X-643-68-368
PREPRINT

NASA TM X-63355

A UNIFIED FORM OF LAMBERT'S THEOREM

E. R. LANCASTER
R. C. BLANCHARD

SEPTEMBER 1968

GSFC GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

X-643-68-368
PREPRINT

A UNIFIED FORM OF LAMBERT'S THEOREM

E. R. Lancaster
R. C. Blanchard

September 1968

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

A UNIFIED FORM OF LAMBERT'S THEOREM

E. R. Lancaster
R. C. Blanchard

ABSTRACT

A unified form of Lambert's theorem is presented which is valid for elliptic, hyperbolic, and parabolic orbits. The key idea involves the selection of an independent variable x and a parameter q such that the normalized time of flight T is a single-valued function of x for each value of q. The parameter q depends only upon known quantities. If T is less than the time of flight for one complete revolution, it is a monotonic function of x for each q, making possible the construction of a simple algorithm for finding x, given T and q. Detailed sketches are given for T(x, q) and formulas developed for the velocity vectors at the initial and final times. Also included is a careful derivation of the classical form of Lambert's equations, including the multi-revolution cases.

CONTENTS

A UNIFIED FORM OF LAMBERT'S THEOREM

INTRODUCTION

Lambert's problem, as it arises in most applications, is concerned with the determination of an orbit from two position vectors and the time of flight. It has important applications in the areas of rendezvous, targeting, guidance, and preliminary orbit determination. In this paper a unified form of Lambert's theorem will be presented which is valid for elliptic, hyperbolic, and parabolic orbits.

The key idea involves the selection of an independent variable x and a parameter q such that the normalized time of flight is a single valued function of x for each value of q. The parameter q depends only upon known quantities. The problem then is to find x for given values of q and the time of flight. If the time of flight T is less than that for one complete revolution, T is a monotonic function of x for each value of q. Thus it is an easy task to design an algorithm for finding x. For multirevolution cases T(x) has a single well-defined minimum for each q.

This idea was presented in a previous paper [1], where a unified formula was given for the computation of T from x and q. Detailed sketches were given for T(x, q) and a simple formula developed for the time derivative of the magnitude of the radius vector at the initial time in terms of x and given quantities.

The present paper expands upon the previous one by

1. giving complete derivations which were only sketched before,
2. giving a careful derivation of the classical form of Lambert's equations, including the multirevolution cases,
3. deriving a number of useful auxiliary formulas, e.g., for the semilatus rectum and for the velocity vectors at the initial and final times in terms of the two given position vectors.

THE PROBLEM

Suppose a particle in a gravitational inverse-square central force field has distances r_1 and r_2 from the center of attraction at times t_1 and t_2. Let c be the distance and θ the central angle between the positions of the particle at the two times, where $0 \le \theta \le 2\pi$.

Lambert's problem is that of finding the semimajor axis or some related quantity for the orbit of the particle, given t_1, r_1, t_2, r_2, and θ. Having solved Lambert's problem, other quantities associated with the orbit are easily found, as will be later discussed. Using the law of cosines, we can express c in terms of r_1, r_2, and θ:

$$c^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta \quad . \tag{1}$$

We define

G = universal gravitational constant

M = mass of attracting body

μ = GM

a = semimajor axis of transfer orbit

e = eccentricity of transfer orbit.

We will follow the common sign convention for a, i.e., $a > 0$ for elliptic orbits and $a < 0$ for hyperbolic orbits. Definitions of other symbols will be given as they are introduced.

THE CLASSICAL FORM OF LAMBERT'S EQUATIONS

The path of a particle in an inverse-square central force field is an ellipse, parabola, or hyperbola. With origin at the center of attraction, we have, for elliptic motion

$$r_1 = a\left(1 - e\cos\phi_1\right) \quad , \tag{2}$$

$$r_2 = a\left(1 - e\cos\phi_2\right) \quad , \tag{3}$$

$$n\left(t_1 - t_p\right) = \phi_1 - e\sin\phi_1 \quad , \tag{4}$$

$$n\left(t_2 - t_p\right) = \phi_2 - e\sin\phi_2 \quad , \tag{5}$$

where ϕ_1 and ϕ_2 are the eccentric anomalies at times t_1 and t_2, t_p is the time at pericenter, and

$$n = (\mu/a^3)^{1/2} .$$

If $\vec{\epsilon}_1$ is a unit vector pointing towards periapsis and $\vec{\epsilon}_2$ is a unit vector in the plane of motion 90° ahead of $\vec{\epsilon}_1$ in the direction of motion, then for the position vectors $\vec{r}_1$ and $\vec{r}_2$ at times t_1 and t_2,

$$\vec{r}_2 = a(\cos\phi_2 - e)\vec{\epsilon}_1 + a(1-e^2)^{1/2}(\sin\phi_2)\vec{\epsilon}_2 ,$$

$$\vec{r}_1 = a(\cos\phi_1 - e)\vec{\epsilon}_1 + a(1-e^2)^{1/2}(\sin\phi_1)\vec{\epsilon}_2 .$$

Substituting these equations in

$$c^2 = r_1^2 + r_2^2 - 2\vec{r}_1 \cdot \vec{r}_2 ,$$

where the dot indicates scalar product, we have

$$\begin{aligned} c^2 &= a^2(\cos\phi_2 - \cos\phi_1)^2 + a^2(1-e^2)(\sin\phi_2 - \sin\phi_1)^2 \\ &= 4a^2\left[1 - e^2\cos^2\left(\frac{1}{2}\right)(\phi_1+\phi_2)\right]\sin^2\left(\frac{1}{2}\right)(\phi_2-\phi_1) . \end{aligned} \tag{6}$$

Adding (2) and (3),

$$r_1 + r_2 = 2a\left[1 - e\cos\left(\frac{1}{2}\right)(\phi_1+\phi_2)\cos\left(\frac{1}{2}\right)(\phi_2-\phi_1)\right] . \tag{7}$$

Subtracting (4) from (5),

$$n(t_2 - t_1) = \phi_2 - \phi_1 - 2e\cos\left(\frac{1}{2}\right)(\phi_1+\phi_2)\sin\left(\frac{1}{2}\right)(\phi_2-\phi_1) . \tag{8}$$

Equations (6), (7), and (8) determine the three unknowns a, $\phi_2 - \phi_1$, and e cos (1/2) $(\phi_1 + \phi_2)$. Let

$$\cos\left(\frac{1}{2}\right)(\alpha+\beta) = e\cos\left(\frac{1}{2}\right)(\phi_1+\phi_2) , \qquad 0 \leq \alpha + \beta < 2\pi , \tag{9}$$

$$\alpha - \beta = \phi_2 - \phi_1 - 2m\pi , \qquad 0 \leq \alpha - \beta < 2\pi , \tag{10}$$

where m is the number of complete circuits made by the particle between times t_1 and t_2.

Equations (6), (7), and (8) become

$$c/2a = \sin\left(\frac{1}{2}\right)(\alpha+\beta)\sin\left(\frac{1}{2}\right)(\alpha-\beta) \tag{11}$$

$$\left(r_1 + r_2\right)/2a = 1 - \cos\left(\frac{1}{2}\right)(\alpha+\beta)\cos\left(\frac{1}{2}\right)(\alpha-\beta) \tag{12}$$

$$n\left(t_2 - t_1\right) = 2m\pi + \alpha - \beta - 2\cos\left(\frac{1}{2}\right)(\alpha+\beta)\sin\left(\frac{1}{2}\right)(\alpha-\beta) . \tag{13}$$

The two inequalities in Equations (9) and (10) are geometrically equivalent to the shaded region of Figure 1, from which it is evident that $0 \leq \alpha < 2\pi$ and $-\pi \leq \beta < \pi$. We can also obtain $0 \leq \alpha < 2\pi$ by adding the inequalities in (9) and (10); and if we add $\beta - \alpha$ to each part of inequality (9) and divide the result by 2, we obtain $-(\alpha-\beta)/2 \leq \beta < \pi + (\alpha-\beta)/2$ or $-\pi \leq \beta < \pi$.

With appropriate trigonometric identities, Equations (11) and (12) become

$$\cos\beta - \cos\alpha = c/a$$

$$\cos\beta + \cos\alpha = 2 - \left(r_1 + r_2\right)/a .$$

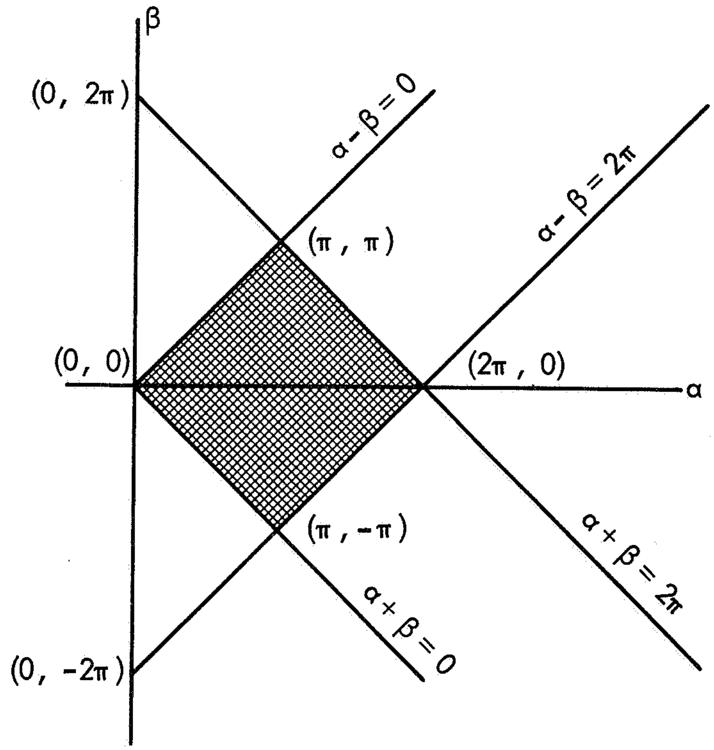


Figure 1.

Solving these two equations,

$$\cos\alpha = 1 - s/a = 1 + 2E , \tag{14}$$

$$\cos\beta = 1 + 2KE , \tag{15}$$

where we have defined

$$s = \left(r_1 + r_2 + c\right)/2 ,$$

$$E = - s/2a ,$$

$$K = 1 - c/s .$$

Since $\cos \alpha = 1 - 2 \sin^2 (\alpha/2)$, (14) can be changed to

$$E = - \sin^2 (\alpha/2) , \qquad 0 \leq \alpha < 2\pi . \tag{16}$$

Similarly (15) becomes

$$KE = - \sin^2 (\beta/2) , \qquad - \pi \leq \beta < \pi \tag{17}$$

$$K = (s - c)/s = \left(r_1 + r_2 - c\right)\Big/2s$$

$$= \left[\left(r_1 + r_2\right)^2 - c^2\right]\Big/4s^2 .$$

Introducing (1) we have

$$K = \left(r_1 \, r_2/2s^2\right)(1 + \cos \theta)$$

$$= \left(r_1 \, r_2/s^2\right) \cos^2 (\theta/2) .$$

Substituting (16) in (17),

$$\sin (\beta/2) = q \sin (\alpha/2) , \qquad - \pi \leq \beta < \pi \tag{18}$$

$$q = \pm \sqrt{K} = \left[\left(r_1 \, r_2\right)^{1/2}\Big/s\right] \cos (\theta/2) . \tag{19}$$

Note that the sign of q is taken care of by the angle θ:

$$1 \geq q \geq 0 \quad \text{if} \quad 0 \leq \theta \leq \pi ,$$

$$0 \geq q \geq - 1 \quad \text{if} \quad \pi \leq \theta \leq 2\pi .$$

We can introduce E into (13) since

$$n\left(t_2 - t_1\right) = \left(\mu/a^3\right)^{1/2}\left(t_2 - t_1\right) = (-E)^{3/2}\,T$$

where

$$T = (8\mu/s)^{1/2}\left(t_2 - t_1\right)/s \; . \tag{20}$$

$$T = (-E)^{-3/2}\left[2m\pi + \alpha - \beta - 2\cos\left(\frac{1}{2}\right)(\alpha+\beta)\,\sin\left(\frac{1}{2}\right)(\alpha-\beta)\right] . \tag{21}$$

This can also be written as

$$T = (-E)^{-3/2}\left[2m\pi + \alpha - \beta - (\sin\alpha - \sin\beta)\right] . \tag{22}$$

Substituting (16) into (22),

$$T\sin^3(\alpha/2) = 2m\pi + \alpha - \beta - \sin\alpha + \sin\beta \; . \tag{23}$$

Equations (18) and (23) with $0 \leq \alpha < 2\pi$ are Lambert's equations for elliptic motion. Given T and q, they are to be solved for α and β, after which it is a simple matter to find all other quantities associated with the orbit.

It is customary in the literature [e.g., 2] to consider T as a function of E (or a) and break the elliptic case of Lambert's theorem into four cases, depending upon the sign of q and whether from (16) $\alpha/2$ is taken in the first or second quadrant. The choice of E as the independent variable makes T a double valued function. This problem can be avoided by choosing α as the independent variable. However, an even better choice will be discussed in the next section.

By a derivation very similar to that for the elliptic case, one finds for the hyperbolic case,

$$T = -E^{-3/2}\left[\gamma - \delta - (\sinh\gamma - \sinh\delta)\right] , \tag{24}$$

$$E = \sinh^2(\gamma/2) , \tag{25}$$

$$\sinh(\delta/2) = q\sinh(\gamma/2) . \tag{26}$$

When $m = 0$, Equations (22) and (24) break down for $E = 0$ and suffer from a critical loss of significant digits in the neighborhood of $E = 0$. To remedy this (22) is written in the form

$$T = \sigma(-E) - qK\sigma(-KE) \tag{27}$$

$$\sigma(u) = 2\left[\arcsin u^{1/2} - u^{1/2}(1-u)^{1/2}\right]/u^{3/2}$$

Replacing $\arcsin u^{1/2}$ and $(1-u)^{1/2}$ by series [3]

$$\sigma(u) = 4/3 + \sum_{n=1}^{\infty} a_n u^n , \qquad |u| < 1$$

$$a_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)/2^{n-2}(2n+3)n!$$

A similar procedure produces the same series for the hyperbolic case. For the parabolic case we have $E = 0$, in which case the series gives

$$T = (4/3)\left(1 - q^3\right) .$$

Thus with $m = 0$ we have a series which is valid for elliptic, hyperbolic, and parabolic transfer provided $0 < x < 2^{1/2}$.

A UNIFIED FORM OF LAMBERT'S EQUATIONS

As mentioned in the previous section, T is a single-valued function of α. However, a better behaved function is obtained if we choose as the independent variable

$$x = \cos(\alpha/2) , \qquad -1 \leq x < 1 ,$$

$$= \cosh(\gamma/2) , \qquad x > 1 .$$

We then have, for both elliptic and hyperbolic transfer,

$$E = x^2 - 1 .$$

For parabolic transfer, it is obvious that we should let $x = 1$.

For the elliptic case let

$$y = \sin(\alpha/2) = (-E)^{1/2}$$

$$z = \cos(\beta/2) = (1 + KE)^{1/2}$$

$$f = \sin\left(\frac{1}{2}\right)(\alpha - \beta) = y(z - qx)$$

$$g = \cos\left(\frac{1}{2}\right)(\alpha - \beta) = xz - qE$$

$$h = \left(\frac{1}{2}\right)(\sin\alpha - \sin\beta) = y(x - qz)$$

$$\lambda = \arctan(f/g) , \qquad 0 \leq \lambda \leq \pi .$$

It then follows from (21) for the elliptic case that

$$T = 2(m\pi + \lambda - h)/y^3 \ .$$

For the hyperbolic case let

$$y = \sinh(\gamma/2) = E^{1/2}$$

$$z = \cosh(\gamma/2) = (1 + KE)^{1/2}$$

$$f = \sinh\left(\frac{1}{2}\right)(\gamma - \delta) = y(z - qx)$$

$$g = \cosh\left(\frac{1}{2}\right)(\gamma - \delta) = xz - qE \ .$$

Note that $0 \leq \gamma - \delta < \infty$ since $0 \leq f < \infty$. Let

$$h = \left(\frac{1}{2}\right)(\sinh\gamma - \sinh\delta) = y(x - qz) \ .$$

It follows that

$$\left(\frac{1}{2}\right)(\gamma - \delta) = \operatorname{arctanh}(f/g)$$

$$= \left(\frac{1}{2}\right)\ell n \left[(f + g)/(g - f)\right]$$

$$= \left(\frac{1}{2}\right)\ell n \left[(f + g)^2/(g^2 - f^2)\right]$$

$$= \ell n\,(f + g) \ .$$

Thus for the hyperbolic case

$$T = 2\left[h - \ln(f+g)\right]/y^3 .$$

It is now apparent that, given q and x, the following steps produce T for all cases:

1. $K = q^2$

2. $E = x^2 - 1$

3. $\rho = |E|$

4. If ρ is near 0, compute T from (27)

5. $y = \rho^{1/2}$

6. $z = (1 + KE)^{1/2}$

7. $f = y(z - qx)$

8. $g = xz - qE$

9. If $E < 0, \lambda = \arctan(f/g)$, $d = m\pi + \lambda$, $0 \leq \lambda \leq \pi$
   If $E > 0, d = \ln(f + g)$

10. $T = 2(x - qz - d/y)/E.$

The following formula for the derivative holds for all cases except for $x = 0$ with $K = 1$ and for $x = 1$.

$$dT/dx = (4 - 4qKx/z - 3xT)/E .$$

If x is near 1, differentiate (27) to obtain

$$dT/dx = 2x\left[qK^2 \sigma'(-KE) - \sigma'(-E)\right] ,$$

$$\sigma'(u) = d\sigma/du = \sum_{n=1}^{\infty} n a_n u^{n-1} .$$

The derivative in the case of $x = 0$ with $k = 1$ will be discussed in the next section.

AUXILIARY FORMULAS

In this section we will show how to obtain a number of useful quantities associated with the two-body orbit, assuming Lambert's problem has been solved for x.

In the derivation of Lambert's equation for the elliptic case, α and β are defined in such a way that

$$\phi_2 - \phi_1 = \alpha - \beta + 2m\pi = 2(\lambda + m\pi) .$$

Using (22) the eccentric anomaly difference can also be written in the form

$$\begin{aligned}\phi_2 - \phi_1 &= (-E)^{3/2} T + \sin\alpha - \sin\beta \\ &= y^3 T + 2y(x - qz) . \end{aligned} \tag{28}$$

From (28) we have

$$\begin{aligned}\sin(\phi_2 - \phi_1) &= 2y(z - qx)(xz - qE) \\ &= 2y(x - qz) + 4y^3 q(z - qx)\end{aligned}$$

$$\phi_2 - \phi_1 - \sin(\phi_2 - \phi_1) = y^3 T - 4y^3 q(z - qx) \tag{29}$$

$$1 - \cos(\phi_2 - \phi_1) = 2y^2 (z - qx)^2 . \tag{30}$$

We now obtain a formula for the time derivative $\dot{r}$ at time t_1. Kepler's equation in the elliptic case can be written in the form [4]

$$\left(\mu/a^3\right)^{3/2}\left(t_2 - t_1\right) = \phi_2 - \phi_1 + r_1\dot{r}_1\left[1 - \cos\left(\phi_2 - \phi_1\right)\right]\Big/(\mu a)^{1/2}$$

$$- \left(1 - r_1/a\right)\sin\left(\phi_2 - \phi_1\right).$$

Substituting $1/a = 2y^2/s$, $t_2 - t_1 = s^{3/2}T/(8\mu)^{1/2}$, and making use of (29) and (30) gives

$$(2/\mu s)^{1/2}(z - qx)\, r_1\dot{r}_1 = 2q - \left(2r_1/s\right)(xz - qE).$$

Multiplying through by $z + qx$, we have, since $(z - qx)(z + qx) = 1 - K = c/s$ and $(z + qx)(xz - qE) = x + qz$,

$$(2/\mu s)^{1/2} c\, r_1\dot{r}_1 = 2qs(z + qx) - 2r_1(x + qz)$$

$$= 2qz(s - r_1) + 2x(Ks - r_1)$$

$$Ks - r_1 = (1 - c/s)s - r_1 = s - c - r_1 = r_2 - s.$$

Thus we have finally that

$$\dot{r}_1 = (2\mu s)^{1/2}\left[qz\left(s - r_1\right) - x\left(s - r_2\right)\right]\Big/cr_1. \tag{31}$$

At time t_2 we find

$$\dot{r}_2 = (2\mu s)^{1/2}\left[x\left(s - r_1\right) - qz\left(s - r_2\right)\right]\Big/cr_2. \tag{32}$$

By a similar procedure we can show that (31) and (32) hold also for the hyperbolic and parabolic cases.

Having x we can find the semimajor axis a or its reciprocal

$$1/a = 2y^2/s .$$

We know that

$$e \cos\phi = 1 - r/a ,$$

$$e \sin\phi = r\dot{r}/(\mu a)^{1/2} ,$$

where r is the magnitude of the position vector at time t.

Thus for the eccentricity we have

$$e^2 = (1 - r/a)^2 + (r\dot{r})^2/\mu a .$$

For the semilatus rectum p we have

$$p = a\left(1 - e^2\right) = 2r - r^2/a - (r\dot{r})^2/\mu$$

and for the value of r at the point of closest approach to the center of attraction we have

$$r_p = \frac{p}{1 + e} .$$

For the speed we have

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) .$$

For the component of velocity perpendicular to a radius vector,

$$v_\theta = (\mu p)^{1/2}/r \ .$$

The above formulas hold for any type of two-body motion. Either of the subscripts 1 or 2 can be placed on r, $\dot{r}$, and v.

If $\dot{r}_1 < 0$ and $\dot{r}_2 > 0$ or if $\dot{r}_1$ and $\dot{r}_2$ have the same sign with $\theta > \pi$, periapsis passage will occur between times t_1 and t_2, in which case it may be of interest to compute r_p.

If the plane of motion is known, the velocity $\vec{v}$ at either time t_1 or time t_2 can be written in any convenient coordinate system, since the components $\dot{r}$ and v_θ are known. The plane of motion can be found from the position vectors $\vec{r}_1$ and $\vec{r}_2$ at times t_1 and t_2, provided they are not parallel. Since this is a common case, we will express $\vec{v}_1$ and $\vec{v}_2$ (the velocities at t_1 and t_2) in terms of $\vec{r}_1$ and $\vec{r}_2$.

For the velocity $\vec{v}_1$ we have

$$\vec{v}_1 = \vec{v}_{r1} + \vec{v}_{\theta 1} \ ,$$

where $\vec{v}_{r1}$ is along $\vec{r}_1$ and $\vec{v}_{\theta 1}$ is in the plane of motion perpendicular to $\vec{r}_1$ and in the direction of motion, i.e., in the direction of increasing true anomaly. We have

$$\vec{v}_{r1} = \left(\dot{r}_1/r_1\right) \vec{r}_1 \ ,$$

$$\vec{v}_{\theta 1} = c_1 \vec{r}_1 + c_2 \vec{r}_2 \ ,$$

where c_1 and c_2 are to be determined.

$$\vec{r}_1 \cdot \vec{v}_{\theta 1} = 0 = c_1 r_1^2 + c_2 \vec{r}_1 \cdot \vec{r}_2$$

$$\vec{r}_2 \cdot \vec{v}_{\theta 1} = r_2 v_{\theta 1} \sin\theta = c_1 \vec{r}_1 \cdot \vec{r}_2 + c_2 r_2^2 \ .$$

Since $\vec{r}_1 \cdot \vec{r}_2 = r_1 r_2 \cos\theta$ we have

$$r_1 c_1 + \left(r_2 \cos\theta\right) c_2 = 0$$

$$\left(r_1 \cos\theta\right) c_1 + r_2 c_2 = v_\theta \sin\theta \quad .$$

Solving for c_1 and c_2 we have

$$\vec{v}_1 = \left(\dot{r}_1 - v_{\theta 1} \cot\theta\right)\left(\vec{r}_1/r_1\right) + \left(v_{\theta 1} \csc\theta\right)\left(\vec{r}_2/r_2\right) .$$

In a similar way we find

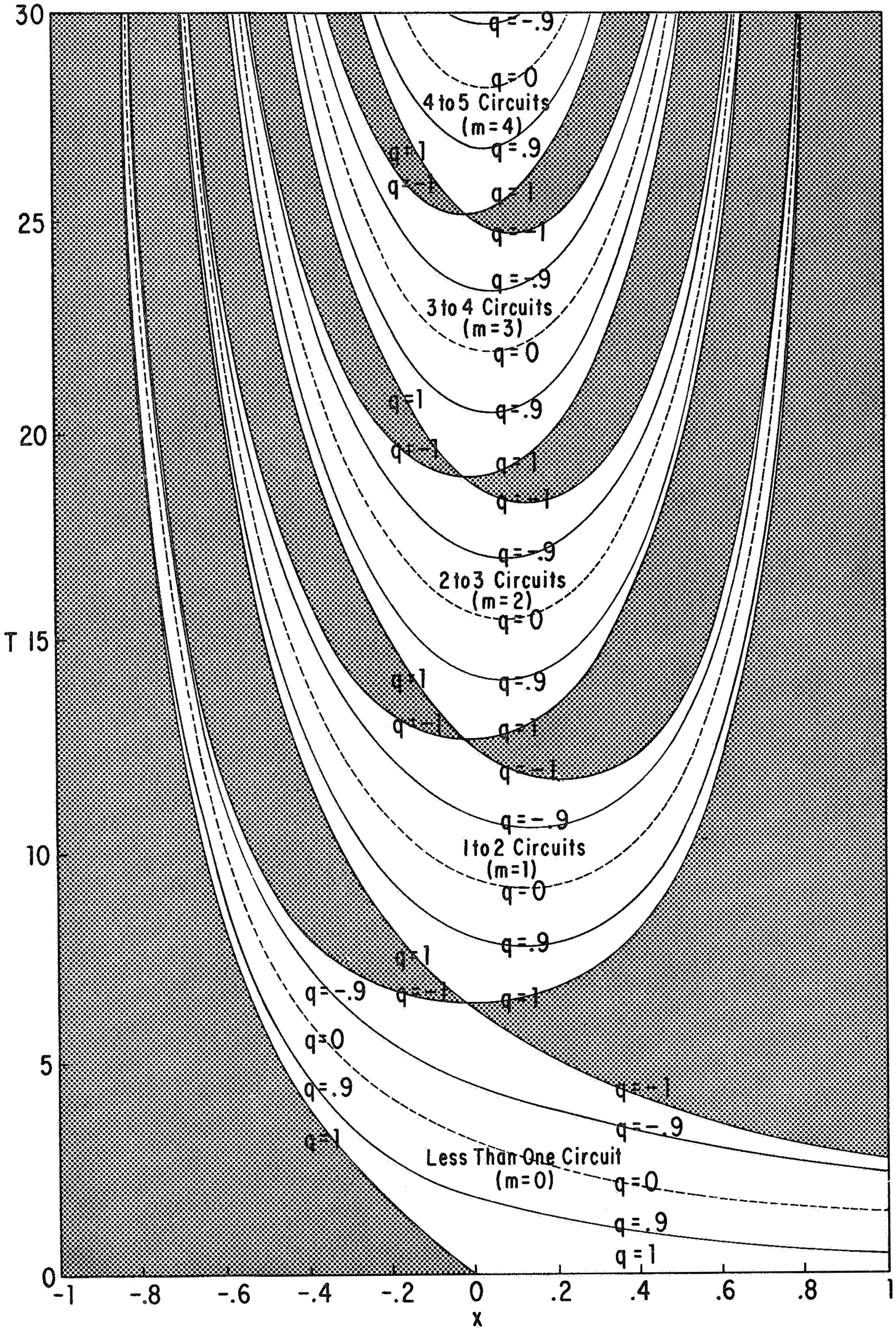
$$\vec{v}_2 = -\left(v_{\theta 2} \csc\theta\right)\left(\vec{r}_1/r_1\right) + \left(\dot{r}_2 + v_{\theta 2} \cot\theta\right)\left(\vec{r}_2/r_2\right) .$$

Figure 2 shows T as a function of x. Note the discontinuity in the slope for $x = 0$ with $K = 1$. For $K = 1$, $z = |x|$. Thus we are led to consider four cases: $q = \pm 1$ with $x \geq 0$, $q = \pm 1$ with $x \leq 0$. Examination of the formulas for dT/dx in these cases reveals that if $q = 1$ we have a left-hand derivative of -8 and a right-hand derivative of 0 at $x = 0$. If $q = 1$ we have a left-hand derivative of 0 and a right-hand derivative of -8 at $x = 0$.

With further analysis we find that the cases ($q = 1$, $x \leq 0$) and ($q = -1$, $x \geq 0$) represent rectilinear orbits.

For $m = 0$, T is a monotone function of x, making possible a simple numerical procedure for solving Lambert's problem. Figure 2 is for the elliptic case, Figure 3 for the hyperbolic case, the parabolic case occurring at $x = 1$ in both figures. Figure 4 shows a small region of Figure 3 where d^2T/dx^2 is negative. If the Newton-Raphson method is being used to find x, a switch should be made in this region to the secant (regula falsi) method.

No solutions of Lambert's problem exist in the shaded regions of Figures 2 and 3. $x = 1$ ($m > 0$) and $x = -1$ are vertical asymptotes. $T \to 0$ as $x \to \infty$.



q=-.9
q=0
4 to 5 Circuits
(m=4)
q=.9
q=1
q=-1
q=-.9
3 to 4 Circuits
(m=3)
q=0
q=.9
2 to 3 Circuits
(m=2)
1 to 2 Circuits
(m=1)
Less Than One Circuit
(m=0)
T
x
30
25
20
15
10
5
0
-1
-.8
-.6
-.4
-.2
.2
.4
.6
.8
1

Figure 2.

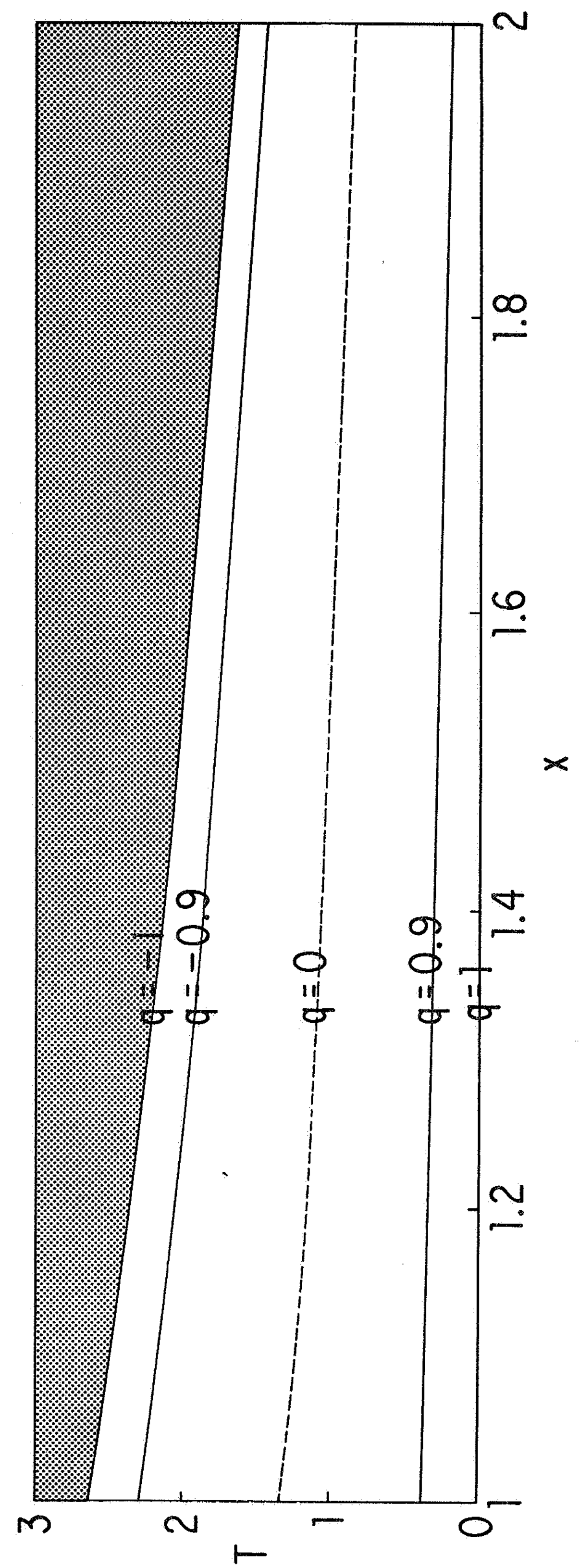


Figure 3.

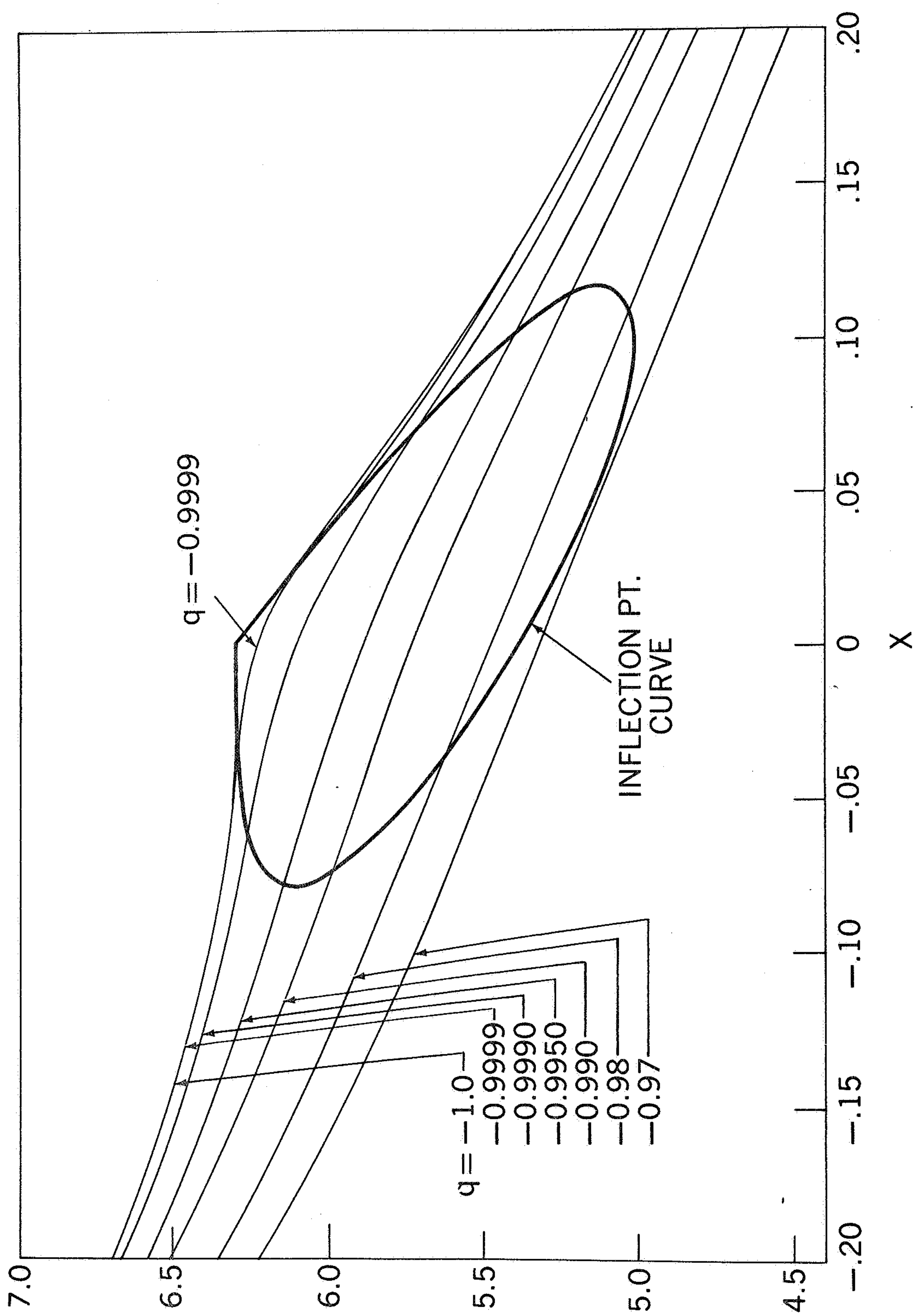

T
X
7.0
6.5
6.0
5.5
5.0
4.5
−.20
−.15
−.10
−.05
0
.05
.10
.15
.20
q = −0.9999
INFLECTION PT. CURVE
q = −1.0
−0.9999
−0.9990
−0.9950
−0.990
−0.98
−0.97

Figure 4.

REFERENCES

1. Lancaster, E. R., Blanchard, R. C., and Devaney, R. A., "A Note on Lambert's Theorem," X-643-66-35, Goddard Space Flight Center, February 1966. Also AIAA Journal, Vol. 3, No. 9, 1436-1437 (Sept. 1966).

2. Breakwell, John V., Gillespie, Rollin W., and Ross, Stanley, "Researches in Interplanetary Transfer," ARS J. 31, 201-208 (Feb. 1961).

3. Gedeon, G. S., "Lambertian Mechanics," Proceedings of the XII International Astronautical Congress (Springer-Verlag, Wien, and Academic Press, Inc., New York, 1963).

4. Battin, Richard H., "Astronautical Guidance," McGraw-Hill Book Co., Inc., 1964, page 46.