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by M. C. Bailey, S. N. Samaddar, and Calvin T. Swift

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ELECTROMAGNETIC PROPERTIES OF A CIRCULAR APERTURE IN A DIELECTRIC-COVERED OR UNCOVERED GROUND PLANE

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SUMMARY

The problem of an aperture fed by a circular waveguide excited in the dominant transverse electric mode (TE_{11}) is considered. The input admittance in the aperture reference plane for the case of a homogeneous lossy dielectric cover of finite thickness and the radiation patterns for the case of no cover are computed from integral transforms, for which a TE_{11} mode variation of the electric field is assumed at the aperture. Computations for the uncoated antenna show that the patterns and aperture admittance are similar to those of a square aperture with a field distribution given by the TE_{01} rectangular mode.

It is noted that the uncoated aperture is well matched above midband of the circular waveguide in contrast to the open-end rectangular waveguide. This inherent property would be advantageous for possible use of the open-end circular waveguide as an antenna.

INTRODUCTION

The antenna under consideration consists of a circular waveguide, opening onto a flat dielectric-covered ground plane as shown in figure 1. The dominant mode is incident upon the aperture, and reflected modes of higher order are assumed to be negligible. The fields inside the waveguide are therefore uniquely defined by a discrete TE (transverse electric) vector potential containing one unknown, the reflection coefficient Γ.

The fields inside the dielectric slab are represented by standing waves and are described by integral transforms of TE and TM (transverse magnetic) potentials, each containing two unknown coefficients and each potential exterior to the dielectric slab containing one unknown coefficient. The boundary conditions at the aperture and at the outer surface of the dielectric slab are used to solve for the unknowns. This solution results in an integral expression for Γ, or equivalently, the input admittance at the aperture

*Raytheon Company, Space and Information Systems Division. (Dr. Samaddar's collaboration with the Langley authors resulted from recognition during their separate endeavors of mutual interest in the same problem.)

reference plane, $z = 0$. (Further mention of input admittance is referred to this plane.) When the permittivity of the dielectric medium ϵ_1 equals the permittivity of free space ϵ_o, or when the thickness of the dielectric coating d is zero, the expression correctly reduces to that derived by Mishustin (ref. 1), who used a Green's function formalism; however, the integral transform method appears to be more easily applied to the slab problem.

The radiation fields for the case of no dielectric coating are derived by using the method of stationary phase, and they are in agreement with the patterns computed by modifying the expressions given by Risser (ref. 2). Numerical values of patterns and admittance for the aperture with no coating are plotted and compared with those of a square aperture of equal area, excited in the TE_{01} rectangular waveguide mode.

SYMBOLS

A_z	z-component of vector potential, electric type
A_z^*	z-component of vector potential, magnetic type
a	waveguide radius
b_{in}	normalized input aperture susceptance
d	thickness of dielectric coating
$\vec{E}$	electric field intensity
g_{in}	normalized input aperture conductance
g_s	normalized surface wave conductance
$\vec{H}$	magnetic field intensity
$j = \sqrt{-1}$	
$J_1(\xi\rho)$	Bessel function of first kind of order 1
$k_z{}^{11}$	axial TE_{11} wave number in the guide, $k_o\sqrt{1 - \left(\frac{x_{11}}{k_o a}\right)^2}$

k_{z1}	axial wave number in the dielectric medium
k_{zo}	axial wave number exterior to the structure
k_o	propagation constant in free space, $2\pi/\lambda_o$
k_1	propagation constant in the dielectric medium, $\frac{2\pi}{\lambda_o}\sqrt{\frac{\epsilon_1}{\epsilon_o}}$
N	index of refraction of dielectric slab, k_1/k_o
P	complex power
r, θ, ϕ	spherical coordinates
t	time
$\hat{u}_z$	unit vector in z direction
x,y,z	Cartesian coordinates
x_{11}	first root of $J_1'(x_{11}) = 0$
y_{in}	normalized input admittance of aperture, $g_{in} - jb_{in}$
y_o	admittance of free space
β	normalized radial wave number, ξ/k_o
β_n	location of surface wave poles in complex β-plane
Γ	reflection coefficient
$\vec{\nabla}$	differential vector operator
$\delta(\xi-\zeta)$	Dirac delta function
ϵ_o	permittivity of free space

ϵ_1	complex permittivity of the dielectric medium
ζ	radial wave number exterior to waveguide (used as a continuous dummy variable)
λ_o	wavelength in free space
μ_o	permeability of free space
ξ	radial wave number exterior to waveguide
ρ,ϕ,z	cylindrical coordinates
ω	angular frequency

Intermediate functions:

$I_1(\xi),R_1(\xi)$	coefficients of magnetic vector potential in the dielectric slab
$I_2(\xi),R_2(\xi)$	coefficients of electric vector potential in the dielectric slab
$T_1(\xi)$	coefficient of magnetic vector potential external to the structure
$T_2(\xi)$	coefficient of electric vector potential external to the structure
$X_1(\xi),X_2(\xi),X_3(\xi),X_4(\xi),W(\xi)$	expressions given by equations (14) to (18) to simplify the solution to the boundary-value problem

Subscripts:

ρ,ϕ,z	vector components in cylindrical coordinates
r,θ,ϕ	vector components in spherical coordinates
n	order of surface wave mode

Superscripts:

I	denotes field components and vector potentials inside waveguide

II denotes field components and vector potentials in the dielectric region

III denotes field components and vector potentials in the region outside the dielectric slab

* complex conjugate, except when used in connection with the magnetic vector potentials

THEORY

Statement of Problem and Method of Approach

The geometry of the problem is shown in figure 1. A circular waveguide, fed in the dominant TE_{11} mode opens onto an infinite, perfectly conducting flat ground plane coated with dielectric material of thickness d and complex index of refraction N. The medium inside the waveguide and external to the structure is assumed to be free space. It is also assumed that no higher-order waveguide modes are excited at the aperture and that the fields everywhere vary in time as $e^{-j\omega t}$.

In order to derive the input admittance expression for the circular waveguide aperture, the fields in each region are constructed from vector potentials which are solutions to the wave equation in cylindrical coordinates. Implicit relations between the unknown coefficients are obtained by equating tangential $\vec{E}$ and $\vec{H}$ fields at the boundary, $z = d$, and requiring that tangential $\vec{E}$ be continuous at the aperture, $z = 0$. The integrals are evaluated with the use of the Dirac delta function δ for Hankel transforms. The input admittance is then found by equating the complex power expressions at the aperture, and the solution is expressed in a form more convenient for numerical integration.

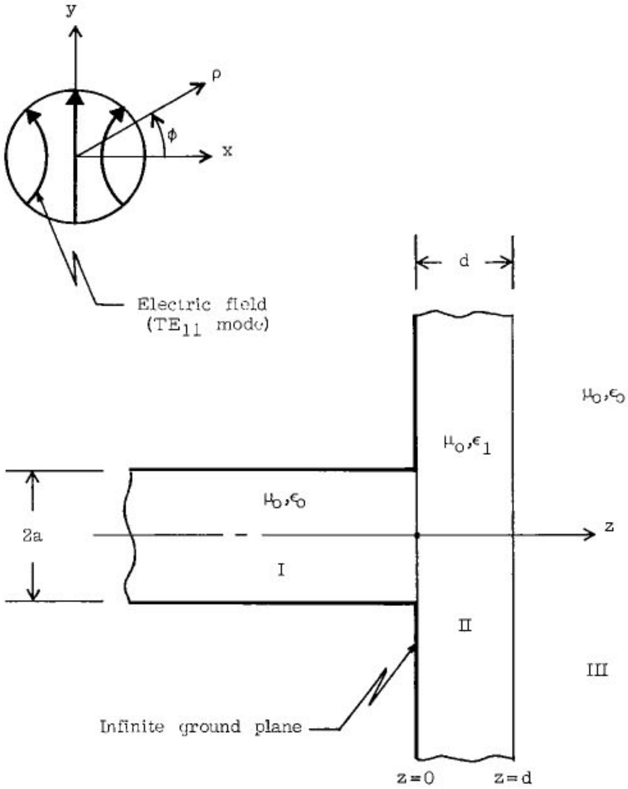

Figure 1.- Circular aperture coated with homogeneous lossy dielectric.

Fields and Vector Potentials

Since only the dominant TE_{11} mode is assumed to exist inside the waveguide, the fields for $z \leqq 0$ can be uniquely described in terms of a single vector potential as follows:

$$\left.\begin{aligned} \vec{E}^{I} &= -\vec{\nabla} \times \left(\frac{A_z^{*I}}{\epsilon_o}\right)\hat{u}_z \\ \vec{H}^{I} &= -\frac{1}{j\omega\mu_o}\vec{\nabla} \times \left[\vec{\nabla} \times \left(\frac{A_z^{*I}}{\epsilon_o}\right)\hat{u}_z\right] \end{aligned}\right\} \tag{1}$$

whereas, for $z > 0$, the fields must be described by both TE and TM potentials such that

$$\left.\begin{aligned} \vec{E}^{II} &= -\vec{\nabla} \times \left(\frac{A_z^{*II}}{\epsilon_1}\right)\hat{u}_z - \frac{1}{j\omega\mu_o\epsilon_1}\vec{\nabla} \times \left[\vec{\nabla} \times \left(A_z^{II}\right)\hat{u}_z\right] \\ \vec{H}^{II} &= -\frac{1}{j\omega\mu_o}\vec{\nabla} \times \left[\vec{\nabla} \times \left(\frac{A_z^{*II}}{\epsilon_1}\right)\hat{u}_z\right] + \frac{1}{\mu_o}\vec{\nabla} \times \left(A_z^{II}\right)\hat{u}_z \end{aligned}\right\} \tag{2}$$

and

$$\left.\begin{aligned} \vec{E}^{III} &= -\vec{\nabla} \times \left(\frac{A_z^{*III}}{\epsilon_o}\right)\hat{u}_z - \frac{1}{j\omega\mu_o\epsilon_o}\vec{\nabla} \times \left[\vec{\nabla} \times \left(A_z^{III}\right)\hat{u}_z\right] \\ \vec{H}^{III} &= -\frac{1}{j\omega\mu_o}\vec{\nabla} \times \left[\vec{\nabla} \times \left(\frac{A_z^{*III}}{\epsilon_o}\right)\hat{u}_z\right] + \frac{1}{\mu_o}\vec{\nabla} \times \left(A_z^{III}\right)\hat{u}_z \end{aligned}\right\} \tag{3}$$

where A_z and A_z^*, each of which satisfies the wave equation, are defined by Stratton (ref. 3).

Since all higher-order TE and TM reflected modes are assumed negligible, the solution for $z < 0$ is given by

$$\frac{A_z^{*I}}{\epsilon_o} = A\left(e^{jk_z^{11}z} + \Gamma e^{-jk_z^{11}z}\right) J_1\left(\frac{x_{11}\rho}{a}\right)\cos\phi \tag{4}$$

where the argument of the radial dependence of equation (4) follows from the requirement that tangential $\vec{E} = 0$ at $\rho = a$. The general solution to the wave equation in regions II and III requires that both sets of potentials contain a Fourier series in ϕ. However, from the boundary conditions, all terms except the coefficient of $\cos\phi$ in the series for $\frac{A_z^{*II}}{\epsilon_1}$ and $\frac{A_z^{*III}}{\epsilon_o}$ and the coefficient of $\sin\phi$ in the series for A_z^{II} and A_z^{III} must vanish.

With this a priori information, a straightforward separation of variables and an integration over radial modes ξ leads to solutions of the form

$$\left.\begin{aligned} \frac{A_z^{*II}}{\epsilon_1} &= \cos\phi \int_0^\infty \xi J_1(\xi\rho)\left[I_1(\xi)e^{jk_{z1}z} + R_1(\xi)e^{-jk_{z1}z}\right]d\xi \\ A_z^{II} &= \sin\phi \int_0^\infty \xi J_1(\xi\rho)\left[I_2(\xi)e^{jk_{z1}z} + R_2(\xi)e^{-jk_{z1}z}\right]d\xi \end{aligned}\right\} \quad (5)$$

and

$$\left.\begin{aligned} \frac{A_z^{*III}}{\epsilon_o} &= \cos\phi \int_0^\infty \xi T_1(\xi)J_1(\xi\rho)e^{jk_{zo}z}d\xi \\ A_z^{III} &= \sin\phi \int_0^\infty \xi T_2(\xi)J_1(\xi\rho)e^{jk_{zo}z}d\xi \end{aligned}\right\} \quad (6)$$

where the Bessel function of the first kind is required to assure finite fields at $\rho = 0$. Equations (5) and (6) can also be obtained by performing the Hankel transform of the wave equations satisfied by A_z and A_z^*/ϵ with respect to the kernel $J_1(\xi\rho)$.

The radiation condition is guaranteed if

$$\left.\begin{aligned} k_{zo} &= \sqrt{k_o^2 - \xi^2} \qquad & (k_o^2 > \xi^2) \\ k_{zo} &= j\sqrt{\xi^2 - k_o^2} \qquad & (k_o^2 < \xi^2) \end{aligned}\right\} \quad (7)$$

where $k_o^2 = \omega^2\mu_o\epsilon_o$ and

$$\left.\begin{aligned} k_{z1} &= \sqrt{k_1^2 - \xi^2} \qquad & (k_1^2 > \xi^2) \\ k_{z1} &= j\sqrt{\xi^2 - k_1^2} \qquad & (k_1^2 < \xi^2) \end{aligned}\right\} \quad (8)$$

where $k_1^2 = \omega^2\mu_o\epsilon_1$.

When equations (4), (5), and (6) are substituted into equations (1), (2), and (3), respectively, explicit expressions for the fields in terms of the unknown coefficients are obtained. These field equations are given in the appendix.

Evaluation of Coefficients

The unknown coefficients now remain to be solved. For this purpose, appropriate boundary conditions are applied at $z = 0$ and $z = d$. Continuity of the tangential components of the fields at $z = d$ leads to the four equations:

(1) Continuity of E_ϕ:

$$e^{jk_{z1}d}\left[I_1(\xi)\frac{d}{d\rho}J_1(\xi\rho) - I_2(\xi)\frac{k_{z1}}{\omega\mu_o\epsilon_1}\frac{J_1(\xi\rho)}{\rho}\right]$$

$$+ e^{-jk_{z1}d}\left[R_1(\xi)\frac{d}{d\rho}J_1(\xi\rho) + R_2(\xi)\frac{k_{z1}}{\omega\mu_o\epsilon_1}\frac{J_1(\xi\rho)}{\rho}\right] = e^{jk_{zo}d}\left[T_1(\xi)\frac{dJ_1(\xi\rho)}{d\rho} - T_2(\xi)\frac{k_{zo}}{\omega\mu_o\epsilon_o}\frac{1}{\rho}J_1(\xi\rho)\right] \quad (9a)$$

(2) Continuity of E_ρ:

$$e^{jk_{z1}d}\left[I_1(\xi)\frac{J_1(\xi\rho)}{\rho} - I_2(\xi)\frac{k_{z1}}{\omega\mu_o\epsilon_1}\frac{d}{d\rho}J_1(\xi\rho)\right]$$

$$+ e^{-jk_{z1}d}\left[R_1(\xi)\frac{J_1(\xi\rho)}{\rho} + R_2(\xi)\frac{k_{z1}}{\omega\mu_o\epsilon_1}\frac{d}{d\rho}J_1(\xi\rho)\right] = e^{jk_{zo}d}\left[T_1(\xi)\frac{J_1(\xi\rho)}{\rho} - T_2(\xi)\frac{k_{zo}}{\omega\mu_o\epsilon_o}\frac{d}{d\rho}J_1(\xi\rho)\right] \quad (9b)$$

(3) Continuity of H_ϕ:

$$e^{jk_{z1}d}\left[I_1(\xi)\frac{k_{z1}}{\omega\mu_o}\frac{J_1(\xi\rho)}{\rho} - I_2(\xi)\frac{1}{\mu_o}\frac{d}{d\rho}J_1(\xi\rho)\right]$$

$$- e^{-jk_{z1}d}\left[R_1(\xi)\frac{k_{z1}}{\omega\mu_o}\frac{J_1(\xi\rho)}{\rho} + R_2(\xi)\frac{1}{\mu_o}\frac{d}{d\rho}J_1(\xi\rho)\right] = e^{jk_{zo}d}\left[T_1(\xi)\frac{k_{zo}}{\omega\mu_o}\frac{J_1(\xi\rho)}{\rho} - T_2(\xi)\frac{1}{\mu_o}\frac{d}{d\rho}J_1(\xi\rho)\right] \quad (9c)$$

(4) Continuity of H_ρ:

$$e^{jk_{z1}d}\left[I_1(\xi)\frac{k_{z1}}{\omega\mu_o}\frac{d}{d\rho}J_1(\xi\rho) - I_2(\xi)\frac{1}{\mu_o}\frac{J_1(\xi\rho)}{\rho}\right]$$

$$- e^{-jk_{z1}d}\left[R_1(\xi)\frac{k_{z1}}{\omega\mu_o}\frac{d}{d\rho}J_1(\xi\rho) + R_2(\xi)\frac{1}{\mu_o}\frac{J_1(\xi\rho)}{\rho}\right] = e^{jk_{zo}d}\left[T_1(\xi)\frac{k_{zo}}{\omega\mu_o}\frac{d}{d\rho}J_1(\xi\rho) - T_2(\xi)\frac{1}{\mu_o}\frac{J_1(\xi\rho)}{\rho}\right] \quad (9d)$$

Solving equations (9a) through (9d) by the method of determinants yields

$$R_1(\xi) = \frac{X_1(\xi)}{W(\xi)}\Big[I_1(\xi) + R_1(\xi)\Big] \tag{10}$$

$$R_2(\xi) = \frac{X_2(\xi)}{W(\xi)}\Big[I_2(\xi) - R_2(\xi)\Big] \tag{11}$$

$$T_1(\xi) = \frac{X_3(\xi)}{W(\xi)}\Big[I_1(\xi) + R_1(\xi)\Big] \tag{12}$$

$$T_2(\xi) = \frac{X_4(\xi)}{W(\xi)}\Big[I_2(\xi) - R_2(\xi)\Big] \tag{13}$$

where

$$W(\xi) = k_{z1}k_{zo}\left[\epsilon_1\left(e^{jk_{z1}d} + e^{-jk_{z1}d}\right)^2 + \epsilon_o\left(e^{jk_{z1}d} - e^{-jk_{z1}d}\right)^2\right] - \left(\epsilon_o k_{z1}{}^2 + \epsilon_1 k_{zo}{}^2\right)\left(e^{2jk_{z1}d} - e^{-2jk_{z1}d}\right) \tag{14}$$

$$X_1(\xi) = \left(k_{z1} - k_{zo}\right)\left[e^{2jk_{z1}d}\left(\epsilon_1 k_{zo} - \epsilon_o k_{z1}\right) + \epsilon_1 k_{zo} + \epsilon_o k_{z1}\right] \tag{15}$$

$$X_2(\xi) = \left(\epsilon_o k_{z1} - \epsilon_1 k_{zo}\right)\left[e^{2jk_{z1}d}\left(k_{z1} - k_{zo}\right) + k_{z1} + k_{zo}\right] \tag{16}$$

$$X_3(\xi) = k_{z1}e^{jk_{z1}d}e^{-jk_{zo}d}\left[\epsilon_1 k_{zo}\left(e^{jk_{z1}d} + e^{-jk_{z1}d}\right)^2 + \epsilon_o k_{z1}\left(e^{jk_{z1}d} - e^{-jk_{z1}d}\right)^2 - \left(\epsilon_o k_{z1} + \epsilon_1 k_{zo}\right)\left(e^{2jk_{z1}d} - e^{-2jk_{z1}d}\right)\right] \tag{17}$$

$$X_4(\xi) = \epsilon_o k_{z1}e^{jk_{z1}d}e^{-jk_{zo}d}\left[k_{z1}\left(e^{jk_{z1}d} + e^{-jk_{z1}d}\right)^2 + k_{zo}\left(e^{jk_{z1}d} - e^{-jk_{z1}d}\right)^2 - \left(k_{zo} + k_{z1}\right)\left(e^{2jk_{z1}d} - e^{-2jk_{z1}d}\right)\right] \tag{18}$$

Continuity of E_ϕ at $z = 0$ gives

$$A(1 + \Gamma)\frac{d}{d\rho} J_1\left(\frac{x_{11}\rho}{a}\right) = \int_0^\infty d\xi \ \xi\left\{\left[I_1(\xi) + R_1(\xi)\right]\frac{d}{d\rho} J_1(\xi\rho) - \frac{\left[I_2(\xi) - R_2(\xi)\right]k_{z1}J_1(\xi\rho)}{\omega\mu_o\epsilon_1\rho}\right\} \tag{19}$$

and continuity of E_ρ at $z = 0$ gives

$$A(1 + \Gamma)\frac{J_1\left(\frac{x_{11}\rho}{a}\right)}{\rho} = \int_0^\infty d\xi \ \xi\left\{\left[I_1(\xi) + R_1(\xi)\right]\frac{J_1(\xi\rho)}{\rho} - \frac{\left[I_2(\xi) - R_2(\xi)\right]k_{z1}}{\omega\mu_o\epsilon_1}\frac{dJ_1(\xi\rho)}{d\rho}\right\} \tag{20}$$

The coefficients $I_1(\xi)$ and $R_1(\xi)$ may be eliminated by first multiplying equation (19) by $J_1(\zeta\rho)\,d\rho$ and equation (20) by $\rho\,\frac{dJ_1(\zeta\rho)}{d\rho}\,d\rho$, where ζ is introduced as a dummy variable. Then, integrating each equation with respect to ρ and adding them gives

$$A(1 + \Gamma)\int_0^\infty \frac{d}{d\rho}\left[J_1\left(\frac{x_{11}\rho}{a}\right)J_1(\xi\rho)\right]d\rho$$

$$= -\int_0^\infty d\xi\ \frac{\xi k_{z1}\left[I_2(\xi) - R_2(\xi)\right]}{\omega\mu_o\epsilon_1}\int_0^\infty\left[\frac{J_1(\xi\rho)J_1(\zeta\rho)}{\rho^2} + \frac{dJ_1(\xi\rho)}{d\rho}\frac{dJ_1(\zeta\rho)}{d\rho}\right]\rho\,d\rho \tag{21}$$

where the integral over ρ on the right-hand side of equation (21) was extended from symbol a to ∞ because of the boundary condition, which requires tangential $\vec{E} = 0$ on the ground plane.

The integration of the left-hand side of equation (21) is trivial. However, a few manipulations are necessary to integrate the right-hand side. First, Bessel's equation is used in the following form:

$$\rho\frac{d}{d\rho}\left[\rho\frac{dJ_1(\xi\rho)}{d\rho}\right] + \left(\xi^2\rho^2 - 1\right)J_1(\xi\rho) = 0 \tag{22}$$

Equation (22) is then multiplied by $\frac{J_1(\zeta\rho)}{\rho}\,d\rho$ and integrated from 0 to ∞. Second derivatives are eliminated by integrating by parts, and the following relationship is obtained:

$$\int_0^\infty \left[\frac{J_1(\xi\rho)J_1(\zeta\rho)}{\rho^2} + \frac{dJ_1(\xi\rho)}{d\rho}\,\frac{dJ_1(\zeta\rho)}{d\rho}\right]\rho\,d\rho = \xi^2 \int_0^\infty \rho J_1(\xi\rho)J_1(\zeta\rho)d\rho$$

$$= \xi\delta(\xi - \zeta) \tag{23}$$

where the definition of the Dirac delta function for Hankel transforms is given elsewhere (ref. 4).

With the use of equation (23), it is easy to show that the coefficients $I_2(\xi)$, $R_2(\xi)$, and Γ are implicitly given by

$$\left[I_2(\xi) - R_2(\xi)\right] = \frac{-A(1 + \Gamma)\omega\mu_o\epsilon_1}{\xi^2 k_{z1}} J_1\left(x_{11}\right)J_1(\xi a) \tag{24}$$

When equation (19) is multiplied by $\rho\,\dfrac{dJ_1(\zeta\rho)}{d\rho}\,d\rho$ and equation (20) by $J_1(\zeta\rho)d\rho$ and both equations are added and integrated with respect to ρ, an implicit expression for $I_1(\xi)$, $R_1(\xi)$, and Γ is obtained in a manner analogous to the previous development of equation (24). The result is

$$\left[I_1(\xi) + R_1(\xi)\right] = \frac{A(1 + \Gamma)\left(x_{11}\right)^2}{\xi a\left[\left(\frac{x_{11}}{a}\right)^2 - \xi^2\right]} J_1\left(x_{11}\right)J_1'(\xi a) \tag{25}$$

where the prime on the Bessel function denotes differentiation with respect to the argument.

Continuity of complex power across the aperture completes the boundary conditions. The complex power expression in region II, at the aperture, is found by integration of the Poynting vector over the aperture:

$$P^{*II} = \frac{1}{2}\int_0^{2\pi}\int_0^\infty d\rho\,d\phi\,\rho\left(E_\rho^{*II}H_\phi^{II} - E_\phi^{*II}H_\rho^{II}\right) \tag{26}$$

Substitution of the field equations from the appendix, integrating over ϕ, and combining terms yields

$$P^{*II} = \frac{\pi}{2}\int_0^\infty \int_0^\infty d\xi\ d\zeta\ \xi\zeta\left(\left\{\frac{k_{z1}}{\omega\mu_o}\left[I_1^*(\xi) + R_1^*(\xi)\right]\left[I_1(\zeta) + R_1(\zeta)\right]\left[1 - \frac{2X_1(\zeta)}{W(\zeta)}\right]\right.\right.$$

$$\left. + \frac{k_{z1}^*}{\omega\mu_o{}^2\epsilon_1^*}\left[I_2^*(\xi) - R_2^*(\xi)\right]\left[I_2(\zeta) - R_2(\zeta)\right]\left[1 + \frac{2X_2(\zeta)}{W(\zeta)}\right]\right\}\int_0^\infty\left[\frac{J_1(\xi\rho)J_1(\zeta\rho)}{\rho^2} + \frac{dJ_1(\xi\rho)}{d\rho}\frac{dJ_1(\zeta\rho)}{d\rho}\right]\rho\ d\rho$$

$$- \left\{\frac{1}{\mu_o}\left[I_1^*(\xi) + R_1^*(\xi)\right]\left[I_2(\zeta) - R_2(\zeta)\right]\left[1 + \frac{2X_2(\zeta)}{W(\zeta)}\right]\right.$$

$$\left.\left. + \frac{|k_{z1}|^2}{\omega^2\mu_o{}^2\epsilon_1^*}\left[I_2^*(\xi) - R_2^*(\xi)\right]\left[I_1(\zeta) + R_1(\zeta)\right]\left[1 - \frac{2X_1(\zeta)}{W(\zeta)}\right]\right\}\int_0^\infty \frac{d}{d\rho}\left[J_1(\xi\rho)J_1(\zeta\rho)\right]d\rho\right) \tag{27}$$

The first integral over ρ gives ξ times the delta function, $\delta(\xi - \zeta)$, and the second integral over ρ vanishes; hence, when equations (24) and (25) are used, equation (27) reduces to

$$P^{*II} = \frac{\pi}{2}|A|^2(1 + \Gamma^*)(1 + \Gamma)\left[J_1(x_{11})\right]^2\int_0^\infty d\xi\left\{\left(\frac{x_{11}}{a}\right)^4\left(\frac{a^2\xi k_{z1}}{\omega\mu_o}\right)\frac{\left[J_1{}'(\xi a)\right]^2\left[W(\xi) - 2X_1(\xi)\right]}{W(\xi)\left[\left(\frac{x_{11}}{a}\right)^2 - \xi^2\right]^2}\right.$$

$$\left. + \left(\frac{\omega\epsilon_1}{\xi k_{z1}}\right)\frac{\left[J_1(\xi a)\right]^2\left[W(\xi) + 2X_2(\xi)\right]}{W(\xi)}\right\} \tag{28}$$

The complex power in region I at $z = 0$ is

$$P^{*I} = \frac{1}{2}\int_0^{2\pi}\int_0^a d\rho\ d\phi\rho\left(E_\rho^{*I}H_\phi^I - E_\phi^{*I}H_\rho^I\right)$$

$$= \frac{\pi}{2}|A|^2(1 + \Gamma^*)(1 - \Gamma)\frac{k_z{}^{11}}{\omega\mu_o}\int_0^a\left\{\left[\frac{d}{d\rho}J_1\left(\frac{x_{11}\rho}{a}\right)\right]^2 + \left[\frac{J_1\left(\frac{x_{11}\rho}{a}\right)}{\rho}\right]^2\right\}\rho\ d\rho \tag{29}$$

which integrates to

$$P^{*I} = \frac{\pi}{4}|A|^2(1+\Gamma^*)(1-\Gamma)\sqrt{\frac{\epsilon_o}{\mu_o}}\sqrt{1-\left(\frac{x_{11}}{k_oa}\right)^2}\left[(x_{11})^2-1\right]\left[J_1(x_{11})\right]^2 \tag{30}$$

Input Admittance

The input admittance, normalized to the waveguide characteristic admittance, is found by setting $P^{*I} = P^{*II}$ at $z = 0$. This substitution immediately gives

$$\begin{aligned} y_{in} &= g_{in} - jb_{in} \\ &= \frac{1-\Gamma}{1+\Gamma} \\ &= \left\{\frac{2(x_{11})^4}{a^2k_o\left[(x_{11})^2-1\right]\sqrt{1-\left(\frac{x_{11}}{k_oa}\right)^2}}\right\}\int_0^\infty d\xi\left\{\frac{\xi k_{z1}\left[J_1'(\xi a)\right]^2\left[W(\xi)-2X_1(\xi)\right]}{W(\xi)\left[\left(\frac{x_{11}}{a}\right)^2-\xi^2\right]^2}\right\} \\ &+ \left\{\frac{2\omega^2\mu_o\epsilon_1}{k_o\left[(x_{11})^2-1\right]\sqrt{1-\left(\frac{x_{11}}{k_oa}\right)^2}}\right\}\int_0^\infty d\xi\left\{\frac{\left[J_1(\xi a)\right]^2\left[W(\xi)+2X_2(\xi)\right]}{\xi k_{z1}W(\xi)}\right\} \end{aligned} \tag{31}$$

Equation (31) can be simplified by examining the expressions for $X_1(\xi)$, $X_2(\xi)$, and $W(\xi)$ (eqs. (14) to (16)). This simplification, after being factored, gives:

$$\frac{W(\beta)-2X_1(\beta)}{W(\beta)} = \frac{\sqrt{1-\beta^2}\cos\left(k_od\sqrt{N^2-\beta^2}\right) - j\sqrt{N^2-\beta^2}\sin\left(k_od\sqrt{N^2-\beta^2}\right)}{\sqrt{N^2-\beta^2}\cos\left(k_od\sqrt{N^2-\beta^2}\right) - j\sqrt{1-\beta^2}\sin\left(k_od\sqrt{N^2-\beta^2}\right)} \tag{32}$$

and

$$\frac{W(\beta)+2X_2(\beta)}{W(\beta)} = \frac{\sqrt{N^2-\beta^2}\cos\left(k_od\sqrt{N^2-\beta^2}\right) - jN^2\sqrt{1-\beta^2}\sin\left(k_od\sqrt{N^2-\beta^2}\right)}{N^2\sqrt{1-\beta^2}\cos\left(k_od\sqrt{N^2-\beta^2}\right) - j\sqrt{N^2-\beta^2}\sin\left(k_od\sqrt{N^2-\beta^2}\right)} \tag{33}$$

where $\beta = \xi/k_o$ and $N = k_1/k_o$.

The input admittance expression then becomes

$$y_{in} = \left\{\frac{2(x_{11})^2\left(\frac{x_{11}}{k_oa}\right)^2}{\left[(x_{11})^2 - 1\right]\sqrt{1 - \left(\frac{x_{11}}{k_oa}\right)^2}}\right\}\int_0^{\infty} d\beta \left\{\frac{\beta\sqrt{N^2 - \beta^2}\left[J_1{}'(k_oa\beta)\right]^2\left[\frac{\sqrt{1 - \beta^2}}{\sqrt{N^2 - \beta^2}} - j\tan\left(k_od\sqrt{N^2 - \beta^2}\right)\right]}{\left[\left(\frac{x_{11}}{k_oa}\right)^2 - \beta^2\right]^2\left[1 - j\frac{\sqrt{1 - \beta^2}}{\sqrt{N^2 - \beta^2}}\tan\left(k_od\sqrt{N^2 - \beta^2}\right)\right]}\right\}$$

$$+ \left\{\frac{2N^2}{\left[(x_{11})^2 - 1\right]\sqrt{1 - \left(\frac{x_{11}}{k_oa}\right)^2}}\right\}\int_0^{\infty} d\beta \left\{\frac{\left[J_1(k_oa\beta)\right]^2\left[1 - j\frac{N^2\sqrt{1 - \beta^2}}{\sqrt{N^2 - \beta^2}}\tan\left(k_od\sqrt{N^2 - \beta^2}\right)\right]}{\beta\sqrt{N^2 - \beta^2}\left[\frac{N^2\sqrt{1 - \beta^2}}{\sqrt{N^2 - \beta^2}} - j\tan\left(k_od\sqrt{N^2 - \beta^2}\right)\right]}\right\} \tag{34}$$

Equation (34) is basically a contour integral along the real axis of the complex β-plane. For the case of a lossless dielectric coating, the singularities of the integrands in this equation occur on the real axis; therefore, the path of integration for numerical computation must be deformed around these singularities by semicircular excursions in the complex β-plane. These poles are of first order and occur at the values of $\beta = \beta_n$, which are the roots of the transcendental equations

For TE modes:

$$j\sqrt{1 - \beta_n{}^2} = \sqrt{N^2 - \beta_n{}^2}\cot\left(k_od\sqrt{N^2 - \beta_n{}^2}\right) \tag{35a}$$

For TM modes:

$$-jN^2\sqrt{1 - \beta_n{}^2} = \sqrt{N^2 - \beta_n{}^2}\tan\left(k_od\sqrt{N^2 - \beta_n{}^2}\right) \tag{35b}$$

These pole locations correspond to the eigenvalues for the cutoff conditions of surface wave modes (energy confined within the dielectric coating). Since the dielectric coating which covers the waveguide aperture is located on the ground plane, only the odd TE and the even TM surface wave modes can exist. In addition to these poles there is a branch point at $\beta = 1$; however, it may be noted that $\beta = N$ is not a branch point.

The contribution of the semicircular contours that occur at each surface wave pole may be evaluated by residue theory. The amount of real power confined to the dielectric coating is represented by the surface wave conductance, which may be computed from multiplying πj by the sum of the residues of equation (34). This operation gives, for the TE surface wave conductance,

$$g_s^{TE} = \frac{2\pi (x_{11})^2 \left(\frac{x_{11}}{k_o a}\right)^2}{k_o d\left[(x_{11})^2 - 1\right]\sqrt{1 - \left(\frac{x_{11}}{k_o a}\right)^2}} \sum_n \frac{\left(N^2 - \beta_n^2\right)\left[J_1'(k_o a \beta_n)\right]^2}{\left[\left(\frac{x_{11}}{k_o a}\right)^2 - \beta_n^2\right]^2 \left[1 - \left(\frac{N^2 - 1}{\beta_n^2 - 1}\right)\frac{\sin\left(2k_o d\sqrt{N^2 - \beta_n^2}\right)}{2k_o d\sqrt{N^2 - \beta_n^2}}\right]} \tag{36}$$

and, for the TM surface wave conductance,

$$g_s^{TM} = \frac{2\pi N^2}{k_o d\left[(x_{11})^2 - 1\right]\sqrt{1 - \left(\frac{x_{11}}{k_o a}\right)^2}} \sum_n \frac{\left[J_1(k_o a \beta_n)\right]^2}{\beta_n^2 \left[1 + \left(\frac{N^2 - 1}{\beta_n^2 - 1}\right)\frac{\sin\left(2k_o d\sqrt{N^2 - \beta_n^2}\right)}{2k_o d\sqrt{N^2 - \beta_n^2}}\right]} \tag{37}$$

where the values of β_n for the TE and TM surface wave modes are given by equations (35a) and (35b), respectively.

For the lossy dielectric coating, the values of β_n are complex; thus, no poles are located on the real β-axis and the numerical integration of equation (34) is more straightforward.

PROPERTIES OF THE UNCOATED APERTURE

Input Admittance

A partial check on the admittance expression for the dielectric covered aperture is that it should revert to the free-space solution when either $d = 0$ or $\epsilon_1 = \epsilon_o$.

For the aperture with no dielectric coating ($d = 0$ or $\epsilon_1 = \epsilon_o$), equation (34) reduces to

$$y_{in} = \left\{\frac{2(x_{11})^2\left(\frac{x_{11}}{k_o a}\right)^2}{\left[(x_{11})^2 - 1\right]\sqrt{1 - \left(\frac{x_{11}}{k_o a}\right)^2}}\right\} \int_0^\infty d\beta \left\{\frac{\beta\sqrt{1 - \beta^2}\left[J_1'(k_o a\beta)\right]^2}{\left[\left(\frac{x_{11}}{k_o a}\right)^2 - \beta^2\right]^2}\right\} + \left\{\frac{2}{\left[(x_{11})^2 - 1\right]\sqrt{1 - \left(\frac{x_{11}}{k_o a}\right)^2}}\right\} \int_0^\infty d\beta \left\{\frac{\left[J_1(k_o a\beta)\right]^2}{\beta\sqrt{1 - \beta^2}}\right\} \tag{38}$$

which agrees with the expression derived by Mishustin (ref. 1), in which a Green's function approach was used.

From the phase condition of equations (7), equation (38) may be separated into its real and imaginary parts, which results in

$$\frac{\left[(x_{11})^2 - 1\right]}{2}\sqrt{1 - \left(\frac{x_{11}}{k_o a}\right)^2}\, g_{in} = (x_{11})^2\left(\frac{x_{11}}{k_o a}\right)^2 \int_0^1 d\beta \left\{\frac{\beta\sqrt{1 - \beta^2}\left[J_1'(k_o a\beta)\right]^2}{\left[\left(\frac{x_{11}}{k_o a}\right)^2 - \beta^2\right]^2}\right\} + \int_0^1 d\beta \left\{\frac{\left[J_1(k_o a\beta)\right]^2}{\beta\sqrt{1 - \beta^2}}\right\} \tag{39a}$$

$$\frac{\left[(x_{11})^2 - 1\right]}{2}\sqrt{1 - \left(\frac{x_{11}}{k_o a}\right)^2}\, b_{in} = \int_1^\infty d\beta \left\{\frac{\left[J_1(k_o a\beta)\right]^2}{\beta\sqrt{\beta^2 - 1}}\right\} - (x_{11})^2\left(\frac{x_{11}}{k_o a}\right)^2 \int_1^\infty d\beta \left\{\frac{\beta\sqrt{\beta^2 - 1}\left[J_1'(k_o a\beta)\right]^2}{\left[\left(\frac{x_{11}}{k_o a}\right)^2 - \beta^2\right]^2}\right\} \tag{39b}$$

These integrals must be numerically evaluated.

Asymptotic Expansion of Radiation Fields

For the aperture with no dielectric coating ($d = 0$ or $\epsilon_1 = \epsilon_o$), equations (12) and (13) reduce to

$$T_1(\xi) = A(1 + \Gamma)(x_{11})^2 J_1(x_{11}) \frac{J_1'(\xi a)}{\xi a\left[\left(\frac{x_{11}}{a}\right)^2 - \xi^2\right]} \tag{40}$$

and

$$T_2(\xi) = -A(1 + \Gamma)\omega\mu_o\epsilon_o J_1(x_{11}) \frac{J_1(\xi a)}{\xi^2 k_{zo}} \tag{41}$$

Therefore, for $d = 0$ or $\epsilon_1 = \epsilon_0$, the vector potentials (eqs. (6)) outside the structure become

$$\frac{A_z^{*III}}{\epsilon_0} = aA(1 + \Gamma)J_1(x_{11})\cos\phi \int_0^{\infty} d\xi \left\{ \frac{J_1'(\xi a) J_1(\xi\rho) e^{jk_{zo}z}}{\left[1 - \left(\frac{\xi a}{x_{11}}\right)^2\right]} \right\} \tag{42a}$$

$$A_z^{III} = -\omega\mu_0\epsilon_0 A(1 + \Gamma)J_1(x_{11}) \sin\phi \int_0^{\infty} d\xi \left\{ \frac{J_1(\xi a) J_1(\xi\rho) e^{jk_{zo}z}}{\xi k_{zo}} \right\} \tag{42b}$$

In order to compute the radiation fields from equations (42), the ranges of integration must be extended over the entire real axis in mode space. This computation can be made by first noting that as $\xi\rho \to \infty$, the Bessel function in the integrand of equations (42) can be asymptotically expanded to give

$$J_1(\xi\rho) \approx \frac{e^{-j\frac{3\pi}{4}}}{\sqrt{2\pi}} \left[\frac{e^{j\xi\rho}}{\sqrt{\xi\rho}} + \frac{e^{-j\xi\rho}}{\sqrt{-\xi\rho}} \right] \tag{43}$$

The substitution of equation (43) into equations (42) results in a sum of two integrals with limits over the positive real axis. However, if ξ is replaced by $-\xi$ in the integral containing $e^{j\xi\rho}$, the sums may be combined and the limits become $-\infty$ and $+\infty$. Hence, $\frac{A_z^{*III}}{\epsilon_0}$ and A_z^{III} may be written as follows:

$$\frac{A_z^{*III}}{\epsilon_0} = A(1 + \Gamma)J_1(x_{11}) \frac{a e^{-j\frac{3\pi}{4}}}{\sqrt{2\pi}} \cos\phi \int_{-\infty}^{\infty} d\xi \left\{ \frac{J_1'(\xi a) e^{jr(k_{zo}\cos\theta + \xi\sin\theta)}}{\left[1 - \left(\frac{\xi a}{x_{11}}\right)^2\right] \sqrt{\xi r \sin\theta}} \right\} \tag{44a}$$

$$A_z^{III} = -A(1 + \Gamma)J_1(x_{11})\frac{\omega\mu_o\epsilon_o e^{-j\frac{3\pi}{4}}}{\sqrt{2\pi}} \sin\phi \int_{-\infty}^{\infty} d\xi \left\{\frac{J_1(\xi a)e^{jr(k_{zo}\cos\theta+\xi\sin\theta)}}{\xi k_{zo}\sqrt{\xi r \sin\theta}}\right\} \tag{44b}$$

where a transformation to spherical coordinates such that $z = r \cos\theta$ and $\rho = r \sin\theta$ was made.

The extension of the integration is required in order to apply stationary phase evaluation of the integral. From Knop (ref. 5), the asymptotic solution of an integral of the form

$$I = \int_{-\infty}^{\infty} g(\xi)e^{jrf(\xi)}d\xi \tag{45}$$

is given by

$$I \approx g(k_o \sin\theta)e^{jk_o r}\sqrt{\frac{-2\pi jk_o}{r}} \cos\theta \tag{46}$$

When equations (44) are expressed in the form given by equation (46), the far-field expansions of the vector potentials become

$$\left.\begin{aligned} \frac{A_z^{*III}}{\epsilon_o} &= -A(1 + \Gamma)J_1(x_{11})\frac{ae^{jk_o r}}{r}\left\{\frac{\cot\theta \cos\phi J_1'(k_o a \sin\theta)}{\left[1 - \left(\frac{k_o a}{x_{11}} \sin\theta\right)^2\right]}\right\} \\ A_z^{III} &= A(1 + \Gamma)J_1(x_{11})\frac{\omega\mu_o\epsilon_o e^{jk_o r}}{k_o^2 r}\left\{\frac{\sin\phi\ J_1(k_o a \sin\theta)}{\sin^2\theta}\right\} \end{aligned}\right\} \tag{47}$$

It is of interest to note that since the integrands of equations (42) are odd functions of ξ, $\int_0^{\infty} d\xi J_1(\xi\rho) \ldots$ may be converted to $\frac{1}{2}\int_{\infty e^{j\pi}}^{\infty} d\xi H_1^{(1)}(\xi\rho) \ldots$. Then equation (16) from Samaddar (ref. 6) with $n = 1$ may be used to obtain equations (47) directly.

For large values of r and $\sin\theta \neq 0$, it is not difficult to show that the radiation fields become

$$\left.\begin{aligned} E_r^{III} &\approx 0 \\ E_\phi^{III} &\approx jk_o \frac{A_z^{*III}}{\epsilon_o} \sin\theta \\ E_\theta^{III} &\approx -j\omega A_z^{III} \sin\theta \end{aligned}\right\} \tag{48}$$

or

$$\left.\begin{aligned} E_\phi^{III} &\approx -jA(1+\Gamma)J_1(x_{11}) \frac{e^{jk_o r}}{r} \left\{ \frac{k_o a \cos\theta \cos\phi J_1'(k_o a \sin\theta)}{\left[1 - \left(\frac{k_o a}{x_{11}} \sin\theta\right)^2\right]} \right\} \\ E_\theta^{III} &\approx -jA(1+\Gamma)J_1(x_{11}) \frac{e^{jk_o r}}{r} \left\{ \frac{\sin\phi J_1(k_o a \sin\theta)}{\sin\theta} \right\} \end{aligned}\right\} \tag{49}$$

The magnitude of the radiation pattern is defined as the square root of the sum of the squares of the terms in braces. Although the method and the geometry for the present problem are somewhat different, the above expressions for the radiation fields resemble those obtained previously (ref. 7). The Kirchhoff integral was used several years ago (ref. 7; see also, ref. 2) to derive the radiation fields of an open-end waveguide with no ground plane. These results can easily be modified to describe the present problem.

Results

The input admittance expression (38) was programed on a data processing machine in which the Gaussian quadrature method was used to evaluate the integrals. The ratio of the diameter to the wavelength $2a/\lambda_o$ was chosen as the independent variable, and calculations were performed over the range $0.5859 < 2a/\lambda_o < 0.9720$ which brackets cutoff of the TE_{11} mode on one end of the band and that of the TE_{21} mode on the other end. Although cutoff of the TM_{01} mode occurs at $2a/\lambda_o = 0.7654$, it is believed that the aperture discontinuity should not induce a reflected TM_{01} component, provided a TM_{01} mode is not incident upon the aperture.

Results of the computations are shown in figure 2. The input admittance is initially large due to the small value of the normalizing guide admittance near cutoff of the TE_{11} mode. However, as $2a/\lambda_o$ increases toward cutoff of the TE_{21} mode, the normalized conductance approaches unity, and the inductive susceptance approaches zero. In other words, the antenna becomes more nearly matched to free space as the frequency increases.

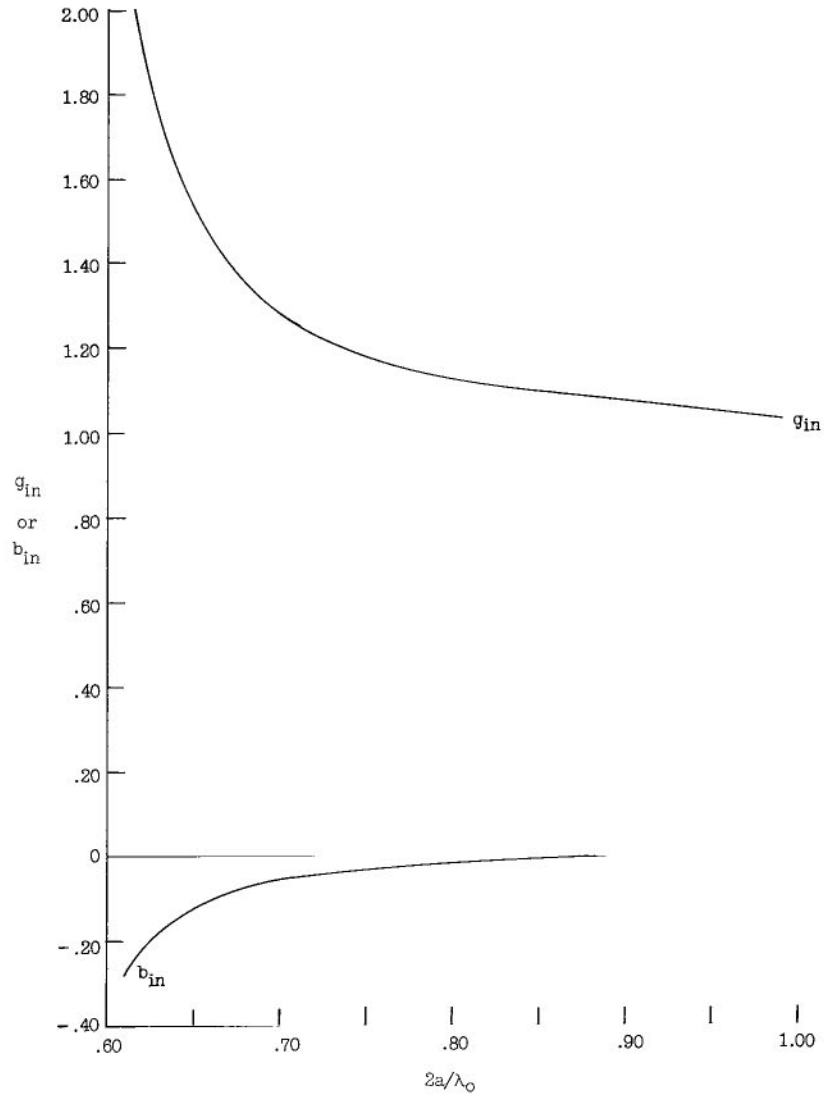

Figure 2.- Input admittance of circular waveguide radiating into free space.

The aperture admittance of the circular waveguide, defined as

$$\frac{y_{ap}}{y_o} = y_{in}\sqrt{1 - \left(\frac{x_{11}}{k_o a}\right)^2}$$

is plotted in figure 3. The calculated values show excellent agreement with Mishustin (ref. 1). Although Mishustin plots his susceptance on the positive axis, the present calculations indicate that the susceptance actually goes inductive at the lower end of the

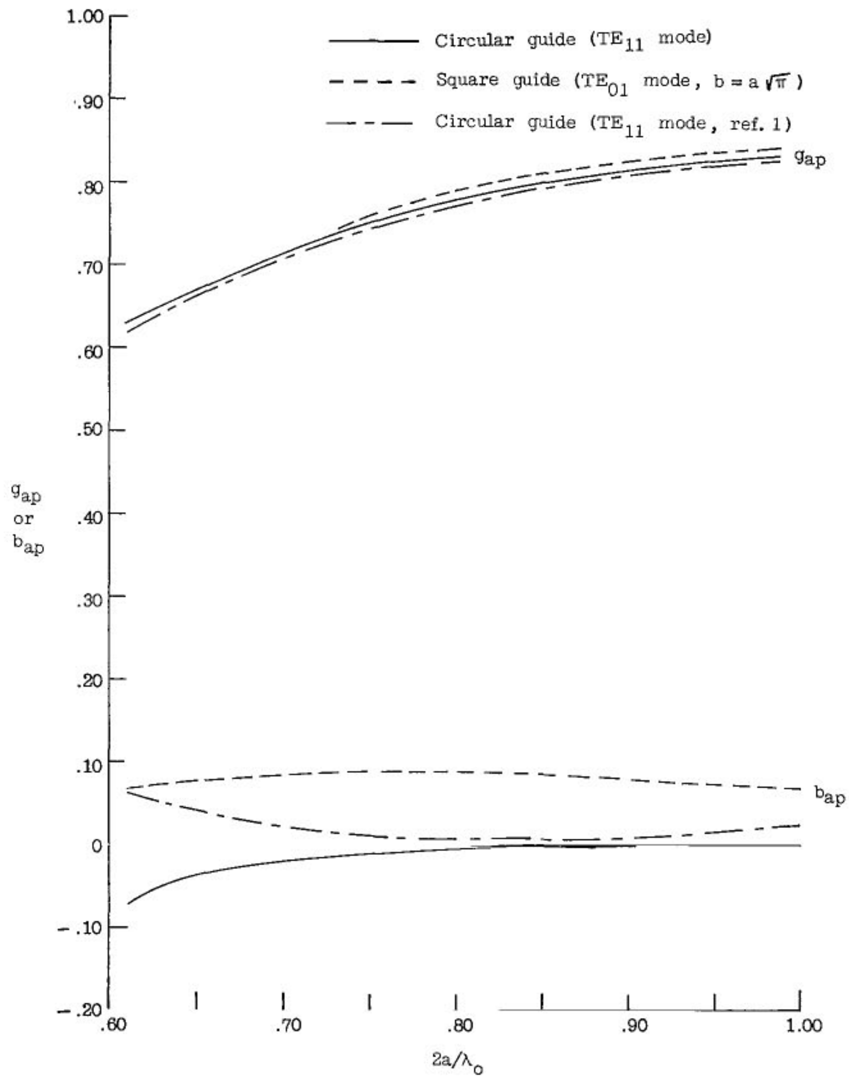

Figure 3.- Admittance of circular and square waveguide apertures of equal areas radiating into free space.

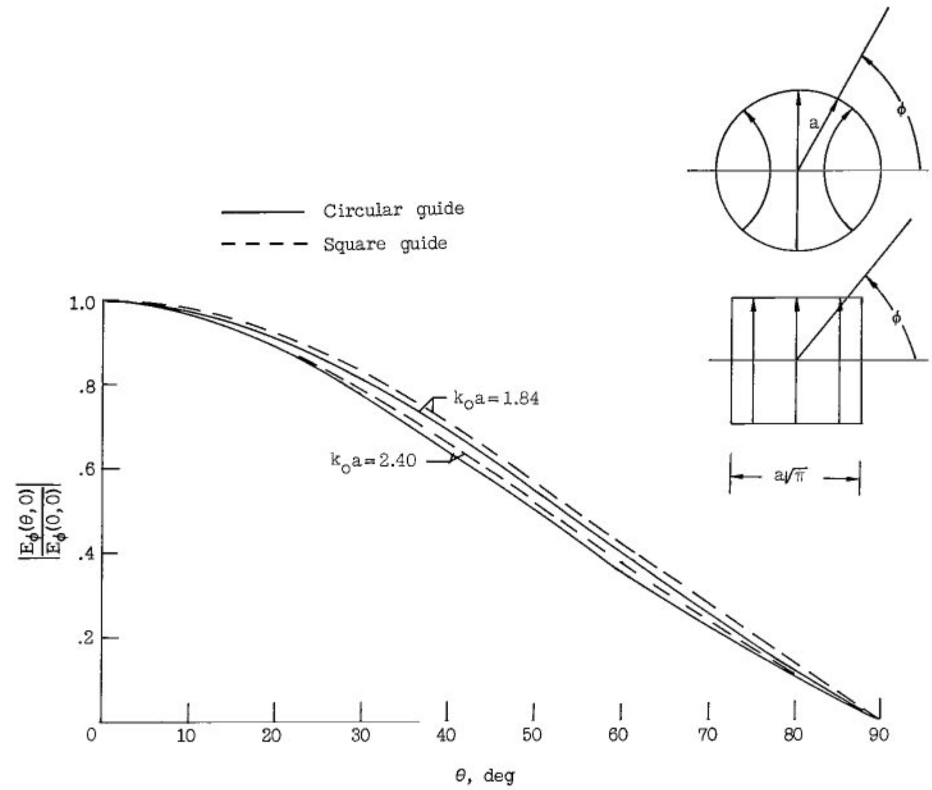

Figure 4.- Radiation patterns in $\Phi = \pi/2$ plane for circular and square waveguide apertures of equal areas.

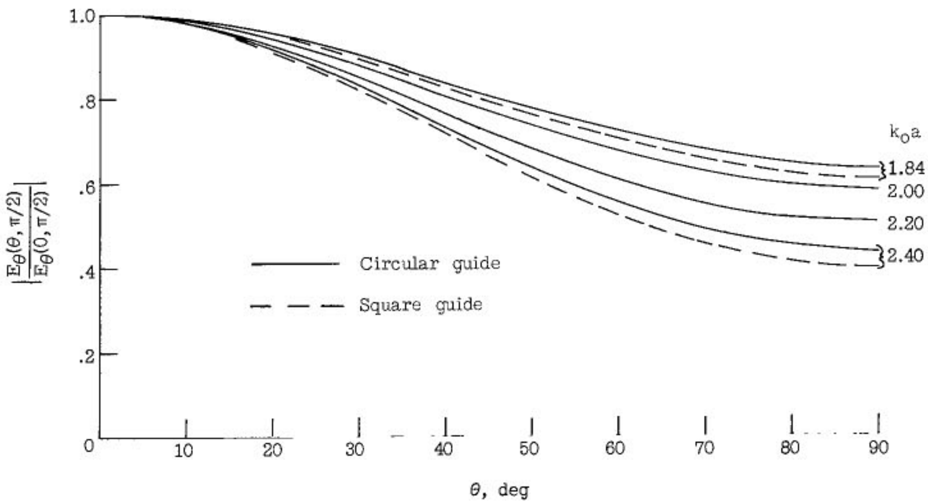

Figure 5.- Radiation patterns in $\Phi = 0$ plane for circular and square waveguide apertures of equal areas.

band as verified by preliminary experimental results. The corresponding values of y_{ap}/y_o are given for a square aperture of equal area, excited in the TE_{01} rectangular waveguide mode (ref. 8). The latter is plotted as a function of the guide dimension in wavelengths b/λ_o where $b = a\sqrt{\pi}$. The conductance of both waveguides is virtually identical, and the susceptance of both is small over the range considered. This similarity in the admittance is somewhat surprising in view of the difference in geometry and field distribution.

The normalized radiation patterns in the two principal planes are given in figures 4 and 5. They, too, are similar to those of the square waveguide (ref. 9), which are indicated by the broken lines.

CONCLUSIONS

The input admittance and radiation patterns of a circular waveguide, opening onto a flat ground plane, have been derived by use of the integral transform methods. The aperture admittance and patterns for no dielectric coating were computed and are quite similar to those of a square waveguide of equal area, excited in the TE_{01} rectangular waveguide mode.

It is of practical importance to note that, for possible use as an antenna, the open-end circular waveguide is well matched to free space above midband of the waveguide.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., April 15, 1968,
125-22-02-02-23.

APPENDIX

FIELD EQUATIONS

Expansion of the fields in equations (1), (2), and (3) into cylindrical coordinates gives, for $z \leqq 0$,

$$\left.\begin{aligned}
E_\rho{}^I &= -\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{A_z^{*I}}{\epsilon_o}\right)\\
E_\phi{}^I &= \frac{\partial}{\partial\rho}\left(\frac{A_z^{*I}}{\epsilon_o}\right)\\
E_z{}^I &= 0\\
H_\rho{}^I &= -\frac{1}{j\omega\mu_o}\frac{\partial^2}{\partial\rho\,\partial z}\left(\frac{A_z^{*I}}{\epsilon_o}\right)\\
H_\phi{}^I &= -\frac{1}{j\omega\mu_o}\frac{1}{\rho}\frac{\partial^2}{\partial\phi\,\partial z}\left(\frac{A_z^{*I}}{\epsilon_o}\right)\\
H_z{}^I &= \frac{1}{j\omega\mu_o}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\left(\frac{A_z^{*I}}{\epsilon_o}\right)\right] + \frac{1}{j\omega\mu_o}\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\left(\frac{A_z^{*I}}{\epsilon_o}\right)
\end{aligned}\right\} \qquad (A1)$$

For $0 \leqq z \leqq d$,

$$\left.\begin{aligned}
E_\rho{}^{II} &= -\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{A_z^{*II}}{\epsilon_1}\right) - \frac{1}{j\omega\mu_o\epsilon_1}\frac{\partial^2}{\partial\rho\,\partial z}\left(A_z{}^{II}\right)\\
E_\phi{}^{II} &= \frac{\partial}{\partial\rho}\left(\frac{A_z^{*II}}{\epsilon_1}\right) - \frac{1}{j\omega\mu_o\epsilon_1}\frac{1}{\rho}\frac{\partial^2}{\partial\phi\,\partial z}\left(A_z{}^{II}\right)\\
E_z{}^{II} &= \frac{1}{j\omega\mu_o\epsilon_1}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\left(A_z{}^{II}\right)\right] + \frac{1}{j\omega\mu_o\epsilon_1}\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\left(A_z{}^{II}\right)\\
H_\rho{}^{II} &= -\frac{1}{j\omega\mu_o}\frac{\partial^2}{\partial\rho\,\partial z}\left(\frac{A_z^{*II}}{\epsilon_1}\right) + \frac{1}{\mu_o}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(A_z{}^{II}\right)\\
H_\phi{}^{II} &= -\frac{1}{j\omega\mu_o}\frac{1}{\rho}\frac{\partial^2}{\partial\phi\,\partial z}\left(\frac{A_z^{*II}}{\epsilon_1}\right) - \frac{1}{\mu_o}\frac{\partial}{\partial\rho}\left(A_z{}^{II}\right)\\
H_z{}^{II} &= \frac{1}{j\omega\mu_o}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\left(\frac{A_z^{*II}}{\epsilon_1}\right)\right] + \frac{1}{j\omega\mu_o}\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\left(\frac{A_z^{*II}}{\epsilon_1}\right)
\end{aligned}\right\} \qquad (A2)$$

and for $z \geqq d$

$$
\left.
\begin{aligned}
E_\rho^{III} &= -\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{A_z^{*III}}{\epsilon_o}\right) - \frac{1}{j\omega\mu_o\epsilon_o}\frac{\partial^2}{\partial\rho\,\partial z}\left(A_z^{III}\right) \\
E_\phi^{III} &= \frac{\partial}{\partial\rho}\left(\frac{A_z^{*III}}{\epsilon_o}\right) - \frac{1}{j\omega\mu_o\epsilon_o}\frac{1}{\rho}\frac{\partial^2}{\partial\phi\,\partial z}\left(A_z^{III}\right) \\
E_z^{III} &= \frac{1}{j\omega\mu_o\epsilon_o}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\left(A_z^{III}\right)\right] + \frac{1}{j\omega\mu_o\epsilon_o}\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\left(A_z^{III}\right) \\
H_\rho^{III} &= -\frac{1}{j\omega\mu_o}\frac{\partial^2}{\partial\rho\,\partial z}\left(\frac{A_z^{*III}}{\epsilon_o}\right) + \frac{1}{\mu_o}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(A_z^{III}\right) \\
H_\phi^{III} &= -\frac{1}{j\omega\mu_o}\frac{1}{\rho}\frac{\partial^2}{\partial\phi\,\partial z}\left(\frac{A_z^{*III}}{\epsilon_o}\right) - \frac{1}{\mu_o}\frac{\partial}{\partial\rho}\left(A_z^{III}\right) \\
H_z^{III} &= \frac{1}{j\omega\mu_o}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\left(\frac{A_z^{*III}}{\epsilon_o}\right)\right] + \frac{1}{j\omega\mu_o}\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\left(\frac{A_z^{*III}}{\epsilon_o}\right)
\end{aligned}
\right\}
\tag{A3}
$$

Substitution of the vector potentials from equations (4), (5), and (6) yields, for the vector components of the fields for $z \leqq 0$,

$$
\left.
\begin{aligned}
E_\rho^{I} &= \frac{A}{\rho}\left(e^{jk_z^{11}z} + \Gamma e^{-jk_z^{11}z}\right)\sin\phi J_1\left(\frac{x_{11}\rho}{a}\right) \\
E_\phi^{I} &= A\left(e^{jk_z^{11}z} + \Gamma e^{-jk_z^{11}z}\right)\cos\phi\,\frac{d}{d\rho}J_1\left(\frac{x_{11}\rho}{a}\right) \\
E_z^{I} &= 0 \\
H_\rho^{I} &= -\frac{Ak_z^{11}}{\omega\mu_o}\left(e^{jk_z^{11}z} - \Gamma e^{-jk_z^{11}z}\right)\cos\phi\,\frac{d}{d\rho}J_1\left(\frac{x_{11}\rho}{a}\right) \\
H_\phi^{I} &= \frac{k_z^{11}}{\omega\mu_o}\frac{A}{\rho}\left(e^{jk_z^{11}z} - \Gamma e^{-jk_z^{11}z}\right)\sin\phi J_1\left(\frac{x_{11}\rho}{a}\right) \\
H_z^{I} &= \frac{A}{j\omega\mu_o\rho}\cos\phi\left(e^{jk_z^{11}z} + \Gamma e^{-jk_z^{11}z}\right)\left\{\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_1\left(\frac{x_{11}\rho}{a}\right)\right] - \frac{J_1\left(\frac{x_{11}\rho}{a}\right)}{\rho}\right\}
\end{aligned}
\right\}
\tag{A4}
$$

for $0 \leqq z \leqq d$

$$
\left.
\begin{aligned}
E_\rho^{\,II} &= \sin\phi \int_0^\infty \xi e^{jk_{z1}z}\left[I_1(\xi)\frac{J_1(\xi\rho)}{\rho} - \frac{k_{z1}I_2(\xi)}{\omega\mu_o\epsilon_1}\frac{d}{d\rho}J_1(\xi\rho)\right]d\xi \\
&\quad + \sin\phi \int_0^\infty \xi e^{-jk_{z1}z}\left[R_1(\xi)\frac{J_1(\xi\rho)}{\rho} + \frac{k_{z1}R_2(\xi)}{\omega\mu_o\epsilon_1}\frac{d}{d\rho}J_1(\xi\rho)\right]d\xi \\
E_\phi^{\,II} &= \cos\phi \int_0^\infty \xi e^{jk_{z1}z}\left[I_1(\xi)\frac{d}{d\rho}J_1(\xi\rho) - \frac{k_{z1}I_2(\xi)}{\omega\mu_o\epsilon_1}\frac{J_1(\xi\rho)}{\rho}\right]d\xi \\
&\quad + \cos\phi \int_0^\infty \xi e^{-jk_{z1}z}\left[R_1(\xi)\frac{d}{d\rho}J_1(\xi\rho) + \frac{k_{z1}R_2(\xi)}{\omega\mu_o\epsilon_1}\frac{J_1(\xi\rho)}{\rho}\right]d\xi \\
E_z^{\,II} &= \frac{\sin\phi}{j\omega\mu_o\epsilon_1\rho}\int_0^\infty \xi\left[I_2(\xi)e^{jk_{z1}z} + R_2(\xi)e^{-jk_{z1}z}\right]\left\{\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_1(\xi\rho)\right] - \frac{J_1(\xi\rho)}{\rho}\right\}d\xi \\
H_\rho^{\,II} &= \cos\phi \int_0^\infty \xi e^{jk_{z1}z}\left[-\frac{k_{z1}I_1(\xi)}{\omega\mu_o}\frac{d}{d\rho}J_1(\xi\rho) + \frac{I_2(\xi)}{\mu_o}\frac{J_1(\xi\rho)}{\rho}\right]d\xi \\
&\quad + \cos\phi \int_0^\infty \xi e^{-jk_{z1}z}\left[\frac{k_{z1}R_1(\xi)}{\omega\mu_o}\frac{d}{d\rho}J_1(\xi\rho) + \frac{R_2(\xi)}{\mu_o}\frac{J_1(\xi\rho)}{\rho}\right]d\xi \\
H_\phi^{\,II} &= \sin\phi \int_0^\infty \xi e^{jk_{z1}z}\left[\frac{k_{z1}I_1(\xi)}{\omega\mu_o}\frac{J_1(\xi\rho)}{\rho} - \frac{I_2(\xi)}{\mu_o}\frac{d}{d\rho}J_1(\xi\rho)\right]d\xi \\
&\quad - \sin\phi \int_0^\infty \xi e^{-jk_{z1}z}\left[\frac{k_{z1}R_1(\xi)}{\omega\mu_o}\frac{J_1(\xi\rho)}{\rho} + \frac{R_2(\xi)}{\mu_o}\frac{d}{d\rho}J_1(\xi\rho)\right]d\xi \\
H_z^{\,II} &= \frac{\cos\phi}{j\omega\mu_o\rho}\int_0^\infty \xi\left[I_1(\xi)e^{jk_{z1}z} + R_1(\xi)e^{-jk_{z1}z}\right]\left\{\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_1(\xi\rho)\right] - \frac{J_1(\xi\rho)}{\rho}\right\}d\xi
\end{aligned}
\right\} \quad \text{(A5)}
$$

and for $z \geqq d$

$$
\left.
\begin{aligned}
E_\rho^{III} &= \sin\phi \int_0^\infty \xi e^{jk_{zo}z}\left[T_1(\xi)\frac{J_1(\xi\rho)}{\rho} - \frac{k_{zo}T_2(\xi)}{\omega\mu_o\epsilon_o}\frac{d}{d\rho}J_1(\xi\rho)\right]d\xi \\
E_\phi^{III} &= \cos\phi \int_0^\infty \xi e^{jk_{zo}z}\left[T_1(\xi)\frac{d}{d\rho}J_1(\xi\rho) - \frac{k_{zo}T_2(\xi)}{\omega\mu_o\epsilon_o}\frac{J_1(\xi\rho)}{\rho}\right]d\xi \\
E_z^{III} &= \frac{\sin\phi}{j\omega\mu_o\epsilon_o\rho}\int_0^\infty \xi T_2(\xi)e^{jk_{zo}z}\left\{\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_1(\xi\rho)\right] - \frac{J_1(\xi\rho)}{\rho}\right\}d\xi \\
H_\rho^{III} &= \cos\phi \int_0^\infty \xi e^{jk_{zo}z}\left[-\frac{k_{zo}T_1(\xi)}{\omega\mu_o}\frac{d}{d\rho}J_1(\xi\rho) + \frac{T_2(\xi)}{\mu_o}\frac{J_1(\xi\rho)}{\rho}\right]d\xi \\
H_\phi^{III} &= \sin\phi \int_0^\infty \xi e^{jk_{zo}z}\left[\frac{k_{zo}T_1(\xi)}{\omega\mu_o}\frac{J_1(\xi\rho)}{\rho} - \frac{T_2(\xi)}{\mu_o}\frac{d}{d\rho}J_1(\xi\rho)\right]d\xi \\
H_z^{III} &= \frac{\cos\phi}{j\omega\mu_o\rho}\int_0^\infty \xi T_1(\xi)e^{jk_{zo}z}\left\{\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_1(\xi\rho)\right] - \frac{J_1(\xi\rho)}{\rho}\right\}d\xi
\end{aligned}
\right\}
\qquad \text{(A6)}
$$

REFERENCES

1. Mishustin, B. A.: Radiation From the Aperture of a Circular Waveguide With an Infinite Flange. Sov. Radiophy., vol. 8, no. 6, Nov.-Dec. 1965, pp. 852-858.

2. Risser, J. R.: Waveguide and Horn Feeds. Microwave Antenna Theory and Design, Samuel Silver, ed., Dover Publ., Inc., 1965, pp. 334-387.

3. Stratton, Julius Adam: Electromagnetic Theory. McGraw-Hill Book Co., Inc., 1941, pp. 23-32.

4. Morse, Philip M.; and Feshbach, Herman: Methods of Theoretical Physics. Pt. I, McGraw-Hill Book Co., Inc., 1953, p. 943.

5. Knop, Charles M.: The Radiation Fields From a Circumferential Slot on a Metal Cylinder Coated With a Lossy Dielectric. IRE, Trans. Antenna Propagation, vol. AP-9, no. 6, Nov. 1961, pp. 535-545.

6. Samaddar, S. N.: Radiation From a Vertical Magnetic Dipole in Inhomogeneous Stratified Media Above a Horizontal Conducting Plane. Int. J. Electron., vol. 19, no. 3, Sept. 1965, pp. 279-299.

7. Chu, L. J.: Calculation of the Radiation Properties of Hollow Pipes and Horns. J. Appl. Phys., vol. 11, no. 9, Sept. 1940, pp. 603-610.

8. Cohen, M. H.; Crowley, T. H.; and Levis, C. A.: The Aperture Admittance of a Rectangular Waveguide Radiating Into Half-Space. Proj. Rep. 339-22 (Contract W 33-038 ac21114), Ohio State Univ. Res. Found., Nov. 14, 1951.

9. Harrington, Roger F.: Time-Harmonic Electromagnetic Fields. McGraw-Hill Book Co., Inc., 1961, pp. 139-140.

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