A3)
$$

The poles of g_2 and f_2 given in tables I and II for $N^2 > 1$ are found from the zeros of

$$
\left.
\begin{aligned}
\tan k_0 Z_o\sqrt{N^2 - \beta^2} &= -\frac{\sqrt{N^2 - \beta^2}}{\sqrt{\beta^2 - 1}} && \text{(TE)} \\
\tan k_0 Z_o\sqrt{N^2 - \beta^2} &= \frac{N^2\sqrt{\beta^2 - 1}}{\sqrt{N^2 - \beta^2}} && \text{(TM)}
\end{aligned}
\right\} \quad (A4)
$$

By making the substitution $Z = k_0 Z_o\sqrt{N^2 - \beta^2}$, equations (A4) can be written as

$$
\left.
\begin{aligned}
\cot Z &= -\sqrt{\frac{\left(N^2 - 1\right)k_0{}^2 Z_o{}^2}{Z^2} - 1} && \text{(TE)} \\
\tan Z &= N^2\sqrt{\frac{\left(N^2 - 1\right)k_0{}^2 Z_o{}^2}{Z^2} - 1} && \text{(TM)}
\end{aligned}
\right\} \quad (A5)
$$

By plotting each side of equations (A5) as a function of Z to the same scale, the pole locations are then found at the intersections of these curves. These two equations are of the same form as the equations in Weeks (ref. 12). The TE surface-wave modes will not be present if the proper thickness of material in wavelengths is selected at the onset of surface waves ($\beta = 1$), such that

$$\frac{Z_o}{\lambda_0} < \frac{1}{4\sqrt{N^2 - 1}} \tag{A6}$$

However, cutoff conditions can be adjusted by changing the thickness in wavelengths to correspond to odd multiples of $\pi/2$ $\left(\text{that is,}\ \cot k_0 Z_o \sqrt{N^2 - 1} = \frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2}, \text{and so forth}\right)$, or

$$\frac{Z_o}{\lambda_0} = \frac{1}{4\sqrt{N^2 - 1}}, \frac{3}{4\sqrt{N^2 - 1}}, \frac{5}{4\sqrt{N^2 - 1}}, \ldots \qquad \text{(TE)} \tag{A7}$$

For TM surface-wave modes there is no cutoff frequency because the lowest TM mode is the zero mode. Hence, for the TM mode the thickness in wavelengths corresponds to integer multiples of Z_o/λ_0; that is,

$$\frac{Z_o}{\lambda_0} = 0, \frac{1}{2\sqrt{N^2 - 1}}, \frac{1}{\sqrt{N^2 - 1}}, \frac{3}{2\sqrt{N^2 - 1}}, \ldots \qquad \text{(TM)} \tag{A8}$$

Generally, equations (A7) and (A8) are expressed as $\frac{Z_o}{\lambda_0} = \frac{n}{4\sqrt{N^2 - 1}}$ with $n = 0, 1, 2, 3,$ and so forth with TM having even modes and TE having odd modes. These cutoff conditions are computed at the onset of surface waves ($\beta = 1$). Typical sketches exemplifying the technique of determining the roots (zeros) of equations (A5) are shown in figure 10 for two arbitrary values of $k^2 = (N^2 - 1)k_0^2 Z_o^2$. The cutoff conditions for the TE modes are at the zeros of $\cot Z$ as shown in figure 10(a). The cutoff conditions for the TM modes are at the zeros of $\tan Z$ as shown in figure 10(b). These sketches are similar to the ones given by Weeks (ref. 12).

For the case $N^2 < -1$, only a single surface-wave pole occurs for the TM mode. Figure 9(b) shows the path of integration and the surface-wave pole. No surface-wave poles occur for the TE mode. This case discusses a lossless overdense plasma for which the contribution of the residue to the TM admittance expression is (ref. 6)

$$g_{s_{lm}}^{TM} = -\pi j \operatorname{Res}\left(y_{lm}^{TM}\right)_{N^2<-1} = -j\pi \frac{\left|N^2\right| \int_0^{2\pi} A_{lm} \cos^2\alpha \, d\alpha}{k_0 Z_o \left[1 - \frac{\left(\left|N^2\right| + 1\right)\sinh\left(2k_0 Z_o \sqrt{\left|N^2\right| + \beta_o^2}\right)}{\left(\beta_o^2 - 1\right) 2k_0 Z_o \sqrt{\left|N^2\right| + \beta_o^2}}\right]} \tag{A9}$$

where β_o is a root.

APPENDIX

For the lossy case N^2 complex no poles occur on the real β axis, therefore no difficulties arise in the integrations. However, a proper root change at $\beta = 1$ in accordance with equations (A1) must be taken into account since this is a branch point. The complex index of refraction N is written, in general, as

$$N^2 = u - jv \tag{A10}$$

where

$$\left.\begin{aligned} u &= \frac{\epsilon_R}{\epsilon_0} - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} \\ v &= \frac{\epsilon_I}{\epsilon_0} + \frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} \end{aligned}\right\} \tag{A11}$$

For a lossy dielectric $\omega_p = \nu = 0$; therefore,

$$N^2 = \frac{\epsilon_R}{\epsilon_0} - j\,\frac{\epsilon_I}{\epsilon_0} \qquad \left(\frac{\epsilon_R}{\epsilon_0} > 1\right) \tag{A12}$$

then

$$\left.\begin{aligned} &\sqrt{N^2 - \beta^2} = P - jQ \\ &\tan k_0 Z_0 \sqrt{N^2 - \beta^2} = \frac{\tan k_0 Z_0 P - j \tanh k_0 Z_0 Q}{1 + j \tan k_0 Z_0 P \tanh k_0 Z_0 Q} \end{aligned}\right\} \tag{A13}$$

where

$$\left.\begin{aligned} P &= \frac{1}{\sqrt{2}}\left[\frac{\epsilon_R}{\epsilon_0} - \beta^2 + \sqrt{\left(\frac{\epsilon_R}{\epsilon_0} - \beta^2\right)^2 + \left(\frac{\epsilon_I}{\epsilon_0}\right)^2}\right]^{1/2} \\ Q &= \frac{1}{\sqrt{2}}\left[-\left(\frac{\epsilon_R}{\epsilon_0} - \beta^2\right) + \sqrt{\left(\frac{\epsilon_R}{\epsilon_0} - \beta^2\right)^2 + \left(\frac{\epsilon_I}{\epsilon_0}\right)^2}\right]^{1/2} \end{aligned}\right\} \tag{A14}$$

For a lossy plasma $\epsilon_R/\epsilon_0 = 1$ and $\epsilon_I/\epsilon_0 = 0$; therefore,

$$\frac{\epsilon_1}{\epsilon_0} = N^2 = 1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} - j\,\frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} \tag{A15}$$

then

$$\sqrt{N^2 - \beta^2} = P - jQ$$

$$\tan k_0 Z_o \sqrt{N^2 - \beta^2} = \frac{\tan k_0 Z_o P - j \tanh k_0 Z_o Q}{1 + j \tan k_0 Z_o P \tanh k_0 Z_o Q}$$

where

$$P = \frac{1}{\sqrt{2}} \left\{ \left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} - \beta^2 \right] + \sqrt{\left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} - \beta^2 \right]^2 + \left[\frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} \right]^2} \right\}^{1/2}$$

$$Q = \frac{1}{\sqrt{2}} \left\{ - \left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} - \beta^2 \right] + \sqrt{\left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} - \beta^2 \right]^2 + \left[\frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} \right]^2} \right\}^{1/2}$$

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TABLE I.- TE COMPONENT OF

	y_{11}	y_{13}	y_{33}
$N^2 = u - jv$	$-\int_0^1 g_1 \int_0^{2\pi} A_{11} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_0^1 g_1 \int_0^{2\pi} A_{13} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_0^1 g_1 \int_0^{2\pi} A_{33} \sin^2\alpha d\alpha\, \beta d\beta$
	$-\int_1^\infty g_2 \int_0^{2\pi} A_{11} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_1^\infty g_2 \int_0^{2\pi} A_{13} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_1^\infty g_2 \int_0^{2\pi} A_{33} \sin^2\alpha d\alpha\, \beta d\beta$
$N^2 > 1$	$-\int_0^1 g_1 \int_0^{2\pi} A_{11} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_0^1 g_1 \int_0^{2\pi} A_{13} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_0^1 g_1 \int_0^{2\pi} A_{33} \sin^2\alpha d\alpha\, \beta d\beta$
	$-\int_1^N g_2 \int_0^{2\pi} A_{11} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_1^N g_2 \int_0^{2\pi} A_{13} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_1^\infty g_2 \int_0^{2\pi} A_{33} \sin^2\alpha d\alpha\, \beta d\beta$
	$-\int_N^\infty g_3 \int_0^{2\pi} A_{11} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_N^\infty g_3 \int_0^{2\pi} A_{13} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_N^\infty g_3 \int_0^{2\pi} A_{33} \sin^2\alpha d\alpha\, \beta d\beta$
$0 < N^2 < 1$	$-\int_0^N g_1 \int_0^{2\pi} A_{11} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_0^N g_1 \int_0^{2\pi} A_{13} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_0^N g_1 \int_0^{2\pi} A_{33} \sin^2\alpha d\alpha\, \beta d\beta$
	$-\int_N^1 g_2 \int_0^{2\pi} A_{11} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_N^1 g_2 \int_0^{2\pi} A_{13} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_N^1 g_2 \int_0^{2\pi} A_{33} \sin^2\alpha d\alpha\, \beta d\beta$
	$-\int_1^\infty g_3 \int_0^{2\pi} A_{11} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_1^\infty g_3 \int_0^{2\pi} A_{13} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_1^\infty g_3 \int_0^{2\pi} A_{33} \sin^2\alpha d\alpha\, \beta d\beta$
$N^2 < 0$	$-\int_0^1 g_1 \int_0^{2\pi} A_{11} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_0^1 g_1 \int_0^{2\pi} A_{13} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_0^1 g_1 \int_0^{2\pi} A_{33} \sin^2\alpha d\alpha\, \beta d\beta$
	$-\int_1^\infty g_2 \int_0^{2\pi} A_{11} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_1^\infty g_2 \int_0^{2\pi} A_{13} \sin^2\alpha d\alpha\, \beta d\beta$	$-\int_1^\infty g_2 \int_0^{2\pi} A_{33} \sin^2\alpha d\alpha\, \beta d\beta$

$$N^2 = u - jv \quad \text{where} \quad u = \frac{\epsilon_R}{\epsilon_0} - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} \quad \text{and} \quad v = \frac{\epsilon_I}{\epsilon_0} + \frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2}$$

$$A_{11}(\alpha,\beta) = -j \frac{2k_0{}^2 ab}{\frac{Y_{01}}{Y_0}} \left[\frac{\sin\left(\frac{k_0\beta a \cos\alpha}{2}\right)}{\frac{k_0\beta a \cos\alpha}{2}}\right]^2 \left[\frac{\cos\left(\frac{k_0\beta b \sin\alpha}{2}\right)}{\pi^2 - \left(k_0\beta b \sin\alpha\right)^2}\right]^2$$

$$A_{13}(\alpha,\beta) = -j \frac{2k_0{}^2 ab}{\frac{Y_{01}}{Y_0}} \left[\frac{\sin\left(\frac{k_0\beta a \cos\alpha}{2}\right)}{\frac{k_0\beta a \cos\alpha}{2}}\right]^2 \left\{\frac{-3 \cos^2\left(\frac{k_0\beta b \sin\alpha}{2}\right)}{\left[(3\pi)^2 - \left(k_0\beta b \sin\alpha\right)^2\right]\left[(\pi)^2 - \left(k_0\beta b \sin\alpha\right)^2\right]}\right\}$$

$$A_{33}(\alpha,\beta) = -j \frac{2k_0{}^2 ab}{\frac{Y_{01}}{Y_0}} \left[\frac{\sin\left(\frac{k_0\beta a \cos\alpha}{2}\right)}{\frac{k_0\beta a \cos\alpha}{2}}\right]^2 \left[\frac{3 \cos\left(\frac{k_0\beta b \sin\alpha}{2}\right)}{(3\pi)^2 - \left(k_0\beta b \cos\alpha\right)^2}\right]^2$$

g_1	g_2	g_3
$\dfrac{\sqrt{N^2-\beta^2}\left(\tan k_0 Z_o\sqrt{N^2-\beta^2} - j\dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\right)}{1 + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\tan k_0 Z_o\sqrt{N^2-\beta^2}}$	$\dfrac{\sqrt{N^2-\beta^2}\left(\tan k_0 Z_o\sqrt{N^2-\beta^2} - \dfrac{\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}\right)}{1 + \dfrac{\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}\tan k_0 Z_o\sqrt{N^2-\beta^2}}$	
$\dfrac{\sqrt{N^2-\beta^2}\left(\tan k_0 Z_o\sqrt{N^2-\beta^2} - j\dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\right)}{1 + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\tan k_0 Z_o\sqrt{N^2-\beta^2}}$	$\dfrac{\sqrt{N^2-\beta^2}\left(\tan k_0 Z_o\sqrt{N^2-\beta^2} - \dfrac{\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}\right)}{1 + \dfrac{\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}\tan k_0 Z_o\sqrt{N^2-\beta^2}}$	$-\dfrac{\sqrt{\beta^2-N^2}\left(\tanh k_0 Z_o\sqrt{\beta^2-N^2} + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\right)}{1 + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\tanh k_0 Z_o\sqrt{\beta^2-N^2}}$
$\dfrac{\sqrt{N^2-\beta^2}\left(\tan k_0 Z_o\sqrt{N^2-\beta^2} - j\dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\right)}{1 + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\tan k_0 Z_o\sqrt{N^2-\beta^2}}$	$-\dfrac{\sqrt{\beta^2-N^2}\left(\tanh k_0 Z_o\sqrt{\beta^2-N^2} + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{\beta^2-N^2}}\right)}{1 + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{\beta^2-N^2}}\tanh k_0 Z_o\sqrt{\beta^2-N^2}}$	$-\dfrac{\sqrt{\beta^2-N^2}\left(\tanh k_0 Z_o\sqrt{\beta^2-N^2} + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\right)}{1 + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\tanh k_0 Z_o\sqrt{\beta^2-N^2}}$
$-\dfrac{\sqrt{\lvert N^2\rvert+\beta^2}\left(\tanh k_0 Z_o\sqrt{\lvert N^2\rvert+\beta^2} + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{\lvert N^2\rvert+\beta^2}}\right)}{1 + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{\lvert N^2\rvert+\beta^2}}\tanh k_0 Z_o\sqrt{\lvert N^2\rvert+\beta^2}}$	$-\dfrac{\sqrt{\lvert N^2\rvert+\beta^2}\left(\tanh k_0 Z_o\sqrt{\lvert N^2\rvert+\beta^2} + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\lvert N^2\rvert+\beta^2}}\right)}{1 + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\lvert N^2\rvert+\beta^2}}\tanh k_0 Z_o\sqrt{\lvert N^2\rvert+\beta^2}}$	

Lossy dielectric case $\left(\omega_p = \nu = 0\right)$:

$$\sqrt{N^2-\beta^2} = P - jQ$$

$$\tan k_0 Z_o\sqrt{N^2-\beta^2} = \frac{\tan k_0 Z_o P - j\tanh k_0 Z_o Q}{1 + j\tan k_0 Z_o P\tanh k_0 Z_o Q}$$

$$P = \frac{1}{\sqrt{2}}\left[\frac{\epsilon_R}{\epsilon_0} - \beta^2 + \sqrt{\left(\frac{\epsilon_R}{\epsilon_0} - \beta^2\right)^2 + \left(\frac{\epsilon_I}{\epsilon_0}\right)^2}\right]^{1/2}$$

$$Q = \frac{1}{\sqrt{2}}\left[-\frac{\epsilon_R}{\epsilon_0} + \beta^2 + \sqrt{\left(\frac{\epsilon_R}{\epsilon_0} - \beta^2\right)^2 + \left(\frac{\epsilon_I}{\epsilon_0}\right)^2}\right]^{1/2}$$

Lossy plasma case $\left(\frac{\epsilon_R}{\epsilon_0} = 1,\ \frac{\epsilon_I}{\epsilon_0} = 0\right)$:

$$\sqrt{N^2-\beta^2} = P - jQ$$

$$\tan k_0 Z_o\sqrt{N^2-\beta^2} = \frac{\tan k_0 Z_o P - j\tanh k_0 Z_o Q}{1 + j\tan k_0 Z_o P\tanh k_0 Z_o Q}$$

$$P = \frac{1}{\sqrt{2}}\left\{\left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} - \beta^2\right] + \sqrt{\left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} - \beta^2\right]^2 + \left[\frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2}\right]^2}\right\}^{1/2}$$

$$Q = \frac{1}{\sqrt{2}}\left\{-\left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} - \beta^2\right] + \sqrt{\left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} - \beta^2\right]^2 + \left[\frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2}\right]^2}\right\}^{1/2}$$

	y_{11}	y_{13}	y_{33}
$N^2 = u - jv$	$\int_0^1 f_1 \int_0^{2\pi} A_{11} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^\infty f_2 \int_0^{2\pi} A_{11} \cos^2\alpha d\alpha \, \beta d\beta$	$\int_0^1 f_1 \int_0^{2\pi} A_{13} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^\infty f_2 \int_0^{2\pi} A_{13} \cos^2\alpha d\alpha \, \beta d\beta$	$\int_0^1 f_1 \int_0^{2\pi} A_{33} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^\infty f_2 \int_0^{2\pi} A_{33} \cos^2\alpha d\alpha \, \beta d\beta$
$N^2 > 1$	$\int_0^1 f_1 \int_0^{2\pi} A_{11} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^N f_2 \int_0^{2\pi} A_{11} \cos^2\alpha dE \, \beta d\beta$ $+ \int_N^\infty f_3 \int_0^{2\pi} A_{11} \cos^2\alpha d\alpha \, \beta d\beta$	$\int_0^1 f_1 \int_0^{2\pi} A_{13} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^N f_2 \int_0^{2\pi} A_{13} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_N^\infty f_3 \int_0^{2\pi} A_{13} \cos^2\alpha d\alpha \, \beta d\beta$	$\int_0^1 f_1 \int_0^{2\pi} A_{33} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^N f_2 \int_0^{2\pi} A_{33} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_N^\infty f_3 \int_0^{2\pi} A_{33} \cos^2\alpha d\alpha \, \beta d\beta$
$0 < N^2 < 1$	$\int_0^N f_1 \int_0^{2\pi} A_{11} \cos^2\alpha dE \, \beta d\beta$ $+ \int_N^1 f_2 \int_0^{2\pi} A_{11} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^\infty f_3 \int_0^{2\pi} A_{11} \cos^2\alpha d\alpha \, \beta d\beta$	$\int_0^N f_1 \int_0^{2\pi} A_{13} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_N^1 f_2 \int_0^{2\pi} A_{13} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_N^\infty f_3 \int_0^{2\pi} A_{13} \cos^2\alpha d\alpha \, \beta d\beta$	$\int_0^N f_1 \int_0^{2\pi} A_{33} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_N^1 f_2 \int_0^{2\pi} A_{33} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^\infty f_3 \int_0^{2\pi} A_{33} \cos^2\alpha d\alpha \, \beta d\beta$
$N^2 < 0$	$\int_0^1 f_1 \int_0^{2\pi} A_{11} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^\infty f_2 \int_0^{2\pi} A_{11} \cos^2\alpha d\alpha \, \beta b\beta$	$\int_0^1 f_1 \int_0^{2\pi} A_{13} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^\infty f_2 \int_0^{2\pi} A_{13} \cos^2\alpha d\alpha \, \beta d\beta$	$\int_0^1 f_1 \int_0^{2\pi} A_{33} \cos^2\alpha d\alpha \, \beta d\beta$ $+ \int_1^\infty f_2 \int_0^{2\pi} A_{33} \cos^2\alpha d\alpha \, \beta d\beta$

$$N^2 = u - jv \quad \text{where} \quad u = \frac{\epsilon_R}{\epsilon_0} - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} \quad \text{and} \quad v = \frac{\epsilon_I}{\epsilon_0} + \frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2}$$

$$A_{11}(\alpha,\beta) = -j \frac{2k_0{}^2 ab}{\frac{Y_{01}}{Y_0}} \left[\frac{\sin\left(\frac{k_0\beta a \cos\alpha}{2}\right)}{\frac{k_0\beta a \cos\alpha}{2}}\right]^2 \left[\frac{\cos\left(\frac{k_0\beta b \sin\alpha}{2}\right)}{\pi^2 - \left(k_0\beta b \sin\alpha\right)^2}\right]^2$$

$$A_{13}(\alpha,\beta) = -j \frac{2k_0{}^2 ab}{\frac{Y_{01}}{Y_0}} \left[\frac{\sin\left(\frac{k_0\beta a \cos\alpha}{2}\right)}{\frac{k_0\beta a \cos\alpha}{2}}\right]^2 \left\{\frac{-3\cos^2\left(\frac{k_0\beta b \sin\alpha}{2}\right)}{\left[(3\pi)^2 - \left(k_0\beta b \sin\alpha\right)^2\right]\left[(\pi)^2 - \left(k_0\beta b \sin\alpha\right)^2\right]}\right\}$$

$$A_{33}(\alpha,\beta) = -j \frac{2k_0{}^2 ab}{\frac{Y_{01}}{Y_0}} \left[\frac{\sin\left(\frac{k_0\beta a \cos\alpha}{2}\right)}{\frac{k_0\beta a \cos\alpha}{2}}\right]^2 \left[\frac{3\cos\left(\frac{k_0\beta b \sin\alpha}{2}\right)}{(3\pi)^2 - \left(k_0\beta b \cos\alpha\right)^2}\right]^2$$

ADMITTANCE EXPRESSION

f_1	f_2	f_3
$\dfrac{N^2\left(1 + jN \dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}} \tan k_0 Z_o \sqrt{N^2-\beta^2}\right)}{\sqrt{N^2-\beta^2}\left(\tan k_0 Z_o \sqrt{N^2-\beta^2} - jN^2 \dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\right)}$	$\dfrac{N^2\left(1 + N^2 \dfrac{\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}} \tan k_0 Z_o \sqrt{N^2-\beta^2}\right)}{\sqrt{N^2-\beta^2}\left(\tan k_0 Z_o \sqrt{N^2-\beta^2} - N^2 \dfrac{\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}\right)}$	
$\dfrac{N^2\left(1 + jN^2 \dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}} \tan k_0 Z_o \sqrt{N^2-\beta^2}\right)}{\sqrt{N^2-\beta^2}\left(\tan k_0 Z_o \sqrt{N^2-\beta^2} - jN^2 \dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\right)}$	$\dfrac{N^2\left(1 + N^2 \dfrac{\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}} \tan k_0 Z_o \sqrt{N^2-\beta^2}\right)}{\sqrt{N^2-\beta^2}\left(\tan k_0 Z_o \sqrt{N^2-\beta^2} - N^2 \dfrac{\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}\right)}$	$-\dfrac{N^2\left(1 + N^2 \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}} \tanh k_0 Z_o \sqrt{\beta^2-N^2}\right)}{\sqrt{\beta^2-N^2}\left(\tanh k_0 Z_o \sqrt{\beta^2-N^2} + N^2 \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\right)}$
$\dfrac{N^2\left(1 + jN^2 \dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}} \tan k_0 Z_o \sqrt{N^2-\beta^2}\right)}{\sqrt{N^2-\beta^2}\left(\tan k_0 Z_o \sqrt{N^2-\beta^2} - jN^2 \dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\right)}$	$-\dfrac{N^2\left(1 + jN^2 \dfrac{\sqrt{1-\beta^2}}{\sqrt{\beta^2-N^2}} \tanh k_0 Z_o \sqrt{\beta^2-N^2}\right)}{\sqrt{\beta^2-N^2}\left(\tanh k_0 Z_o \sqrt{\beta^2-N^2} + jN^2 \dfrac{\sqrt{1-\beta^2}}{\sqrt{\beta^2-N^2}}\right)}$	$-\dfrac{N^2\left(1 + N^2 \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}} \tanh k_0 Z_o \sqrt{\beta^2-N^2}\right)}{\sqrt{\beta^2-N^2}\left(\tanh k_0 Z_o \sqrt{\beta^2-N^2} + N^2 \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\right)}$
$\dfrac{\lvert N^2\rvert\left(1 - j\lvert N^2\rvert \dfrac{\sqrt{1-\beta^2}}{\sqrt{\lvert N^2\rvert+\beta^2}} \tanh k_0 Z_o \sqrt{\lvert N^2\rvert+\beta^2}\right)}{\sqrt{\lvert N^2\rvert+\beta^2}\left(\tanh k_0 Z_o \sqrt{\lvert N^2\rvert+\beta^2} - j\lvert N^2\rvert \dfrac{\sqrt{1-\beta^2}}{\sqrt{\lvert N^2\rvert+\beta^2}}\right)}$	$\dfrac{\lvert N^2\rvert\left(1 - \lvert N^2\rvert \dfrac{\sqrt{\beta^2-1}}{\sqrt{\lvert N^2\rvert+\beta^2}} \tanh k_0 Z_o \sqrt{\lvert N^2\rvert+\beta^2}\right)}{\sqrt{\lvert N^2\rvert+\beta^2}\left(\tanh k_0 Z_o \sqrt{\lvert N^2\rvert+\beta^2} - \lvert N^2\rvert \dfrac{\sqrt{\beta^2-1}}{\sqrt{\lvert N^2\rvert+\beta^2}}\right)}$	

Lossy dielectric case $\left(\omega_p = \nu = 0\right)$:

$$\sqrt{N^2-\beta^2} = P - jQ$$

$$\tan k_0 Z_o \sqrt{N^2-\beta^2} = \frac{\tan k_0 Z_o P - j \tanh k_0 Z_o Q}{1 + j \tan k_0 Z_o P \tanh k_0 Z_o Q}$$

$$P = \frac{1}{\sqrt{2}}\left[\frac{\epsilon_R}{\epsilon_0} - \beta^2 + \sqrt{\left(\frac{\epsilon_R}{\epsilon_0} - \beta^2\right)^2 + \left(\frac{\epsilon_I}{\epsilon_0}\right)^2}\right]^{1/2}$$

$$Q = \frac{1}{\sqrt{2}}\left[-\frac{\epsilon_R}{\epsilon_0} + \beta^2 + \sqrt{\left(\frac{\epsilon_R}{\epsilon_0} - \beta^2\right)^2 + \left(\frac{\epsilon_I}{\epsilon_0}\right)^2}\right]^{1/2}$$

Lossy plasma case $\left(\frac{\epsilon_R}{\epsilon_0} = 1,\ \frac{\epsilon_I}{\epsilon_0} = 0\right)$:

$$\sqrt{N^2-\beta^2} = P - jQ$$

$$\tan k_0 Z_o \sqrt{N^2-\beta^2} = \frac{\tan k_0 Z_o P - j \tanh k_0 Z_o Q}{1 + j \tan k_0 Z_o P \tanh k_0 Z_o Q}$$

$$P = \frac{1}{\sqrt{2}}\left\{\left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\frac{\nu}{\omega}\right)^2} - \beta^2\right] + \sqrt{\left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\frac{\nu}{\omega}\right)^2} - \beta^2\right]^2 + \left[\frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\frac{\nu}{\omega}\right)^2}\right]^2}\right\}^{1/2}$$

$$Q = \frac{1}{\sqrt{2}}\left\{-\left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\frac{\nu}{\omega}\right)^2} - \beta^2\right] + \sqrt{\left[1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\frac{\nu}{\omega}\right)^2} - \beta^2\right]^2 + \left[\frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\frac{\nu}{\omega}\right)^2}\right]^2}\right\}^{1/2}$$

TABLE III.- NORMALIZED ADMITTANCE CALCULATIONS FOR PLASMA SLABS

Z_o, cm	$\left(\frac{\omega_p}{\omega}\right)^2$	ν/ω					
		0.004		0.04		0.4	
		Self	Aperture	Self	Aperture	Self	Aperture
0.5	0.6	0.3920 - j0.0310	0.3979 - j0.0289	0.4087 - j0.0443	0.4144 - j0.0423	0.5929 - j0.0990	0.5980 - j0.0989
1.0	0.6	0.2553 - j0.0281	0.2599 - j0.0239	0.2755 - j0.0414	0.2801 - j0.0375	0.4785 - j0.0950	0.4825 - j0.0928
2.0	0.6	0.2274 + j0.0703	0.2329 - j0.0767	0.2517 + j0.0496	0.2572 + j0.0555	0.4813 - j0.0388	0.4868 - j0.0364
3.0	0.6	0.2789 + j0.0343	0.2851 + j0.0295	0.2946 + j0.0187	0.3006 + j0.0232	0.4865 - j0.0550	0.4919 - j0.0529
0.5	1.2	0.1154 - j0.6376	0.1162 - j0.6362	0.1528 - j0.6361	0.1537 - j0.6348	0.4921 - j0.5824	0.4940 - j0.5819
1.0	1.2	0.0287 - j0.6255	0.0289 - j0.6235	0.0646 - j0.6260	0.0649 - j0.6240	0.3875 - j0.5759	0.3891 - j0.5745
2.0	1.2	0.0059 - j0.6155	0.0059 - j0.6133	0.0429 - j0.6151	0.0431 - j0.6129	0.3756 - j0.5550	0.3772 - j0.5533
3.0	1.2	0.0043 - j0.6143	0.0043 - j0.6121	0.0416 - j0.6138	0.0418 - j0.6116	0.3772 - j0.5544	0.3788 - j0.5527
0.5	4.0	0.0628 - j1.9783	0.0629 - j1.9782	0.1339 - j1.9663	0.1341 - j1.9661	0.5879 - j1.8670	0.5881 - j1.8669
1.0	4.0	0.0067 - j1.9995	0.0067 - j1.9993	0.0572 - j1.9981	0.0572 - j1.9979	0.5136 - j1.8598	0.5137 - j1.8596
2.0	4.0	0.0056 - j1.9999	0.0056 - j1.9997	0.0558 - j1.9983	0.0558 - j1.9981	0.5128 - j1.8587	0.5129 - j1.8585
3.0	4.0	0.0056 - j1.9989	0.0056 - j1.9997	0.0558 - j1.9984	0.0558 - j1.9982	0.5128 - j1.8587	0.5129 - j1.8585

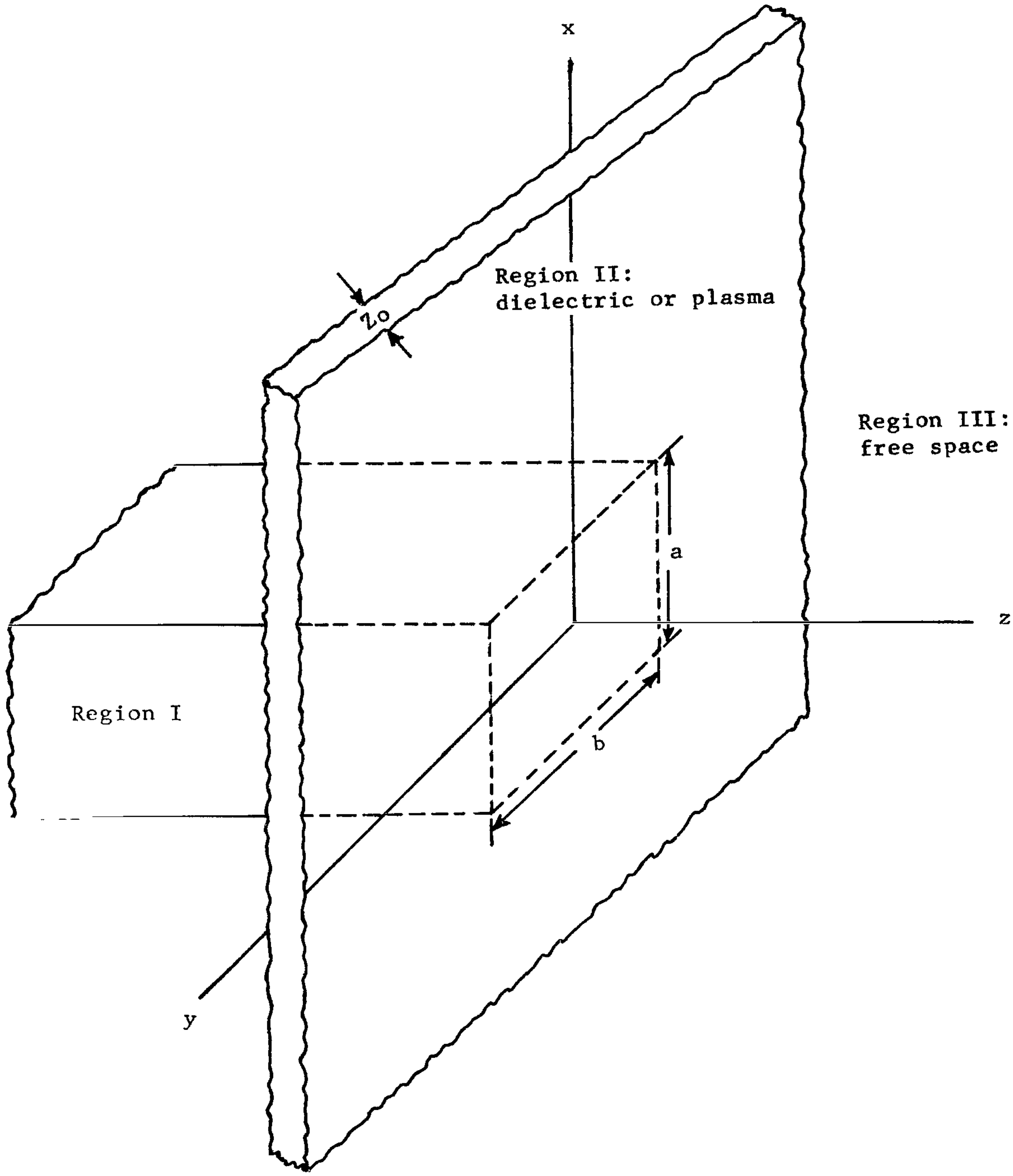
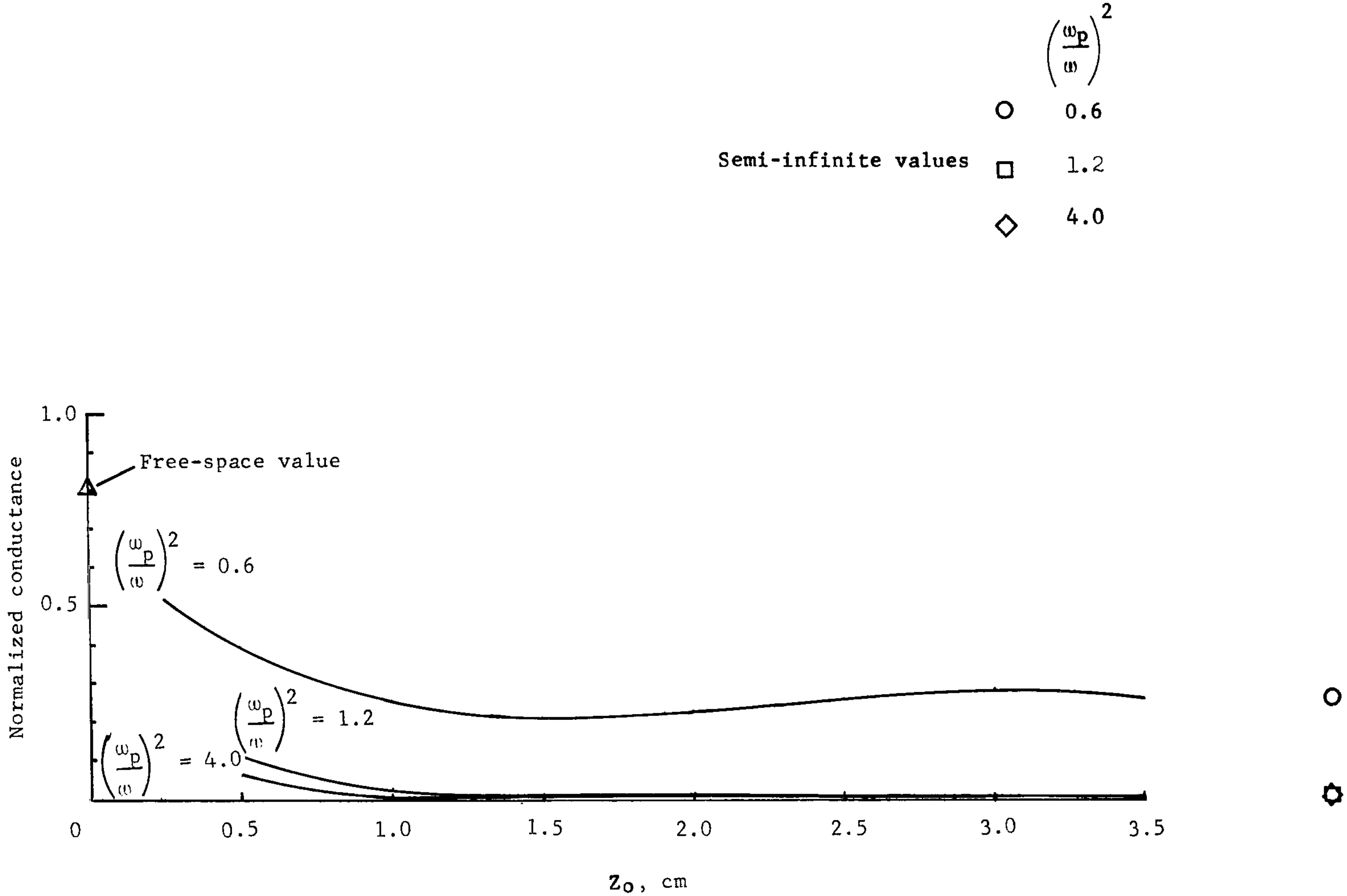
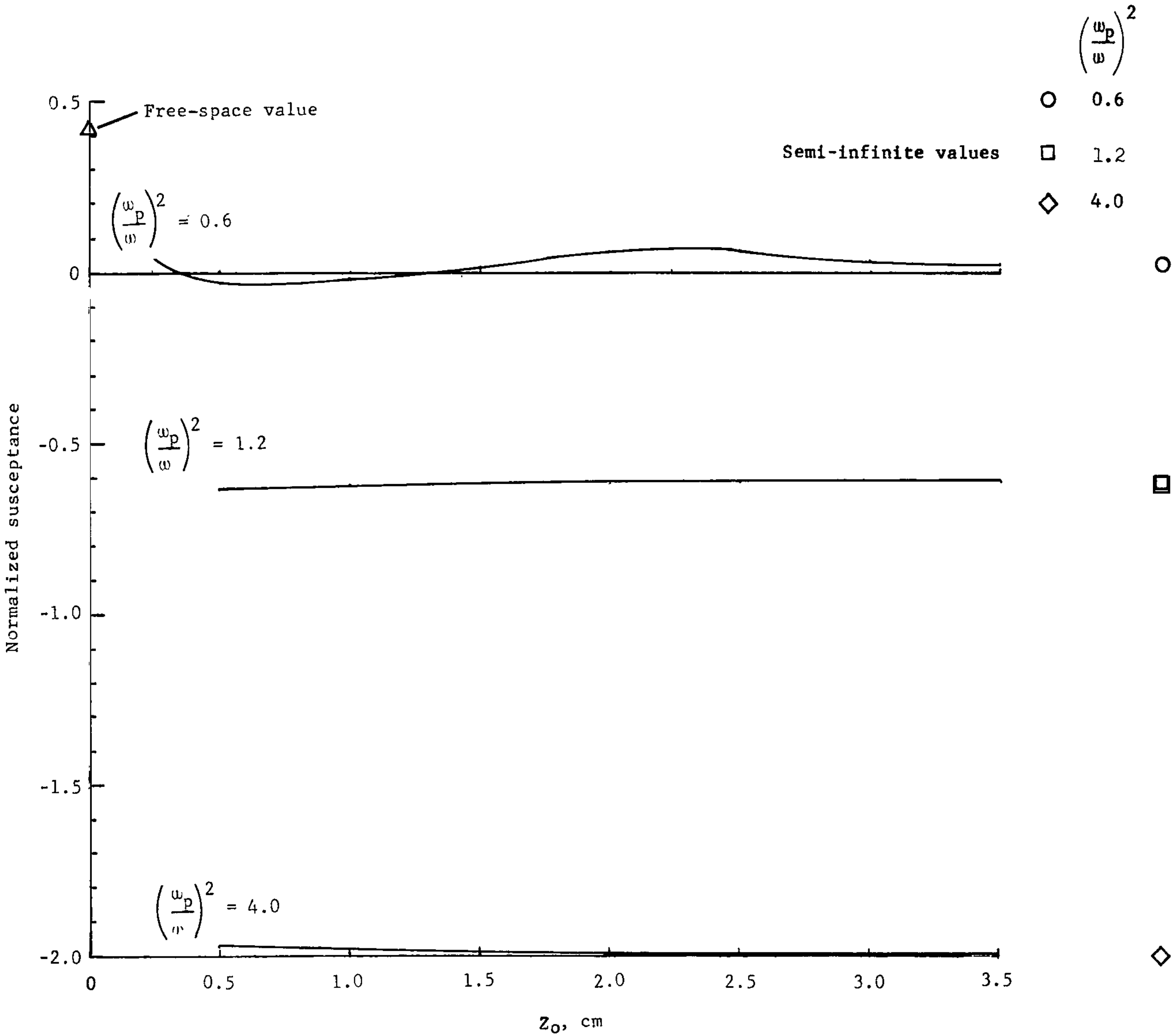


Figure 1.- Rectangular waveguide covered with a slab of homogeneous material of infinite extent in the x and y directions.



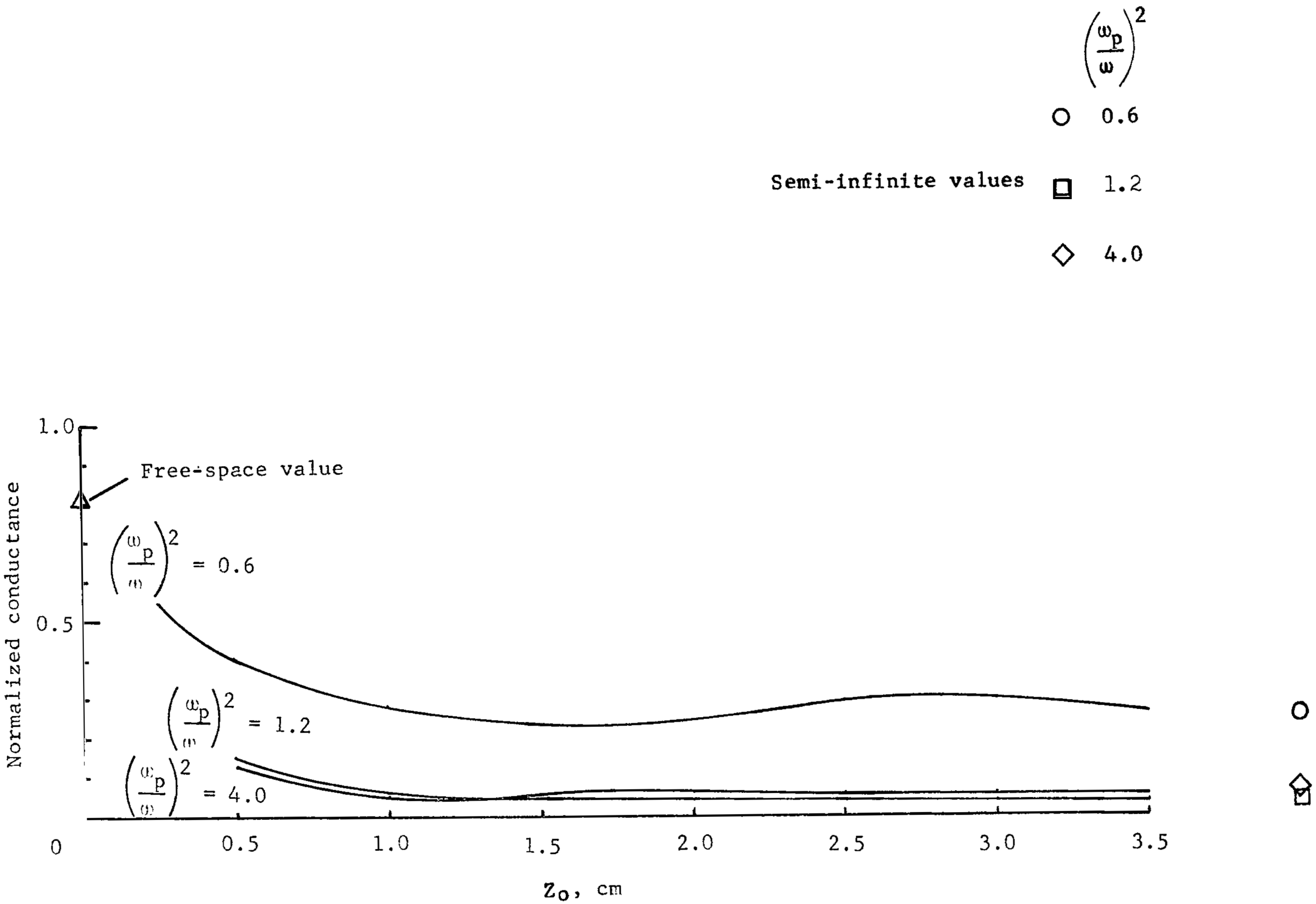
(a) Normalized conductance.

Figure 2.- Admittance calculations for a rectangular aperture, covered by varying thicknesses of plasma slabs having a collision-frequency ratio ν/ω of 0.004.



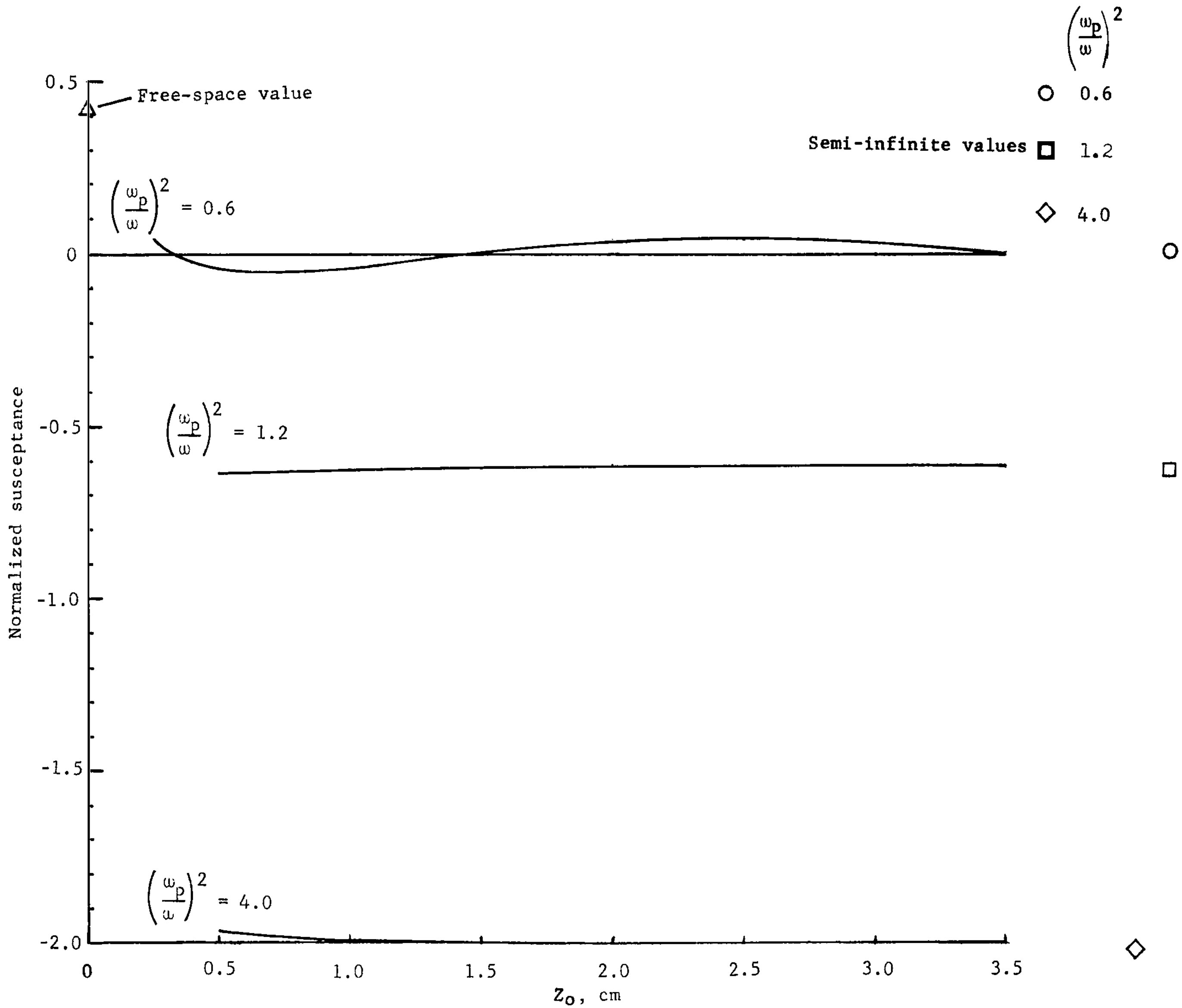
(b) Normalized susceptance.

Figure 2.- Concluded.



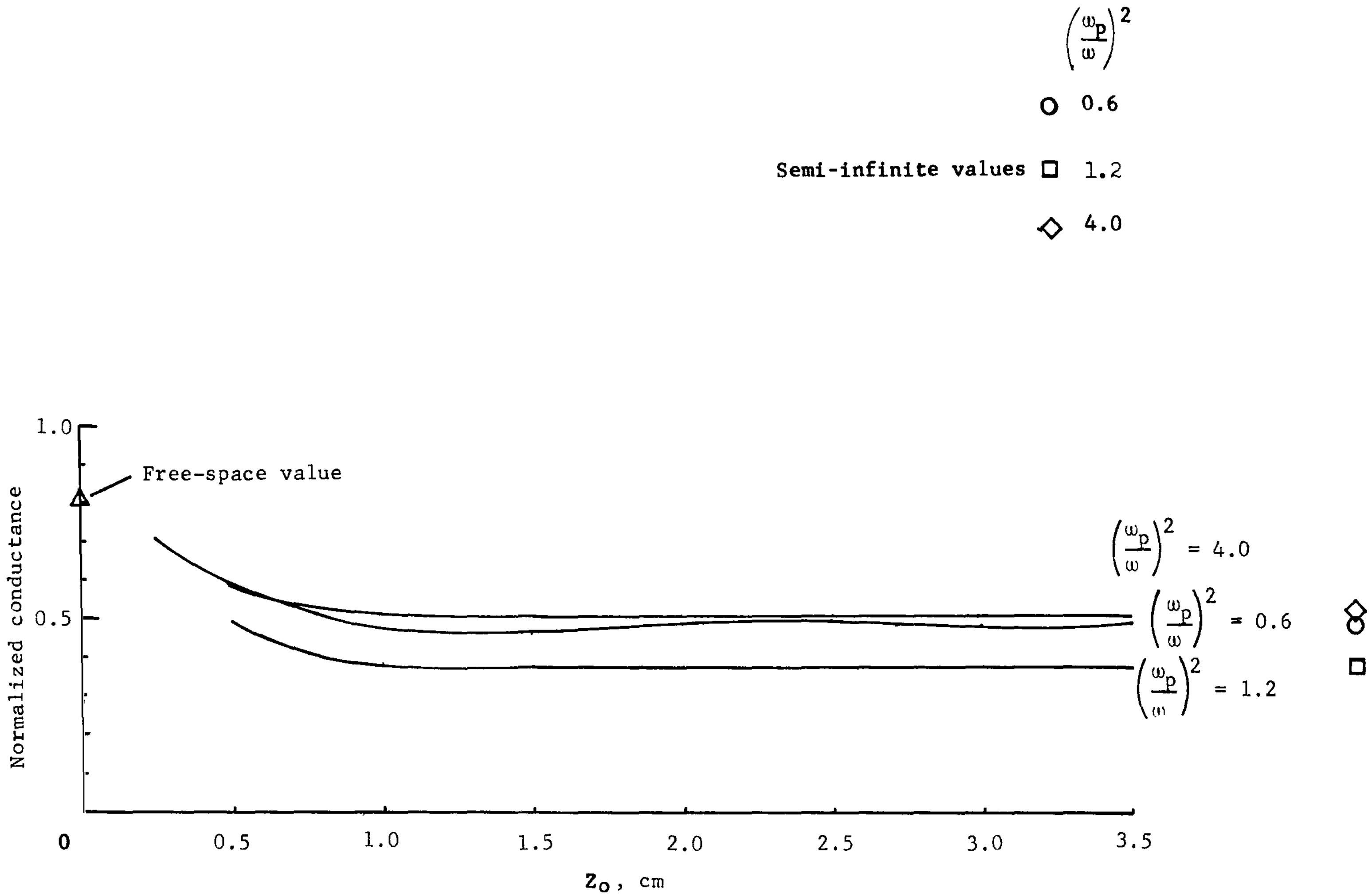
(a) Normalized conductance.

Figure 3.- Admittance calculations for a rectangular aperture, covered by varying thicknesses of plasma slabs having a collision-frequency ratio ν/ω of 0.04.



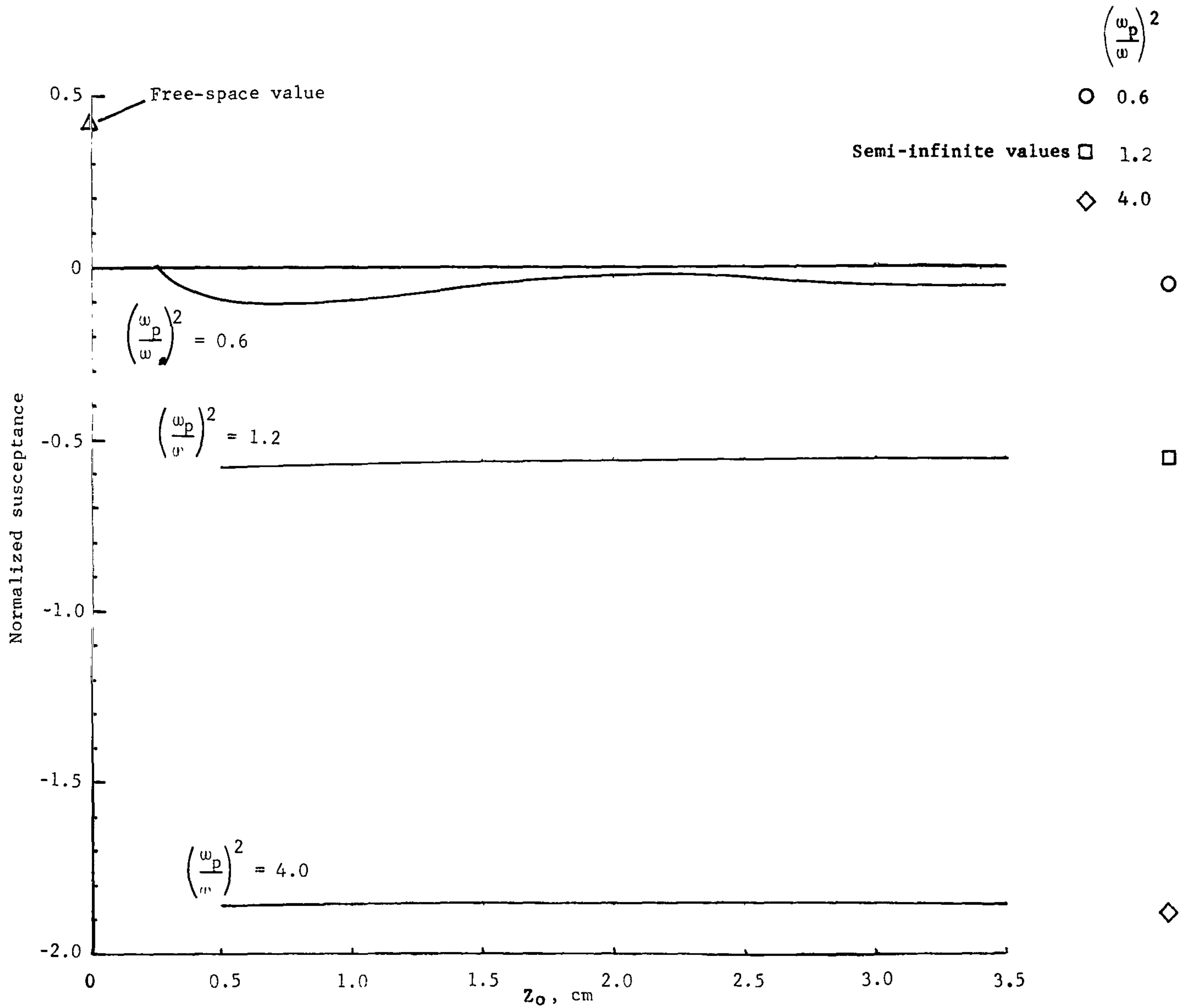
(b) Normalized susceptance.

Figure 3.- Concluded.



(a) Normalized conductance.

Figure 4.- Admittance calculations for a rectangular aperture, covered by varying thicknesses of plasma slabs having a collision-frequency ratio ν/ω of 0.4.



(b) Normalized susceptance.

Figure 4.- Concluded.

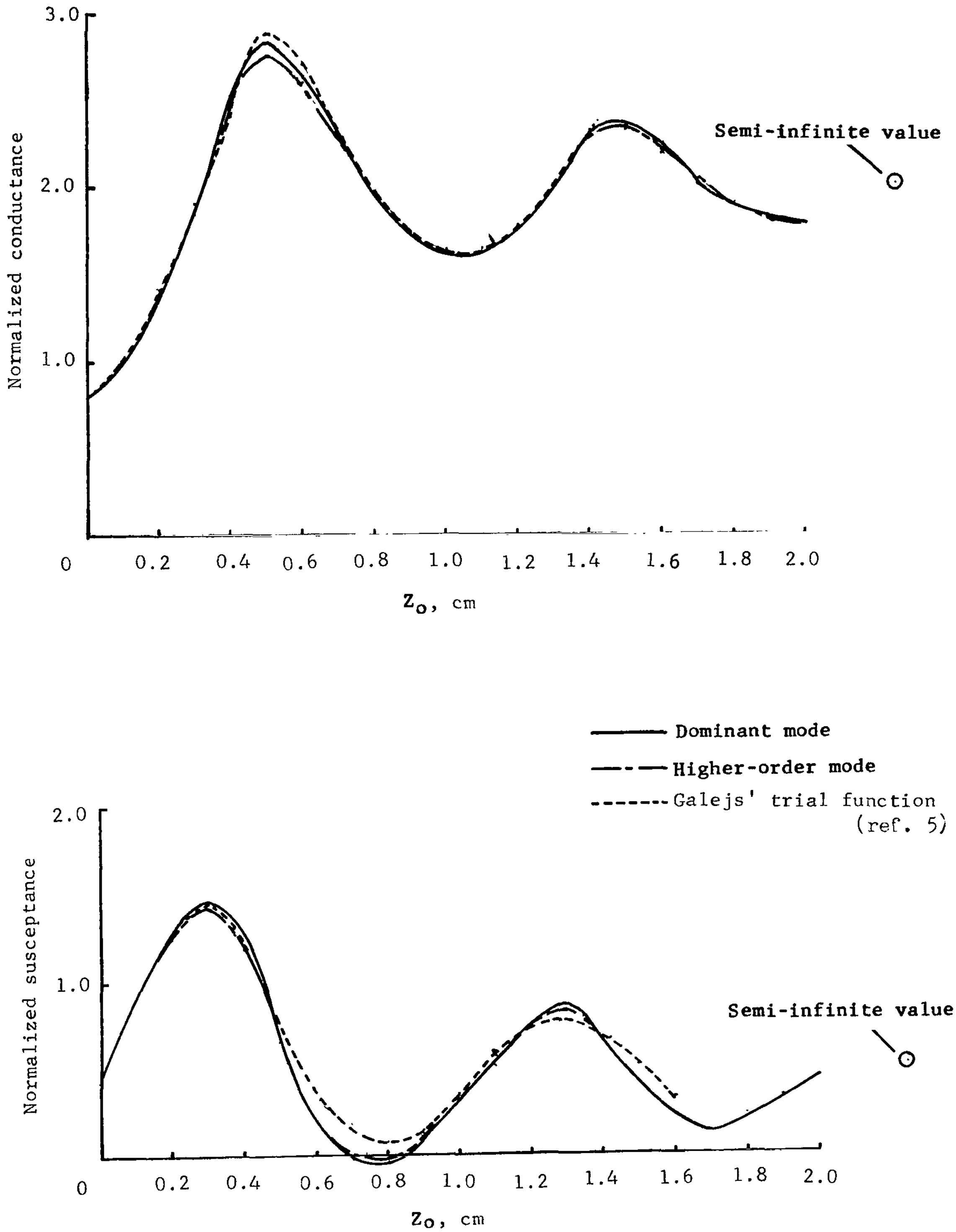


Figure 5.- Admittance values as a function of slab thickness for polystyrene material.

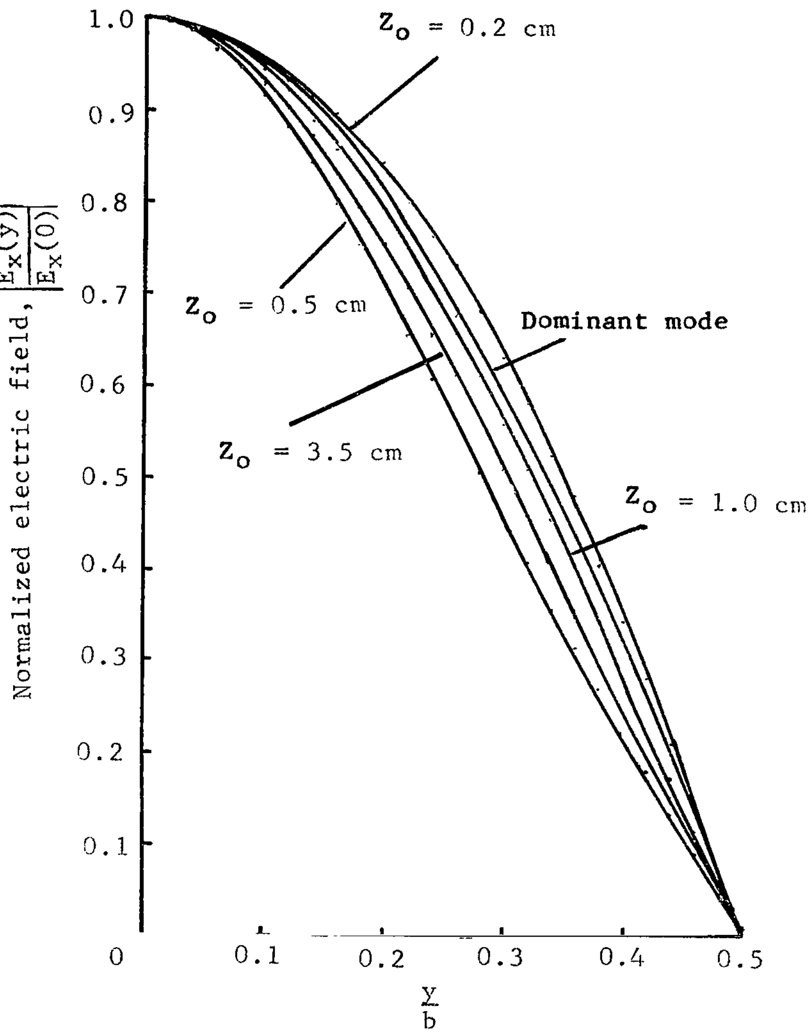
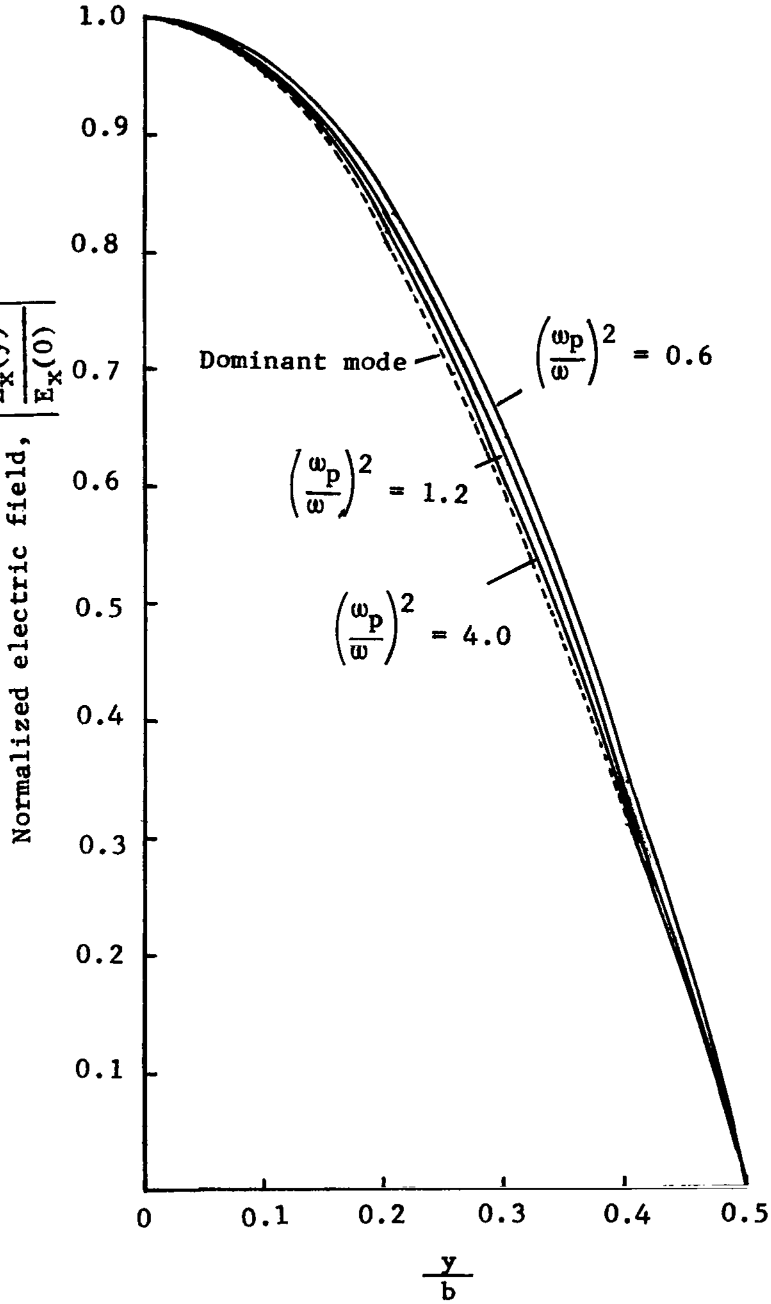
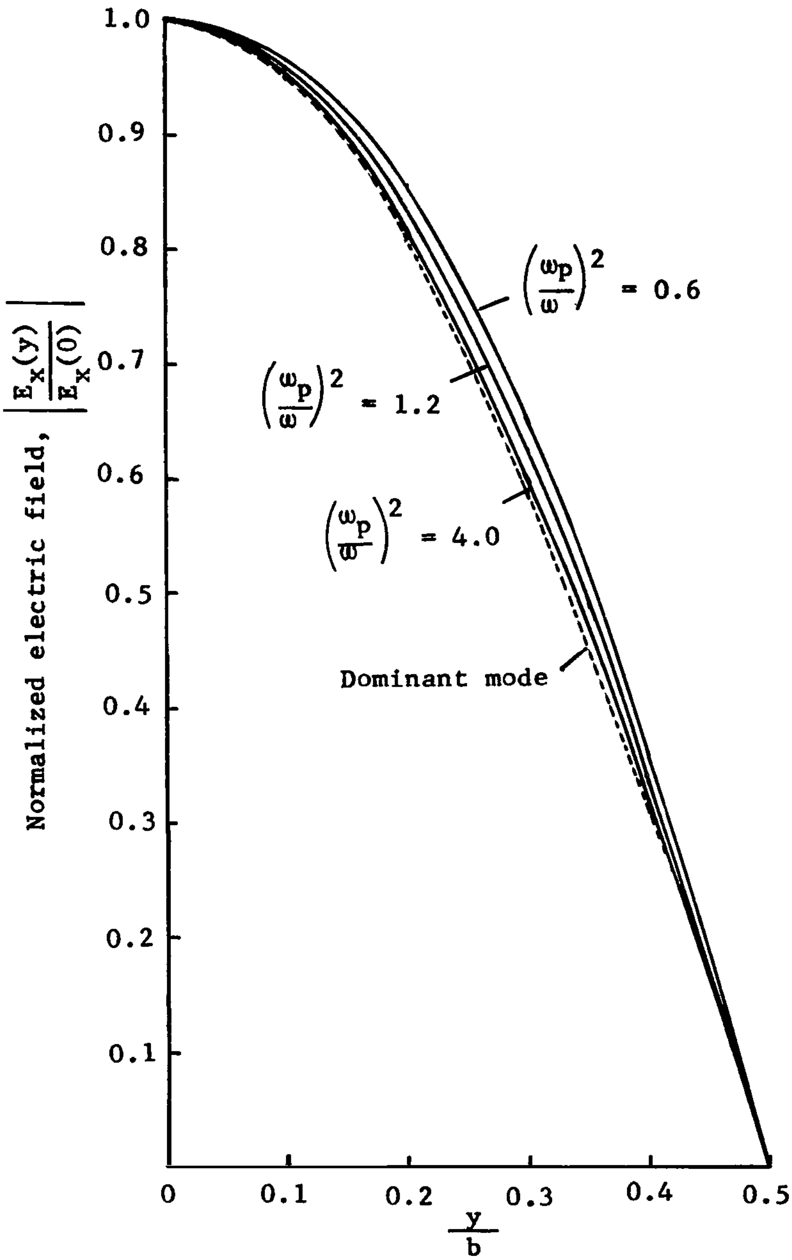


Figure 6.- Normalized electric field distribution across the rectangular aperture for varying thicknesses of polystyrene slabs.



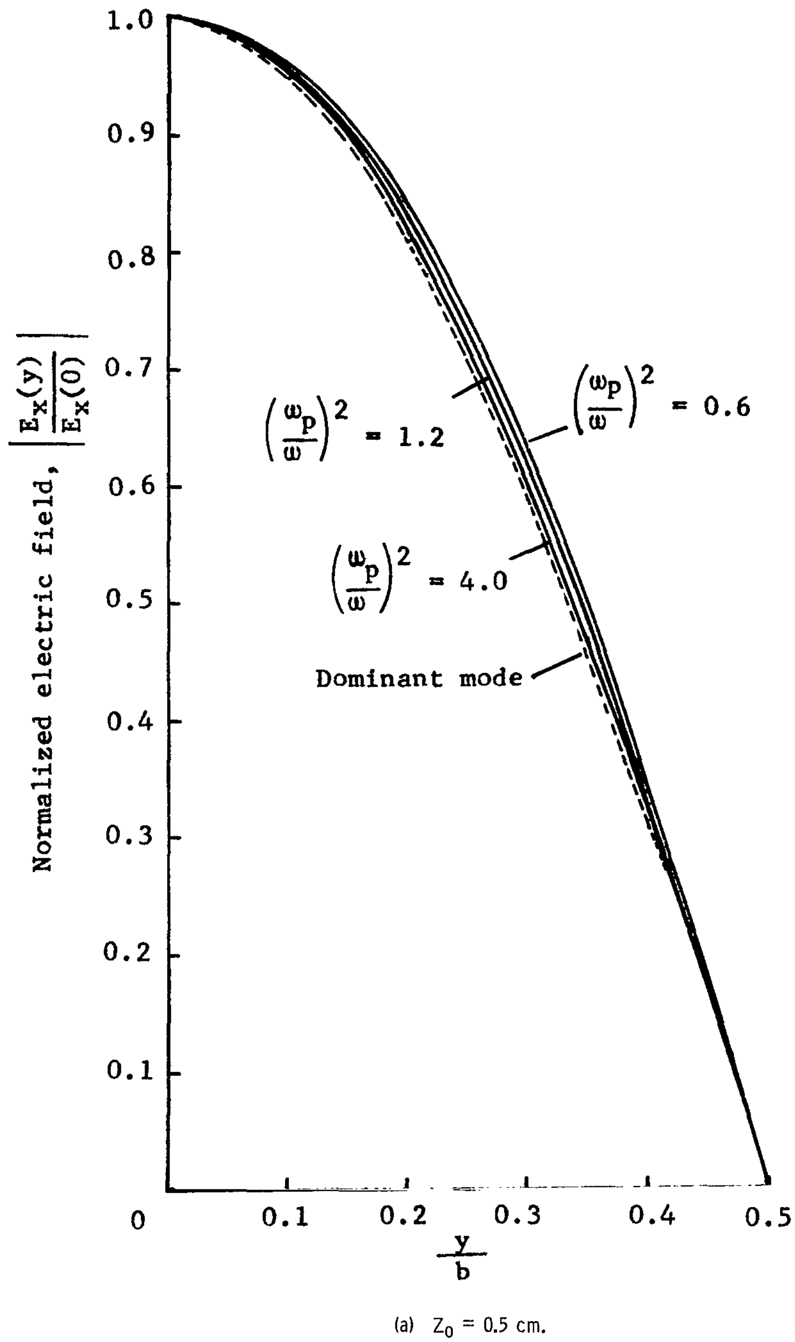
(a) $Z_0 = 0.5$ cm.

Figure 7.- Normalized electric field distribution across the rectangular aperture for varying electron densities $\left(\frac{\omega_p}{\omega}\right)^2$ with a collision-frequency ratio ν/ω of 0.004.



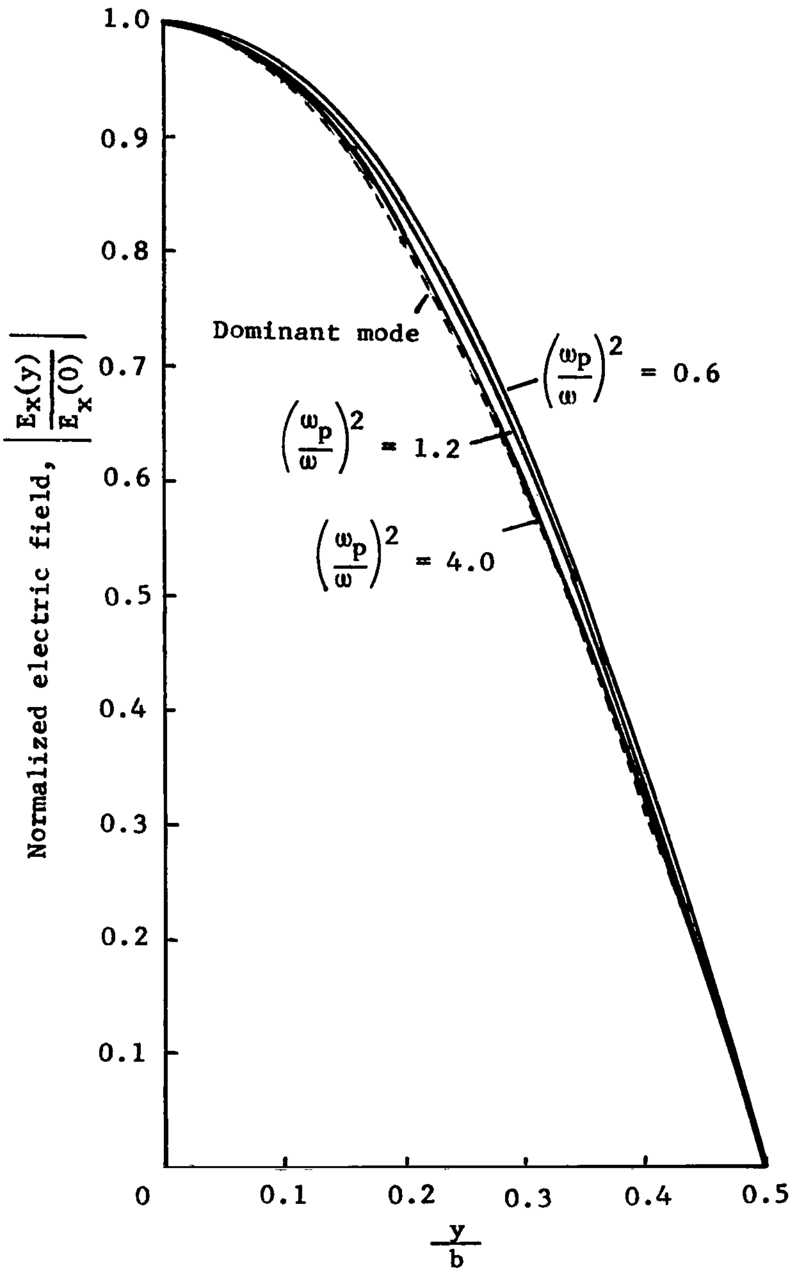
(b) $Z_0 = 3.0$ cm.

Figure 7.- Concluded.



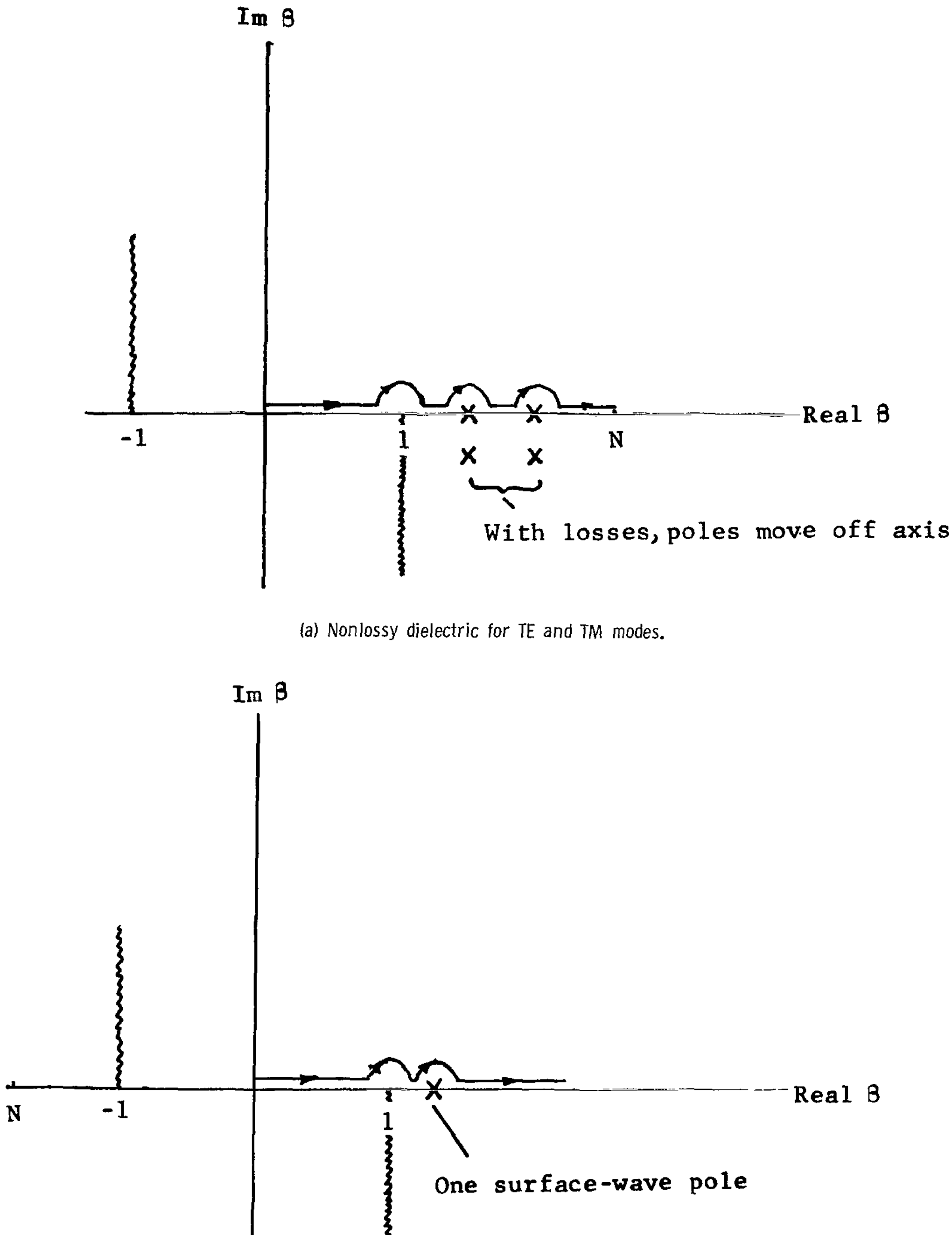
(a) $Z_0 = 0.5$ cm.

Figure 8.- Normalized electric field distribution across the rectangular aperture for varying electron densities $\left(\frac{\omega_p}{\omega}\right)^2$ with a collision-frequency ratio ν/ω of 0.4.



(b) $Z_0 = 3.0$ cm.

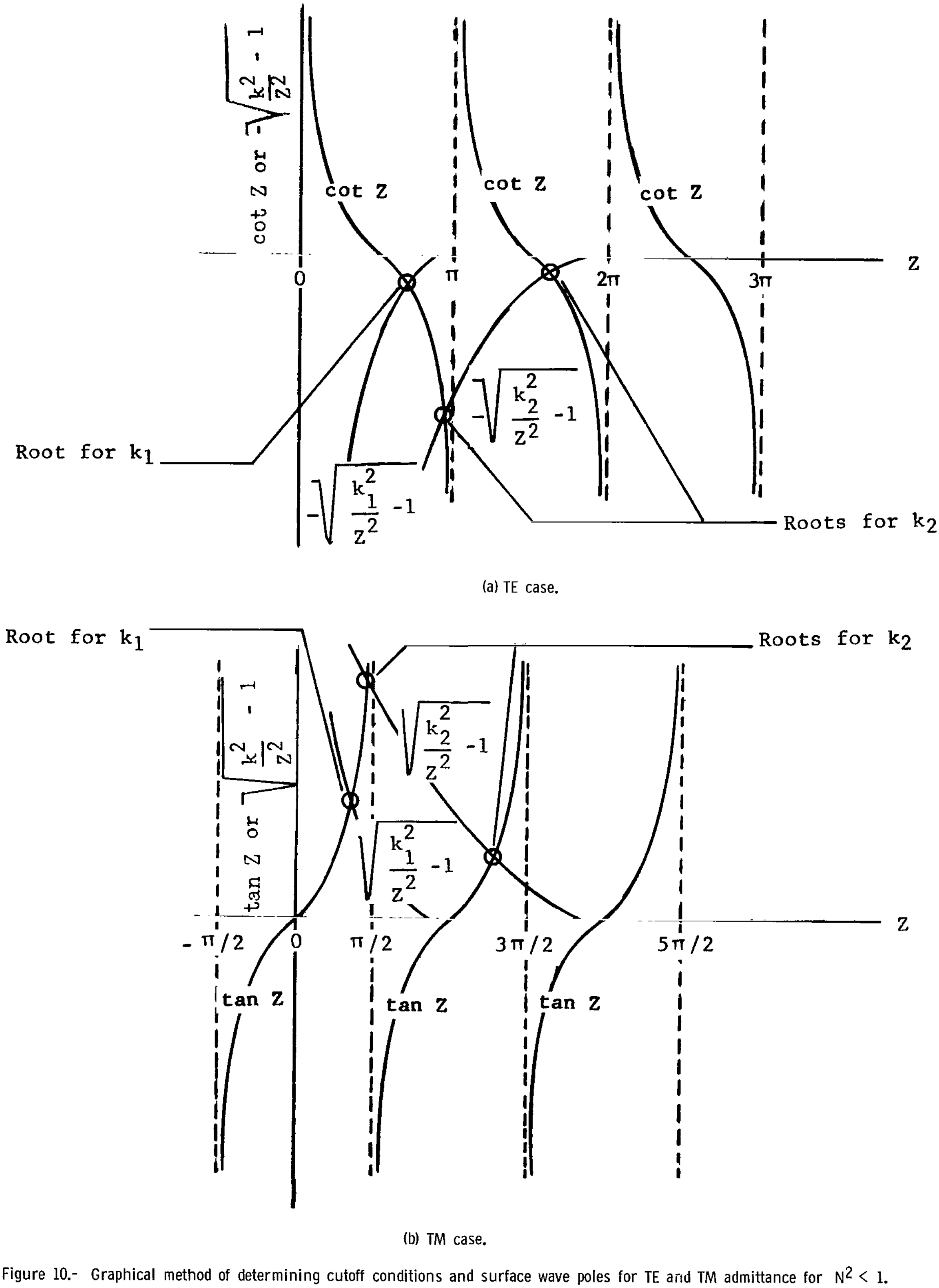
Figure 8.- Concluded.



(a) Nonlossy dielectric for TE and TM modes.

(b) Overdense nonlossy plasma for TM mode.

Figure 9.- Contour plots showing path and surface-wave poles for $N^2 > 1$ and $N^2 < -1$.



(a) TE case.

(b) TM case.

Figure 10.- Graphical method of determining cutoff conditions and surface wave poles for TE and TM admittance for $N^2 < 1$.

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