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FACILITY FORM 602

N 68-35613

ACCESSION NUMBER 246 (THRU) 1

PAGES 30 (CODE) 30

896954 (CATEGORY)

(NASA CR OR TAX OR AD NUMBER)

GPO PRICE \$ _____

CSFTI PRICE(S) \$ _____

Hard copy (HC) 3.00

Microfiche (MF) .65

653 July 65

DERIVATION OF EQUATIONS

GENERALIZED SPACECRAFT SIMULATION

06464-6004-T000

Volume I DYNAMIC EQUATIONS

15 FEBRUARY 1967

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ABSTRACT

This document is one in a series covering the development and use of the Generalized Spacecraft Simulation (GSS). The formulation of the basic equations used in programming the simulation is presented in two volumes. Volume I, which supersedes Reference 5, contains the derivation of a set of nonlinear, large-angle dynamics equations for an N-body system, including gravity gradient and disturbance torques and forces. Volume II defines and presents the equations for a generalized active control system.

The basic spacecraft configuration consists of N connected bodies. Body 1 is considered the main spacecraft structure wherein the orbital experiments are housed. Each of the N-1 auxiliary bodies is assumed to be attached to Body 1 at an arbitrary single point, to have up to three degrees of rotational freedom relative to Body 1, and no interconnection with the other auxiliary bodies. The relative rotation of the auxiliary bodies, with respect to the main body, is assumed to take place about a gimballed hinge. For axes with degrees of freedom, the reaction torques may be zero for a frictionless pin, may be proportional to displacement or velocity to represent springs and dampers, or may be control torques for array drives. In addition, the motion about any gimbal axis is limited by a system of elastic gimbal stops.

The disturbance parameters considered include solar radiation forces and torques, thermal bending torques, aerodynamic forces and torques, and magnetic torques. Magnetic disturbances are based upon a simple dipole magnetic field representation. Thermal bending, both in plane and out of plane, is computed for all booms. Solar and aerodynamic disturbances include the effects of the thermally deflected boom, and are based upon up to four basic shapes for each body. The shapes considered are spheres, flat plates, cylinders, and parallelepipeds.

In addition, the equations necessary to incorporate the effects of solar eclipse, boom deployment, and magnetic hysteresis dampers, have been derived and presented in the form required for digital programming.

ACKNOWLEDGEMENT

This document covers only a portion of the continuing development and use of the Generalized Spacecraft Simulation (GSS). Additional documentation of the current version C is contained in References 2 and 3. The primary areas of responsibility were assigned as follows:

Spacecraft Configurations and Basic Equations

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In exercising their responsibilities in connection with the present and previous versions of the program, Dr. Tarnove and Mr. Jones have made numerous suggestions and recommendations which have been incorporated in this present revision of Reference 5. The reformulation of the dynamic equations for digital programming presented in Appendix A represents a small portion of Dr. Tarnove's many contributions to the development of the simulation.

Grateful acknowledgement is given to the technical staff members for the suggestions and comments resulting from their use of the program, in particular, Dr. P. C. Wheeler of TRW Systems and Mr. H. R. Freeman of Goddard Space Flight Center, for their many contributions in adapting the current version for the Nimbus Control Systems Design Studies.

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1. INTRODUCTION

This document is one in a series of documents covering the development and use of the Generalized Spacecraft Simulation (GSS). This simulation is a general purpose digital computer program designed to produce a time history of the motion of a generalized multi-body spacecraft. The program input is provided in general form such that the majority of existing and proposed spacecraft may be simulated in an arbitrary orbit about any preselected body. The output is in both tabular and plotted form. Provisions are made to plot up to 48 of up to 256 preselected variables.

The basic document covering the simulation is the User's Handbook, Reference 3, which contains a complete description of the program and its uses. The present document is a reference document to the User's Handbook and consists of two volumes. Volume I, which supersedes Reference 5, contains the derivation of a set of non-linear large-angle dynamic equations for an N body system including gravity gradient and disturbance torques and forces. The disturbances considered are due to aerodynamics, solar radiation, the magnetic field, thermal bending, eclipse, and boom deployment. The reaction torques between bodies may result from springs, viscous dampers, stops, magnetic hysteresis dampers, and positioning control torques. Volume II of the present document defines and presents the equations for a generalized active control system. The sensor routine provides for two rate gyros, two horizon scanners, and two two-axis sun sensors complete with sensor dynamics. The sensors may be patched to the control systems through saturating lead-lag compensation networks. The control systems consist of three gas jet systems, four reaction wheel systems, and two array drive systems, where each control system is provided with 6 possible control laws. Reaction wheel speed control and unloading loops are provided.

The digital programming of the simulation is covered in a fourth document, Reference 2, which contains the program listings and flow diagrams.

2. SPACECRAFT CONFIGURATIONS

The basic satellite configuration under investigation consists of N connected bodies. Body 1 is considered the main satellite structure where in the orbital experiments are housed. Each of the $N-1$ auxiliary bodies is assumed to be attached to body 1 at a single point, to have up to three degrees of rotational freedom relative to body 1 and no interconnection with the other auxiliary bodies.

2.1 DYNAMIC REPRESENTATION

The relative positions of the mass elements of each of the N bodies are defined with respect to corresponding sets of body coordinates. It is assumed that the mass elements of the bodies may be in motion with respect to each other and that this motion is defined with respect to the corresponding body coordinates.

The main body of the satellite, designated as body 1, is the only body which is required to have some degree of rigidity. The relative positions of the attach points or hinge points for the auxiliary bodies are assumed to be fixed in body 1 coordinates. The vector location of the hinge point for the i^{th} body is given by $\bar{u}_i$ expressed as three constant components in the body 1, $\bar{x}_1$, $\bar{y}_1$, and $\bar{z}_1$ coordinate system. In general the allowable relative motion of the mass elements of a given body with respect to its body axes is used to simulate the effects of flexibility (thermal bending), changes in shape (boom deployment), and momentum changes (reaction wheels). These effects are expressed in the dynamic equations as momentum vectors and time rates of change in momentum defined with respect to the corresponding body coordinates.

The relative rotational positions of the N bodies are defined by the relative positions of the corresponding body coordinates. As noted above, each auxiliary body is assumed to have three degrees of rotational freedom about its point of attachment to body 1. The relative rotation of the i^{th} body with respect to body 1, where i may have values from 2 to N , is assumed to take place about a gimballed hinge nominally aligned with the three orthogonal body axes of body i . The components of the hinge torque about each of the gimbal axis is a function of the reaction torques at these

axes. For axes with degrees of freedom these reaction torques may be zero for a frictionless pin, may be proportional to displacement or velocity to represent springs and dampers, or may be control torques for array drives. In addition, the motion about any gimbal axis is limited by a system of elastic gimbal stops.

Body 1 is assumed to be rigid with respect to the hinge points of the auxiliary bodies and with regard to the mass moments of inertia. However, the body dynamics may include an arbitrarily oriented angular momentum vector which may have programmed changes in the magnitudes of its components. This momentum vector is used to simulate reaction wheels in the main body. The inertia of the wheels is included in the inertia matrix for the body and the momentum is defined with respect to the body axes. The mass properties of body 1 are defined by its total mass, M_1 , and a constant nine element inertia matrix, $[I_1]$. The three diagonal terms represent the mass moments of inertia with respect to the body coordinates while the off diagonal terms are the negative of the corresponding products of inertia.

Each auxiliary body i is defined as k masses, m_{ik} , supported by uniform rods each of which is oriented, in body i axes, by the arbitrary position vector $\bar{R}_{ik}$. The value of k may be 1, 2, 3, or 4. The auxiliary bodies are allowed to change shape due to thermal bending and deployment of the k rods or booms, and may contain arbitrary oriented and controlled reaction wheels. These effects are represented dynamically by changing linear and angular momentum vectors. In order to generalize the mass properties of body i , represented by the masses m_{ik} , each mass is given a constant inertia matrix defined with respect to the end of rod $\bar{R}_{ik}$. The total mass of body i is given by

$$M_i = \sum_{k=1}^4 m_{ik} \quad (2.1-1)$$

Due to changes in shape the expressions for the location of the center of mass and the total inertia matrix for body i are time varying functions.

2.2 DISTURBANCE REPRESENTATION

The disturbance parameters considered include solar radiation forces and torques, thermal bending torques, aerodynamic forces and torques, and magnetic torques. Magnetic torques are computed upon the bases of an arbitrary oriented magnetic moment on body i . Thermal bending both in plane and out of plane, together with solar and aerodynamic forces and torques may be computed for all booms. Solar and aerodynamic disturbances include the effects of the thermally deflected boom. The following solar and aerodynamic shapes are considered for the main body and tip masses.

0. Point Mass
1. Sphere
2. Flat Plate
3. Cylinder
4. Parallelepiped

Each of the six bodies of the system may be represented by four shapes. For the auxiliary bodies any one of the possible shapes may be placed at the tip of each boom with any desired orientation. It is assumed that the centroid of the shape is at the tip of the vector $\bar{R}_{ik}$. In the case of the main body the four shapes may have any orientation and the location of each centroid is defined by an arbitrary input vector.

3. COORDINATE SYSTEMS

In describing the motion of body 1 and the auxiliary bodies a number of right-hand orthogonal coordinate systems are used. These coordinate systems are defined and described below.

3.1 INERTIALLY FIXED COORDINATE SYSTEM

The inertially fixed coordinate system has its origin at the center of the earth and is defined by unit vectors $\bar{x}_I$, $\bar{y}_I$, and $\bar{z}_I$. The $\bar{y}_I$ axis is normal to the equatorial plane, positive northward, and $\bar{z}_I$ is along the autumnal equinox. The coordinate matrix for this system as defined by Equation 3-10 of Reference 1 is given by

$$\{\bar{Q}_I\} = \begin{Bmatrix} \bar{x}_I \\ \bar{y}_I \\ \bar{z}_I \end{Bmatrix} \quad (3.1-1)$$

Since the coordinate system is inertially fixed its rotational velocity vector is given by

$$\bar{\omega}_I = 0 \quad (3.1-2)$$

3.2 ORBITAL REFERENCE COORDINATE SYSTEM

The orbital reference coordinate system has its origin at the center of mass of the total satellite and is defined by unit vectors $\bar{x}_M$, $\bar{y}_M$, $\bar{z}_M$. The $\bar{z}_M$ axis is directed toward the center of mass of the earth, the $\bar{x}_M$ axis lies in the orbit plane and forms an acute angle with the tangential velocity vector and the $\bar{y}_M$ axis is normal to the orbit plane, see Figure 3.2-1. The vector distance from the center of the earth to the center of mass of the total satellite is $\bar{S}_M$.

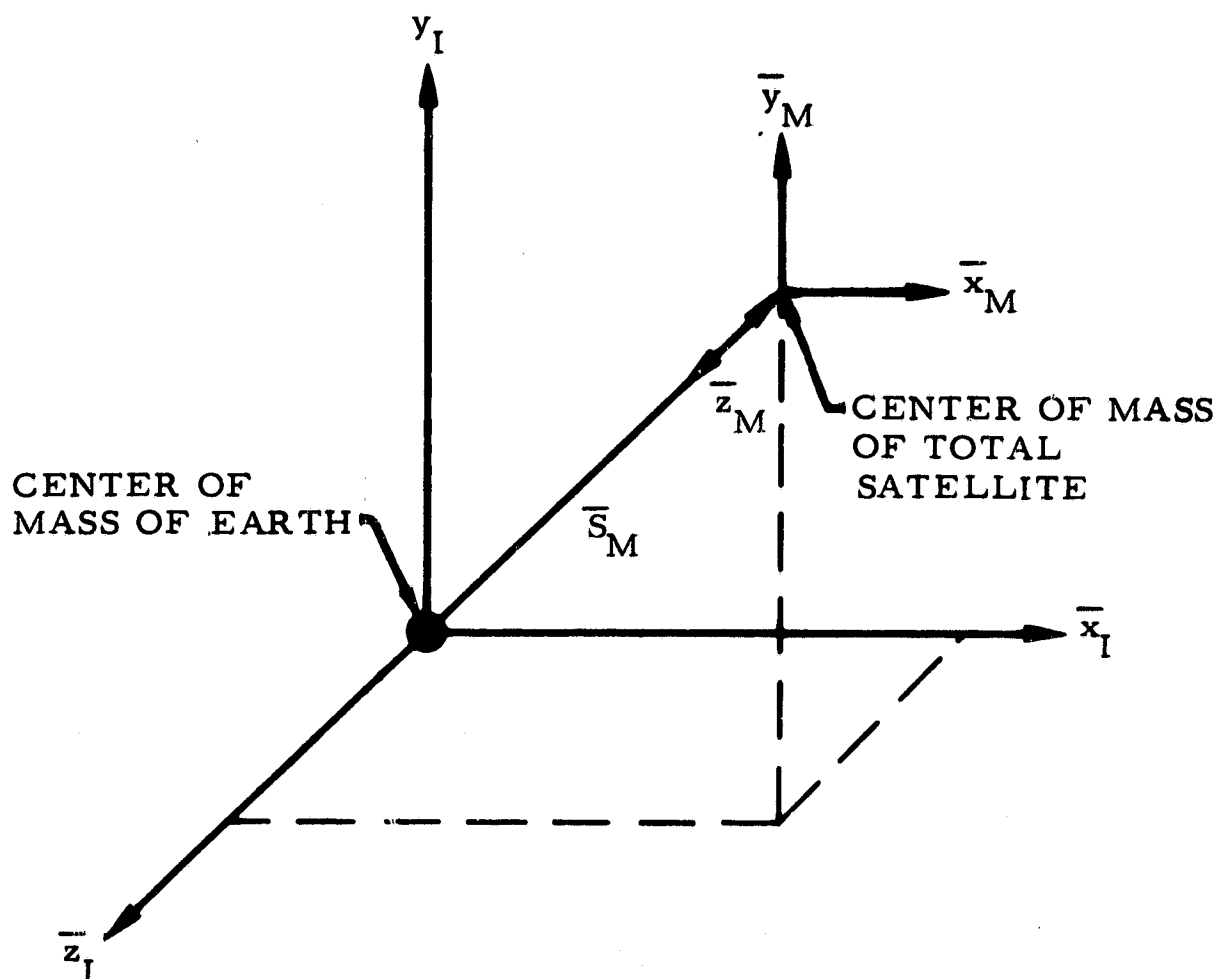


Figure 3.2-1. Inertial and Orbital Reference Systems

A second orbital reference system, designated as the body 1 reference system, has its origin at the center of mass of body 1 and is defined by unit vectors $\bar{x}_R$, $\bar{y}_R$, and $\bar{z}_R$. This coordinate system is produced by pure translation of the $\bar{x}_M$, $\bar{y}_M$ and $\bar{z}_M$ system to the center of mass of body 1, therefore, the corresponding coordinate axes are parallel. From Equation 3-14 of Reference 1,

$$\bar{s}_M = \{s_{MM}\}_T \{\bar{Q}_M\} = \{s_{MR}\}_T \{\bar{Q}_R\} \quad (3.2-1)$$

The coordinate matrices are

$$\{\bar{Q}_M\} = \begin{Bmatrix} \bar{x}_M \\ \bar{y}_M \\ \bar{z}_M \end{Bmatrix} = \{\bar{Q}_R\} = \begin{Bmatrix} \bar{x}_R \\ \bar{y}_R \\ \bar{z}_R \end{Bmatrix} \quad (3.2-2)$$

From Figure 3.2-1 it may be seen that

$$\{S_{MM}\} = \{S_{MR}\} = \begin{Bmatrix} 0 \\ 0 \\ -S_M \end{Bmatrix} \quad (3.2-3)$$

where S_M is the magnitude of the vector location of the total spacecraft mass with respect to the center of mass of the earth. The rotational velocity vectors for the orbital reference systems are

$$\bar{\omega}_R = \{\omega_{MM}\}_T \{\bar{Q}_M\} = \{\omega_{RR}\}_T \{\bar{Q}_R\} \quad (3.2-4)$$

and

$$\{\omega_{MM}\} = \{\omega_{RR}\} \quad (3.2-5)$$

The coordinate transformation matrix between $\{\bar{Q}_I\}$ and the orbital reference systems is $[E]$ such that

$$\{\bar{Q}_M\} = [E] \{\bar{Q}_I\} \quad (3.2-6)$$

$$\{\bar{Q}_R\} = [E] \{\bar{Q}_I\} \quad (3.2-7)$$

3.3 BODY 1 COORDINATE SYSTEM

The body 1 coordinate system has its origin at the center of mass of body 1 to coincide with the origin of the body 1 reference coordinate system, and is defined by the unit vectors $\bar{x}_1$, $\bar{y}_1$, and $\bar{z}_1$. The axes are oriented in body 1 such that they coincide with the corresponding axes of

the reference system when body 1 is in its nominal position. The coordinate matrix for the body 1 coordinate system is

$$\{\bar{Q}_1\} = \begin{Bmatrix} \bar{x}_1 \\ \bar{y}_1 \\ \bar{z}_1 \end{Bmatrix} \quad (3.3-1)$$

and the rotational velocity vector for the body 1 coordinate system is $\bar{\omega}_1$ where

$$\bar{\omega}_1 = \{\omega_{11}\}_T \{\bar{Q}_1\} \quad (3.3-2)$$

The coordinate transformation matrix between $\{\bar{Q}_R\}$ and $\{\bar{Q}_1\}$ is $[A]$ such that

$$\{\bar{Q}_1\} = [A] \{\bar{Q}_R\} \quad (3.3-3)$$

using Equation 3.2-7

$$\{\bar{Q}_1\} = [A] [E] \{\bar{Q}_I\} \quad (3.3-4)$$

3.4 BODY i NOMINAL COORDINATE SYSTEM

The body i nominal coordinate system has its origin at the attach point between body i and body 1 and is defined by the unit vectors $\bar{x}_{ni}$, $\bar{y}_{ni}$, and $\bar{z}_{ni}$. The body i nominal coordinates are fixed relative to the body 1 coordinates and determine the nominal location of the body i coordinates relative to body 1. The coordinate matrix for the nominal body i coordinate system is

$$\{\bar{Q}_{ni}\} = \begin{Bmatrix} \bar{x}_{ni} \\ \bar{y}_{ni} \\ \bar{z}_{ni} \end{Bmatrix} \quad (3.4-1)$$

The coordinate transformation matrix between $\{\bar{Q}_1\}$ and $\{\bar{Q}_{ni}\}$ is $[B_i]$ such that

$$\{\bar{Q}_{ni}\} = [B_i] \{\bar{Q}_1\} \quad (3.4-2)$$

where $[B_i]$ is a three by three matrix of constants. The rotational velocity vector for the $\{\bar{Q}_{ni}\}$ system is the same as for the body 1 coordinate system

$$\bar{\omega}_{ni} = \bar{\omega}_1 = \{\omega_{11}\}_T \{\bar{Q}_1\} = \{\omega_{1ni}\}_T \{\bar{Q}_{ni}\} = \left\{ [B_i] \{\omega_{11}\} \right\}_T \{\bar{Q}_{ni}\} \quad (3.4-3)$$

Let the vector location of the body i attach point with respect to the body 1 center of mass be $\bar{u}_i$ where the components of $\bar{u}_i$ are defined in the body 1 coordinate system

$$\bar{u}_i = \{u_{i1}\}_T \{\bar{Q}_1\} = \left\{ [B_i] \{u_{i1}\} \right\}_T \{\bar{Q}_{ni}\} \quad (3.4-4)$$

where the three components $\{u_{i1}\}$ are constants.

3.5 BODY i COORDINATE SYSTEM

The body i coordinate system has its origin at the attach point between body i and body 1 coincide with the origin of $\{\bar{Q}_{ni}\}$ system, and is defined by the unit vectors $\bar{x}_i$, $\bar{y}_i$, and $\bar{z}_i$. The axes are oriented in body i such that they coincide with the corresponding axes of the $\{Q_{ni}\}$ system when body i is in its nominal position. It should be noted that the orientation of the $\{Q_{ni}\}$ system is arbitrarily defined by the selection of the constant $[B_i]$ transformation matrix which in turn defines the orientation of the body i axes with respect to body i. The coordinate matrix for the body i coordinate system is

$$\{\bar{Q}_i\} = \begin{Bmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{z}_i \end{Bmatrix} \quad (3.5-1)$$

and the rotational velocity vector for the body i coordinate system is $\bar{\omega}_i$ where

$$\bar{\omega}_i = \{\omega_{ii}\}_T \{\bar{Q}_i\} \quad (3.5-2)$$

The coordinate transformation matrix between $\{\bar{Q}_{ni}\}$ and $\{\bar{Q}_i\}$ is $[C_i]$ such that

$$\{\bar{Q}_i\} = [C_i] \{\bar{Q}_{ni}\} \quad (3.5-3)$$

From Equation 3.4-2

$$\{\bar{Q}_i\} = [C_i] [B_i] \{\bar{Q}_1\}$$

Let

$$[D_i] = [C_i] [B_i] \quad (3.5-4)$$

then

$$\{\bar{Q}_i\} = [D_i] \{\bar{Q}_1\} \quad (3.5-5)$$

The body i coordinate system $\{\bar{Q}_i\}$ has its origin at the attach point to body 1 and not at the body center mass. Let the vector position of the center of mass of body i with respect to the attach point be $\bar{v}_i$ where the components of $\bar{v}_i$ are defined with respect to the $\{\bar{Q}_i\}$ coordinate system. Then

$$\bar{v}_i = \{v_{ii}\}_T \{\bar{Q}_i\} \quad (3.5-6)$$

where the three components $\{v_{ii}\}$ are variables.

4. METHOD

The equations of motion are sufficiently complexed to warrant an initial discussion of the general approach to be utilized. The ultimate objective of this derivation is to provide a set of differential equations of motion to be programmed for a digital computer. When provided with a set of initial conditions and disturbances the program will then produce a time history of the motion.

Considering each of the N bodies as a free body in space, it will have six rigid body degrees of freedom neglecting any internal elastic degrees of freedom. In general, the six degrees of freedom can be expressed by three force equations and three moment equations, however, for this derivation the matrix notation established in Reference 1 will be used and the three force equations will be expressed as a single matrix force equation and the three moment equations by a single matrix moment equation. The $6N$ degrees of freedom of the N bodies considered as free bodies may then be expressed by N matrix force equations and N matrix moment equations. At the attach point between an auxiliary body and body 1 the reaction forces between the bodies may be equated thereby eliminating three degrees of freedom and one matrix force equation. Since all of the auxiliary bodies are attached to body 1, the system will have $3(N+1)$ degrees of freedom and the motion may be represented by N matrix moment equations and one matrix force equation. Since the reaction forces between the bodies are equal and opposite, the force equation must equate the sum of the external forces to the product of the total satellite mass and the acceleration of the center of mass. The solution of the force equation will describe the motion of the center of mass of the spacecraft and will, therefore, give the orbital equations for the spacecraft from which the location of the center of mass and the position of the orbital reference axes can be computed for any instant of time. The remaining N matrix moment equations are the dynamic equations describing the motion of the N bodies with respect to a reference system at the total center of mass. The $3N$ independent variables involved in the equations express the relative rotational positions of the $N-1$

auxiliary bodies with respect to body 1 and the relative rotational position of body 1 with respect to the orbital reference axes.

It is assumed that for the time interval to be studied the external forces, other than gravity forces, do not appreciably effect the orbit of the center of mass. Orbital parameters are not integrated to determine the satellite position and the matrix force equation is reduced to a set of orbit equations. An arbitrary orbit is established and orbit position is computed as a function of time.

The N matrix moment equations may be written in the form

$$[M_{11}]\{\dot{\sigma}_1\} + [M_{12}]\{\dot{\sigma}_2\} + [M_{13}]\{\dot{\sigma}_3\} + \dots\dots[M_{1N}]\{\dot{\sigma}_N\} = \{b_1\} \quad (4.0-1)$$

$$[M_{21}]\{\dot{\sigma}_1\} + [M_{22}]\{\dot{\sigma}_2\} + [M_{23}]\{\dot{\sigma}_3\} + \dots\dots[M_{2N}]\{\dot{\sigma}_N\} = \{b_2\} \quad (4.0-2)$$

$$[M_{31}]\{\dot{\sigma}_1\} + [M_{32}]\{\dot{\sigma}_2\} + [M_{33}]\{\dot{\sigma}_3\} + \dots\dots[M_{3N}]\{\dot{\sigma}_N\} = \{b_3\} \quad (4.0-3)$$

$$\dots\dots\dots$$

$$[M_{N1}]\{\dot{\sigma}_1\} + [M_{N2}]\{\dot{\sigma}_2\} + [M_{N3}]\{\dot{\sigma}_3\} + \dots\dots[M_{NN}]\{\dot{\sigma}_N\} = \{b_N\} \quad (4.0-4)$$

where

$\{\sigma_i\}$ = a three element column matrix representing the three variables associated with body i.

$\{b_i\}$ = a three element column matrix representing the three components of the driving torque on body i.

$[M]$ = 3 x 3 matrix of coefficients

Equations 4.0-1 through 4.0-4 may be written as the single matrix equation

$$\begin{bmatrix} [M_{11}][M_{12}][M_{13}]\dots[M_{1N}] \\ [M_{21}][M_{22}][M_{23}]\dots[M_{2N}] \\ [M_{31}][M_{32}][M_{33}]\dots[M_{3N}] \\ \dots\dots\dots \\ [M_{N1}][M_{N2}][M_{N3}]\dots[M_{NN}] \end{bmatrix} \begin{Bmatrix} \{\dot{\sigma}_1\} \\ \{\dot{\sigma}_2\} \\ \{\dot{\sigma}_3\} \\ \vdots \\ \{\dot{\sigma}_N\} \end{Bmatrix} = \begin{Bmatrix} \{b_1\} \\ \{b_2\} \\ \{b_3\} \\ \vdots \\ \{b_N\} \end{Bmatrix} \quad (4.0-5)$$

or simply as

$$[M] \{\ddot{\sigma}\} = \{b\} \quad (4.0-6)$$

where $[M]$ is the $3N \times 3N$ system matrix, $\{\ddot{\sigma}\}$ is a column matrix of $3N$ unknown variables and $\{b\}$ is a column matrix of driving functions. The first three elements of $\{\ddot{\sigma}\}$ are $\{\ddot{\sigma}_1\}$ representing the components of the angular acceleration of body 1 with respect to the orbital reference frame. In the case of auxiliary body i ($2 \leq i \leq N$), with three degrees of rotational freedom with respect to body 1, $\{\ddot{\sigma}_i\}$ represents the angular accelerations about the three gimbal axes. As noted in Section 2.0 the auxiliary bodies may have 0, 1, 2, or 3 degrees of freedom with respect to the main body. For each degree of freedom eliminated the independent variable representing the corresponding gimbal deflection is eliminated and a dependent variable representing the locked gimbal reaction torque is introduced. One of the approaches to the solution of Equation 4.0-6 for a locked gimbal consists of eliminating the locked gimbal torques by substitution and reducing the order of the system matrix, however, this approach will destroy the matrix form of the equations and may greatly complicate the computer programming. A second approach consists of maintaining the order of Equation 4.0-6 at $3N$ and introducing the locked gimbal reaction torque as a dependent variable on the left side of the equation. This approach introduces $3(N-1)$ additional variables such that for every possible gimbal axis there is a gimbal deflection and a locked gimbal reaction torque one of which must be set to zero. The factors involved in determining the most suitable method of solution with locked gimbals are based upon digital programming considerations. Therefore, the dynamic equations presented in Section 12.0 of this report contain both the gimbal deflections as independent variables and the locked gimbal reaction torques as dependent variables. The formulation of these equations in the format desired for digital programming is presented in Appendix for reference. The method of solution selected consists of elimination of the locked gimbal reaction torques by substitution. The order of the system matrix is maintained at $3N$ by placing unity on the diagonal to replace the equations eliminated. In order to further simplify

the programming it is assumed that for bodies with one degree of freedom, the degree of freedom is about the x gimbal axis, and for bodies with two degrees of freedom the freedom is about the x and y gimbal axes. This assumption does not result in any loss in generality of the program since the location of the auxiliary body axes is arbitrary.

At a given time, t , Equation 4.0-6 may be solved for the accelerations by

$$\{\ddot{\sigma}\} = [M]^{-1} \{b\} \quad (4.0-7)$$

The accelerations may be integrated to obtain velocities which in turn may be integrated to obtain the relative angular position of the bodies. In order to provide a better understanding of the detailed derivation and solution of Equation 4.0-6, a number of the factors involved in the detailed approach are discussed below:

1. Integration is performed by a Runge-Kutta, Adams-Moulton digital subroutine using a fourth order predictor corrector to control the integration interval. Basically the integration is performed by fitting a curve to the present and passed values of each variable and integrating the curve in closed form. As a result the effect of continuous varying parameters is obtained using step by step integration, and both the accelerations and the velocities at time t are integrated during a single pass through the subroutine. A more detailed description of the integration package is given in Reference 2. In Equation 4.0-6 the driving functions $\{b\}$ and the elements of the system matrix $[M]$ are functions of position, velocities, and time.
2. The accelerations used in Equation 4.0-6 are relative accelerations while the accelerations used in the dynamic equations are inertial accelerations. As a result those portions of the inertial accelerations which are functions of position, velocity, and time are separated and placed on the driving function side of Equation 4.0-6.
3. Integration of the three angular accelerations components for body 1, $\{\ddot{\sigma}_1\}$, will result in the three angular velocity components relative to the orbital reference frame. Normally the velocity components are integrated to obtain relative position, however, from Equation 3.3-3 the relative position of body 1 with respect to the orbital reference frame is given by the

nine elements transformation matrix $[A]$. Two methods of integrating the three velocity components to obtain nine position parameters have been programmed and used. In the first method the nine elements of the $[\dot{A}]$ matrix can be expressed in terms of the velocity components and the $[A]$ matrix since all nine elements are not independent. A minimum of four elements of $[\dot{A}]$ may be integrated to obtain four elements of $[A]$ from which the other five elements may be computed by the relationships given in Reference 1. The approach used consists of integrating the six elements in the first two rows of $[\dot{A}]$ to obtain the first two rows of $[A]$. The third row of $[A]$ is then computed from the first two. The second method of obtaining the $[A]$ matrix involves the integration of normal Euler parameters. The time rate of change of the Euler parameters are expressed in terms of the angular velocity components and integrated. The $[A]$ matrix is then computed from the Euler parameters, see Reference 1. The first method involving the integration of six elements of $[\dot{A}]$ is currently used in the standard milestone program.

4. The $[A]$ matrix defining the angular position of body 1 is an orthogonal transformation matrix. Since it is obtained for each time step as a result of a double integration process, a means of insuring orthogonality is provided. The orthogonality is checked, compared to limits, and corrected if necessary after each integration step. Details of the orthogonality checks and corrections are given in Reference 2.

4.1 BASIC DYNAMIC EQUATIONS

The basic moment equation for a body with three degrees of rotational freedom is given by Equation 6-18 of Reference 1. The general subscript i has been used to denote the body being considered where i equal unity is used to denote the main body of the spacecraft.

$$\begin{aligned} \{T_{Eii}\} = & M_i [J_{vii}]\{G_{nii}\} + [I_{iH}]\{\dot{\omega}_{ii}\} + [J_{\omega ii}][I_{iH}]\{\omega_{ii}\} \\ & + [I'_{iH}]\{\omega_{ii}\} + [J_{\omega ii}]\{H_{ii}\} + \{H'_{ii}\} \end{aligned} \quad (4.1-1)$$

where

$\{T_{Eii}\}$ = components of external torque on body i
in body i coordinates

$\{v_{ii}\}$ = components of center of mass location in body i
coordinates

$\{G_{nii}\}$ = components of inertial acceleration of body i reference point in body i coordinates

$[I_{iH}]$ = inertia matrix for body i with respect to body i coordinates

$\{\omega_{ii}\}$ = components of inertial angular velocity of body i in body i coordinates

$\{\dot{\omega}_{ii}\}$ = components of inertial angular acceleration of body i in body i coordinates

$\{H_{ii}\}$ = components of angular momentum of body i with respect to the body i coordinates. $\bar{H}_i$ results from motion due to thermal bending boom deployment, and reaction wheel rotation.

$\{\dot{H}_{ii}\}$ = components of the time rate of change of $\{H_{ii}\}$ in body i coordinates

M_i = mass of body i in slugs

The $[J]$ matrix is a 3 x 3 matrix obtained by operating on a three element vector such as $\{a\}$ where

$$[J_a] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \quad (4.1-2)$$

A term such as $[J_a] \{b\}$ is the matrix representation for the vector product $a \times b$, see Reference 1, Section 3.2. The inertia matrices are defined as

$$[I_{iH}] = \begin{bmatrix} I_{ix} & -I_{ixy} & -I_{ixz} \\ -I_{ixy} & I_{iy} & -I_{iyz} \\ -I_{ixz} & -I_{iyz} & I_{iz} \end{bmatrix} \quad (4.1-3)$$

where

$$[I] = - \int_M [J_R][J_R] dm \quad (4-1-4)$$

from Equation 3-30 of Reference 1.

If Equation 4.1-1 is written for a rigid body $[I_{iH}]$ and $\{H_{ii}\}$ are both zero. If in addition, the torques are taken about the center of mass of the body, $\{v_{ii}\} = 0$, or about a point with no inertial acceleration $\{G_{nii}\} = 0$, the equation reduces to

$$\{T_{Eii}\} = [I_{iH}]\{\dot{\omega}_{ii}\} + [J_{\omega ii}][I_{iH}]\{\omega_{ii}\}$$

If it is further assumed that the body axes are principal axes of inertia such that $[I_{iH}]$ is a diagonal matrix, the above equation expands into the normal basic Euler equations for a rigid body.

The basic force equation for a body with three degrees of translational freedom is simply

$$\bar{F}_{Ei} = M_i \bar{G}_i \quad (4.1-5)$$

$$\{F_{Eii}\} = M_i \{G_{ii}\} \quad (4.1-6)$$

where

- $\bar{G}_i$ = inertial acceleration of center of mass of body i
- $\{F_{Eii}\}$ = components of external force in body i coordinates
- $\{G_{ii}\}$ = components of inertia acceleration of body i center of mass in body i coordinates

5. RELATIVE MOTION BETWEEN BODIES

The relationships between the relative positions, velocities, and accelerations of the bodies are presented in this section.

5.1 ROTATIONAL POSITIONS

The relative rotational positions of the bodies are defined by the relative positions of the corresponding body axes. These axes and the transformations between axes are defined in Section 3.0 and are presented here for reference.

The reference axes are:

$|\bar{Q}_I|$ = inertial reference system at center of mass of the earth
(Equation 3.1-1)

$|\bar{Q}_M|$ = orbital reference system at center of mass of the spacecraft (Equation 3.2-6)

$|\bar{Q}_R|$ = orbital reference system at center of mass of body 1
(Equation 3.2-7)

$|\bar{Q}_1|$ = body 1 reference system at center of mass of body 1
(Equation 3.3-1)

$|\bar{Q}_{ni}|$ = auxiliary body i nominal reference system at the hinge point of body i on body 1. These axes are fixed relative to body 1 and are the reference for motions of body i
(Equation 3.4-2)

$|\bar{Q}_i|$ = auxiliary body i reference system at the hinge point of body i on body 1 (Equation 3.5-1)

The transformation matrices are

$$|\bar{Q}_M| = [E] |\bar{Q}_I| \quad (\text{Equation 3.2-6})$$

$$|\bar{Q}_R| = [E] |\bar{Q}_I| \quad (\text{Equation 3.2-7})$$

$$|\bar{Q}_1| = [A] |\bar{Q}_R| \quad (\text{Equation 3.3-3})$$

$$|\bar{Q}_{ni}| = [B_i] |\bar{Q}_1| \quad (\text{Equation 3.4-2})$$

$$\begin{bmatrix} \bar{Q}_i \end{bmatrix} = \begin{bmatrix} C_i \end{bmatrix} \begin{bmatrix} \bar{Q}_{ni} \end{bmatrix} \quad (\text{Equation 3.5-3})$$

$$\begin{bmatrix} \bar{Q}_i \end{bmatrix} = \begin{bmatrix} D_i \end{bmatrix} \begin{bmatrix} \bar{Q}_i \end{bmatrix} \quad (\text{Equation 3.5-5})$$

$$\begin{bmatrix} D_i \end{bmatrix} = \begin{bmatrix} C_i \end{bmatrix} \begin{bmatrix} B_i \end{bmatrix} \quad (\text{Equation 3.5-4})$$

$$\begin{bmatrix} E \end{bmatrix} = \text{function of orbital parameters and therefore a function of time only}$$

$$\begin{bmatrix} A \end{bmatrix} = \text{function of the position of body 1 relative to the orbital reference frame}$$

$$\begin{bmatrix} B_i \end{bmatrix} = \text{constant matrix}$$

$$\begin{bmatrix} C_i \end{bmatrix} = \text{function of the body i gimbal deflections}$$

5.2 ROTATIONAL VELOCITIES

The inertial rotational velocity vector for each coordinate system has been defined in Section 3.0 as:

$$\bar{\omega}_I = 0 = \text{inertial rotational velocity of } \begin{bmatrix} \bar{Q}_I \end{bmatrix} \quad (\text{Equation 3.1-2})$$

$$\bar{\omega}_R = \text{inertial rotational velocity of } \begin{bmatrix} \bar{Q}_M \end{bmatrix} \text{ and } \begin{bmatrix} \bar{Q}_R \end{bmatrix} \quad (\text{Equation 3.2-4})$$

$$\bar{\omega}_1 = \text{inertial rotational velocity of } \begin{bmatrix} \bar{Q}_1 \end{bmatrix} \quad (\text{Equation 3.3-2})$$

$$\bar{\omega}_{ni} = \bar{\omega}_1 = \text{inertial rotational velocity of } \begin{bmatrix} \bar{Q}_{ni} \end{bmatrix} \quad (\text{Equation 3.4-3})$$

$$\bar{\omega}_i = \text{inertial rotational velocity of } \begin{bmatrix} \bar{Q}_i \end{bmatrix} \quad (\text{Equation 3.5-2})$$

Since the independent variables of the system are relative positions, the relative angular velocities will now be defined.

$$\bar{\Omega}_R = \text{the angular velocity of } \begin{bmatrix} \bar{Q}_R \end{bmatrix} \text{ with respect to } \begin{bmatrix} \bar{Q}_I \end{bmatrix}$$

$$\bar{\Omega}_R = \bar{\omega}_R - \bar{\omega}_I = \bar{\omega}_R \quad (5.2-1)$$

$\bar{\Omega}_R$ will be defined in the orbital reference systems where $|\Omega_{RR}|$ is the magnitude matrix giving the angular velocity components of the orbital reference frame with respect to the inertial system $|\bar{Q}_I|$. The components of $|\Omega_{RR}|$ are expressed in terms of the orbital reference coordinates and are functions of the orbit parameters and time.

$$\bar{\Omega}_1 = \begin{array}{l} \text{the angular velocity of the body 1 coordinate system} \\ |\bar{Q}_1| \text{ with respect to the orbital reference system } |\bar{Q}_R| \end{array}$$

$$\bar{\Omega}_1 = \bar{\omega}_1 - \bar{\Omega}_R \quad (5.2-2)$$

$\bar{\Omega}_1$ will be defined in the body 1 coordinate system where $|\Omega_{11}|$ is the magnitude matrix giving the angular velocity components of the body 1 coordinate axes with respect to the orbital reference system $|\bar{Q}_R|$. The components of $|\Omega_{11}|$ are expressed in terms of body 1 coordinates

$$\bar{\Omega}_i = \begin{array}{l} \text{the angular velocity of the body i coordinate system } |\bar{Q}_i| \\ \text{with respect to the body 1 reference system } |\bar{Q}_1| \text{ and} \\ \text{the body i nominal reference system } |\bar{Q}_{ni}| \end{array}$$

$$\bar{\Omega}_i = \bar{\omega}_i - \bar{\omega}_1 \quad (5.2-3)$$

$\bar{\Omega}_i$ will be defined in the body i coordinate system where $|\Omega_{ii}|$ is the magnitude matrix giving the angular velocity components of the body i coordinate axes with respect to the body 1 coordinate axes. The components of $|\Omega_{ii}|$ are expressed in terms of body i coordinates.

Using Equation 4-32 of Reference 1, the time rate of change of the transformation matrices between coordinate axes may be written in terms of the relative angular velocity components

$$\begin{bmatrix} \dot{E} \end{bmatrix} = - \begin{bmatrix} J_{\Omega_{RR}} \end{bmatrix} \begin{bmatrix} E \end{bmatrix} \quad (5.2-4)$$

$$\begin{bmatrix} \dot{A} \end{bmatrix} = - \begin{bmatrix} J_{\Omega_{11}} \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \quad (5.2-5)$$

$$\begin{bmatrix} \dot{D}_i \end{bmatrix} = - \begin{bmatrix} J_{\Omega ii} \end{bmatrix} \begin{bmatrix} D_i \end{bmatrix} \quad (5.2-6)$$

$$\begin{bmatrix} \dot{C}_i \end{bmatrix} = - \begin{bmatrix} J_{\Omega ii} \end{bmatrix} \begin{bmatrix} C_i \end{bmatrix} \quad (5.2-7)$$

where $\begin{bmatrix} J_a \end{bmatrix}$ is given by Equation 4.1-2.

The inertial angular velocity of body 1 may be written in terms of the relative velocities and the transformation matrices. From Equation 5.2-2

$$\bar{\omega}_1 = \bar{\Omega}_R + \bar{\Omega}_1 \quad (5.2-8)$$

$\bar{\omega}_1$ in body 1 coordinates has components

$$\begin{bmatrix} \omega_{11} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \Omega_{RR} \end{bmatrix} + \begin{bmatrix} \Omega_{11} \end{bmatrix} \quad (5.2-9)$$

$\bar{\omega}_1$ in body i coordinate has components

$$\begin{bmatrix} \omega_{1i} \end{bmatrix} = \begin{bmatrix} D_i \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \Omega_{RR} \end{bmatrix} + \begin{bmatrix} D_i \end{bmatrix} \begin{bmatrix} \Omega_{11} \end{bmatrix} \quad (5.2-10)$$

From Equation 5.2-3

$$\bar{\omega}_i = \bar{\omega}_1 + \bar{\Omega}_i \quad (5.2-11)$$

$\bar{\omega}_i$ in body i coordinates has components

$$\begin{bmatrix} \omega_{ii} \end{bmatrix} = \begin{bmatrix} \omega_{1i} \end{bmatrix} + \begin{bmatrix} \Omega_{ii} \end{bmatrix}$$

$$\begin{bmatrix} \omega_{ii} \end{bmatrix} = \begin{bmatrix} D_i \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \Omega_{RR} \end{bmatrix} + \begin{bmatrix} D_i \end{bmatrix} \begin{bmatrix} \Omega_{11} \end{bmatrix} + \begin{bmatrix} \Omega_{ii} \end{bmatrix} \quad (5.2-12)$$

5.3 ROTATIONAL ACCELERATIONS

The relative angular accelerations between the reference systems are simply defined as $\left| \dot{\Omega}_{RR} \right|$, $\left| \dot{\Omega}_{11} \right|$, and $\left| \dot{\Omega}_{ii} \right|$ where

$\left| \dot{\Omega}_{RR} \right|$ = angular acceleration components of the orbital reference frame with respect to the inertial axes $\left| \bar{Q}_I \right|$. The components are expressed in terms of the orbital reference axes and are a function of the orbital parameters and time.

$\left| \dot{\Omega}_{11} \right|$ = angular acceleration components of the body 1 axes with respect to the orbital reference system $\left| \bar{Q}_R \right|$. The components are expressed in terms of the body 1 coordinates and represent three of the independent variables of the system.

$\left| \dot{\Omega}_{ii} \right|$ = angular acceleration components of the body i axes with respect to the body i nominal axes $\left| \bar{Q}_{ni} \right|$. The components are expressed in terms of the body i coordinates.

The inertial angular acceleration components of body 1 are found by differentiating Equation 5.2-9.

$$\left| \dot{\omega}_{11} \right| = \left[A \right] \left| \dot{\Omega}_{RR} \right| + \left[\dot{A} \right] \left| \Omega_{RR} \right| + \left| \dot{\Omega}_{11} \right|$$

using Equation 5.2-5

$$\left| \dot{\omega}_{11} \right| = \left[A \right] \left| \dot{\Omega}_{RR} \right| - \left[J_{\Omega 11} \right] \left[A \right] \left| \Omega_{RR} \right| + \left| \dot{\Omega}_{11} \right| \quad (5.3-1)$$

Note that the first two terms of Equation 5.3-1 are functions of positive, velocities, and time while the third term represents an independent unknown acceleration of the system. Torques resulting from the first two terms are placed on the driving function side of Equation 4.0-6 while torques resulting from the third term are placed on the unknown side of Equation 4.0-6. Therefore, as discussed in Section 4.0 the inertial accelerations will be separated such that

$$\left| \dot{\omega}_{11} \right| = \left[A^{\dot{\omega}} \right] \left| \dot{\Omega}_{RR} \right| + \left[N^{\dot{\omega}} \right] \left| \dot{\Omega}_{11} \right| \quad (5.3-2)$$

where

$$\left\{ \dot{A}_{11} \right\} = \text{unknown independent accelerations of the system}$$

$$\left\{ \dot{N}_{11} \right\} = \text{accelerations which are a function of positions, velocities, and time}$$

then

$$\left\{ \dot{A}_{11} \right\} = \left\{ \dot{\Omega}_{11} \right\} \quad (5.3-3)$$

$$\left\{ \dot{N}_{11} \right\} = \left[A \right] \left\{ \dot{\Omega}_{RR} \right\} - \left[J_{\Omega 11} \right] \left[A \right] \left\{ \Omega_{RR} \right\} \quad (5.3-4)$$

The inertial acceleration components of body i ($2 \leq i \leq N$) are found by differentiating Equation 5.3-12.

$$\begin{aligned} \left\{ \ddot{\omega}_{ii} \right\} = & \left[\dot{D}_i \right] \left[A \right] \left\{ \Omega_{RR} \right\} + \left[D_i \right] \left[\dot{A} \right] \left\{ \Omega_{RR} \right\} + \left[D_i \right] \left[A \right] \left\{ \dot{\Omega}_{RR} \right\} \\ & + \left[\dot{D}_i \right] \left\{ \Omega_{11} \right\} + \left[D_i \right] \left\{ \dot{\Omega}_{11} \right\} + \left\{ \dot{\Omega}_{ii} \right\} \end{aligned}$$

using Equations 5.2-5 and 5.2-6

$$\begin{aligned} \left\{ \ddot{\omega}_{ii} \right\} = & - \left[J_{\Omega ii} \right] \left[D_i \right] \left\{ \left[A \right] \left\{ \Omega_{RR} \right\} + \left\{ \Omega_{11} \right\} \right\} + \left[D_i \right] \left\{ \left[A \right] \left\{ \dot{\Omega}_{RR} \right\} - \left[J_{\Omega 11} \right] \left[A \right] \left\{ \Omega_{RR} \right\} \right\} \\ & + \left[D_i \right] \left\{ \dot{\Omega}_{11} \right\} + \left\{ \dot{\Omega}_{ii} \right\} \end{aligned} \quad (5.3-5)$$

Then

$$\left\{ \dot{A}_{ii} \right\} = \left[D_i \right] \left\{ \dot{\Omega}_{11} \right\} + \left\{ \dot{\Omega}_{ii} \right\} \quad (5.3-6)$$

$$\begin{aligned} \{N_{ii}^{\dot{\omega}}\} = & - \{J_{\Omega ii}\} \{D_i\} \left\{ \{A\} \{\Omega_{RR}\} + \{\Omega_{11}\} \right\} + \{D_i\} \left\{ \{A\} \{\dot{\Omega}_{RR}\} \right. \\ & \left. - \{J_{\Omega 11}\} \{A\} \{\Omega_{RR}\} \right\} \end{aligned} \quad (5.3-7)$$

$$\{N_{ii}^{\dot{\omega}}\} = - \{J_{\Omega ii}\} \{D_i\} \{\omega_{11}\} + \{D_i\} \{N_{11}^{\dot{\omega}}\} \quad (5.3-8)$$

5.4 TRANSLATION POSITION

The relative positions of the centers of mass of body i and body 1 ($2 \leq i \leq N$) are shown in Figure 5.4-1. The vector location of the body i attach or reference point with respect to the body 1 center of mass has been defined in Section 2.0 as $\bar{u}_i$ and the vector location of the center of

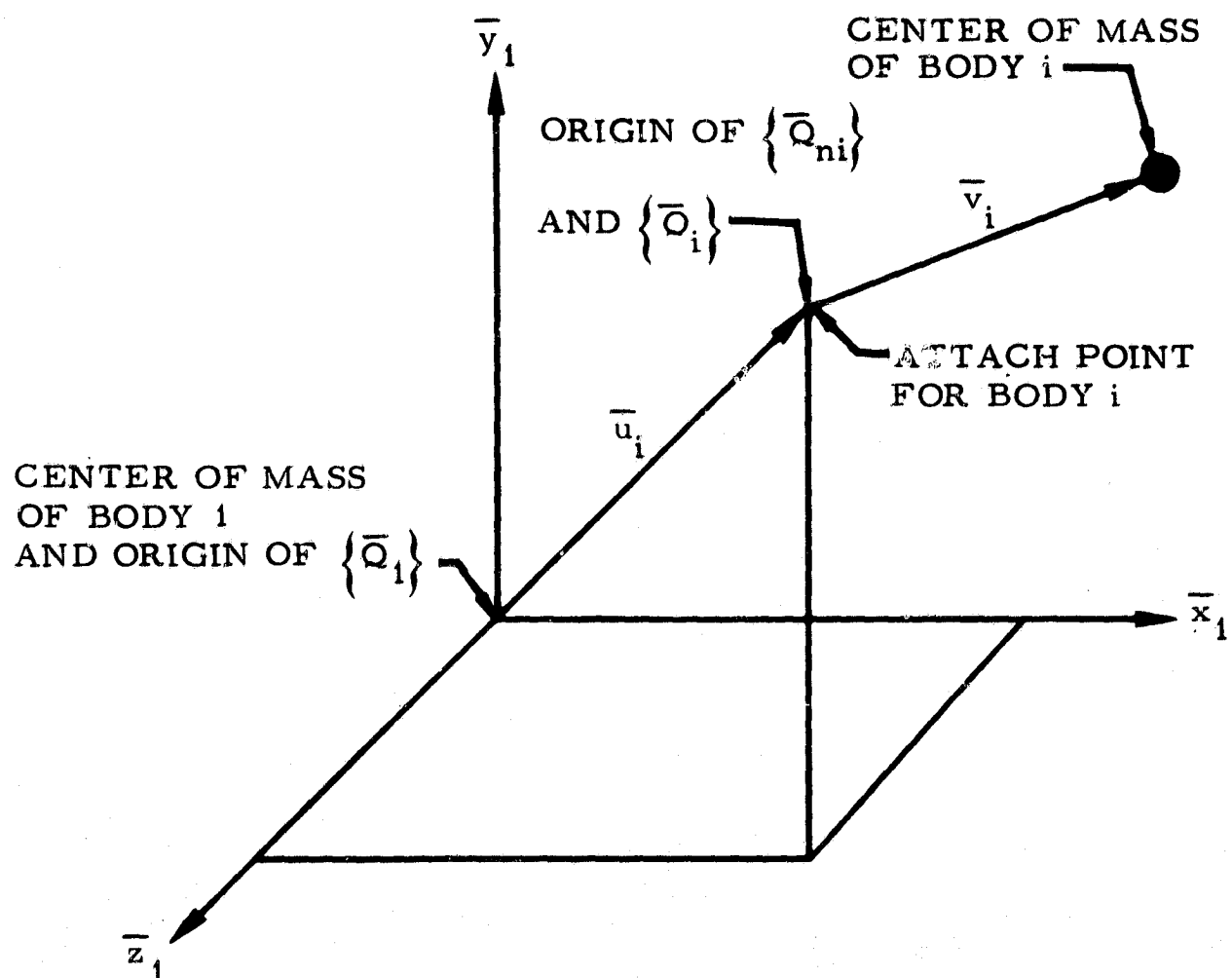


Figure 5.4-1. Center of Mass Locations

mass of body i with respect to the reference point has been defined as $\bar{v}_i$. Let $\bar{p}_i$ be the vector location of the centers of mass of body i with respect to the body 1 center of mass. Then

$$\bar{p}_i = \bar{u}_i + \bar{v}_i \quad (5.4-1)$$

The vector $\bar{u}_i$ is defined in body 1 coordinates such that its components in body 1 coordinates are $\{u_{i1}\}$. The vector $\bar{v}_i$ is defined in body i coordinates such that its components in body i coordinates are $\{v_{ii}\}$. The components $\{u_{i1}\}$ are input constants while the components $\{v_{ii}\}$ are computed time varying functions of thermal bending or boom deployment. The components of $\bar{v}_i$ in body 1 coordinates are given by

$$\{v_{i1}\} = [D_i]_T \{v_{ii}\} \quad (5.4-2)$$

The components of $\bar{p}_i$ in body 1 coordinates may be computed from

$$\{p_{i1}\} = [D_i]_T \{v_{ii}\} + \{u_{i1}\} \quad (5.4-3)$$

Let $\bar{R}_M$ be the vector location of the total system center of mass with respect to the body 1 center of mass. Then

$$\bar{R}_M = \sum_{i=2}^N \frac{M_i}{M_T} \bar{p}_i \quad (5.4-4)$$

where

$$M_T = \sum_{i=1}^N M_i \quad (5.4-5)$$

Then in body 1 coordinates

$$|R_{M1}| = \sum_{i=2}^N \frac{M_i}{M_T} |p_{i1}| \quad (5.4-6)$$

The vector positions of all of the reference systems are shown in Figure 5.4-2.

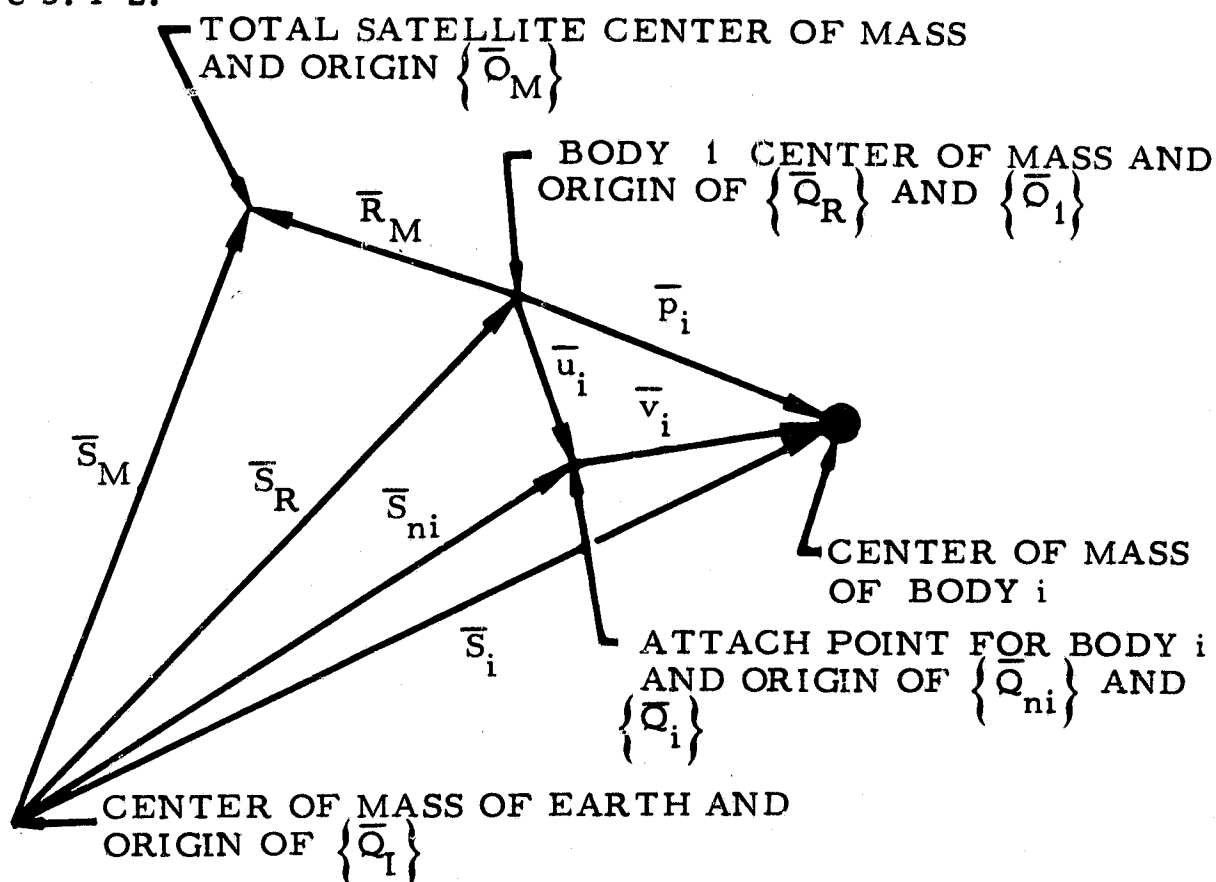


Figure 5.4-2. Relative Position Vectors

The vectors $\bar{R}_M$, $\bar{u}_i$, $\bar{v}_i$, and $\bar{p}_i$ are in feet while the vector $\bar{S}_M$, $\bar{S}_R$, $\bar{S}_{ni}$, and $\bar{S}_i$ are in nautical miles. Then from Figure 5.4-2

$$\bar{S}_R = \bar{S}_M - \frac{1}{K_N} \bar{R}_M \quad (\text{nautical miles}) \quad (5.4-7)$$

$$\bar{S}_{ni} = \bar{S}_R + \frac{1}{K_N} \bar{u}_i \quad (\text{nautical miles}) \quad (5.4-8)$$

$$\bar{S}_i = \bar{S}_R + \frac{1}{K_N} \bar{p}_i \quad (\text{nautical miles}) \quad (5.4-9)$$

where K_N is the number of feet in a nautical miles.

The basic form of the displacement vectors are given in Table 5-1.

Table 5-1

Basic Forms of Displacement Vectors

Vector	Components	Coordinate System	Dimension
$\bar{u}_i$	$ u_{i1} $	body 1	feet
$\bar{v}_i$	$ v_{ii} $	body i	feet
$\bar{p}_i$	$ p_{i1} $	body 1	feet
$\bar{r}_M$	$ R_{M1} $	body 1	feet
$\bar{s}_M$	$ S_{MR} $	orbital reference	nautical miles
$\bar{s}_R$	$ S_{R1} $	body 1	nautical miles
$\bar{s}_{ni}$	$ S_{ni1} $	body 1	nautical miles
$\bar{s}_i$	$ S_{ii} $	body i	nautical miles

Since the $\bar{z}_R$ axis is directed toward the center of the earth it can be seen from Figure 5.4-2 that the components of $\bar{s}_M$ in orbital reference coordinates are

$$|S_{MR}| = \begin{Bmatrix} 0 \\ 0 \\ -S_M \end{Bmatrix} \quad (5.4-10)$$

where S_M is the magnitude of the orbital radius vector in nautical miles and may be computed at any instant of time independent of the relative motion and positions of the spacecraft bodies. Then from Equations 5.4-7, 5.4-8, and 5.4-9.

$$|S_{R1}| = [A] |S_{MR}| - \frac{1}{K_N} |R_{M1}| \quad (5.4-11)$$

$$|S_{ni1}| = |S_{R1}| + \frac{1}{K_N} |u_{i1}| \quad (5.4-12)$$

$$|S_{ii}| = |D_i| |S_{R1}| + \frac{1}{K_N} |p_{i1}| \quad (5.4-13)$$

5.5 TRANSLATIONAL ACCELERATIONS

The translational acceleration vectors are defined as

$$\begin{aligned} \bar{H}_{ui} &= \ddot{u}_i & \text{ft/sec}^2 \\ \bar{H}_{vi} &= \ddot{v}_i & \text{ft/sec}^2 \\ \bar{H}_{pi} &= \ddot{p}_i & \text{ft/sec} \\ \bar{H}_M &= \ddot{R}_M & \text{ft/sec}^2 \\ \bar{G}_M &= K_N \ddot{S}_M & \text{ft/sec}^2 \\ \bar{G}_R &= K_N \ddot{S}_R & \text{ft/sec}^2 \\ \bar{G}_{ni} &= K_N \ddot{S}_{ni} & \text{ft/sec}^2 \\ \bar{G}_i &= K_N \ddot{S}_i & \text{ft/sec}^2 \end{aligned}$$

The translational acceleration components required for the dynamic equations are the components of the acceleration of the reference point and the center of mass of each body, $\bar{G}_{ni}$ and $\bar{G}_i$ ($1 \leq i \leq N$). Since the reference point for body 1 is at its center of mass, which in turn is at the origin of the orbital reference system

$$\bar{G}_{n1} = \bar{G}_1 = \bar{G}_R \quad (5.5-1)$$

From Equation 5.4-1 the acceleration of the body i center of mass with respect to the body 1 reference point is

$$\ddot{p}_i = \ddot{u}_i + \ddot{v}_i$$

and

$$\bar{H}_{pi} = \bar{H}_{ui} + \bar{H}_{vi} \quad (5.5-2)$$

From Equation 5.4-4 the acceleration of the total spacecraft center of mass with respect to the body 1 center of mass is

$$\ddot{\bar{R}}_M = \sum_{i=2}^N \frac{M_i}{M_T} \ddot{\bar{p}}_i$$

and

$$\bar{H}_M = \sum_{i=2}^N \frac{M_i}{M_T} \bar{H}_{pi} \quad (5.5-3)$$

From Equation 5.4-7 the acceleration of the body 1 center of mass with respect to the inertial origin is

$$\ddot{\bar{S}}_R = \ddot{\bar{S}}_M - \frac{1}{K_N} \ddot{\bar{R}}_M$$

and

$$\bar{G}_R = \bar{G}_M - \bar{H}_M \quad (5.5-4)$$

From equation 5.4-8 the acceleration of the body i reference point with respect to the inertial origin is

$$\ddot{\bar{S}}_{ni} = \ddot{\bar{S}}_R + \frac{1}{K_N} \ddot{\bar{u}}_i$$

and

$$\bar{G}_{ni} = \bar{G}_R + \bar{H}_{ui} \quad (5.5-5)$$

From Equation 5.4-9 the acceleration of the body i center of mass with respect to the inertial origin is

$$\ddot{\bar{S}}_i = \ddot{\bar{S}}_R + \frac{1}{K_N} \ddot{\bar{p}}_i$$

and

$$\bar{G}_i = \bar{G}_R + \bar{H}_{pi} \quad (5.5-6)$$

The components of the translational acceleration vectors will be defined in the same coordinate system as the corresponding displacement vectors. Therefore, the components of the inertial translational acceleration of the spacecraft center of mass $\bar{G}_M$, defined in the orbital reference system, are $\{G_{MR}\}$, the components of the acceleration of the body i hinge point relative to the center of mass of body 1, defined in the body 1 system, are $\{H_{uil}\}$ and the components of the acceleration of the total center of mass relative to the center of mass of body 1 defined in body 1 coordinates are $\{H_{M1}\}$.

The components of the reference point inertial accelerations and the center of mass inertial accelerations defined in body 1 coordinates are

$$\{G_{ni1}\} = \{G_{R1}\} + \{H_{uil}\} \quad (5.5-7)$$

$$\{G_{i1}\} = \{G_{R1}\} + \{H_{pi1}\} \quad (5.5-8)$$

where

$$\{G_{R1}\} = [A] \{G_{MR}\} - \{H_{M1}\} \quad (5.5-9)$$

$$\{H_{M1}\} = \sum_{i=2}^N \frac{M_i}{M_T} \{H_{pi1}\} \quad (5.5-10)$$

From Equation 5.5-1

$$\left| G_{n11} \right| = \left| G_{11} \right| = \left| G_{R1} \right| \quad (5.5-11)$$

and from Equation 5.5-2

$$\left| H_{pi1} \right| = \left| H_{ui1} \right| + \left[D_i \right]_T \left| H_{vii} \right| \quad (5.5-12)$$

From Equation 3-41 of Reference 1 the total inertial acceleration of the components $\left| u_{i1} \right|$ defined in body 1 coordinates are given by

$$\left| H_{ui1} \right| = \left| \ddot{u}_{i1} \right| + 2 \left[J_{\omega 11} \right] \left| \dot{u}_{i1} \right| + \left[\dot{J}_{\omega 11} \right] \left| u_{i1} \right| + \left[J_{\omega 11} \right] \left[J_{\omega 11} \right] \left| u_{i1} \right|$$

Since the components $\left| u_{i1} \right|$ are constant in the body 1 coordinate system, the derivations $\left| \dot{u}_{i1} \right|$ and $\left| \ddot{u}_{i1} \right|$ are zero and the acceleration components are due only to the motion of the coordinate system and

$$\left| H_{ui1} \right| = \left[J_{\omega 11} \right] \left[J_{\omega 11} \right] + \left[\dot{J}_{\omega 11} \right] \left| u_{i1} \right| \quad (5.5-13)$$

The inertial acceleration of the components $\left| v_{ii} \right|$ defined in body i coordinates are given by

$$\left| H_{vii} \right| = \left| \ddot{v}_{ii} \right| + 2 \left[J_{\omega ii} \right] \left| \dot{v}_{ii} \right| + \left[\dot{J}_{\omega ii} \right] \left| v_{ii} \right| + \left[J_{\omega ii} \right] \left[J_{\omega ii} \right] \left| v_{ii} \right| \quad (5.5-14)$$

The acceleration and velocity components of the body i center of mass relative to the body coordinate frame $\left| \dot{v}_{ii} \right|$ and $\left| \ddot{v}_{ii} \right|$ are the result of thermal bending and boom deployment motions.

The translational acceleration components for the total center of mass $\left| G_{MR} \right|$ will be defined in connection with the discussion of the motion of the orbital reference frame in Section 9.0.

6. FORCES AND TORQUES ON SPACECRAFT BODIES

The basic force and torque Equations, 4.1-1 and 4.1-5, are

$$\begin{aligned}\{T_{Eii}\} &= M_i [J_{vii}] \{G_{nii}\} + [I_{iH}] \{\dot{\omega}_{ii}\} + [J_{\omega ii}] [I_{iH}] \{\omega_{ii}\} \\ &\quad + [I_{iH}^*] \{\omega_{ii}\} + [J_{\omega ii}] \{H_{lii}\} + \{H_{lii}^*\} \\ \{F_{Eii}\} &= M_i \{G_{ii}\}\end{aligned}$$

where the forces and torques are summed at the origin of the body i reference system. The body 1 reference point has been defined as the center of mass of the body while the auxiliary body ($2 \leq i \leq N$) reference point has been defined as the attach point on the main body.

6.1 EXTERNAL APPLIED LOADS

The external loads are assumed to result from the earth's gravity, the disturbances, and the reaction at the attach points between bodies. In addition, it is assumed that the main body, i equal to 1, may have applied control forces and torques. The disturbances to be considered are solar radiation forces and torques, aerodynamic forces and torques, magnetic torques, and the effects of changes in shape of the auxiliary bodies.

The applied loads are defined as follows:

$\bar{F}_{gi}$ = external gravity force on body i applied at its center of gravity

$\bar{T}_{gi}$ = torque about the center of mass of body i due to gravity force $\bar{F}_{gi}$. $\bar{T}_{gi}$ is the external torque that must be applied to body i if the gravity force $\bar{F}_{gi}$ is assumed to act at the center of mass instead of at the center of gravity.

$\bar{v}_i \times \bar{F}_{gi}$ = torque about the reference point of body i due to the gravity force $\bar{F}_{gi}$ applied at the center of mass. $\bar{v}_i \times \bar{F}_{gi}$ is the additional external torque that must be applied to body i if the gravity force $\bar{F}_{gi}$ is assumed to act at the reference point ($\bar{v}_1 = 0$ for body 1).

$\bar{F}_{Si}$ = solar radiation force on body i applied at the solar center of pressure.

$\bar{T}_{Si}$ = torque about the body i reference point due to $\bar{F}_{Si}$. $\bar{T}_{Si}$ is the external torque that must be applied when $\bar{F}_{Si}$ is assumed to act at the body i reference point.

$\bar{F}_{Ai}$ = aerodynamic force on body i applied at the aerodynamic center of pressure.

$\bar{T}_{Ai}$ = torque about the body i reference point due to $\bar{F}_{Ai}$. $\bar{T}_{Ai}$ is the external torque that must be applied when $\bar{F}_{Ai}$ is assumed to act at the body i reference point.

$\bar{T}_{Mi}$ = magnetic torque on body i assumed to be a pure couple.

$\bar{F}_{C1}$ = external control force on body 1

$\bar{T}_{C1}$ = moment of $\bar{F}_{C1}$ about the body 1 center of mass (reference point).

$\bar{F}_{Ri}$ = internal reaction force on body 1 due to body i; applied at the body i attach point ($2 \leq i \leq N$.)

$\bar{T}_{Hi}$ = internal reaction torque on body 1 due to body i; applied at the body i attach point ($2 \leq i \leq N$)

$\bar{F}_{Ei}$ = the net applied force on body i, at the reference point, to be reacted by inertia forces.

$\bar{T}_{Ei}$ = the net applied torque on body i, at the reference point, to be reacted by inertia torques.

6.1.1 External Loads on Main Body

The external applied loads on the main body ($i = 1$) are

$$\bar{F}_{E1} = \bar{F}_{g1} + \bar{F}_{S1} + \bar{F}_{A1} + \bar{F}_{C1} + \sum_{i=2}^N \bar{F}_{Ri} \quad (6.1-1)$$

$$\bar{T}_{E1} = \bar{T}_{g1} + \bar{T}_{S1} + \bar{T}_{A1} + \bar{T}_{M1} + \bar{T}_{C1} + \sum_{i=2}^N (\bar{u}_i \times \bar{F}_{Ri}) + \sum_{i=2}^N \bar{T}_{Hi} \quad (6.1-2)$$

It should be noted that the reaction loads bdtween the bodies have been defined as positive for loads transferred from the auxiliary bodies to the main body. The vector $\bar{u}_i$ has been defined in Section 2.1 as the vector location of the body i attach point with respect to the body 1 center of mass. Therefore, the moment $\bar{u}_i \times \bar{F}_{Ri}$ is the moment of the body i reaction force on the main body. Equations (6.1-1) and 6.1-2) may be written in matrix notation as

$$\{F_{E11}\} = \{F_{g11}\} + \{F_{S11}\} + \{F_{A11}\} + \{F_{C11}\} + \sum_{i=2}^N [D_i]_T \{F_{Rii}\} \quad (6.1-3)$$

$$\begin{aligned} \{T_{E11}\} = & \{T_{g11}\} + \{T_{S11}\} + \{T_{A11}\} + \{T_{M11}\} + \{T_{C11}\} \\ & + \sum_{i=2}^N [J_{uil}] [D_i]_T \{F_{Rii}\} + \sum_{i=2}^N [D_i]_T \{T_{Hii}\} \end{aligned} \quad (6.1-4)$$

The components of the reaction loads between bodies $\{F_{Rii}\}$ and $\{T_{Hii}\}$ will be computed in body i coordinates, and transformed to the body 1 system by $[D_i]_T$. The vector $\bar{u}_i$ is defined by constant components in body 1 coordinates.

6.1.2 External Loads on Auxiliary Bodies

The external applied loads on the auxiliary bodies ($2 \leq i \leq N$) are

$$\bar{F}_{Ei} = \bar{F}_{gi} + \bar{F}_{Si} + \bar{F}_{Ai} - \bar{F}_{Ri} \quad (6.1-5)$$

$$\bar{T}_{Ei} = \bar{v}_i \times \bar{F}_{gi} + \bar{T}_{gi} + \bar{T}_{Si} + \bar{T}_{Ai} + \bar{T}_{Mi} - \bar{T}_{Hi} \quad (6.1-6)$$

It should be noted that the torque equations for the auxiliary body are written about the reference or attach point and not the center of mass. The vector $\bar{v}_i$ has been defined in Section 3.5 as the vector location of the center of mass of body i with respect to the reference point. The moment $\bar{v}_i \times \bar{F}_{gi}$ is the additional hinge moment since $\bar{T}_{gi}$ has been written about the center of mass. These terms have not been combined since $\bar{v}_i \times \bar{F}_{gi}$ will be canceled by an identical term in the expansion of $\bar{v}_i \times \bar{G}_{ni}$. All other terms in 6.1-6 represent moments about the reference point. In matrix notation Equations 6.1-5 and 6.1-6 become

$$\{F_{Eii}\} = \{F_{gii}\} + \{F_{Sii}\} + \{F_{Aii}\} - \{F_{Rii}\} \quad (6.1-7)$$

$$\{T_{Eii}\} = [J_{vii}] \{F_{gii}\} + \{T_{gii}\} + \{T_{Sii}\} + \{T_{Aii}\} + \{T_{Mii}\} - \{T_{Hii}\} \quad (6.1-8)$$

6.2 TORQUES DUE TO BODY MOMENTUM

The basic torque Equation (4.1-1) contains the terms $[J_{\omega ii}] \{H_{ii}\}$ and $\{H_{ii}\}$. These torques result from the fact that the body axes are not rigid body axes and the mass elements are assumed to have motion with respect to the axes. The motion is assumed to be due to reaction wheel rotation and changes in shape of the auxiliary bodies, where the changes in shape result from thermal boom bending and boom deployment.

Let

$\{H_{wii}\}$ = the components of the body i reaction wheel angular momentum in body i coordinates.

$\{H_{Rii}\}$ = the components of the body i angular momentum due to changes in shape expressed in body i coordinates (for body 1 $\{H_{R11}\} = 0$).

then

$$\{H_{ii}\} = \{H_{wii}\} + \{H_{Rii}\} \quad (1 \leq i \leq N) \quad (6.2-1)$$

$$\{\dot{H}_{ii}\} = \{\dot{H}_{wii}\} + \{\dot{H}_{Rii}\} \quad (1 \leq i \leq N) \quad (6.2-2)$$

6.3 FORCE EQUATION

As noted in Section 4.0 on method and approach, the matrix force equation for the complete spacecraft, should simply equate the sum of the external forces to the product of the total mass and the acceleration of the total center of mass. This result will be verified by equating the reaction forces at the attach points.

From Equation (6.1-5) and (4.1-5) the body i force transmitted to body 1 is

$$\bar{F}_{Ri} = \bar{F}_{gi} + \bar{F}_{Si} + \bar{F}_{Ai} - M_i \bar{G}_i \quad (6.3-1)$$

Defining the forces as components in the body i coordinate system

$$\{F_{Rii}\} = \{F_{gii}\} + \{F_{Sii}\} + \{F_{Aii}\} - M_i \{G_{ii}\} \quad (2 \leq i \leq N) \quad (6.3-2)$$

and the equation for body 1 is given by Equation (6.1-1)

$$M_1 \bar{G}_1 = \bar{F}_{g1} + \bar{F}_{S1} + \bar{F}_{A1} + \bar{F}_{C1} + \sum_{i=2}^M \bar{F}_{Ri} \quad (6.3-3)$$

Substituting (6.3-1) into (6.3-3)

$$M_1 \bar{G}_1 + \sum_{i=2}^N M_i \bar{G}_i = \bar{F}_{C1} + \sum_{i=1}^N \bar{F}_{gi} + \bar{F}_{S1} + \bar{F}_{A1} \quad (6.3-4)$$

Since $\bar{S}_M$ is the vector location of the spacecraft center of mass with respect to the inertial origin and $\bar{S}_i$ is the vector location of the center of mass of body i

$$M_T \bar{S}_M = \sum_{i=1}^N M_i \bar{S}_i \quad (6.3-5)$$

Then

$$M_T \bar{G}_M = \sum_{i=1}^N M_i \bar{G}_i = M_1 \bar{G}_1 + \sum_{i=2}^N M_i G_i \quad (6.3-6)$$

and Equation (6.3-4) becomes

$$M_T \bar{G}_M = F_{Cl} + \sum_{i=1}^N (\bar{F}_{gi} + F_{si} + \bar{F}_{Ai}) \quad (6.3-7)$$

7. HINGE TORQUES AND MOTION

The hinge torque, $\bar{T}_{Hi}$, at the attach point between body i and body 1, has been defined as the torque on body 1 due to body i . The components of $\bar{T}_{Hi}$ in the body i coordinate system appear in the torque equations for all bodies (Equations 6.1-4 and 6.1-8). The magnitudes of these components are functions of the reaction torques developed at the hinge. They may be zero for a frictionless hinge, may be proportional to displacement or velocity to represent springs and dampers or may be functions of control torques. For this analysis it is assumed that a gimballed hinge is used and that the reaction elements are located on a set of gimbal axes nominally aligned with the three orthogonal body axes. As a result the motion at the hinge points must be resolved into components about the gimbal axes. For a given auxiliary body i , let the displacements about the gimbal axes be θ_{ix} , θ_{iy} , and θ_{iz} , the transformation matrix from the nominal body i coordinates $\{\bar{Q}_{ni}\}$ to the body i coordinates $\{\bar{Q}_i\}$ has been defined by Equation 3.5-3 as $[C_i]$. Using Equation 4-54 and Section 4.2 of Reference 1, $[C_i]$ is given by

$$[C_i] = \begin{bmatrix} (C\theta_{iy} C\theta_{iz})(S\theta_{ix} S\theta_{iy} C\theta_{iz} + C\theta_{ix} S\theta_{iz})(-C\theta_{ix} S\theta_{iy} C\theta_{iz} + S\theta_{ix} S\theta_{iz}) \\ (-C\theta_{iy} S\theta_{iz})(-S\theta_{ix} S\theta_{iy} S\theta_{iz} + C\theta_{ix} C\theta_{iz})(C\theta_{ix} S\theta_{iy} S\theta_{iz} + S\theta_{ix} C\theta_{iz}) \\ (S\theta_{iy})(-S\theta_{ix} C\theta_{iy})(C\theta_{ix} C\theta_{iy}) \end{bmatrix} \quad (7.0-1)$$

where

$$S(\) = \text{sine}(\)$$

$$C(\) = \text{cosine}(\)$$

The transformation matrix from the body 1 coordinates $\{\bar{Q}_1\}$ to the nominal i coordinates $\{\bar{Q}_{ni}\}$ has been defined by Equation 3.4-2 as $[B_i]$. The elements of $[B_i]$ are constants which are functions of input parameters defining the fixed orientation of the body i nominal reference system with respect to the body 1 coordinate system. Rather than input all nine elements of $[B_i]$ which are not all independent, the input parameters are the three ordered x , y , z Euler rotations τ_{ix} , τ_{iy} , τ_{iz} . The nominal angular orientation of

the body i axes is produced by performing the ordered rotations on the body 1 coordinate system. The transformation matrix $[B_i]$ is found by replacing θ_{ix} , θ_{iy} , and θ_{iz} in Equation 7.0-1 by τ_{ix} , τ_{iy} , τ_{iz} .

Then from Equation 3.5-4

$$[D_i] = [C_i][B_i]$$

7.1 VELOCITY AND ACCELERATION TRANSFORMATIONS

The three gimbal deflections θ_{ix} , θ_{iy} , and θ_{iz} may be arranged in a 3×1 column matrix $\{\theta_i\}$ such that

$$\{\theta_i\} = \begin{Bmatrix} \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \end{Bmatrix} \quad (7.1-1)$$

where $\{\theta_i\}$ is not a magnitude matrix for a vector quantity. Then from Equation 4-61 and 4-62 of Reference 1

$$\{\Omega_{ii}\} = [G_i]\{\dot{\theta}_i\} \quad (7.1-2)$$

where

$$[G_i] = \begin{bmatrix} (\cos\theta_{iy}\cos\theta_{iz}) & (\sin\theta_{iz}) & (0) \\ (-\cos\theta_{iy}\sin\theta_{iz}) & (\cos\theta_{iz}) & (0) \\ (\sin\theta_{iy}) & (0) & (1) \end{bmatrix} \quad (7.1-3)$$

Differentiating Equation 7.0-3

$$\{\dot{\Omega}_{ii}\} = [\dot{G}_i]\{\dot{\theta}_i\} + [G_i]\{\ddot{\theta}_i\} \quad (7.1-4)$$

where

$$[\dot{G}_i] = \begin{bmatrix} -(\dot{\theta}_{iy}\sin\theta_{iy}\cos\theta_{iz} + \dot{\theta}_{iz}\cos\theta_{iy}\sin\theta_{iz}) & (\dot{\theta}_{iz}\cos\theta_{iz})(0) \\ (\dot{\theta}_{iy}\sin\theta_{iy}\sin\theta_{iz} - \dot{\theta}_{iz}\cos\theta_{iy}\cos\theta_{iz}) & (-\dot{\theta}_{iz}\sin\theta_{iz})(0) \\ (\dot{\theta}_{iy}\cos\theta_{iy}) & (0) & (0) \end{bmatrix} \quad (7.1-5)$$

It should be noted that for a system for which body i has only one degree of rotational freedom, the variables θ_{iy} and θ_{iz} are zero and $[G_i]$ is equal to the unit matrix $[U]$ and $[\dot{G}_i]$ is zero. The elements of $\{\theta_i\}$ are the independent variables of the system associated with the auxiliary body degrees of freedom and are the parameters computed and printed out giving the time history of the auxiliary body motion with respect to the main body.

7.2 TORQUE TRANSFORMATIONS

Let T_{gix} , T_{giy} , and T_{giz} be the gimbals reaction torques about the x , y , z gimbal axes of body i . These three gimbal torques may be arranged in a 3×1 column matrix such that

$$\{T_{gi}\} = \begin{Bmatrix} T_{gix} \\ T_{giy} \\ T_{giz} \end{Bmatrix} \quad (7.2-1)$$

where the elements of $\{T_{gi}\}$ are not orthogonal components of a resultant torque vector.

From Equation 5-6 of Reference 1 the torque components of the total gimbal reaction torque, transmitted from body i to body 1, defined in body coordinates is

$$\{T_{Hii}\} = [G_i]_T^{-1} \{T_{gi}\} \quad (7.2-2)$$

where

$$[G_i]_T^{-1} = \begin{bmatrix} \frac{\cos \theta_{iz}}{\cos \theta_{iy}} & \sin \theta_{iz} & -\frac{\sin \theta_{iy} \cos \theta_{iz}}{\cos \theta_{iy}} \\ -\frac{\sin \theta_{iz}}{\cos \theta_{iy}} & \cos \theta_{iz} & \frac{\sin \theta_{iy} \sin \theta_{iz}}{\cos \theta_{iy}} \\ 0 & 0 & 1 \end{bmatrix} \quad (7.2-3)$$

It is assumed that the gimbal stops on the y gimbal axis will always be at a value less than 90° in order that $\cos \theta_{iy} \neq 0$.

Similar transformations may be defined to transform the motion and torques from the gimbal axes to body 1 coordinates, however, in order to reduce the number of matrices to be computed and stored the transformations will always be made to the body i system and then transformed to the body 1 system by the transformation matrix $[D_i]_T$.

Since

$$\{T_{Hil}\} = [D_i]_T \{T_{Hii}\}$$

$$\{T_{Hil}\} = [D_i]_T [G_i]_T^{-1} \{T_{gi}\} \quad (7.2-4)$$

7.3 GIMBAL TORQUES

The gimbal torques about the x, y, and z gimbal axes of body i are defined as

$$\{T_{gi}\} = [K_i] \{\theta_i\} + [P_i] \{\dot{\theta}_i\} + \{T_{Bi}\} + \{T_{Ni}\} + \{T_{Vi}\} + \{T_{Li}\} \quad (7.3-1)$$

where

$[K_i] \{\theta_i\}$ = the spring restraint torques

$[P_i] \{\dot{\theta}_i\}$ = viscous damping torques

$\{T_{Bi}\}$ = magnetic hysteresis damping torques defined in Section 15.0

$\{T_{Ni}\}$ = stop torques

$\{T_{Vi}\}$ = reaction torques resulting from the elimination of degrees of freedom

$\{T_{Li}\}$ = motor torques used in connection with controlled auxiliary bodies

$$[K_i] = \begin{bmatrix} k_{ix} & 0 & 0 \\ 0 & k_{iy} & 0 \\ 0 & 0 & k_{iz} \end{bmatrix}$$

k_{ix} , k_{iy} , and k_{iz} are the spring constants associated with the x, y, and z gimbal axes of body i (ft-lb per radians)

$$[P_i] = \begin{bmatrix} b_{ix} & 0 & 0 \\ 0 & b_{iy} & 0 \\ 0 & 0 & b_{iz} \end{bmatrix}$$

b_{ix} , b_{iy} , and b_{iz} are the damping constants associated with the x, y, and z gimbal axes of body i (ft-lb per radians/sec)

The elastic stops used in the program are implemented by introducing and addition spring restraint and viscous damping when the gimbal deflection is greater than the stop position. The introduction of the additional torque represents a discontinuity of the system and an integration step must end at the point of contact and the integration must be restarted with the corresponding initial conditions. The stop torques are then computed as the sum of the products of the stop spring times the deflection past the stop angle and the stop damping coefficient times the gimbal velocity. Since the stop torques are large, the integration step size must be kept small during the time any auxiliary body is into its stops. The exit point must also be determined, the step size reset, and the integration restarted.

The elements of $\{T_{yi}\}$ represent the locked gimbal reaction torques and are non-zero only when a degree of freedom has been eliminated, in which case all of the other corresponding elements in Equation 7.3-1 are set to zero. The non-zero elements of $\{T_{yi}\}$ then become the dependent unknowns of the system and provisions must be made to eliminate them from the equations. The selection of the elements of $\{\theta_i\}$ and $\{T_{yi}\}$, based upon the specified degrees of freedom of body i, is made at the start of the program and the formulation of the $[M]$ matrix is then set.

The auxiliary body control torques $\{T_{Li}\}$ are computed in the control routines and can be non-zero only for those gimbal axis which are given a degree of freedom.

Equation 7.2-1 may be written as

$$\{T_{gi}\} = \{T_{sgi}\} + \{T_{vi}\} \quad (7.3-2)$$

where the elements of $\{T_{sgi}\}$ are zero for axes without degrees of freedom and the elements of $\{T_{vi}\}$ are zero for axes with degrees of freedom.

8. ORBIT EQUATIONS

As noted in Section 4.0, the solution of the force equation will describe the motion of the center of mass of the spacecraft. The vector force equation given by Equation 5.3-6 is

$$M_T \bar{G}_M = \bar{F}_{C1} + \sum_{i=1}^N (\bar{F}_{gi} + \bar{F}_{Si} + \bar{F}_{Ai})$$

The effects of the disturbance forces and the control force will be neglected in the computation of the orbital position of the spacecraft. Therefore, the force equation for orbital computations reduces to

$$M_T \bar{G}_{MO} \approx \sum_{i=1}^N \bar{F}_{gi} = \bar{F}_{gM} \quad (8.0-1)$$

In addition, it is assumed that the magnitude of the gravitational force at the center of mass is equal to the force that would result if the total mass were concentrated at the center of mass. Then

$$\bar{F}_{gM} \approx -K_N \frac{\gamma M_T}{S_M^2} \left(\frac{\bar{S}_M}{S_M} \right) \quad (8.0-2)$$

where

$$\gamma = \text{earth gravitational constant} = 62746.8 \frac{(\text{n. mi.})^3}{\text{sec}^2}$$

$$S_M = \text{magnitude of orbital radius vector } \bar{S}_M \text{ in nautical miles}$$

$$M_T = \text{total mass of spacecraft in slugs}$$

Equation 8.0-1 represents the motion of the mass M_T in an elliptical orbit about a spherical earth, Figure 8.0-1. The orbital equations are:

$$P = \text{orbital period in hrs}$$

$$e = \text{eccentricity of orbit (N. D.)}$$

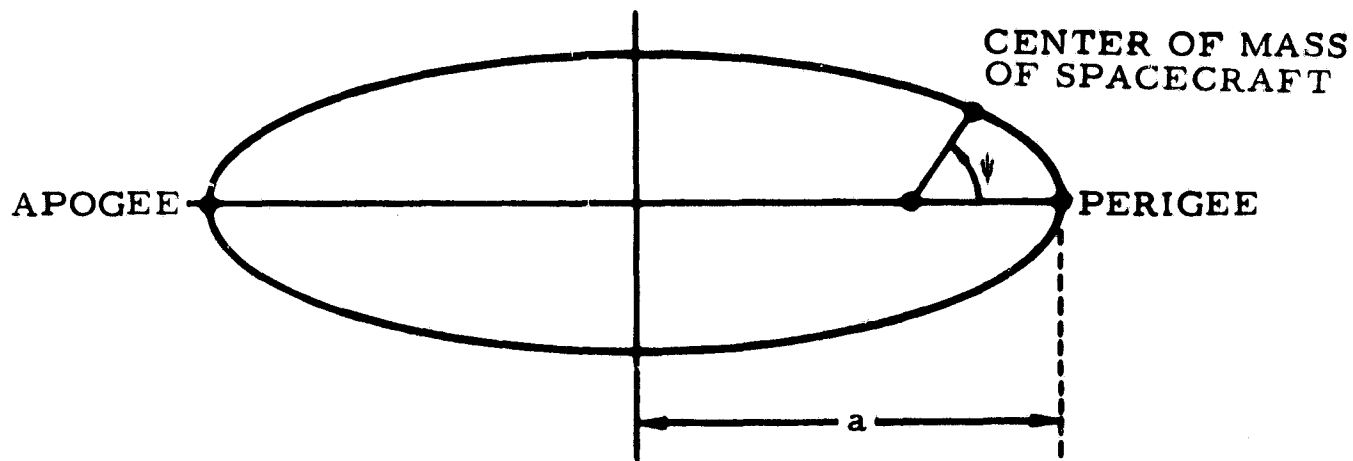


Figure 8.0-1. Orbit Parameters

The semi-major axis of the orbit a is given by

$$a = \left(\frac{\sqrt{Y}}{\omega_o} \right)^{2/3} = \text{nautical miles} \quad (8.0-3)$$

where

$$\omega_o = \frac{\pi}{1800 P} = \text{equivalent circular orbital frequency in radians per second} \quad (8.0-4)$$

$$\psi = 2 \tan^{-1} \left[\sqrt{\frac{1+\epsilon}{1-\epsilon}} \tan \frac{1}{2} E \right] \quad (8.0-5)$$

where E must be found from

$$E - \epsilon \sin E = \omega_o (t - t_p) \quad (8.0-6)$$

The angle ψ is the orbit angle from perigee at time t and t_p is the time at perigee.

$$S_M = \frac{a (1 - \epsilon^2)}{1 + \epsilon \cos \psi} = \text{nautical miles} \quad (8.0-7)$$

$$\dot{\psi} = \frac{\sqrt{a \gamma (1 - \epsilon^2)}}{S_M^2} = \text{radians per second} \quad (8.0-8)$$

by differentiating

$$\dot{S}_M = K_N \sqrt{\frac{\gamma}{a(1-\epsilon^2)}} \epsilon \sin \psi = \text{ft/sec} \quad (8.0-9)$$

$$S_M = K_N \frac{\epsilon \gamma \cos \psi}{S_M^2} = \text{ft/sec}^2 \quad (8.0-10)$$

$$\ddot{\psi} = - \frac{2\epsilon \gamma \sin \psi}{S_M^2} = \text{radians/sec}^2 \quad (8.0-11)$$

9. MOTION OF ORBITAL REFERENCE SYSTEM

The inertial reference system $\{\bar{Q}_I\}$, the orbital reference system $\{\bar{Q}_M\}$, and the transformation between them $[E]$ have been defined in Section 3.0. The components of the relative angular velocity $\{\Omega_{RR}\}$ and the linear acceleration components $\{G_{MR}\}$ have been used in the equations of Section 5.0. These parameters are independent of the relative motion of the bodies of the spacecraft and can be written as functions of the orbit parameters and time.

9.1 THE INERTIAL REFERENCE AXES

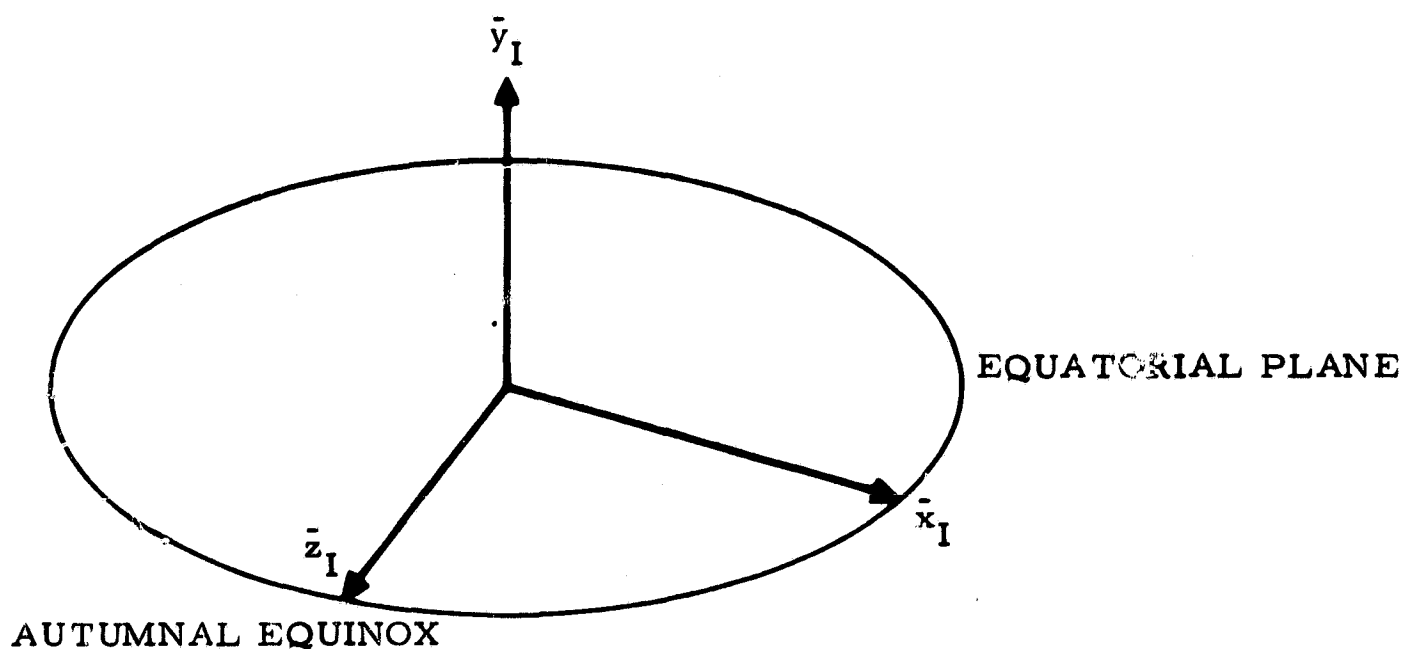


Figure 9.1-1. Inertial Reference System

The inertial reference axes have their origin at the center of mass of the earth. The $\bar{y}_I$ axis is normal to the equatorial plane and is positive in the northward direction and the $\bar{z}_I$ axis lies along the autumnal equinox. The coordinate matrix of the inertial reference axes is given by Equation 3.1-1 as

$$\{\bar{Q}_I\} = \begin{Bmatrix} \bar{x}_I \\ \bar{y}_I \\ \bar{z}_I \end{Bmatrix}$$

9.2 THE ORBITAL REFERENCE AXES

A rotation of the inertial reference axes through an angle β about the $\bar{y}_I$ axis produces the $\{\bar{Q}_\beta\}$ system in which $\bar{z}_\beta$ lies along the line of nodes (intersection of the orbit plane and the equatorial plane, Figure 9.2-1). The positive direction of the $\bar{z}_\beta$ axis is toward the ascending node.

where

$$\{\bar{Q}_\beta\} = \begin{Bmatrix} \bar{x}_\beta \\ \bar{y}_\beta \\ \bar{z}_\beta \end{Bmatrix} \quad (9.2-1)$$

Let $[E_{\beta I}]$ be the transformation matrix which transforms the $\{\bar{Q}_I\}$ system into the $\{\bar{Q}_\beta\}$ system such that

$$\{\bar{Q}_\beta\} = [E_{\beta I}] \{\bar{Q}_I\} \quad (9.2-2)$$

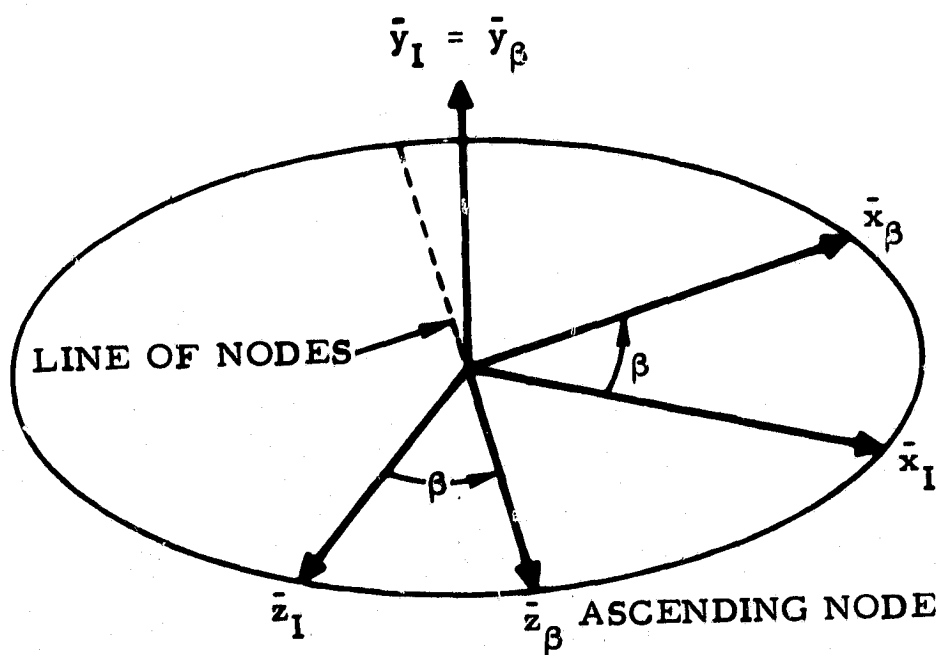


Figure 9.2-1. Line of Nodes

where

$$[E_{\beta I}] = \begin{bmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{bmatrix} \quad (9.2-3)$$

and

$$[E_{\beta I}] = \beta \begin{bmatrix} -\sin \beta & 0 & -\cos \beta \\ 0 & 0 & 0 \\ \cos \beta & 0 & -\sin \beta \end{bmatrix} \quad (9.2-4)$$

Rotation of the $\{\bar{Q}_\beta\}$ system, Figure 9.2-1, about $\bar{z}_\beta$ through an angle ξ , the orbit inclination angle, produces the $\{\bar{Q}_0\}$ system Figure 9.2-2 where $\bar{x}_0$ lies in the orbit plane.

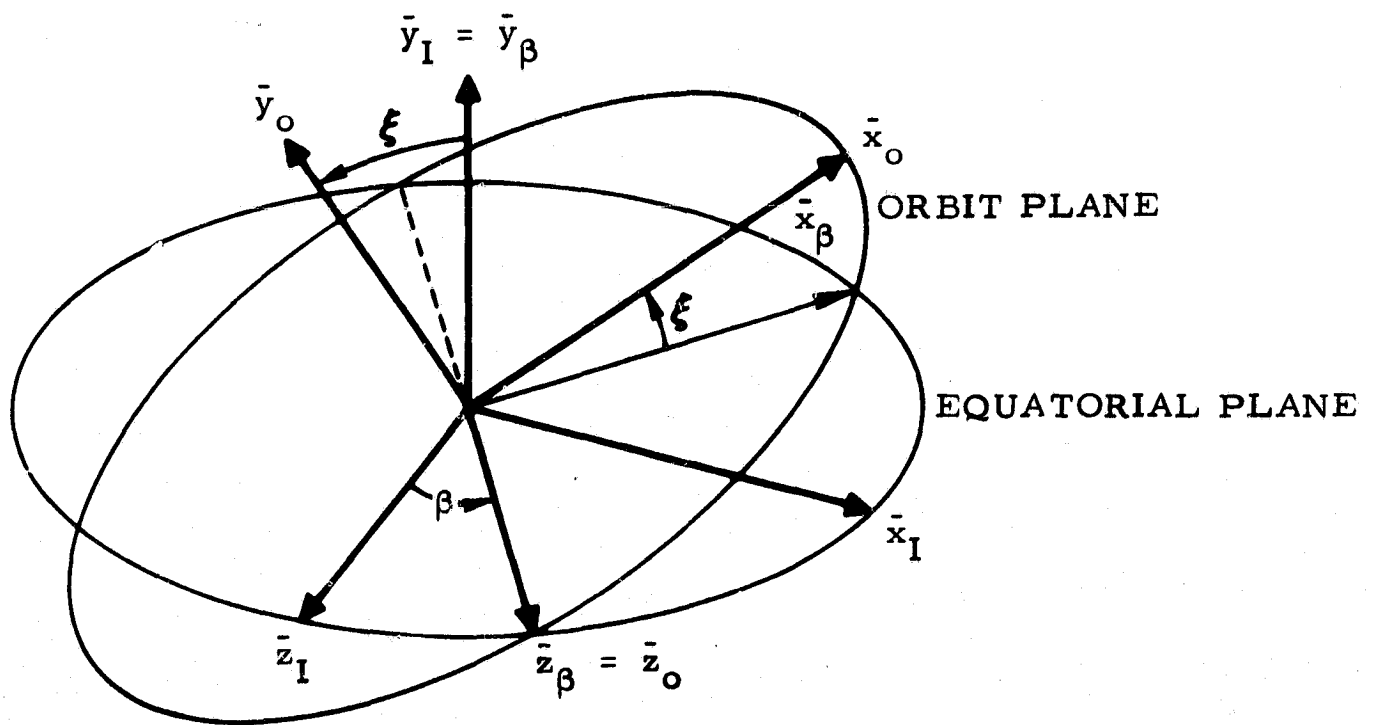


Figure 9.2-2. Orbit Inclination

then

$$\{Q_0\} = \begin{bmatrix} \bar{x}_0 \\ \bar{y}_0 \\ \bar{z}_0 \end{bmatrix} \quad (9.2-5)$$

Let $[E_{o\beta}]$ be the transformation matrix which transforms the $\{\bar{Q}_\beta\}$ system into the $\{\bar{Q}_o\}$ system such that

$$\{\bar{Q}_o\} = [E_{o\beta}] \{\bar{Q}_\beta\} \quad (9.2-6)$$

where

$$[E_{o\beta}] = \begin{bmatrix} \cos \xi & \sin \xi & 0 \\ -\sin \xi & \cos \xi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (9.2-7)$$

Since ξ is a constant $\dot{\xi} = 0$ and

$$[\dot{E}_{o\beta}] = 0 \quad (9.2-8)$$

Rotation of the $\{\bar{Q}_o\}$ system, Figure 9.2-2, about $\bar{y}_o$ through an angle α produces the $\{\bar{Q}_\alpha\}$ system, Figure 9.2-3, where $\bar{z}_\alpha$ passes through the orbital position of the center of mass of the spacecraft.

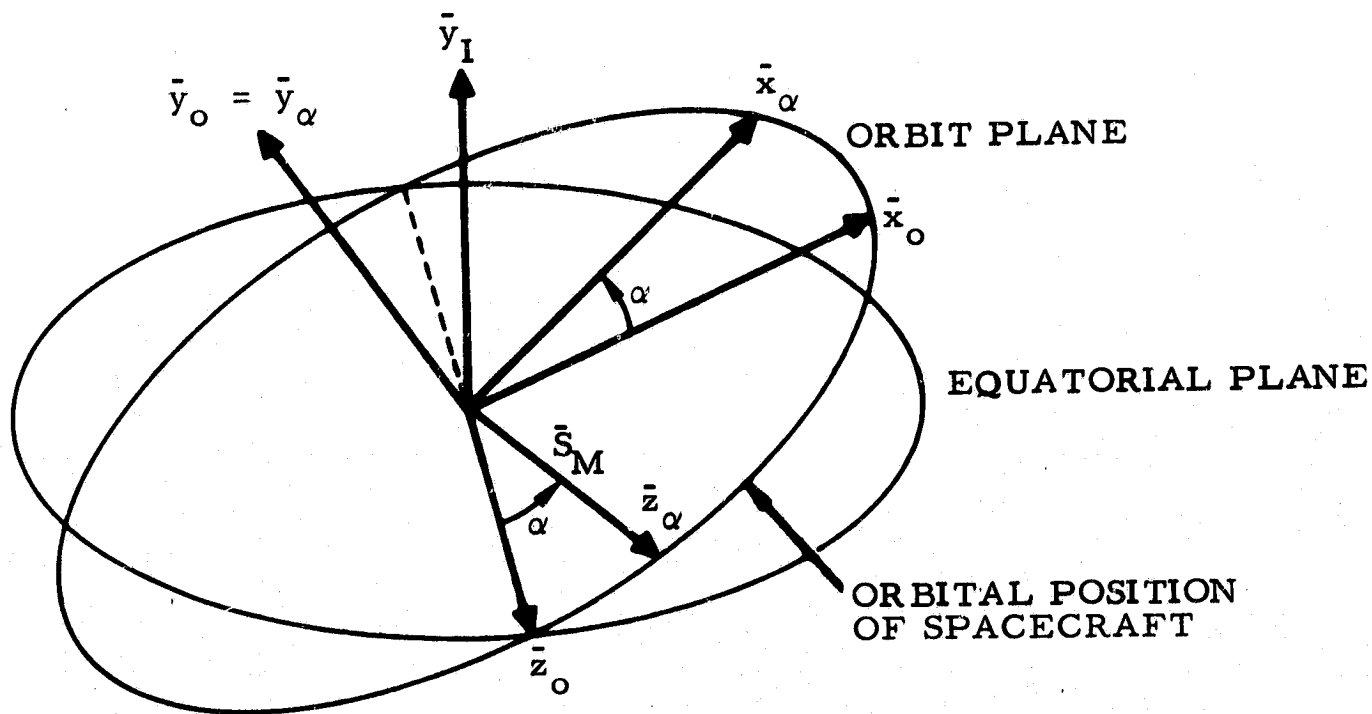


Figure 9.2-3. Orbit Angles

Then

$$\{\bar{Q}_\alpha\} = \begin{Bmatrix} \bar{x}_\alpha \\ \bar{y}_\alpha \\ \bar{z}_\alpha \end{Bmatrix} \quad (9.2-9)$$

Let $[E_{\alpha 0}]$ be the transformation matrix which transforms the $\{\bar{Q}_0\}$ system into the $\{\bar{Q}_\alpha\}$ system such that

$$\{\bar{Q}_\alpha\} = [E_{\alpha 0}] \{\bar{Q}_0\} \quad (9.2-10)$$

where

$$[E_{\alpha 0}] = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \quad (9.2-11)$$

and

$$[\dot{E}_{\alpha 0}] = \dot{\alpha} \begin{bmatrix} -\sin \alpha & 0 & -\cos \alpha \\ 0 & 0 & 0 \\ \cos \alpha & 0 & -\sin \alpha \end{bmatrix} \quad (9.2-12)$$

The orbital reference system $\{\bar{Q}_M\}$, Figure 9.2-4, with unit vectors $\bar{x}_M$, $\bar{y}_M$, and $\bar{z}_M$ is defined in Section 3.0 with its origin at the center of the mass of the total satellite, with the $\bar{z}_M$ axis pointing toward the earth's center of mass, the $\bar{x}_M$ in the orbit plane forming an acute angle with the tangential velocity vector, and with $\bar{y}_M$ normal to the orbit plane.

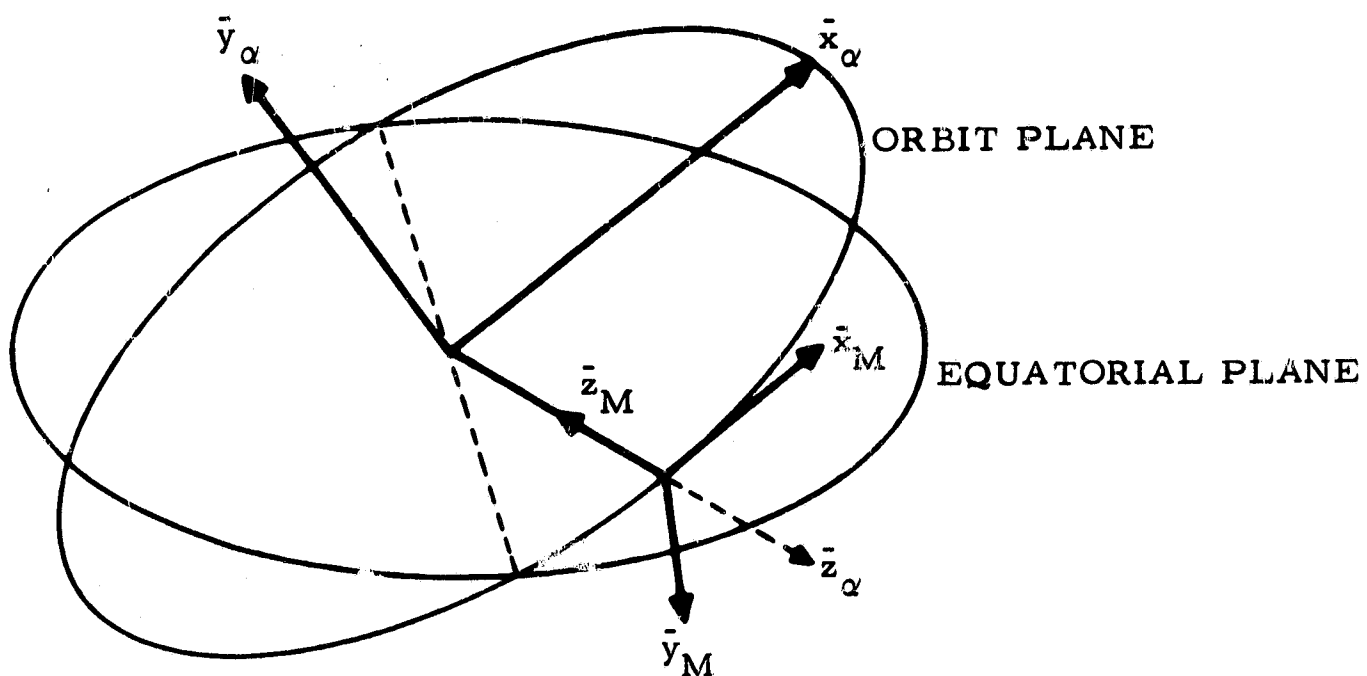


Figure 9.2-4. Orbital Reference System

Then

$$\bar{x}_M = \bar{x}_\alpha$$

$$\bar{y}_M = -\bar{y}_\alpha$$

$$\bar{z}_M = -\bar{z}_\alpha$$

Let $[E_{M\alpha}]$ be the transformation matrix which transposes the $\{\bar{Q}_\alpha\}$ system into the $\{\bar{Q}_M\}$ system such that

$$\{\bar{Q}_M\} = [E_{M\alpha}]\{\bar{Q}_\alpha\} \quad (9.2-13)$$

where

$$[E_{M\alpha}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (9.2-14)$$

and

$$[\dot{E}_{M\alpha}] = 0 \quad (9.2-15)$$

Combining Equations 9.2-13 and 9.2-10

$$\{\bar{Q}_M\} = [E_{M\alpha}][E_{\alpha O}]\{\bar{Q}_O\}$$

and letting $[E_{Mo}] = [E_{M\alpha}][E_{\alpha O}]$

$$\{\bar{Q}_M\} = [E_{Mo}]\{\bar{Q}_O\} \quad (9.2-16)$$

and

$$[E_{Mo}] = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & -1 & 0 \\ -\sin \alpha & 0 & -\cos \alpha \end{bmatrix} \quad (9.2-17)$$

and

$$[\dot{E}_{Mo}] = \dot{\alpha} \begin{bmatrix} -\sin \alpha & 0 & -\cos \alpha \\ 0 & 0 & 0 \\ -\cos \alpha & 0 & \sin \alpha \end{bmatrix} \quad (9.2-18)$$

from Equations 9.2-16, 9.2-6 and 9.2-2

$$\{\bar{Q}_M\} = [E_{Mo}][E_{O\beta}][E_{\beta I}]\{\bar{Q}_I\} \quad (9.2-19)$$

Since the transformation from $\{\bar{Q}_I\}$ to $\{\bar{Q}_M\}$ has been defined in Equation 3.2-6 as $[E]$

$$[E] = [E_{Mo}][E_{O\beta}][E_{\beta I}] \quad (9.2-20)$$

Substituting Equations 9.2-3, 9.2-7, and 9.2-17, the transformation matrix [E] defining the angular position of the orbital reference frame may be written as

$$[E] = \begin{bmatrix} (\cos \alpha \cos \xi \cos \beta - \sin \alpha \sin \beta) (\cos \alpha \sin \xi) (-\cos \alpha \cos \xi \sin \beta - \sin \alpha \cos \beta) \\ (\sin \xi \cos \beta) (-\cos \xi) (-\sin \xi \sin \beta) \\ (-\sin \alpha \cos \xi \cos \beta - \cos \alpha \sin \beta) (-\sin \alpha \sin \xi) (\sin \alpha \cos \xi \sin \beta - \cos \alpha \cos \beta) \end{bmatrix} \quad (9.2-21)$$

where [E] is a function of the three parameters α , ξ , and β . The angle α has been defined as the orbital position of the spacecraft with respect to the ascending node while ψ has been defined in Section 8.0 as the orbital position with respect to perigee. Therefore, if α_p is the orbit angle between the ascending node and perigee

$$\alpha = \alpha_p + \psi \quad (9.2-22)$$

from Equations 8.0-8 and 8.0-11

$$\dot{\alpha} = \dot{\psi} = \frac{\sqrt{a} \gamma (1 - e^2)}{S_M^2} = \text{radian/sec} \quad (9.2-23)$$

$$\ddot{\alpha} = \ddot{\psi} = - \frac{2e \gamma \sin \psi}{S_M^3} = \text{radian/sec}^2 \quad (9.2-24)$$

The orbital inclination of ξ is an input constant. The nodal regression rate $\dot{\beta}$ is assumed to be a constant value given by

$$\dot{\beta} = -1.637 \times 10^{-3} \left(\frac{R_e^2}{a} \right) \omega_o \cos \xi \quad (9.2-25)$$

where R_e is the radius of the earth in nautical miles. If β_0 is the value of β at time zero the parameter β is simply defined by

$$\beta = \beta_0 + \dot{\beta} t \quad (9.2-26)$$

Since the matrix $[E_{o\beta}]$ is zero (Equation 9.2-8) differentiating Equation 9.2-20 gives

$$[\dot{E}] = [\dot{E}_{Mo}] [E_{o\beta}] [E_{\beta I}] + [E_{Mo}] [E_{o\beta}] [\dot{E}_{\beta I}]$$

and from Equation 5.2-4

$$[\dot{E}] = -[J_{\Omega RR}] [E]$$

Then

$$[J_{\Omega RR}] = -[\dot{E}_{Mo}] [E_M]_T - [E_{Mo}] [E_{o\beta}] [\dot{E}_{\beta I}] [E_{\beta I}]_T [E_{o\beta}]_T [E_{Mo}]_T \quad (9.2-27)$$

expanding

$$[J_{\Omega RR}] = \dot{\beta} \begin{bmatrix} J \begin{pmatrix} \sin \xi \cos \alpha \\ -\cos \xi \\ -\sin \xi \sin \alpha \end{pmatrix} \end{bmatrix} + \dot{\alpha} \begin{bmatrix} J \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \end{bmatrix}$$

therefore,

$$\{\Omega_{RR}\} = \begin{Bmatrix} \dot{\beta} \sin \xi \cos \alpha \\ -\dot{\beta} \cos \xi - \dot{\alpha} \\ -\dot{\beta} \sin \xi \sin \alpha \end{Bmatrix} \quad (9.2-28)$$

and since ξ is constant

$$\{\dot{\Omega}_{RR}\} = \begin{Bmatrix} -\dot{\beta} \dot{\alpha} \sin \xi \sin \alpha \\ -\ddot{\alpha} \\ -\dot{\beta} \dot{\alpha} \sin \xi \cos \alpha \end{Bmatrix} \quad (9.2-29)$$

9.3 LINEAR ACCELERATION OF SPACECRAFT CENTER OF MASS

Since the moment equations are written about reference points which do not coincide with the system center of mass, the acceleration of the reference points appear in the equation. The expressions for these accelerations contain the linear translational acceleration components of the spacecraft center of mass $\{G_{MR}\}$. In defining the orbit equation of Section 8.0, the control and disturbance forces were neglected and the acceleration of the total center of mass was assumed to result from only the gravity force. This assumption is not valid for the dynamic equations defining the relative angular motion of the spacecraft and Equation 6.3-7 will be used

$$M_T \bar{G}_M = \bar{F}_{C1} + \sum_{i=1}^N (\bar{F}_{gi} + \bar{F}_{Si} + \bar{F}_{Ai})$$

From Equations 8.0-1 and 8.0-2

$$\sum_{i=1}^N \bar{F}_{gi} = -K_N \frac{Y M_T}{S_M^2} \left(\frac{\bar{S}_M}{S_M} \right)$$

Letting

$$g_o = K_N \frac{Y}{S_M^2} \quad (9.3-1)$$

where g_o = magnitude of the acceleration of gravity at the center of mass of the spacecraft. Then

$$\sum_{i=1}^N \bar{F}_{gi} = -g_o M_T \frac{\bar{S}_M}{S_M} \quad (9.3-2)$$

Equation 6.3-7 may now be written as

$$\bar{G}_M = -g_o \frac{\bar{S}_M}{S_M} + \frac{\bar{F}_{C1}}{M_T} + \frac{1}{M_T} \sum_{i=1}^N (\bar{F}_{Si} + F_{Ai}) \quad (9.3-3)$$

If the forces are defined in the corresponding body axes and $[D_i]$ for body 1 is taken as the unit matrix $[U]$, the required acceleration components in the reference coordinate system are

$$\{G_{MR}\} = g_o \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \frac{1}{M_T} [A]_T \{F_{C11}\} + \frac{1}{M_T} [A]_T \sum_{i=1}^N [D_i]_T \left\{ \{F_{Sii}\} + \{F_{Aii}\} \right\} \quad (9.3-4)$$

and in the body 1 coordinates

$$\{G_{M1}\} = [A] \{G_{MR}\} = g_o [A] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \frac{1}{M_T} \{F_{C11}\} + \frac{1}{M_T} \sum_{i=1}^N [D_i]_T \left\{ \{F_{Sii}\} + \{F_{Aii}\} \right\} \quad (9.3-5)$$

From Equations 8.0-1 and 9.3-2 the accelerations components assumed in defining the orbit equations are

$$\{G_{MOR}\} = g_o \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad (9.3-6)$$

where $\{G_{MOR}\}$ are the components of the total center of mass acceleration due to orbital motion. In defining the motion of the center of mass in Section 9.0, Equation 9.3-6 was violated in that a constant nodal regression rate $\dot{\beta}$ was introduced with no consideration given to the forces required to maintain $\dot{\beta}$ constant. The magnitude of this neglected force will now be determined.

From Equation 3.41 of Reference 1

$$\{G_{MOR}\} = \{S_{MR}''\} + 2[J_{\Omega RR}]\{S_{MR}'\} + K_N \left\{ [J_{\Omega RR}][J_{\Omega RR}] + [J_{\Omega RR}'] \right\} \{S_{MR}\} \quad (9.3-7)$$

where

$$\{S_{MR}\} = -S_M \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \text{ nautical miles} \quad (9.3-8)$$

$$\{S_{MR}'\} = -\dot{S}_M \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \text{ ft/sec} \quad (9.3-9)$$

$$\{S_{MR}''\} = -\ddot{S}_M \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \text{ ft/sec}^2 \quad (9.3-10)$$

$$[J_{\Omega RR}] = \begin{bmatrix} 0 & -\Omega_{RRz} & \Omega_{RRy} \\ \Omega_{RRz} & 0 & -\Omega_{RRx} \\ -\Omega_{RRy} & \Omega_{RRx} & 0 \end{bmatrix} \quad (9.3-11)$$

$$[\dot{\Omega}_{RR}] = \begin{bmatrix} 0 & -\dot{\Omega}_{RRz} & \dot{\Omega}_{RRy} \\ \dot{\Omega}_{RRz} & 0 & -\dot{\Omega}_{RRx} \\ -\dot{\Omega}_{RRy} & \dot{\Omega}_{RRx} & 0 \end{bmatrix} \quad (9.3-12)$$

Substituting Equations 9.3-8, 9.3-9, 9.3-10, 9.3-11, and 9.3-12 into Equation 9.3-7

$$\{G_{MOR}\} = \{\Delta G_{MOR}\} + g_o \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad (9.3-13)$$

where

$$\{\Delta G_{MOR}\} = - \begin{Bmatrix} 2\dot{S}_M \Omega_{RRy} + K_N S_M (\Omega_{RRx} \Omega_{RRz} + \dot{\Omega}_{RRy}) \\ -2\dot{S}_M \Omega_{RRx} + K_N S_M (\Omega_{RRy} \Omega_{RRz} - \dot{\Omega}_{RRx}) \\ -K_N S_M (\Omega_{RRx}^2 + \dot{\beta}^2 \cos^2 \xi + 2\dot{\beta} \alpha \cos \xi) \end{Bmatrix} \quad (9.3-14)$$

$\{\Delta G_{MOR}\}$ = components of the acceleration of the center of mass due to nodal regression.

Substituting for the orbit rates and accelerations from Equations 9.2-28 and 9.2-29

$$\{\Delta G_{MOR}\} = \dot{\beta} \begin{Bmatrix} 2\dot{S}_M \cos \xi + K_N S_M \dot{\beta} \sin^2 \xi \sin \alpha \cos \alpha \\ 2\dot{S}_M \sin \xi \cos \alpha - K_N S_M \dot{\beta} \sin \xi \cos \xi \sin \alpha \\ K_N S_M (\dot{\beta} \sin^2 \xi \cos^2 \alpha + \dot{\beta} \cos^2 \xi + 2\dot{\alpha} \cos \xi) \end{Bmatrix} \quad (9.3-15)$$

and

$$\{F_{\beta}\} = M_T \{\Delta G_{MOR}\} \quad (9.3-16)$$

where $\{F_{\beta}\}$ are the components of the total external force required to produce nodal regression. It is assumed that the acceleration field is uniform such that the force $\{F_{\beta}\}$ on element of mass dm is equal to $\{\Delta G_{MOR}\} dm$. The total external moment on the spacecraft is zero, however, the nodal regression forces create internal moments between the bodies which cause relative motion. It should be noted that these internal moments are neglected in this derivation.

10. DISTURBANCE PARAMETERS

The parameters required in determining the disturbance torques and forces on the bodies are presented in this section. The disturbances to be considered are solar radiation torques and forces, magnetic torques, aerodynamic torques and forces, and the effects of changes in shape due to thermal expansion.

10.1 THE APPARENT MOTION OF THE SUN

It is assumed that the earth is in a circular orbit about the sun and that the sun appears to be in a circular orbit about the earth in the ecliptic plane, Figure 10.1-1.

The angle θ_S is the apparent displacement of the sun from the autumnal equinox measured in the ecliptic plane. Since the period of revolution is 365.24 days,

$$\theta_S = \frac{360}{365.24} D_{AE} \quad (10.1-1)$$

where D_{AE} = days after autumnal equinox.

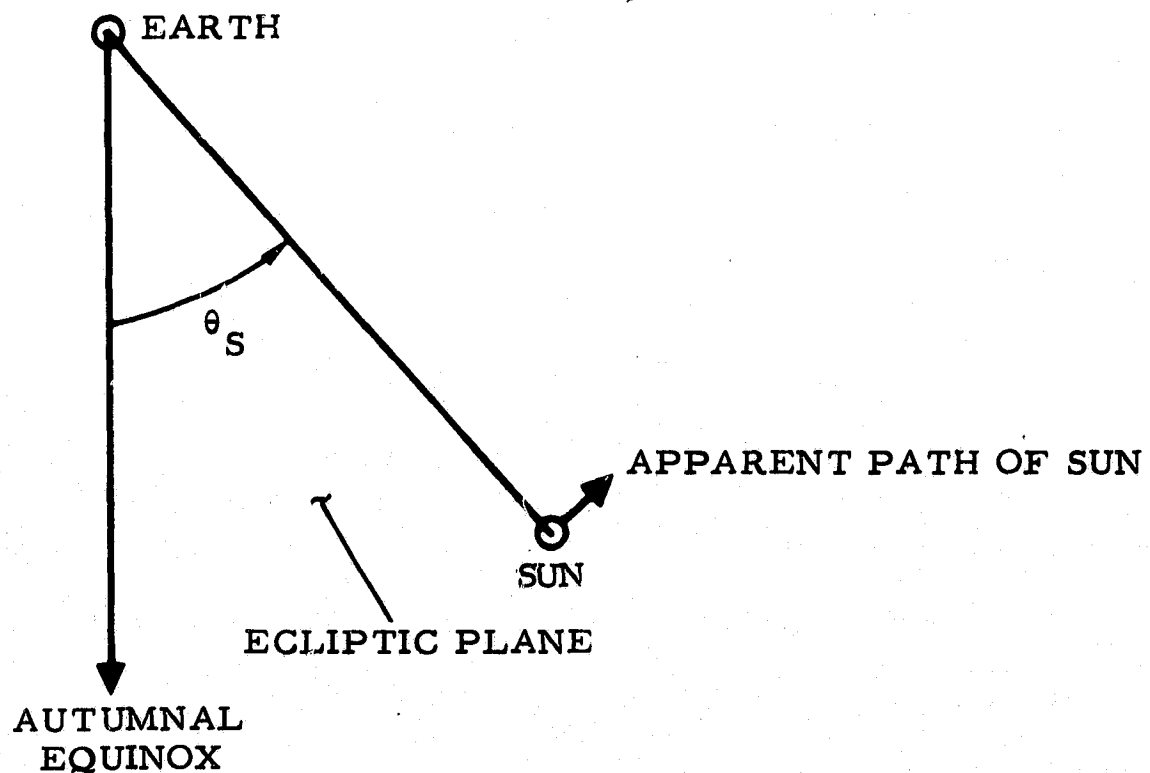


Figure 10.1-1 Sun Angle

It is assumed that θ_S is a constant for the period being considered and D_{AE} is a mean value.

The relationship between the ecliptic plane and the inertial reference system is shown in Figure 10.1-2.

Let $\bar{\delta}_S$ be a unit vector in the direction from the sun toward the center of the earth. In terms of the inertial reference system $|\bar{Q}_I|$, $\bar{\delta}_S$ is given by

$$-\bar{\delta}_S = (\sin \theta_S \cos 23^\circ 27') \bar{x}_I - (\sin \theta_S \sin 23^\circ 27') \bar{y}_I + (\cos \theta_S) \bar{z}_I$$

In matrix notation the components of the sun vector in inertial coordinates are

$$|\bar{\delta}_{SI}| = \begin{Bmatrix} -\sin \theta_S \cos 23^\circ 27' \\ \sin \theta_S \sin 23^\circ 27' \\ -\cos \theta_S \end{Bmatrix} \quad (10.1-2)$$

The components of $|\bar{\delta}_{SI}|$ are constant during the time being considered and may be computed from input data. Transforming to the orbital

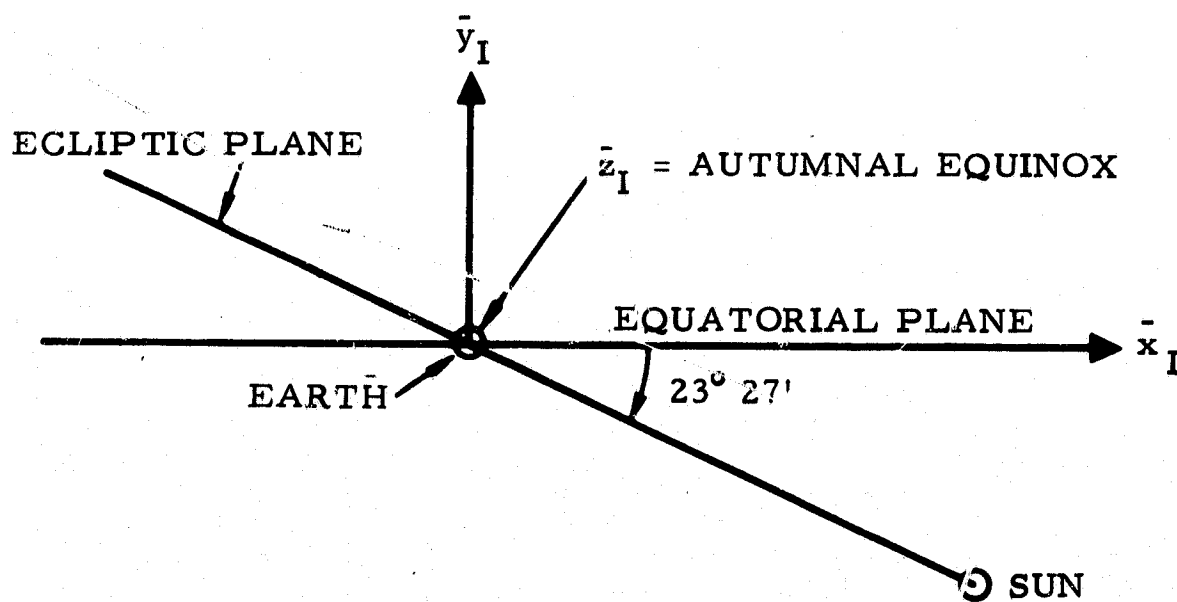


Figure 10.1-2 Ecliptic Plane

reference system through the transformation matrix $[E]$, then to the body 1 axes through the transformation matrix $[A]$, and finally to the body i coordinates through the matrix $[D_i]$, the sun vector components in body i coordinates are

$$[\delta_{Si}] = [D_i] [A] [E] [\delta_{SI}] \quad (10.1-3)$$

Since $[D_i]$ for body 1 has been defined as the unit matrix $[U]$, Equation 10-1-3 will hold for $(1 \leq i \leq N)$.

In order to determine the dynamic torques resulting from the thermal deflection of the auxiliary bodies, it will be necessary to compute the first and second derivatives of the sun vector components with respect to time. Since the components with respect to the inertial axes, $[\delta_{SI}]$, have been taken as constants, the first and second time derivatives with respect to the inertial axes are both zero, and the changes in components in body coordinates result only from rotation of the body axes. From Equation 3-38 of Reference 1

$$0 = [\dot{\delta}_{Si}] + [J_{\omega ii}] [\delta_{Si}]$$

Therefore,

$$[\dot{\delta}_{Si}] = - [J_{\omega ii}] [\delta_{Si}] \quad (10.1-4)$$

and from Equation 3-41 of Reference 1 and 11.1-4 above

$$0 = [\ddot{\delta}_{Si}] + [\dot{J}_{\omega ii}] [\delta_{Si}] - [J_{\omega ii}] [J_{\omega ii}] [\delta_{Si}]$$

therefore,

$$[\ddot{\delta}_{Si}] = \left[[J_{\omega ii}] [J_{\omega ii}] - [\dot{J}_{\omega ii}] \right] [\delta_{Si}] \quad (10.1-5)$$

values of $[\dot{\delta}_{Si}]$ and $[\ddot{\delta}_{Si}]$ need only be computed when the effects of thermal deflections are to be considered, and then only for $(2 \leq i \leq N)$.

The apparent displacement of the sun from the autumnal equinox in the ecliptic plane has been defined as θ_S . Let the apparent displacement of the sun from the autumnal equinox measured in the equatorial plane be θ_S' . Then

$$\theta_S' = \tan^{-1} \frac{\sin \theta_S \cos 23^\circ 27'}{\cos \theta_S} \quad (10.1-6)$$

where θ_S' is a constant angle required in the computation of the magnetic field strength at body i.

10.2 POSITION OF MAGNETIC AXES

Let the $[\bar{O}_G]$ coordinate system be fixed in the earth with its origin at the center of the earth, the $\bar{y}_G$ axes normal to the equatorial plane and coincident with the $\bar{y}_I$ inertial axis and the $\bar{z}_G$ axis passing through the prime meridian, Figure 10.2-1.

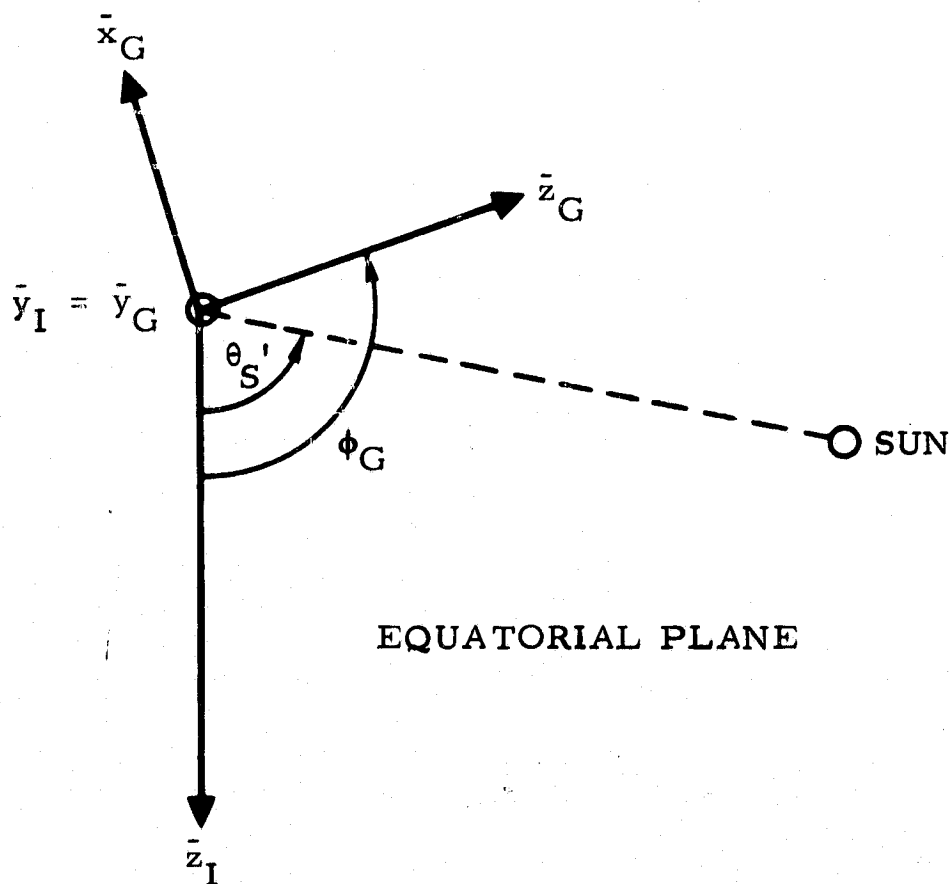


Figure 10.2-1 Earth Fixed Reference System

It is assumed that the sun crosses the prime meridian at noon (GMT) on each day of the year. Then

$$\phi_G = \theta'_S + \frac{360}{1440} t_G = \text{degrees} \quad (10.2-1)$$

where t_G is the elapsed time in minutes after noon (GMT). ϕ_G is a variable and must be computed for each instant of time. The position of the $[\bar{Q}_G]$ coordinate system is obtained by rotating the $[\bar{Q}_I]$ axes about $\bar{y}_I$ by the angle ϕ_G , Figure 10.2-1. Therefore

$$[\bar{Q}_G] = [\phi_G] [\bar{Q}_I] \quad (10.2-2)$$

where

$$[\phi_G] = \begin{bmatrix} \cos \phi_G & 0 & -\sin \phi_G \\ 0 & 1 & 0 \\ \sin \phi_G & 0 & \cos \phi_G \end{bmatrix} \quad (10.2-3)$$

The magnetic coordinate system, $[\bar{Q}_m]$, is obtained by rotating $[\bar{Q}_G]$ about its $\bar{y}_G$ axis through an angle of 20 degrees to form the $[\bar{Q}_G']$ system. The $[\bar{Q}_G']$ system is then rotated about its $\bar{z}_G'$ axis through an angle of 11 degrees to form the $[\bar{Q}_m]$ axes. Then letting

$$\Delta = \begin{bmatrix} \cos 11^\circ \cos 20^\circ & \sin 11^\circ & -\cos 11^\circ \sin 20^\circ \\ -\sin 11^\circ \cos 20^\circ & \cos 11^\circ & \sin 11^\circ \sin 20^\circ \\ \sin 20^\circ & 0 & \cos 20^\circ \end{bmatrix} \quad (10.2-4)$$

$$[\bar{Q}_m] = [\Delta] [\bar{Q}_G] \quad (10.2-5)$$

where $[\Delta]$ is a constant matrix.

Then the position of the magnetic coordinate system is

$$\begin{bmatrix} \bar{Q}_m \end{bmatrix} = \begin{bmatrix} \Delta \end{bmatrix} \begin{bmatrix} \phi_G \end{bmatrix} \begin{bmatrix} \bar{Q}_I \end{bmatrix} \quad (10.2-6)$$

10.3 MAGNETIC FIELD STRENGTH

It is assumed that the earth's magnetic field strength for each body of the satellite is equal to the field strength at the center of mass of the satellite. This field strength is computed in relationship to the magnetic axes of the earth $\begin{bmatrix} \bar{Q}_m \end{bmatrix}$ at time t , Figure 10.3-1.

In terms of the angles, λ_1 and λ_2 the magnetic field strength at the center of mass of the satellite is

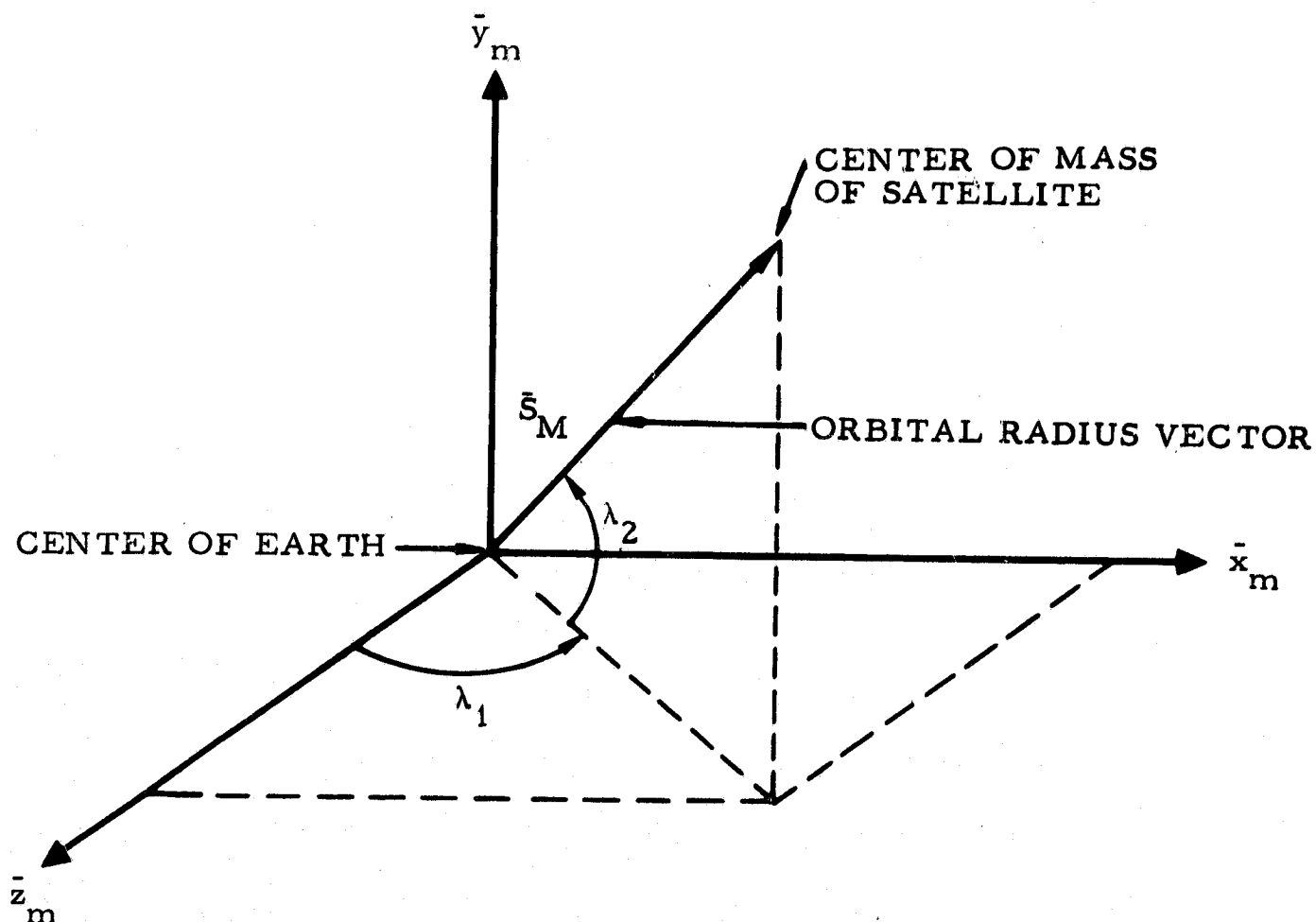


Figure 10.3-1 Magnetic Reference System

$$\vec{B}_m = -\frac{M_e}{S_M} \begin{Bmatrix} 3 \sin\lambda_1 \sin\lambda_2 \cos\lambda_2 \\ 3 \sin^2\lambda_2 - 1 \\ 3 \cos\lambda_1 \sin\lambda_2 \cos\lambda_2 \end{Bmatrix}_T |\vec{a}_m| \quad (10.3-1)$$

where

$$M_e = 1.266 \times 10^{10} \text{ gauss(NM)}^3$$

Let $\vec{a}$ be a unit vector along $\vec{S}_M$ then

$$\vec{a} = \frac{\vec{S}_M}{S_M} \quad (10.3-2)$$

and from Figure 10.3-1 the components of $\vec{a}$ in the magnetic coordinates are

$$|\alpha_m| = \begin{Bmatrix} \alpha_{mx} \\ \alpha_{my} \\ \alpha_{mz} \end{Bmatrix} = \begin{Bmatrix} \cos\lambda_2 \sin\lambda_1 \\ \sin\lambda_2 \\ \cos\lambda_2 \cos\lambda_1 \end{Bmatrix} \quad (10.3-3)$$

and Equation 10.3-1 becomes

$$\vec{B}_m = -\frac{3M_e}{S_M} \begin{Bmatrix} \alpha_{mx} \alpha_{my} \\ \alpha_{my}^2 - 1/3 \\ \alpha_{my} \alpha_{mz} \end{Bmatrix}_T |\vec{a}_m| \quad (10.3-4)$$

defining the magnetic field vector in the magnetic coordinate system

$$|\mathcal{B}_{mm}| = -\frac{3M_e}{S_M^3} \begin{Bmatrix} \alpha_{mx} & \alpha_{my} \\ \alpha_{my}^2 - 1/3 \\ \alpha_{my} & \alpha_{mz} \end{Bmatrix} \quad (10.3-5)$$

Since the $\bar{z}_R$ axis of the orbital reference system is directed toward the center of the earth, the components of $\bar{\alpha}$ in orbital reference coordinates are

$$|\alpha_R| = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} \quad (10.3-6)$$

transforming to the inertial reference system

$$|\alpha_I| = [E]_T |\alpha_R| = - \begin{Bmatrix} E_{31} \\ E_{32} \\ E_{33} \end{Bmatrix} \quad (10.3-7)$$

The matrix product $[E]_T |\alpha_R| = |\alpha_I|$ is a column matrix of the negative of the third row of the $[E]$ matrix. $|\alpha_I|$ are the components of a unit vector from the center of the earth directed toward the center of mass of the spacecraft expressed in inertial coordination. This vector should be computed and stored since it will be used in other computations in the program. From Equation 10.2-6 the components of $\bar{\alpha}$ in the magnetic coordinates are

$$|\alpha_m| = [\Delta] [\phi_G] |\alpha_I| \quad (10.3-8)$$

The elements of $[\alpha_m]$ may now be used to compute the components of $\vec{B}_m$ in magnetic coordinates using Equation 10.3-5. The components are transformed to inertial coordinates by $[[\Delta] [Q_G]]^T$ and then to body i coordinates by $[D_i] [A] [E]$ so that the components of the magnetic field in body i coordinates are

$$[B_{mi}] = [D_i] [A] [E] \left[[\Delta] [\phi_G] \right]^T [B_{mm}] \quad (10.3-9)$$

where

$$[D_i] = [U]$$

10.4 ECLIPSE

For eclipse computations it is assumed that the sun's rays are parallel and that the earth's shadow is cylindrical, see Figure 10.4-1. The axis of the cylindrical shadow lies along the unit sun vector $\vec{\delta}_S$. From Figure 10.4-1 it may be seen that the spacecraft will enter into or exit from eclipse when the quantity $(S_M \sin \sigma_E)$ is equal to the radius of the earth.

The unit sun vector $\vec{\delta}_S$ and the unit vector $\vec{\alpha}$ are both defined in the inertial coordinate system $[\vec{Q}_I]$. From Equation 10.1-2

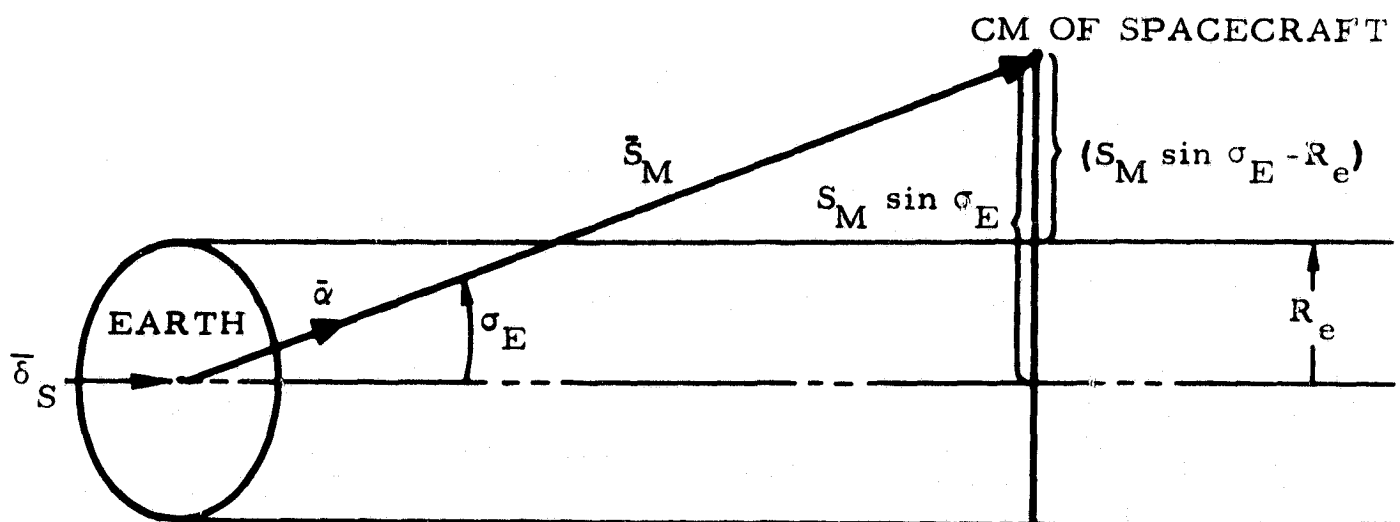


Figure 10.4-1 Earth Shadow for Eclipse

$$\begin{bmatrix} \delta_{SI} \end{bmatrix} = \begin{bmatrix} -\sin \theta_s \cos 23^\circ 27' \\ \sin \theta_s \sin 23^\circ 27' \\ -\cos \theta_s \end{bmatrix} = \begin{bmatrix} x^{\delta SI} \\ y^{\delta SI} \\ z^{\delta SI} \end{bmatrix} = \text{a constant matrix (10.4-1)}$$

From Equation 10.3-7

$$\begin{bmatrix} \alpha_I \end{bmatrix} = - \begin{bmatrix} E_{31} \\ E_{32} \\ E_{33} \end{bmatrix} \quad (10.4-2)$$

The components $\begin{bmatrix} \alpha_I \end{bmatrix}$ are required in nearly all of the disturbance computations and will therefore be computed at each integration step. From Equations 9.2-21 the components $\begin{bmatrix} \alpha_I \end{bmatrix}$ may be expressed as

$$\begin{bmatrix} \alpha_I \end{bmatrix} = \begin{bmatrix} \cos \xi \cos \beta \sin \alpha + \sin \beta \cos \alpha \\ \sin \xi \sin \alpha \\ -\cos \xi \sin \beta \sin \alpha + \cos \beta \cos \alpha \end{bmatrix} \quad (10.4-3)$$

From Figure 10.4-1

$$\cos \sigma_E = \bar{\delta}_S \cdot \bar{\alpha} \quad (10.4-4)$$

$$\cos \sigma_E = \begin{bmatrix} \delta_{SI} \end{bmatrix}_T \begin{bmatrix} \alpha_I \end{bmatrix} \quad (10.4-5)$$

Using 10.4-1 and 10.4-3

$$\cos \sigma_E = D_s \sin \alpha + D_c \cos \alpha \quad (10.4-6)$$

where

$$D_s = (\delta_{SLx} \cos\xi \cos\beta + \delta_{SIy} \sin\xi - \delta_{SIz} \cos\xi \sin\beta) \quad (10.4-7)$$

$$D_c = (\delta_{SLx} \sin\beta + \delta_{SIz} \cos\beta) \quad (10.4-8)$$

Since the components of $|\delta_{SI}|$ are constant and the orbit inclination ξ is constant, D_s and D_c will be constant if β is assumed constant. Then from Equation 10.4-6 $\cos\sigma_E$ will be maximum or a minimum when

$$\alpha_o = \tan^{-1} \frac{D_s}{D_c} \quad (10.4-9)$$

where α_o is the orbit angle for maximum or minimum $\cos\sigma_E$. The two values of α_o found from Equation 10.4-9 represent the orbit angles for the closest approach of the orbit radius to the sun vector; one on the sun side and one on the shadow side of the earth. Substituting either value of α_o into Equation 10.4-6 will give a value of $(\cos\sigma_E)_o$ for minimum approach to $\bar{\delta}_s$. If $(\cos\sigma_E)_o$ is positive the correct value of α_o for minimum approach on the shadow side was used. If $(\cos\sigma_E)_o$ is negative, the other value of α_o , equal to the one used $\pm\pi$ is the correct value. Using the computed value of $(\cos\sigma_E)_o$ the value of $(\sin\sigma_E)_o$ may be computed from

$$(\sin\sigma_E)_o = \sqrt{1 - (\cos\sigma_E)_o^2} \quad (10.4-10)$$

Using Equations 9.2-22 and 8.0-7 together with the value of α_o , the orbit radius at minimum approach, S_{Mo} , may be computed and the quantity $(S_M \sin\sigma_E)_o$ at minimum approach may be found. The value of $(S_M \sin\sigma_E)_o$ will be computed once during each orbit and if greater than 1.1 times the radius of the earth, R_e , it is assumed that eclipse will not occur during the orbit, and no further eclipse computations will be made, see Figures 10.4-1 and 10.4-2. The value of 1.1 is used since β is not constant during

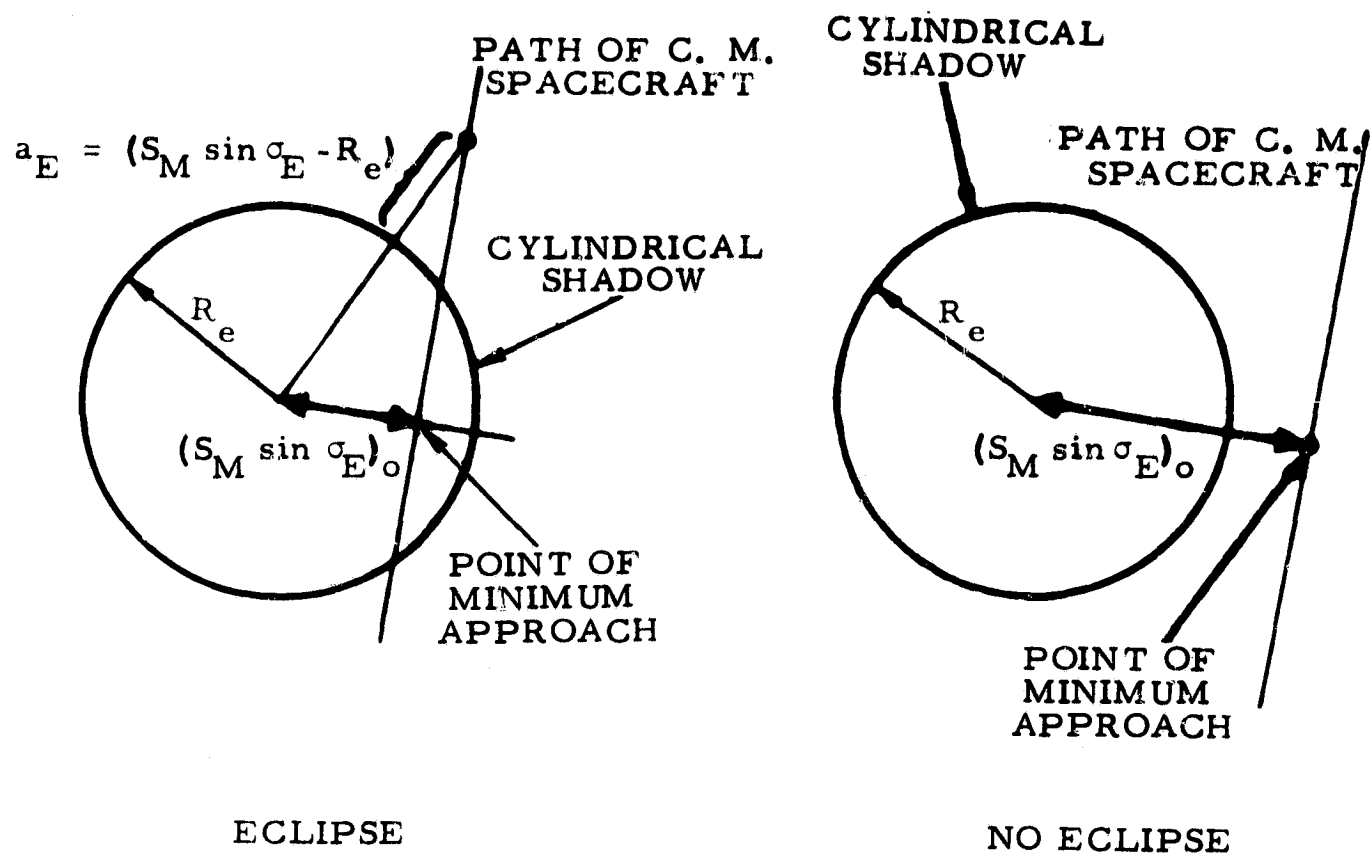


Figure 10.4-2 Minimum Approach to Eclipse

the orbit and β has a small value. If the value of $(S_M \sin \sigma_E)_0$ is less than $1.1 R_e$, eclipse may occur at an orbit angle near α_0 and further eclipse computations must be performed.

Eclipse computations are started at the first integration step after the orbit angle α becomes greater than α_e given by

$$\alpha_e = \alpha_0 - \sin^{-1} \frac{2R_e}{S_{M0}} \quad (10.4-11)$$

where $\sin^{-1} 2R_e/S_{Mo}$ is $\pi/2$ when $2R_e/S_{Mo} > 1.0$. With the current value of β , the values of α_o and α_e are updated and a value of $(S_M \sin \sigma_E)_o$ is computed. If $(S_M \sin \sigma_E)_o$ is greater than $1.1 R_e$ no further eclipse computations are performed until the orbit angle α becomes greater than the revised α_e approximately one orbit later. If $(S_M \cos \sigma_E)_o$ is less than $1.1 R_e$, the current value of the quantity

$$a_E = S_M \sin \sigma_E - R_e \quad (10.4-12)$$

is computed using already available components $|\delta_{SI}|$ and $|\delta_I|$ computed from Equations 10.4-1 and 10.4-2. From Figure 10.4-2 it may be seen that a_E is the distance between the center of mass of the spacecraft and the earth's shadow and the spacecraft will enter and exit from eclipse when a_E is zero. The integration continues computing a value of a_E at each step and retaining one past value. If after a given integration step n , the value of $(a_E)_n$ is negative or zero, it is assumed that eclipse occurred at the end of step n . If, however, the value of $(a_E)_n$ is positive, the quantity

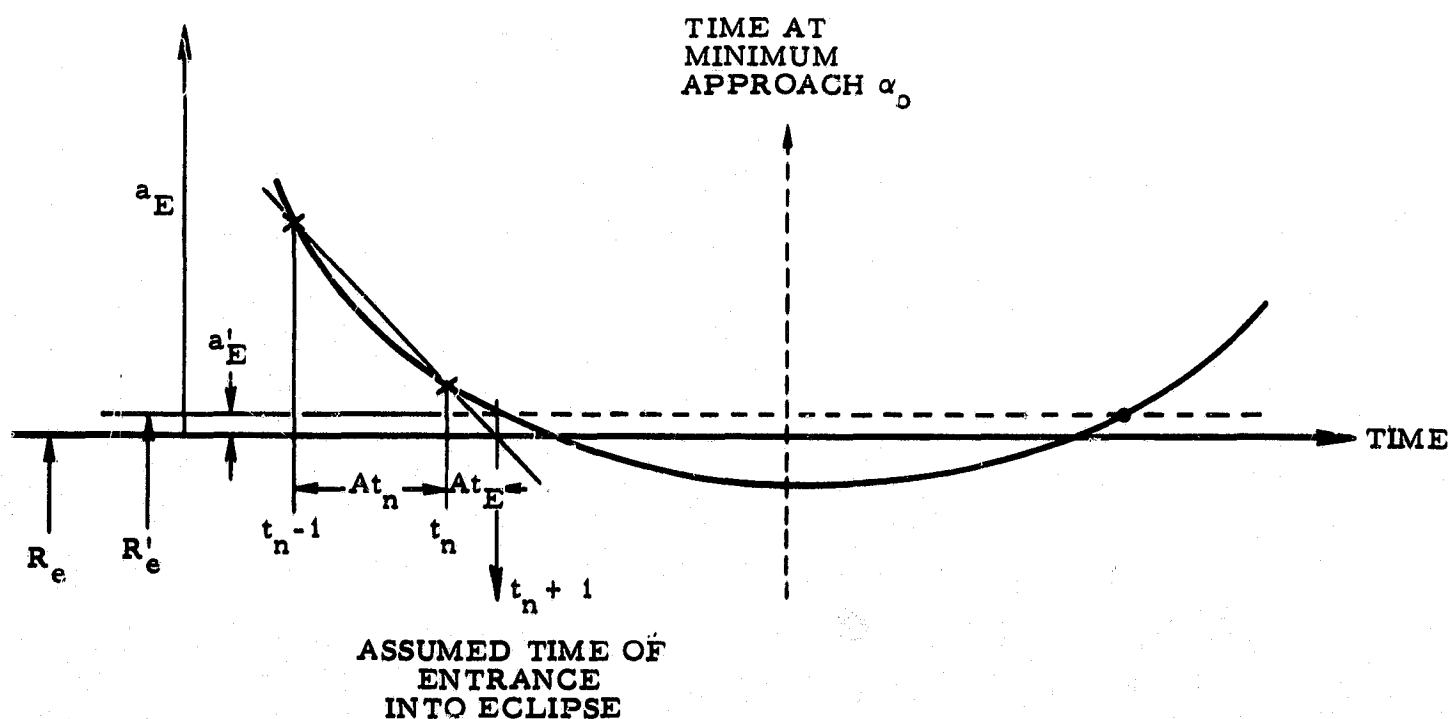


Figure 10.4-3 Prediction of Eclipse

$$\Delta a_E = (a_E)_n - (a_E)_{n-1} \quad (10.4-13)$$

is computed. If Δa_E is positive or zero no further eclipse computations are made until α again becomes greater than α_e . If Δa_E is negative the quantity

$$\Delta t_E = \frac{(a_E)_n \Delta t_n}{\Delta a_E} \quad (10.4-14)$$

is computed. Where Δt_n is the length of integration step n and Δt_E is the linear interpolated length of time to eclipse. If $\Delta t_E > \Delta t_{n+1}$ the integration continues and if $\Delta t_E \leq \Delta t_{n+1}$ the value of Δt_{n+1} is set equal to Δt_E and eclipse is assumed to occur at the end of integration step $n+1$, see Figure 10.4-2. The value of a_E at the time of entrance into eclipse is denoted by a'_E and the quantity

$$R'_e = R_e + a'_E \quad (10.4-15)$$

is computed. Where R'_e is the radius of the earth shadow used to determine exit from eclipse.

After entrance into eclipse the integration is restarted with the solar radiation forces and torques set at zero. However, when thermal bending is being considered there will be an eclipse transition period after entrance into or exit from eclipse. Normally, in sunlight, boom k of auxiliary body i is bent and in motion with respect to the body i coordinate system. When the satellite enters eclipse the boom will return to its nominal undeflected position during the short period of time t_E , after which the deflection and motion remains zero until exit from eclipse. At exit the boom deflects to its normal deflected position in time t_E . The boom motions during these transition periods is represented in the program by allowing the bending coefficients to reduce linearly to zero during time t_E at entrance to eclipse and to increase linearly from zero to their nominal value in time t_E at exit from eclipse.

For each integration step after entrance into eclipse the quantity a_E is computed from

$$a_E = S_M \sin \sigma_E - R'_e \quad (10.4-16)$$

retaining one past value. If after a given integration step n , the value of $(a_E)_n$ is positive or zero, it is assumed that exit occurred at the end of step n . If, however, the value of $(a_E)_n$ is negative, Δa_E and Δt_E are computed from Equations 10.4-13 and 10.4-14. If $\Delta t_E > \Delta t_{n+1}$ the integration continues and if $\Delta t_E \leq \Delta t_{n+1}$ the value of Δt_{n+1} is set equal to Δt_E and exit from eclipse is assumed to occur at the end of integration step $n+1$. After exit from eclipse the solar radiation forces and torques are again computed and the integration is continued. However, no further eclipse computations are performed until the orbit angle α again becomes larger than α_e .

When thermal boom bending is being considered, sufficient data must be computed and stored, during the transition periods, to allow the thermal bending routine to properly modify the bending parameters. The parameter p_E is used to indicate the rate of change of the thermal bending coefficients (see Equations 14.3-1 and 14.3-2). The current value of p_E must be set in accordance with the following logic:

$$\begin{aligned} p_E &= 0 && \text{when not in eclipse transition} \\ p_E &= -\frac{1}{60 t_E} && \text{when in entrance eclipse transition} \\ p_E &= +\frac{1}{60 t_E} && \text{when in exit eclipse transition} \end{aligned}$$

If the input value for the length of eclipse transition, t_E , is zero, the program should set it to four minutes. The parameter k_E is used to compute the current value of the thermal bending coefficients (see

Equations 14.3-3 and 14.3-4). The current value of k_E must be set in accordance with the following logic:

$$k_E = 1.0 \quad \text{for no eclipse and no transition}$$

$$k_E = (1 + p_E \Delta t) \quad \text{for entrance transition}$$

$$k_E = (p_E \Delta t) \quad \text{for exit transition}$$

$$k_E = 0 \quad \text{for eclipse and no transition}$$

where Δt is the time since start of transition. It is assumed that four integration steps during eclipse transition are adequate, therefore, the integration step must be maintained at $1/4 t_E$ during transition. At the start of the program initial values of α_o , α_e and a_E must be computed. If a_E is negative the program has been started in eclipse. If a_E is positive and $\alpha_e \leq \alpha \leq \alpha_o$ the spacecraft is approaching eclipse and if a_E is positive and $\alpha_o \leq \alpha \leq \alpha_e$ no further eclipse computations are made until α exceeds α_e . It will be noted that when the spacecraft enters into or exits from eclipse on the first step after the start of integration, there is no past value of a_E from which to predict the time of transient. However since entrance is assumed if a_E is negative and exit is assumed if a_E is positive, transient will occur at the end of the first step. Since the integration step size small at the start of integration this error may be neglected.

10.5 THERMAL BENDING AND DEPLOYMENT PARAMETERS

Auxiliary body i has been defined in Section 2.0 as k masses m_{ik} oriented at the tip of k arbitrary vectors $\bar{R}_{ik}$ defined in body coordinates ($1 \leq k \leq 4$). Changes in shape are expressed in terms of variations in the components of the vectors $\bar{R}_{ik}$. It is assumed that these variations result from either thermal boom bending or boom deployment which may not occur together. The nominal positions of masses m_{ik} are input as k column matrices $\begin{bmatrix} R_{ik} \end{bmatrix}$. Then the components of $\bar{R}_{ik}$ are given by

$$\begin{bmatrix} R_{ik} \end{bmatrix} = \begin{bmatrix} R_{ik} \end{bmatrix} + \begin{bmatrix} \Delta R_{ik} \end{bmatrix} \quad (10.5-1)$$

and

$$\begin{bmatrix} \dot{R}_{ik} \end{bmatrix} = \begin{bmatrix} \dot{\Delta R}_{ik} \end{bmatrix} \quad (10.5-2)$$

$$\begin{bmatrix} \ddot{R}_{ik} \end{bmatrix} = \begin{bmatrix} \ddot{\Delta R}_{ik} \end{bmatrix} \quad (10.5-3)$$

where $\begin{bmatrix} \Delta R_{ik} \end{bmatrix}$ and $\begin{bmatrix} \dot{\Delta R}_{ik} \end{bmatrix}$ are functions of position and velocity. Since $\begin{bmatrix} \ddot{\Delta R}_{ik} \end{bmatrix}$ is a function of the body i angular acceleration let

$$\begin{bmatrix} \ddot{\Delta R}_{ik} \end{bmatrix} = \begin{bmatrix} \lambda_{ik} \end{bmatrix} + \begin{bmatrix} \lambda_{ik} \end{bmatrix} \begin{bmatrix} \dot{\omega}_{ii} \end{bmatrix} \quad (10.5-4)$$

The 3 x 1 column matrix $\begin{bmatrix} \lambda_{ik} \end{bmatrix}$ contains the components of $\begin{bmatrix} \ddot{\Delta R}_{ik} \end{bmatrix}$ which are functions of position and velocity and the 3 x 3 square matrix $\begin{bmatrix} \lambda_{ik} \end{bmatrix}$ contains the coefficients of the components of $\begin{bmatrix} \dot{\omega}_{ii} \end{bmatrix}$.

The parameters required at any instant of time are:

$$\begin{bmatrix} \Delta R_{ik} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\Delta R}_{ik} \end{bmatrix}$$

$$\begin{bmatrix} \lambda_{ik} \end{bmatrix}$$

$$\begin{bmatrix} \lambda_{ik} \end{bmatrix}$$

These parameters may result from either thermal boom bending or boom deployment and are used to compute the instantaneous inertia and body momentum parameters.

10.6 BODY INERTIA AND MASS PARAMETERS

The inertia matrix for each body represents the inertia elements with respect to the body axes located at the body reference point. The diagonal elements are the mass moments of inertia about the body axes while the off diagonal elements are the negative of the products of inertia. The inertia of any reaction wheels which may be attached to the bodies is assumed to be included in the body inertia matrix. The inertia matrices

and the time rates of change of the inertia matrices are functions of positions and velocities and do not contain the unknown accelerations of the system.

10.6.1 Body 1 Mass and Inertia Parameters

The main body of the spacecraft is assumed to be a rigid body, for which the inertia and mass parameters are constant. Therefore, the mass and inertia parameters are input constants where

$$M_1 = \text{mass of body 1 in slugs}$$

$$\begin{bmatrix} I_1 \end{bmatrix} = 3 \times 3 \text{ inertia matrix for body 1 giving the 9 elements with respect to the } \begin{bmatrix} \bar{Q}_1 \end{bmatrix} \text{ coordinate system in slug-ft}^2$$

It should be noted that the components of the vector location of the center of mass of body 1, $\begin{bmatrix} v_{11} \end{bmatrix}$ are zero.

10.6.2 Auxiliary Body i Mass and Inertia Parameters ($2 \leq i \leq N$)

Each of the auxiliary bodies has been assumed to consist of k point masses supported by rods, where k may have values of 1, 2, 3, or 4.

The mass data is input as k values of m_{ik} , and the total mass of body i is given by

$$M_i = \sum_k m_{ik} = \text{slugs} \quad (10.6-1)$$

where M_i is a constant computed at the start of the program. The location of the k^{th} mass in the $\begin{bmatrix} \bar{Q}_i \end{bmatrix}$ coordinate system is given by the vector $\bar{R}_{ik}$ and the vector location of the center of mass of body from the body i reference point is

$$\bar{v}_i = \frac{1}{M_i} \sum_k m_{ik} \bar{R}_{ik} \quad (10.6-2)$$

From Equation 10.5-1

$$|v_{ii}| = \frac{1}{M_i} \sum_k m_{ik} |{}_o R_{ik}| + \frac{1}{M_i} \sum_k m_{ik} |\Delta R_{ik}| \quad (10.6-3)$$

$$|v_{ii}| = |{}_o v_{ii}| + \frac{1}{M_i} \sum_k m_{ik} |\Delta R_{ik}| \quad (10.6-4)$$

where

$$|{}_o v_{ii}| = \frac{1}{M_i} \sum_k m_{ik} |{}_o R_{ik}| \quad (10.6-5)$$

The components $|{}_o v_{ii}|$ are constants computed at the start of the program and the additional term $1/M_i \sum_k m_{ik} |\Delta R_{ik}|$ need only be computed when thermal bending or deployment is being considered.

From Equation (10.5-3)

$$|\dot{v}_{ii}| = \frac{1}{M_i} \sum_k m_{ik} |\dot{\Delta R}_{ik}| \quad (10.6-6)$$

and

$$|\ddot{v}_{ii}| = \frac{1}{M_i} \sum_k m_{ik} |\ddot{\Delta R}_{ik}|$$

From Equation 10.5-4

$$|\ddot{v}_{ii}| = \frac{1}{M_i} \sum_k m_{ik} |\lambda_{ik}| + \frac{1}{M_i} \sum_k m_{ik} |\lambda_{ik}| |\dot{\omega}_{ii}|$$

Let

$$[\lambda_i] = \frac{1}{M_i} \sum_k m_{ik} [\lambda_{ik}] \quad (10.6-7)$$

$$[\beta_i'] = \frac{1}{M_i} \sum_k m_{ik} [\lambda_{ik}] \quad (10.6-8)$$

Then

$$[\ddot{v}_{ii}] = [\lambda_i] + [\beta_i'] [\dot{\omega}_{ii}] \quad (10.6-9)$$

From Equation 3-30 of reference 1, the inertia matrix for body i with respect to the $[\bar{Q}_i]$ coordinate system with its origin at the hinge point is given by

$$[I_{iH}] = - \sum_k m_{ik} [J_{Rik}] [J_{Rik}] + \sum_k [I_{ik}] \quad (10.6-10)$$

The term $[I_{ik}]$ represents the inertia matrix for mass m_{ik} with respect to body i axes translated to its own center of mass. Provisions are made to input up to four $[I_{ik}]$ matrices for each auxiliary body. The matrix $\sum_k [I_{ik}]$ is the sum of the inertia matrices for each of the k masses m_{ik} with respect to the $[\bar{Q}_i]$ coordinate system when translated to the center of mass of m_{ik} . This provision is made in order that rigid bodies of arbitrary shape may be considered as auxiliary bodies. When considering booms, the inertia of the tip mass about its center of mass is usually small and would normally be neglected. However, if the k masses lie along a coordinate axis the matrix $[I_{iH}]$ is singular and the inverse of the system matrix cannot be found. In order to insure that $[I_{iH}]$ is non-singular the diagonal elements of $\sum_k [I_{ik}]$ are tested for zero. If zeros are found the mass M_i is assumed to be a solid steel sphere and each zero is replaced by $.03(M_i)^{5/3}$. Since the matrix $\sum_k [I_{ik}]$ is a constant matrix computed at the start of the program, this correction is made only once at the start of the program.

By differentiating Equation 10.6-10

$$\left[\dot{I}_{iH} \right] = - \sum_k m_{ik} \left[\left[J_{\Delta Rik} \right] \left[\dot{J}_{Rik} \right] + \left[J_{Rik} \right] \left[\dot{J}_{\Delta Rik} \right] \right] \quad (10.6-11)$$

The matrix $\left[\dot{I}_{iH} \right]$ occurs only as a result of changes in shape and is used to compute the dynamic torque $\left[\dot{I}_{iH} \right] \left[\omega_{ii} \right]$.

The inertia matrix for body i with respect to its center of mass is required in the computation of gravity gradient torques and may be obtained by using Equation 6-19 of reference 1.

$$\left[I_i \right] = \left[I_{iH} \right] + \left[J_{\omega ii} \right] \quad (10.6-12)$$

10.7 BODY ANGULAR MOMENTUM PARAMETERS

The torques due to body angular momentum appear in the dynamic equations as $\left[J_{\omega ii} \right] \left[H_{ii} \right]$ and $\left[\dot{H}_{ii} \right]$ and result from reaction wheel momentum and momentum due to changes in shape. From Equations 6.2-1 and 6.2-2

$$\left[J_{\omega ii} \right] \left[H_{ii} \right] = \left[J_{\omega ii} \right] \left[H_{\omega ii} \right] + \left[J_{\omega ii} \right] \left[H_{Rii} \right] \quad (10.6-13)$$

$$\left[\dot{H}_{ii} \right] = \left[\dot{H}_{\omega ii} \right] + \left[\dot{H}_{Rii} \right] \quad (10.6-14)$$

The body momentum torques due to reaction wheels, $\left[J_{\omega II} \right] \left[H_{wii} \right]$ and $\left[\dot{H}_{wii} \right]$ are computed and introduced into the equations by the control system subroutines. The angular momentum due to changes in shape occurs for the auxiliary bodies only and are given by

$$\left[H_{Rii} \right] = \sum_k m_{ik} \left[J_{Rik} \right] \left[\dot{R}_{ik} \right] = \sum_k m_{ik} \left[J_{Rik} \right] \left[\Delta \dot{R}_{ik} \right] \quad (10.6-15)$$

differentiating

$$\left| \dot{H}_{Rii} \right| = \sum_k m_{ik} \left| J_{Rik} \right| \left| \ddot{R}_{ik} \right| = \sum_k m_{ik} \left| R_{Rik} \right| \left| \Delta \ddot{R}_{ik} \right|$$

Substituting Equation (10.5-4)

$$\left| \dot{H}_{Rii} \right| = \sum_k m_{ik} \left| J_{Rik} \right| \left| \lambda_{ik} \right| + \sum_k m_{ik} \left| J_{Rik} \right| \left| \lambda_{ik} \right| \left| \dot{\omega}_{ii} \right|$$

Let

$$\left| T_{Tii} \right| = \sum_k m_{ik} \left| J_{Rik} \right| \left| \lambda_{ik} \right| \quad (10.6-16)$$

$$\left| W_i \right| = \sum_k m_{ik} \left| J_{Rik} \right| \left| \lambda_{ik} \right| \quad (10.6-17)$$

Then

$$\left| \ddot{H}_{Rii} \right| = \left| T_{Tii} \right| + \left| W_i \right| \left| \dot{\omega}_{ii} \right| \quad (10.6-18)$$

11. GRAVITY FORCES AND TORQUES

The expressions for the gravity force and torque at the center of mass of a rigid body have been presented in a number of published works on gravity gradient systems. Since the forces and torques are functions of positions only, these expressions are also applicable to flexible bodies if the instantaneous position and shape parameters are used. However, when the effects of the gravity forces in producing changes in shape of a flexible body are considered, the forces and moments at points within the body are needed, and additional computations are required. The flexible spacecraft being considered here, consists of N flexibly connected bodies.

It has been assumed that the center of mass of the total spacecraft is in an elliptic earth orbit defined by equating the gravity force to the radial centrifugal force. If all of the mass M_T were concentrated at the center of mass of the system, the centrifugal force would be $K_N \omega_o^2 S_M M_T$ and the gravity force would be $-K_N \gamma M_T \bar{S}_M / S_M^2$, (Figure 11.0-1). However, the mass M_T occupies a finite volume, and the mass elements dm are in a varying centrifugal force field and a varying gravitational force field. The centrifugal force field is linear, the magnitude of the force at any point is a linear function of the distance from the center of mass of the earth, therefore, the net moment at the center of mass is zero. In addition, the principle of superposition is valid for the centrifugal field and the change in forces and moments between points within the body are linear functions of the change in the distance from the system center of mass. The gravitation field, on the other hand, is not linear, the magnitude of the force at any point varies inversely as the square of the distance from the center of the earth, therefore, the net moment at the center of mass is not zero. In addition the principle of superposition is not valid, and the forces at any point within the body must be based upon the distance from the total spacecraft center of mass.

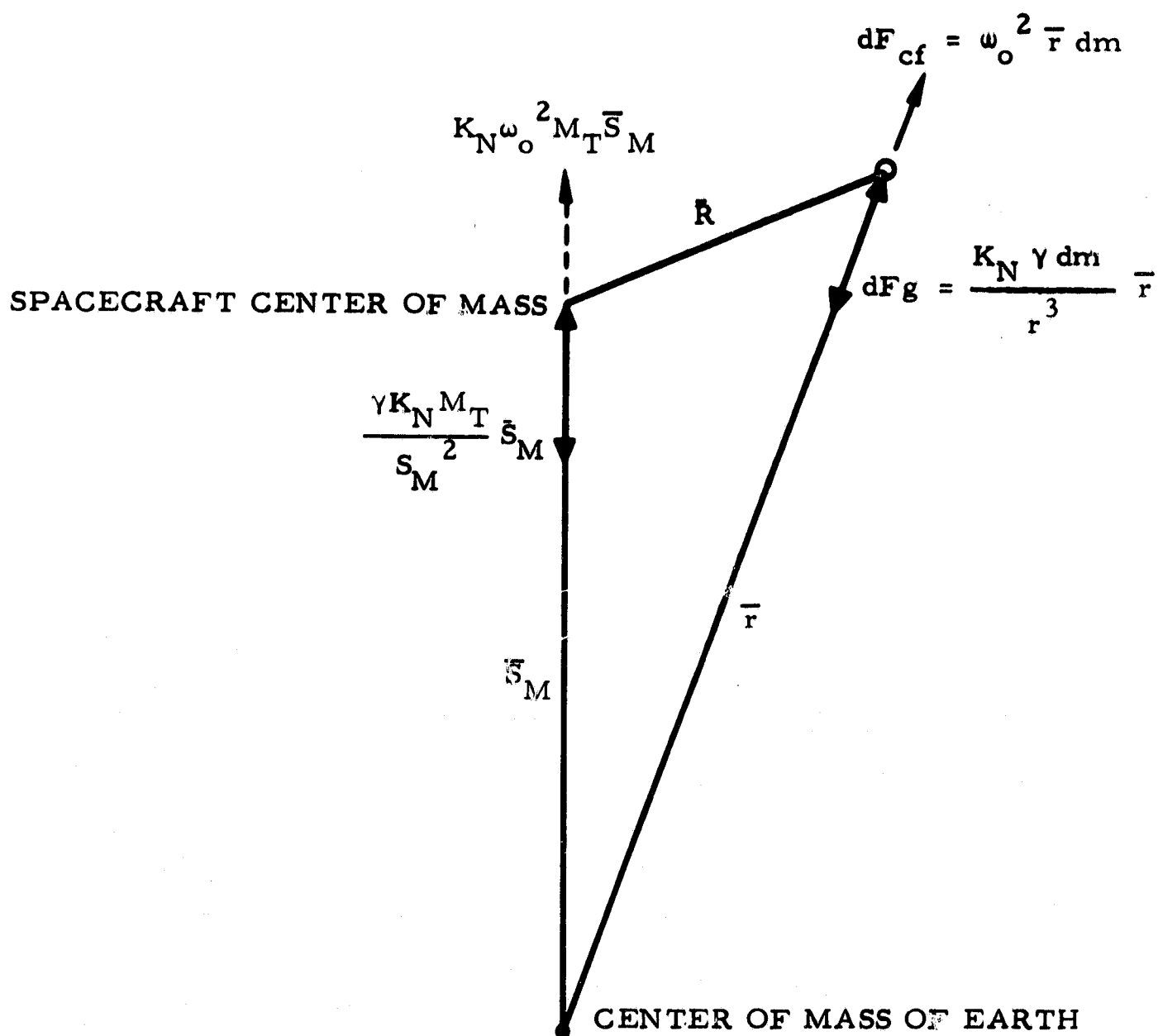


Figure 11.0-1 Inertial Position of Mass Element

Consider a portion of the spacecraft designated as body i with mass M_i and flexibly attached to the remainder of the mass. The center of mass of body i may have an orbital radius different than that of the total spacecraft center of mass, however, the angular orbit rate ω_o is the same and the gravitational forces do not balance the centrifugal forces. The internal forces and torques required to keep all elements of the spacecraft in the same orbit will cancel over the total system and the normal rigid body equations are valid, however, these forces and torques are effective in producing relative displacements between the spacecraft elements. The gravity force and torque equations for body i attached to another body will be derived in this section in order to determine the relative magnitude of these internal forces and torques.

11.1 GRAVITY FORCE EQUATIONS

The gravity force on an element of mass dm is given by

$$d\bar{F}_g = - \frac{K_N \gamma dm}{r^3} \bar{r} \quad (11.1-1)$$

where γ is the earth gravitational constant and r is the distance from the center of the earth to dm .

$$r = \bar{r}$$

from Figure 11.0-1

$$\bar{r} = \bar{S}_M + \bar{R}$$

$$r^2 = \bar{r} \cdot \bar{r} = S_M^2 + 2 (\bar{S}_M \cdot \bar{R}) + R^2$$

where

$$S_M = |\bar{S}_M|$$

$$R = |\bar{R}|$$

Then

$$\frac{r^2}{S_M^2} = 1 + \frac{2 (\bar{S}_M \cdot \bar{R})}{S_M^2} + \frac{R^2}{S_M^2}$$

and

$$\frac{1}{r^3} = \frac{1}{S_M^3} \left(\frac{r^2}{S_M^2} \right)^{-\frac{3}{2}} = \frac{1}{S_M^3} \left(1 + \frac{2 (\bar{S}_M \cdot \bar{R})}{S_M^2} + \frac{R^2}{S_M^2} \right)^{-\frac{3}{2}} \quad (11.1-2)$$

In general $R^2 \ll S_M^2$. A nominal value of R/S_M is 5×10^{-6} and R^2/S_M^2 is 25×10^{-12} , therefore, in comparison to unity the term R^2/S_M^2 may be neglected. Expanding the term $(1 + 2(\bar{S}_M \cdot \bar{R})/S_M^2)^{-3/2}$ into a binomial series and letting

$$\frac{R^2}{S_M^2} = 0 \quad (11.1-3)$$

$$\frac{1}{r^3} \approx \frac{1}{S_M^3} \left(1 - \frac{3 (\bar{S}_M \cdot \bar{R})}{S_M^2} \right) \quad (11.1-4)$$

Substituting Equation 11.1-4 and $r = S_M + \bar{R}$ into Equation 11.1-1, the expression for the gravity force on mass element dm becomes

$$d\bar{F}_g = - \frac{K_N \gamma}{S_M^3} (\bar{S}_M + \bar{R}) dm + \frac{3K_N \gamma}{S_M^5} (\bar{S}_M \cdot \bar{R}) (\bar{S}_M + \bar{R}) dm \quad (11.1-5)$$

Consider some portion of the spacecraft for which the gravity force and torque expressions are desired. Designate the portion as body i and let the mass be M_i and the mass of the total spacecraft be M_T . If the vector location of the center of mass of body i with respect to the total center of mass is $\bar{R}_{Mi}$, it may be seen from Figure 11.1-1 that

$$\int_{M_i} \bar{\zeta}_i dm = 0 \quad (11.1-6)$$

$$\sum_{i=1}^N \bar{R}_{Mi} M_i = 0 \quad (11.1-7)$$

$$\bar{R} = \bar{R}_{Mi} + \bar{\zeta}_i \quad (11.1-8)$$

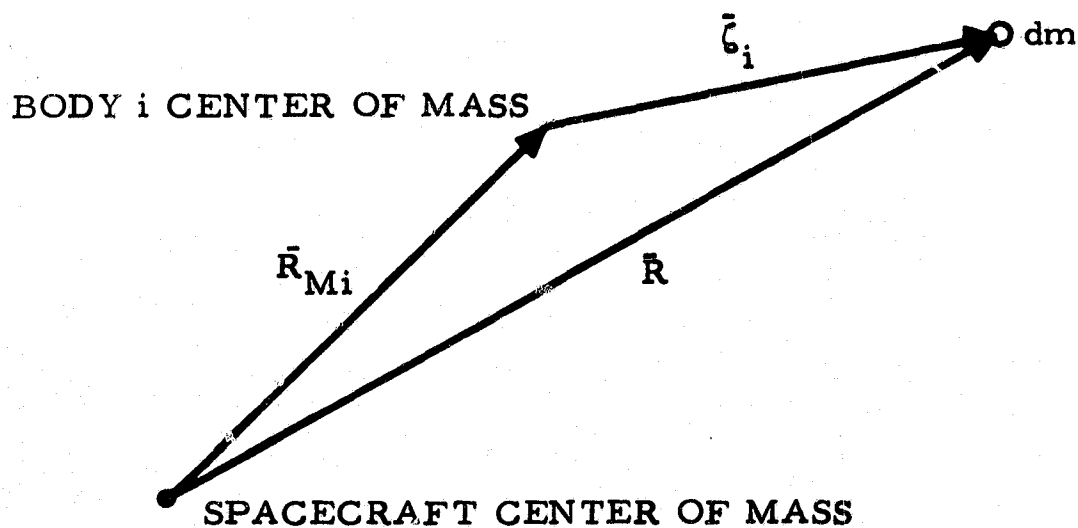


Figure 11.1-1 Relative Position of Mass Element

Substituting Equation 11.1-8 into 11.1-5, the gravity force on mass element dm in body i is

$$\begin{aligned} d\bar{F}_{gi} = & -\frac{K_N \gamma}{S_M} (\bar{S}_M + \bar{R}_{Mi} + \bar{\zeta}_i) dm \\ & + \frac{3K_N \gamma}{S_M} \left[(\bar{S}_M \cdot \bar{R}_{Mi})(\bar{S} + \bar{R}_{Mi} + \bar{\zeta}_i) + (\bar{S}_M \cdot \bar{\zeta}_i)(\bar{S} + \bar{R}_{Mi} + \bar{\zeta}_i) \right] dm \end{aligned} \quad (11.1-9)$$

Integrating over the mass of body i , the gravity force on body i is

$$\bar{F}_{gi} = -\frac{K_N \gamma}{S_M} (\bar{S} + \bar{R}_{Mi}) M_i + \frac{3K_N \gamma}{S_M} \left[(\bar{S} \cdot \bar{R}_{Mi})(\bar{S} + \bar{R}_{Mi}) M_i + \int_{Mi} (\bar{S} \cdot \bar{\zeta}_i) \bar{\zeta}_i dm \right] \quad (11.1-10)$$

From the triple cross product expansion $\bar{A} \times (\bar{A} \times \bar{B})$

$$(\bar{B} \cdot \bar{A}) \bar{A} = \bar{A} \times (\bar{A} \times \bar{B}) + A^2 \bar{B}$$

Then

$$\int_{Mi} (\bar{S} \cdot \bar{\zeta}_i) \bar{\zeta}_i dm = \int_{Mi} \bar{\zeta}_i \times (\bar{\zeta}_i \times \bar{S}) dm + \bar{S} \int_{Mi} \zeta_i^2 dm \quad (11.1-11)$$

Let

$$L_i = K_N^2 \int_{Mi} \zeta_i^2 dm = \frac{1}{2} (I_{xx} + I_{yy} + I_{zz}) \quad (11.1-12)$$

L_i is a constant for a given rigid body and is independent of the orientation of the coordinate system. It is equal to one half the sum of the diagonal elements of the body i inertia matrix

$$\int_{Mi} \bar{\zeta}_i \times (\bar{\zeta}_i \times \bar{S}_M) dm = \frac{1}{K_N} \frac{1}{2} (\bar{I}_i \cdot \bar{S}_M) \quad (11.1-13)$$

where the inertia elements are with respect to the center of mass of body i.

Then

$$\int_{M_i} (\bar{S}_M \cdot \bar{\xi}_i) \bar{\xi}_i dm = \frac{1}{K_N} \left[L_i \bar{S}_M - \bar{I}_i \cdot \bar{S} \right] \quad (11.1-14)$$

The gravity force Equation 11.1-10 then becomes

$$\begin{aligned} \bar{F}_{gi} = & - \frac{K_N \gamma M_i}{S_M^3} \bar{S}_M - \frac{\gamma M_i}{S_M^3} \bar{R}_{Mi} + \frac{3K_N \gamma M_i}{S_M^5} (\bar{S}_M \cdot \bar{R}_{Mi}) (\bar{S}_M + \bar{R}_{Mi}) \\ & + \frac{3K_N \gamma}{K_N^2 S_M^5} \left[L_i \bar{S}_M - \bar{I}_i \cdot \bar{S} \right] \end{aligned} \quad (11.1-15)$$

The first term in Equation 11.1-15 is the gravity force that would exist if the mass M_i were concentrated at the total spacecraft center of mass. The second and third terms, containing $\bar{R}_{Mi}$, represent the additional force due to the fact that the center of mass of body i is not in the same orbit as the total center of mass. The fourth term, not containing $\bar{R}_{Mi}$, is due to the finite size of body i.

Force Equation 11.1-15 may be written as

$$\bar{F}_{gi} = -g_o M_i \left[\left(1 - 3 \frac{\bar{S}_M \cdot \bar{R}_{Mi}}{S_M^2} \right) \left(\frac{\bar{S}_M}{S_M} + \frac{\bar{R}_{Mi}}{S_M} \right) + \frac{3 \left[L_i \frac{\bar{S}_M}{S_M} - \bar{I}_i \cdot \frac{\bar{S}_M}{S_M} \right]}{M_i K_N^2 S_M^2} \right] \quad (11.1-16)$$

The second term in the brackets is approximately three times the radius of gyration of body i divided by the orbital radius squared. This term is less than R^2/S_M^2 which has been neglected in comparison to unity, Equation (11.1-3). Therefore, the additional force due to the finite size of body i will be neglected. The vector $\bar{R}_{Mi}$ will normally have a magnitude of less than 50 feet while the orbital radius has a minimum magnitude of about 3500 nautical miles or 20×10^6 feet. The magnitude of $\frac{R_{Mi}}{S_M}$ is of

the order of magnitude of 10^{-6} which will be neglected in comparison to unity. In other words the assumption that the orbital radius of body i is the same as the orbital radius of the total spacecraft produces negligible errors for size of spacecraft to be considered. The gravity force equation may then be taken as

$$\bar{F}_{gi} \approx - \frac{g_o M_i}{S_M} \bar{S}_M \quad (11.1-17)$$

11.2 GRAVITY TORQUE EQUATION

Let $\bar{T}_{gi}$ be the gravity torque on body i with respect to the body i center of mass. The torque on mass element dm is then

$$d\bar{T}_{gi} = K_N \bar{\xi}_i \times d\bar{F}_{gi} \quad (\# - \text{feet})$$

Substituting Equation 11.1-9 and integrating over the mass M_i

$$\bar{T}_{gi} = \frac{3K_N^2 \gamma}{S_M^5} \left[\int_{M_i} (\bar{S}_M \cdot \bar{\xi}_i) \bar{\xi}_i dm \right] \times (\bar{S}_M + \bar{R}_{Mi}) \quad (11.2-1)$$

using Equation 11.1-14

$$\bar{T}_{gi} = \frac{3\gamma}{S_M^5} \bar{S}_M \times (\bar{I}_i \cdot \bar{S}_M) + \frac{3\gamma}{S_M^5} \bar{R}_{Mi} \times \left[\bar{I}_i \cdot \bar{S}_M - L_i \bar{S}_M \right] \quad (11.2-2)$$

The first term in Equation 11.2-2 is the gravity gradient torque due to the finite size of body i , while the second term is the additional torque due the difference in orbital radii.

The torque equation may be written as

$$\bar{T}_{gi} = \frac{3g_o}{K_N S_M} \left[\left(\frac{\bar{S}_M}{S_M} + \frac{\bar{R}_{Mi}}{S_M} \right) \times \left(\bar{I}_i \cdot \frac{\bar{S}_M}{S_M} \right) - L_i \left(\frac{\bar{S}_M}{S_M} \times \frac{\bar{R}_{Mi}}{S_M} \right) \right] \quad (11.2-3)$$

Neglecting $\bar{R}_{Mi}/S_M$ as in the force equation

$$\bar{T}_{gi} = \frac{3g_o}{K_N S_M} \bar{S}_M \times (\bar{I}_i \cdot \bar{S}_M) \quad (11.2-4)$$

11.3 MATRIX NOTATION

From Equation 9.3-8 the components of the vector from the center of the earth to the spacecraft center of mass in orbital reference coordinates are

$$\{S_{MR}\} = -S_M \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

and in body i coordinates

$$\{S_{MI}\} = -S_M [D_i] [A] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad (11.3-1)$$

Let

$$[F_i] = [D_i] [A] \quad (11.3-2)$$

then

$$\{S_{Mi}\} = -S_M \{F_{i3}\} \quad (11.3-3)$$

where S_M is the magnitude of $\bar{S}_M$ and $\{F_{i3}\}$ is a column matrix formed from the third column of $[F_i]$, $[D_i]$ having been defined as $[U]$ for body 1.

The gravity force equation for body i, 11.1-17, now may be written as

$$\{F_{gii}\} = g_o M_i \{F_{i3}\} = g_o M_i [D_i] [A] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad (11.3-4)$$

The gravity gradient torque equation for body i, 12.2-4, now may be written as

$$\{T_{gii}\} = \frac{3g_o}{K_N S_M} [J_{Fi3}] [I_i] \{F_{i3}\} \quad (11.3-5)$$

where $[I_i]$ is the inertia matrix of body i with respect to a set of body axes translated to the center of mass of body i.

12. DYNAMIC EQUATIONS IN BODY COORDINATES

The dynamic equations in terms of body coordinates are developed in this section. As noted in Section 4.0, the motion of N bodies is defined by N matrix moment equations and N matrix force equations. Since the (N-1) auxiliary bodies are attached to the main body, (N-1) matrix force equations were eliminated in Section 6.3 by equating the reaction forces at the attach points. The remaining matrix force equation, describing the motion of the total spacecraft center of mass, was used in Section 8.0 to define the orbit equations and in Section 9.0 to develop the inertial translational accelerations. The remaining N matrix moment equations are the dynamic equations describing the motion of the N bodies with respect to a reference system at the total center of mass.

12.1 MATRIX MOMENT EQUATION FOR MAIN BODY

The basic moment equation for body i is given by Equation 4.1-1 as

$$\begin{aligned} \{T_{Eii}\} = & M_i [J_{vii}] \{G_{nii}\} + [I_{iH}] \{\dot{\omega}_{ii}\} + [J_{\omega ii}] [I_{iH}] \{\omega_{ii}\} \\ & + [I'_{iH}] \{\omega_{ii}\} + [J_{\omega ii}] \{H_{ii}\} + \{H'_{ii}\} \end{aligned}$$

for the main body ($i = 1.0$) the reference point is at the body center of mass, therefore,

$$\{v_{11}\} = 0 \quad (12.1-1)$$

$$[I_{iH}] = [I_1] \quad (12.1-2)$$

In addition the body is rigid, therefore,

$$[I'_{iH}] = [\dot{I}_i] = [0] \quad (12.1-3)$$

From Equations 6.2-1 and 6.2.2 the body momentum results only from reaction wheel momentum, and

$$\{H_{11}\} = \{H_{w11}\} \quad (12.1-4)$$

$$\{\dot{H}_{11}\} = \{\dot{H}_{w11}\} \quad (12.1-5)$$

The body 1 matrix moment equation reduces to

$$\{T_{E11}\} = [I_1] \{\dot{\omega}_{11}\} + [J_{\omega 11}][I_1] \{\omega_{11}\} + [J_{\omega 11}] \{H_{w11}\} + \{\dot{H}_{w11}\} \quad (12.1-6)$$

From Equation (6.1-4) the external torques on the main body are given by

$$\begin{aligned} \{T_{E11}\} = & \{T_{g11}\} + \{T_{S11}\} + \{T_{A11}\} + \{T_{M11}\} + \{T_{C11}\} \\ & + \sum_{i=2}^N [J_{ui1}][D_i]_T \{F_{Rii}\} + \sum_{i=2}^N [D_i]_T \{T_{Hii}\} \end{aligned}$$

where from Equation 6.3-2

$$\{F_{Rii}\} = \{F_{gii}\} + \{F_{Sii}\} + \{F_{Aii}\} - M_i \{G_{ii}\}$$

and from Equation 7.2-2

$$\{T_{Hii}\} = [G_i]_T^{-1} \{T_{gi}\}$$

The body 1 matrix moment equation may now be written as

$$\begin{aligned} & \{T_{g11}\} + \{T_{S11}\} + \{T_{A11}\} + \{T_{M11}\} + \{T_{C11}\} + \sum_{i=2}^N [J_{ui1}] \left\{ [D_i]_T \{F_{gii}\} - M_i \{G_{ii}\} \right\} \\ & + \sum_{i=2}^N [J_{ui1}][D_i]_T \left\{ \{F_{Sii}\} + \{F_{Aii}\} \right\} + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} \{T_{gi}\} \\ & - [I_1] \{\dot{\omega}_{11}\} - [J_{\omega 11}][I_1] \{\omega_{11}\} - [J_{\omega 11}] \{H_{w11}\} - \{\dot{H}_{w11}\} = 0 \end{aligned} \quad (12.1-7)$$

12.1.1 Translational Accelerations of Body i

The translational acceleration components of the center of mass and of the reference point for each body were derived in general form in Section 5.0. Repeating Equations 5.5-7 through 5.5-14 for reference

$$(5.5-7) \{G_{n1}\} = \{G_{R1}\} + \{H_{u1}\}$$

$$(5.5-8) \{G_{i1}\} = \{G_{R1}\} + \{H_{pi1}\}$$

$$(5.5-9) \{G_{R1}\} = [A] \{G_{MR}\} - \{H_{M1}\}$$

$$(5.5-10) \{H_{M1}\} = \sum_{i=2}^N \frac{M_i}{M_T} \{H_{pi1}\}$$

$$(5.5-11) \{G_{n1}\} = \{G_{11}\} = \{G_{R1}\}$$

$$(5.5-12) \{H_{pi1}\} = \{H_{ui1}\} + [D_i]_T \{H_{vii}\}$$

$$(5.5-13) \{H_{ui1}\} = [J_{\omega 11}][J_{\omega 11}] \{u_{i1}\} - [J_{ui1}] \{\dot{\omega}_{11}\}$$

$$(5.5-14) \{H_{vii}\} = \{\ddot{V}_{ii}\} + 2[J_{\omega ii}]\{\dot{v}_{ii}\} + [J_{\omega ii}][J_{\omega ii}] \{v_{ii}\} - [J_{vii}] \{\dot{\omega}_{ii}\}$$

From Equation 10.6-9 the components of the acceleration of the center of mass of body in body coordinates are

$$\{\ddot{v}_{ii}\} = \{\lambda_i\} + [\beta_i'] \{\dot{\omega}_{ii}\}$$

Substituting 10.6-9 in 5.5-14 and then 5.5-13 and 5.5-14 into 5.5-12, the components of the acceleration of body i center of mass relative to the body 1 center of mass are

$$\begin{aligned} \{H_{pil}\} = & [J_{\omega 11}][J_{\omega 11}]\{u_{i1}\} - [J_{ui1}]\{\dot{\omega}_{i1}\} + [D_i]_T \left\{ \{\lambda_i\} - [\beta_i]\{\dot{\omega}_{ii}\} \right. \\ & \left. + 2[J_{\omega ii}]\{\dot{v}_{ii}\} + [J_{\omega i1}][J_{\omega ii}]\{v_{ii}\} - [J_{vii}]\{\dot{\omega}_{ii}\} \right\} \end{aligned} \quad (12.1-8)$$

The expressions for $\{H_{pil}\}$ contain terms which are functions of only position and velocity and terms which contain the unknown accelerations of the independent variables. Referring to Section 5.3 and Equation 5.3-2 the inertial angular accelerations have been defined as

$$\{\dot{\omega}_{i1}\} = \{N\dot{\omega}_{i1}\} + \{A\dot{\omega}_{i1}\}$$

$$\{\dot{\omega}_{ii}\} = \{N\dot{\omega}_{ii}\} + \{A\dot{\omega}_{ii}\}$$

where the prescript N denotes terms which are not a function of the unknowns while the prescript A denotes terms containing these unknowns. Then

$$\{A^H_{pil}\} = -[J_{ui1}]\{A\dot{\omega}_{i1}\} + [D_i]_T \left[[B'_i] - [J_{vii}] \right] \{A\dot{\omega}_{ii}\} \quad (12.1-9)$$

$$\begin{aligned} \{N^H_{pil}\} = & [J_{\omega 11}][J_{\omega 11}]\{u_{i1}\} - [J_{ui1}]\{N\dot{\omega}_{i1}\} + \\ & [D_i]_T \left\{ \{\lambda_i\} + \left[[B'_i] - [J_{vii}] \right] \{N\dot{\omega}_{ii}\} + 2[J_{\omega ii}]\{\dot{v}_{ii}\} + [J_{\omega ii}][J_{ui1}]\{v_{ii}\} \right\} \end{aligned} \quad (12.1-10)$$

Letting $\{H^H_{pil}\} = k_{i1}$

where $\{K_{i1}\}$ is the symbol used for this term in the digital program, using

$$[D_i]_T [J_{\Omega ii}] [D_i] = [J_{\Omega i1}]$$

$$[J_{\omega 11}] = [J_{\omega i1}] - [J_{\Omega i1}]$$

$$\{N^{\omega i1}\} = [D_i]_T \{N^{\omega ii}\} + [J_{\omega i1}] \{\omega_{11}\}$$

$$\begin{aligned} \{k_{i1}\} = & \left[[D_i]_T [B_i] [D_i] - [J_{pi1}] \right] \left\{ [D_i]_T \{N^{\omega ii}\} \right\} \\ & + 2[J_{\omega ii}] \left\{ \{v_{i1}\} - [J_{\Omega i1}] \{u_{i1}\} \right\} + [J_{\omega i1}] [J_{\omega i1}] \{p_{i1}\} \\ & + [D_i]_T \{\lambda_i\} + [J_{\Omega i1}] [J_{\Omega i1}] \{u_{i1}\} \end{aligned} \quad (12.1-11)$$

Then

$$\{H_{pi1}\} = \{k_{i1}\} + \{A^H_{pi1}\}$$

and from Equation 5.5-10

$$\{H_{M1}\} = \sum_{i=2}^N \frac{M_i}{M_T} \{k_{i1}\} + \sum_{i=2}^N \frac{M_i}{M_T} \{A^H_{pi1}\}$$

Let

$$\sum_{i=2}^N \frac{M_i}{M_T} \{k_{i1}\} = \{k_{T1}\} \quad (12.1-12)$$

Then

$$\{H_{M1}\} = \{k_{T1}\} + \sum_{i=2}^N \frac{M_i}{M_T} \{A^H_{pi1}\} \quad (12.1-13)$$

and from 5.5-9

$$\{G_{R1}\} = [A]\{G_{MR}\} - \{k_{T1}\} - \sum_{i=2}^N \frac{M_i}{M_T} \{A^H_{pi1}\}$$

and from 9.3-5

$$[A]\{G_{MR}\} = g_o [A] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \frac{1}{M_T} \{F_{C11}\} + \frac{1}{M_T} \sum_{i=1}^N [D_i]_T \{F_{Sii}\} + \{F_{Aii}\}$$

Then

$$\begin{aligned} \{G_{R1}\} &= g_o [A] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \frac{1}{M_T} \{F_{C11}\} + \frac{1}{M_T} \sum_{i=1}^N [D_i]_T \{F_{Sii}\} + \{F_{Aii}\} - \{k_{T1}\} \\ &\quad - \sum_{i=2}^N \frac{M_i}{M_T} \{A^H_{pi1}\} \end{aligned}$$

Let

$$\{F\} = \frac{1}{M_T} \{F_{C11}\} + \sum_{i=1}^N [D_i]_T \{F_{Sii}\} + \{F_{Aii}\} - \{k_{T1}\} \quad (12.1-14)$$

then

$$\{G_{R1}\} = g_o [A] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \{F\} - \sum_{i=2}^N \frac{M_i}{M_T} \{A^H_{pi1}\} \quad (12.1-15)$$

and from Equation 5.5-8

$$\{G_{i1}\} = g_o [A] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \{F\} + \{k_{i1}\} - \sum_{i=2}^N \frac{M_i}{M_T} \{A^H_{pi1}\} + \{A^H_{pi1}\} \quad (12.1-16)$$

From Equation 5.5-13 and 5.5-7

$$\{N^H_{uil}\} = [J_{\omega 11}][J_{\omega 11}]\{u_{i1}\} - [J_{ui1}]\{N^{\omega}_{11}\} \quad (12.1-17)$$

$$\{A^H_{uil}\} = -[J_{ui1}]\{A^{\omega}_{11}\} \quad (12.1-18)$$

and

$$\{G_{ni1}\} = g_o [A] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \{F\} + \{N^H_{uil}\} - \sum_{i=2}^N \frac{M_i}{M_T} \{A^H_{pi1}\} + \{A^H_{uil}\} \quad (12.1-19)$$

12.1.2 Main Body Driving Torques in Body Coordinates

Returning to the main body moment Equation 12.1-6 and consider the term

$$\sum_{i=2}^N [J_{ui1}][D_i]_T \{F_{gii}\} - M_i \{G_{i1}\}$$

From Equation 11.3-4 the gravity force on body i is

$$\{F_{gii}\} = M_i g_o [D_i] [A] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

Then using Equation 12.1-15

$$\begin{aligned} \sum_{i=2}^N [J_{ui1}][D_i]_T \{F_{gii}\} - M_i \{G_{i1}\} &= - \sum_{i=2}^N M_i [J_{ui1}] \{F\} + \{k_{i1}\} \\ &+ \sum_{i=2}^N M_i [J_{ui1}] \left\{ \sum_{i=2}^N \frac{M_i}{M_T} \{A^H_{pi1}\} - \{A^H_{pi1}\} \right\} \end{aligned} \quad (12.1-20)$$

Substituting Equation 12.1-20 into 12.1-7 the main body moment equation in body coordinates is

$$\begin{aligned} [I_1] \{\dot{\omega}_{11}\} - \sum_{i=2}^N M_i [J_{ui1}] \left\{ \sum_{i=2}^N \frac{M_i}{M_T} \{A_{pil}^H\} - \{A_{pil}^H\} \right\} \\ = \{q_{11}\} + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} \{T_{gi}\} \end{aligned} \quad (12.1-21)$$

where

$$\begin{aligned} \{q_{11}\} = - [J_{\omega 11}] [I_1] \{\omega_{11}\} - [I_1] \{N \dot{\omega}_{11}\} - \sum_{i=2}^N M_i [J_{ui1}] \left\{ \{F\} + \{k_{i1}\} \right\} \\ + \{T_{g11}\} + \{T_{S11}\} + \{T_{A11}\} + \{T_{M11}\} + \{T_{C11}\} \\ + \sum_{i=2}^N [J_{ui1}] [D_i]_T \left\{ \{F_{Sii}\} + \{F_{Aii}\} \right\} - [J_{\omega 11}] \{H_{w11}\} - \{H_{w11}^{\cdot}\} \end{aligned} \quad (12.1-22)$$

The components of $\{q_{11}\}$ are the main body driving torques in body coordinates (torques about body 1 axes) and are functions of position and velocity.

12.1.3 Coefficients of Unknowns in Main Body Equations

The unknown of left side of Equation 12.1-21 will now be expanded. From Equations 5.3-3 , 5.3-6 and 12.1-8 ,

$$\{\dot{\omega}_{11}\} = \{\dot{\Omega}_{11}^{\circ}\}$$

$$\{A_{ii}^{\omega}\} = [D_i] \{\dot{\Omega}_{11}^{\circ}\} + \{\dot{\Omega}_{ii}^{\circ}\}$$

and

$$\{A_{pil}^H\} = \left[[D_i]_T [\beta_i'] [D_i] - [J_{pil}] \right] \{\dot{\Omega}_{11}^{\circ}\} + [D_i]_T \left[[\beta_i'] - [J_{vii}] \right] \{\dot{\Omega}_{ii}^{\circ}\} \quad (12.1-23)$$

Letting

$$\sum_{i=2}^N \frac{M_i}{M_T} [D_i]_T [\beta_i'] [D_i] = [R_{11}] \quad (12.1-24)$$

and from (5.4-6)

$$\sum_{i=2}^N \frac{M_i}{M_T} \{p_{i1}\} = \{R_{M1}\}$$

Then

$$\begin{aligned} \sum_{i=2}^N \frac{M_i}{M_T} \{A^H_{pi1}\} &= \left[[R_{11}] - [J_{RM1}] \right] \{\dot{\Omega}'_{11}\} \\ &+ \sum_{i=2}^N \frac{M_i}{M_T} [D_i]_T \left[[\beta_i'] - [J_{vii}] \right] \{\dot{\Omega}'_{ii}\} \quad (12.1-25) \end{aligned}$$

and

$$\sum_{i=2}^N \frac{M_i}{M_T} \{A^H_{pi1}\} - \{A^H_{pi1}\} = \left[[R_{11}] - [J_{RM1}] + [J_{pi1}] - [D_i]_T [\beta_i'] [D_i] \right] \{\dot{\Omega}'_{11}\} + \left[\sum_{i=2}^N \frac{M_i}{M_T} [D_i]_T \left[[\beta_i'] - [J_{vii}] \right] - [D_i]_T \left[[\beta_i'] - [J_{vii}] \right] \right] \{\dot{\Omega}'_{ii}\}$$

(12.1-26)

The main body moment equation may now be written as

$$[L_{11}] \{\dot{\Omega}'_{11}\} + \sum_{j=2}^N [L_{ij}] \{\dot{\Omega}'_{jj}\} = \{q_{11}\} + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} \{T_{gi}\} \quad (12.1-27)$$

where

$$[L_{11}] = [I_1] + \sum_{i=2}^N M_i [J_{ui1}] \left[[J_{RM1}] - [J_{pi1}] + [D_i]_T [\beta'_i] [D_i] - [R_{11}] \right] \quad (12.1-28)$$

$$[L_{ij}] = M_j \left[[J_{uj1}] - [J_{uT1}] \right] [D_j]_T \left[[\beta'_j] - [J_{vjj}] \right] \quad (12.1-29)$$

$$\{u_{T1}\} = \frac{1}{M_T} \sum_{j=2}^N M_j \{u_{j1}\} \quad (12.1-30)$$

12.2 MATRIX MOMENT EQUATION FOR AUXILIARY BODIES

The basic torque equation for body i is given by Equation (4.1-1) as

$$\begin{aligned} \{T_{Eii}\} &= M_i [J_{vii}] \{G_{nii}\} + [I_{iH}] \{\dot{\omega}_{ii}\} + [J_{\omega ii}] [I_{iH}] \{\omega_{ii}\} \\ &+ [I'_{iH}] \{\omega_{ii}\} + [J_{\omega ii}] \{H_{ii}\} + \{H'_{ii}\} \end{aligned}$$

From Equation 6.2-1 and 5.2-2 the body momentum results from reaction wheels and changes in shape.

$$\{H_{ii}\} = \{H_{wii}\} + \{H_{Rii}\} \quad (12.2-1)$$

$$\{H'_{ii}\} = \{H'_{wii}\} + \{H'_{Rii}\} \quad (12.2-2)$$

and from Equation 10.6-18

$$\{H'_{Rii}\} = \{T_{Tii}\} + [W_i] \{\dot{\omega}_{ii}\} \quad (12.2-3)$$

From Equation 6.8-8 the external torques on auxiliary body i are given by

$$\{T_{Eii}\} = [J_{vii}]\{F_{gii}\} + \{T_{gii}\} + \{T_{Sii}\} + \{T_{Aii}\} + \{T_{Mii}\} - \{T_{Hii}\} \quad (12.2-4)$$

where from Equation 7.2.2

$$\{T_{Hii}\} = [G_i]_T^{-1} \{T_{gi}\} \quad (12.2-5)$$

The body i matrix torque equation may now be written as

$$\begin{aligned} & \{T_{gii}\} + \{T_{Sii}\} + \{T_{Aii}\} + \{T_{Mii}\} - [G_i]_T^{-1} \{T_{gi}\} + [J_{vii}] \{F_{gii}\} \\ & - M_i [D_i] \{G_{nil}\} - [I_{iH}] \{\dot{\omega}_{ii}\} - [J_{\omega ii}] [I_{iH}] \{\omega_{ii}\} - [I_{iH}^*] \{\omega_{ii}\} \\ & - [J_{\omega ii}] \{H_{wii}\} - \{H_{wii}^*\} - [J_{\omega ii}] \{H_{Rii}\} - \{T_{Tii}\} - [W_i] \{\dot{\omega}_{ii}\} = 0 \end{aligned} \quad (12.2-6)$$

12.2.1 Auxiliary Body Driving Torques

The torques due to the translational acceleration of the body i reference point and due to the gravity force on the body are given in Equation 12.2-6 by the term

$$[J_{vii}] \{F_{gii}\} - M_i [D_i] \{G_{nil}\}$$

Substituting Equations 11.3-4, 12.1-19, and 12.1-17

$$\begin{aligned} [J_{vii}] \{F_{gii}\} - M_i [D_i] \{G_{nil}\} &= -M_i [D_i] \{F\} + [J_{\omega 11}] [J_{\omega 11}] \{u_{i1}\} \\ &- [J_{uil}] \{N_{ii}^{\omega}\} + M_i [D_i] \left\{ \sum_{i=2}^N \frac{M_i}{M_T} \{A_{pil}^H\} \right. \\ &\left. - \{A_{uil}^H\} \right\} \end{aligned}$$

Let

$$\{F_{il}\} = \{F\} + [J_{\omega 11}][J_{\omega 11}]\{\omega_{il}\} - [J_{uil}]\{N\omega_{il}\}$$

Then

$$\{F_{ii}\} = [D_i]\{F_{il}\}$$

and

$$\begin{aligned} [J_{vii}]\left\{\{F_{gii}\} - M_i [D_i]\{G_{nil}\}\right\} &= -M_i [J_{vii}]\{F_{ii}\} \\ &+ M_i [J_{vii}][D_i] \left\{\sum_{i=2}^N \frac{M_i}{M_T} \{A^{H_{pil}}\} - \{A^{H_{uil}}\}\right\} \end{aligned}$$

(12.2-7)

Substituting Equation 12.2-7 into the body i Equation 12.2-6 and letting

$$\begin{aligned} \{q_{ii}\} &= -[I_{iH}]\{\omega_{ii}\} - [J_{\omega ii}][I_{iH}]\{\omega_{ii}\} - [I_{iH}]\{N\omega_{ii}\} - M_i [J_{vii}]\{F_{ii}\} \\ &- [J_{\omega ii}]\{H_{wii}\} - \{H_{wii}\} - [J_{\omega ii}]\{H_{Rii}\} - \{T_{Tii}\} - [W_i]\{N\omega_{ii}\} \\ &+ \{T_{gii}\} + \{T_{Sii}\} + \{T_{Aii}\} + \{T_{Mii}\} \end{aligned} \quad (12.2-8)$$

the body i moment equation becomes

$$[I_{iH}]\{A\omega_{ii}\} + [W_i]\{A\omega_{ii}\} - M_i [J_{vii}][D_i] \left\{\sum_{i=2}^N \frac{M_i}{M_T} \{A^{H_{pil}}\} - \{A^{H_{uil}}\}\right\} = \{q_{ii}\} - [G_i]_T^{-1} \{T_{gi}\} \quad (12.2-9)$$

12.2.2 Coefficients of Unknowns in Body i Equation

The unknown or left side of Equation 12.2-9 will now be expanded. From Equations 5.3-3, 5.3-6, 12.1-18,

$$\{A \dot{\omega}_{11}\} = \{\dot{\Omega}_{11}\}$$

$$\{A \dot{\omega}_{ii}\} = [D_i] \{\dot{\Omega}_{11}\} + \{\dot{\Omega}_{ii}\}$$

$$\{A H_{uil}\} = -[J_{uil}] \{A \dot{\omega}_{11}\} = -[J_{uil}] \{\dot{\Omega}_{11}\}$$

$$\sum_{i=2}^N \frac{M_i}{M_T} \{A H_{pil}\} = [R_{11}] - [J_{RM1}] \{\dot{\Omega}_{11}\} + \sum_{i=2}^N \frac{M_i}{M_T} [D_i]_T [\beta'_i] - [J_{vii}] \{\dot{\Omega}_{ii}\}$$

Then

$$\begin{aligned} \sum_{i=2}^N \frac{M_i}{M_T} \{A H_{pil}\} - \{A H_{uil}\} &= [R_{11}] - [J_{RM1}] + [J_{uil}] \{\dot{\Omega}_{11}\} \\ &+ \sum_{i=2}^N \frac{M_i}{M_T} [D_i]_T [\beta'_i] - [J_{vii}] \{\dot{\Omega}_{ii}\} \end{aligned}$$

(12.2-10)

The auxiliary body i moment equation may now be written as

$$[L_{i1}] \{\dot{\Omega}_{11}\} + \sum_{j=2}^N [L_{ij}] \{\dot{\Omega}_{jj}\} = \{q_{ii}\} - [G_i]_T^{-1} \{T_{gi}\} \quad (12.2-11)$$

where

$$[L_{i1}] = [I_{iH}] + [W_i] [D_i] + M_i [D_i] [J_{vil}] [J_{RM1}] - [R_{11}] - [J_{uil}] \quad (12.2-12)$$

$$[L_{ij}] = \left[[I_{iH}] + [W_i] \right] \hat{\delta}_{ij} - \frac{M_i M_j}{M_T} [J_{vii}] [D_i] [D_j]^T \left[[\beta_j] - [J_{vjj}] \right] \quad (12.2-13)$$

$$\hat{\delta}_{ij} = 0 \quad j \neq i$$

$$\hat{\delta}_{ij} = 1.0 \quad j = i$$

12.3 BASIC DYNAMIC EQUATIONS

The basic dynamic equations expressed in terms of body coordinates are summarized in this section. The main body or body 1 equation given by 12.1-27 is

$$[L_{11}] \{\dot{\Omega}_{11}\} + \sum_{j=2}^N [L_{1j}] \{\dot{\Omega}_{jj}\} = \{q_{11}\} + \sum_{i=2}^N [D_i]^T [G_i]^{-1} \{T_{gi}\} \quad (12.3-1)$$

The auxiliary body or body i equation ($2 \leq i \leq N$) is given by Equation 12.2-11 as

$$[L_{i1}] \{\dot{\Omega}_{11}\} + \sum_{j=2}^N [L_{ij}] \{\dot{\Omega}_{jj}\} = \{q_{ii}\} - [G_i]^T \{T_{gi}\} \quad (12.3-2)$$

From Equation 7.3-1 the gimbal torques $\{T_{gi}\}$ may be written as

$$\{T_{gi}\} = \{T_{sgi}\} + \{T_{vi}\} \quad (12.3-3)$$

where $\{T_{sgi}\}$ are the gimbal reaction torques for gimbal axes which have degrees of freedom while $\{T_{vi}\}$ are the gimbal reaction torques for gimbal axes which do not have degrees of freedom. Therefore, for a given axis the corresponding element in either $\{T_{sgi}\}$ or $\{T_{vi}\}$ must be zero. Substituting Equation 12.3-3 the basic dynamic equations become

$$\begin{aligned}
[L_{11}]\{\dot{\Omega}_{11}\} + \sum_{j=2}^N [L_{1j}]\{\dot{\Omega}_{jj}\} &= \{q_{11}\} + \sum_{i=2}^N [D_i][G_i]_T^{-1} \{T_{sg i}\} \\
&+ \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} \{T_{vi}\} \quad (12.3-4)
\end{aligned}$$

$$\begin{aligned}
[L_{11}]\{\dot{\Omega}_{11}\} + \sum_{j=2}^N [L_{ij}]\{\dot{\Omega}_{jj}\} &= \{q_{ii}\} - [G_i]_T^{-1} \{T_{sg i}\} - [G_i]_T^{-1} \{T_{vi}\} \\
&12.3-5)
\end{aligned}$$

The basic dynamic equations may be written as the single system matrix equation

$$\begin{bmatrix}
[L_{11}] & [L_{12}] & [L_{1j}] & [L_{1n}] \\
[L_{21}] & [L_{22}] & & \\
[L_{i1}] & & [L_{ij}] & \\
[L_{N1}] & & & [L_{NN}]
\end{bmatrix}
\begin{bmatrix}
\{\dot{\Omega}_{11}\} \\
\{\dot{\Omega}_{22}\} \\
\{\dot{\Omega}_{jj}\} \\
\{\dot{\Omega}_{NN}\}
\end{bmatrix}
=
\begin{bmatrix}
\{q_{11}\} + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} \{T_{sg i}\} \\
\{q_{22}\} - [G_2]_T^{-1} \{T_{sg 2}\} \\
\{q_{ii}\} - [G_i]_T^{-1} \{T_{sg i}\} \\
\{q_{NN}\} - [G_N]_T^{-1} \{T_{sg N}\}
\end{bmatrix}
+
\begin{bmatrix}
\sum_{i=2}^N [D_i]_T [G_i]_T^{-1} \{T_{vi}\} \\
-[G_2]_T^{-1} \{T_{vi 2}\} \\
-[G_i]_T^{-1} \{T_{vi}\} \\
-[G_N]_T^{-1} \{T_{vi}\}
\end{bmatrix} \quad (12.3-6)$$

or

$$[L]\{\dot{\Omega}\} = \{q\} + f_1 \{T_{sg i}\} + f_2 \{T_{vi}\} \quad (12.3-7)$$

Equation 12.3-7 can be solved directly for the unknown accelerations only if all auxiliary bodies have three degrees of freedom about their gimbal axes. All of the elements in the $\{T_{vi}\}$ matrices are then zero and since $\{T_{sg i}\}$ is a function of only position and velocity the elements of the $[L]$ matrix and the right hand side of Equation 12.3-7 do not contain

the unknown accelerations. The elements of $\{\dot{\Omega}\}$ can then be found by inverting the $[L]$ matrix. From Equation 7.1-2

$$\{\Omega_{jj}\} = [G_j] \{\dot{\theta}_j\} \quad (12.3-8)$$

$$\{\dot{\Omega}_{jj}\} = [G_j] \{\ddot{\theta}_j\} + [\dot{G}_j] \{\dot{\theta}_j\} \quad (12.3-9)$$

$$\{\ddot{\theta}_j\} = [G_j]^{-1} \left\{ \{\dot{\Omega}_{jj}\} - [\dot{G}_j] \{\dot{\theta}_j\} \right\} \quad (12.3-10)$$

For each degree of freedom eliminated a locked gimbal reaction torque will appear in the $\{T_{yi}\}$ matrices and Equation 12.3-7 cannot be used to directly solve the equations of motion. Methods of solution for locked gimbals were discussed in Section 4.0 covering method of approach. As noted in Section 4.0, the reformulation of Equation 12.3-4 and 12.3-5 in the desired form for digital programming is presented in Appendix A.

13. SOLAR AND AERODYNAMIC FORCES AND TORQUES

Both the aerodynamic and the solar forces on an element of area are assumed to result from a particle flow against the area. Using one of the simple and most commonly used particle-surface interaction models, the same set of force and moment equations will apply for both aerodynamic and solar disturbances. The equations derived in this section are expressed in terms of a unit flow vector $\bar{\delta}_F$, a pressure V_F and the normal and tangential coefficients ν_N and ν_T . For solar disturbances, $\bar{\delta}_F$ is the unit sun vector $\bar{\delta}_S$, V_F is the solar radiation pressure V_S , and ν_N and ν_T are the normal and tangential reflectivity coefficients. For aerodynamic disturbances, $\bar{\delta}_F$ is a unit vector along the negative of the spacecraft velocity vector, V_F is the dynamic pressure V_A , and ν_N and ν_T are the normal and tangential momentum exchange coefficients.

13.1 SOLAR AND AERODYNAMIC REFERENCE SYSTEMS

For the derivation of the force and torque equations, it is convenient to define time variant reference systems for each shape. These reference systems will be designated as flow oriented systems. The general subscript F will be used which may be changed to S for solar disturbance computations and to A for aerodynamic disturbance computations. The solar oriented reference systems are the same as the boom reference system used to describe thermal bending except that the boom reference axis is replaced by a unit vector defining the orientation of the body shape.

Let $\bar{A}_{Nik}$ be a unit vector defining the orientation of a surface of the k^{th} element on the i^{th} body, and let $\bar{\delta}_{Fi}$ be the unit vector defining the direction of partial flow, Figure 13.1-1. The angle η_{Nik} , between $\bar{\delta}_{fi}$ and $\bar{A}_{Nik}$, may have values in the range $0 \leq \eta_{Nik} \leq \pi$; therefore $\sin \eta_{Nik}$ is always positive

$$\cos \eta_{Nik} = \bar{\delta}_{Fik} \cdot \bar{A}_{Nik} \quad (13.1-1)$$

and

$$\sin \eta_{Nik} = +\sqrt{1 - \cos^2 \eta_{Nik}} \quad (13.1-2)$$

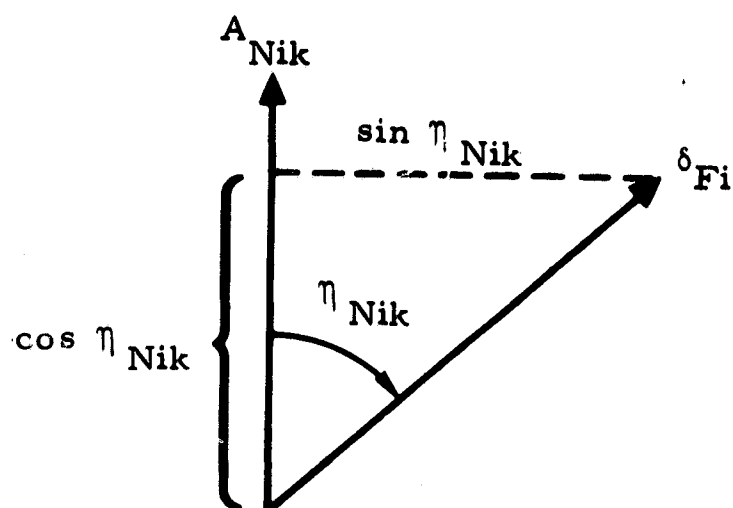


Figure 13.1-1. Relative Position of Flow Vector

The flow oriented reference system is defined such that $\bar{y}_N$ equals $\bar{A}_{Nik}$, $\bar{x}_F$ is in the direction of the component of $\bar{\delta}_{Fi}$ normal to $\bar{A}_{Nik}$, and $\bar{z}_N$ forms a right hand orthogonal system, Figure 13.1-2. The component of $\bar{\delta}_{Fi}$ along $\bar{z}_N$ will always be zero, and the unit flow vector $\bar{\delta}_{Fi}$ may be defined in the flow oriented system as

$$\bar{\delta}_{Fi} = (\sin \eta_{Nik}) \bar{x}_{Nik} + (\cos \eta_{Nik}) \bar{A}_{Nik} \quad (13.1-3)$$

The flow oriented reference system varies with time. From Figure 13.1-2

$$\bar{x}_{Nik} = \frac{\bar{\delta}_{Fi} - (\cos \eta_{Nik}) \bar{A}_{Nik}}{\sin \eta_{Nik}} \quad (13.1-4)$$

$$\bar{y}_{Nik} = \bar{A}_{Nik} \quad (13.1-5)$$

$$\bar{z}_{Nik} = \frac{\bar{\delta}_{Fi} \cdot \bar{A}_{Nik}}{\sin \eta_{Nik}} \quad (13.1-6)$$

If all of the vector quantities in Equations 13.1-4, 13.1-5, and 13.1-6 are expressed as components in the same coordinate system, the equations may be written in matrix form as

$$\begin{bmatrix} x_{Nik} \\ y_{Nik} \\ z_{Nik} \end{bmatrix} = \frac{\begin{bmatrix} \delta_{Fi} \\ 0 \\ 0 \end{bmatrix} - (\cos \eta_{Nik}) \begin{bmatrix} A_{Nik} \\ 0 \\ 0 \end{bmatrix}}{\sin \eta_{Nik}} \quad (13.1-7)$$

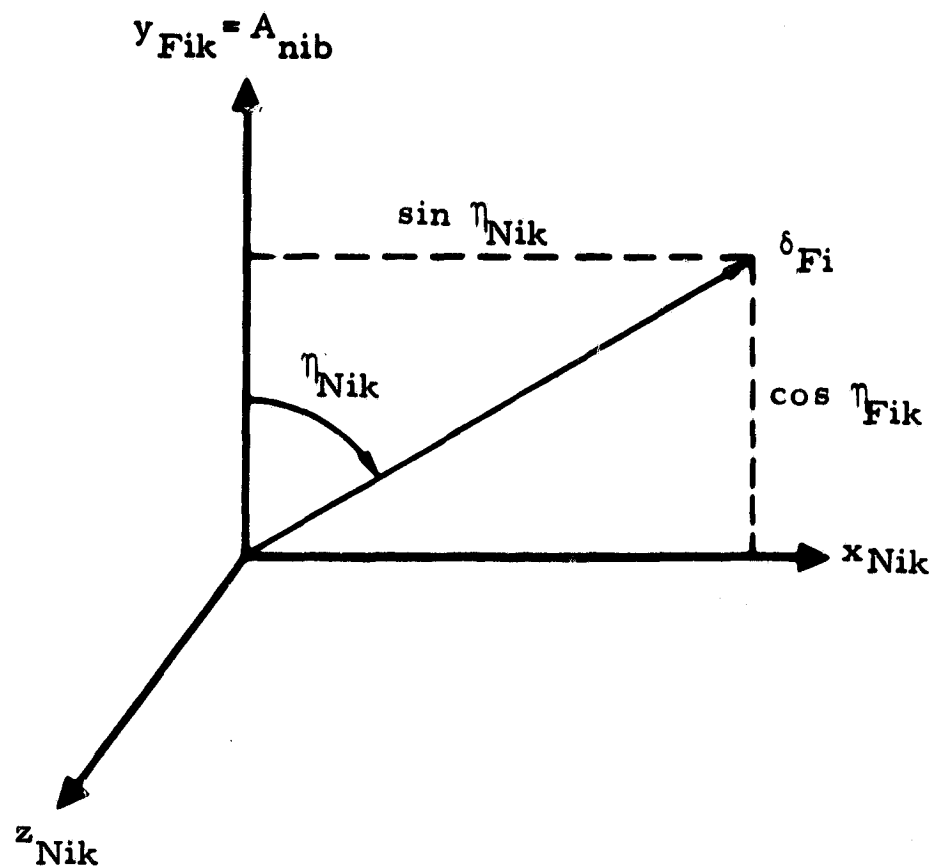


Figure 13.1-2. Flow Oriented Coordinate System

$$\begin{bmatrix} y_{Nik} \end{bmatrix} = \begin{bmatrix} A_{Nik} \end{bmatrix} \quad (13.1-8)$$

$$\begin{bmatrix} z_{Nik} \end{bmatrix} = \frac{\begin{bmatrix} J_{\delta Fi} \end{bmatrix} \begin{bmatrix} A_{Nik} \end{bmatrix}}{\sin \eta_{Nik}} \quad (13.1-9)$$

If δ_{Fi} and $\bar{A}_{Nik}$ are in body i coordinates, Equations 13.1-7, 13.1-8, and 13.1-9 give the components of the flow oriented axes in body i coordinates. The flow vector δ_{Si} in body i coordinates is given in Section 10. In the case of the auxiliary bodies the vector $\bar{A}_{Nik}$ for the tip shapes may change orientation with respect to its body axes due to thermal bending. From Equation 14.2-25 the transformation from the body system to the boom coordinates is given by

$$\begin{bmatrix} \bar{Q}_{Bik} \end{bmatrix} = \begin{bmatrix} \tau_{ik} \end{bmatrix} \begin{bmatrix} \bar{Q}_i \end{bmatrix}.$$

From Equation 14.5-9 the transformation from the boom coordinates to the thermally deflected boom coordinates at the tip is given by

$$\left[\bar{Q}'_{Bik} \right] = \left[\alpha_{ik} \right] \left[\bar{Q}_{Bik} \right].$$

Let $\left[A'_{Nik} \right]$ be constant input components defining the nominal orientation of the k^{th} shape on the i^{th} body in body i coordinates. The components in nominal boom coordinates are $\left[\tau_{ik} \right] \left[A'_{Nik} \right]$ and in deflected boom coordinates they are $\left[\alpha_{ik} \right]_T \left[\tau_{ik} \right] \left[A'_{Nik} \right]$. Transforming back to body i coordinates the deflected orientation is then defined by

$$\left[A_{Nik} \right] = \left[\tau_{ik} \right]_T \left[\alpha_{ik} \right]_T \left[\tau_{ik} \right] \left[A'_{Nik} \right] \quad (13.1-10)$$

In the case of the main body of the spacecraft, there is no thermal bending and $\left[\alpha_{ik} \right] = \left[U \right]$ and since $\left[\tau_{ik} \right]_T \left[\tau_{ik} \right] = \left[U \right]$, $\left[A_{Nik} \right] = \left[A'_{Nik} \right]$ for body 1.

In general the disturbance forces and torques will be computed in the flow oriented coordinate systems and will then be transformed into components in the body axes before substitution into the dynamic equations.

Let the 3×3 matrix $\left[\eta_{Nik} \right]$ be the transformation matrix between the body i coordinates $\left[\bar{Q}_i \right]$ and the flow oriented system $\left[\bar{Q}_{Nik} \right]$ where

$$\left[\bar{Q}_{Nik} \right] = \begin{Bmatrix} \bar{x}_{Nik} \\ \bar{y}_{Nik} \\ \bar{z}_{Nik} \end{Bmatrix} \quad (13.1-11)$$

then

$$\left[\bar{Q}_{Nik} \right] = \left[\eta_{Nik} \right] \left[\bar{Q}_i \right] \quad (13.1-12)$$

and

$$\left[\eta_{Nik} \right]_T = \left[\begin{array}{ccc} \left[\bar{x}_{Nik} \right] & \left[\bar{y}_{Nik} \right] & \left[\bar{z}_{Nik} \right] \end{array} \right] \quad (13.1-13)$$

Using Equations 13.1-7, 13.1-8, and 13.1-9

$$[\eta_{Nik}]_T = \left[\left\{ \frac{|\delta_{Fi}| - \cos \eta_{Nik} |A_{Nik}|}{\sin \eta_{Nik}} \right\} \left\{ A_{Nik} \right\} \left\{ \frac{|J_{\delta Fi}| |A_{Nik}|}{\sin \eta_{Nik}} \right\} \right] \quad (13.1-14)$$

$$\sin \eta_{Nik} [\eta_{Nik}]_T = \left[\left\{ |\delta_{Fi}| - \cos \eta_{Nik} |A_{Nik}| \right\} \left\{ |A_{Nik}| \sin \eta_{Nik} \right\} \left\{ |J_{\delta Fi}| |A_{Nik}| \right\} \right] \quad (13.1-15)$$

where $[\eta_{Nik}]_T$ is the matrix which transforms components in the flow oriented system to body coordinates if $|\delta_{Fi}|$ and $|A_{Nik}|$ are in body i coordinates.

13.2 GENERAL FORCE EQUATION

In this section the general expression for the force on an element of area dA will be defined. Using the unit flow vector $\bar{\delta}_F$ and the equations of Section 13.1, define the flow oriented reference system $\bar{e}_t$, $\bar{e}_n$, and $\bar{e}_m$. Figure 13.2-1. The unit vector $\bar{e}_n$ is normal to the element of area dA and corresponds to the unit vector $\bar{A}_F$ and the $\bar{y}$ axis. $\bar{e}_t$ and $\bar{e}_m$ represent the $\bar{x}$ and $\bar{z}$ axes respectively. Then

$$\cos \eta = \bar{e}_n \cdot \bar{\delta}_F = \bar{\delta}_F \cdot \bar{e}_n \quad (13.2-1)$$

$$\sin \eta = +\sqrt{1 - \cos^2 \eta} \quad (13.2-2)$$

$$\sin \eta \bar{e}_t = \bar{\delta}_F - \cos \eta \bar{e}_n \quad (13.2-3)$$

From Equations 4 and 5 of Reference 4, the force components along $\bar{e}_n$, $\bar{e}_t$, and $\bar{e}_m$ are

$$d\bar{F}_n = V_{Fn} (1 + \nu_N) \cos \eta dA \bar{e}_n \quad (13.2-4)$$

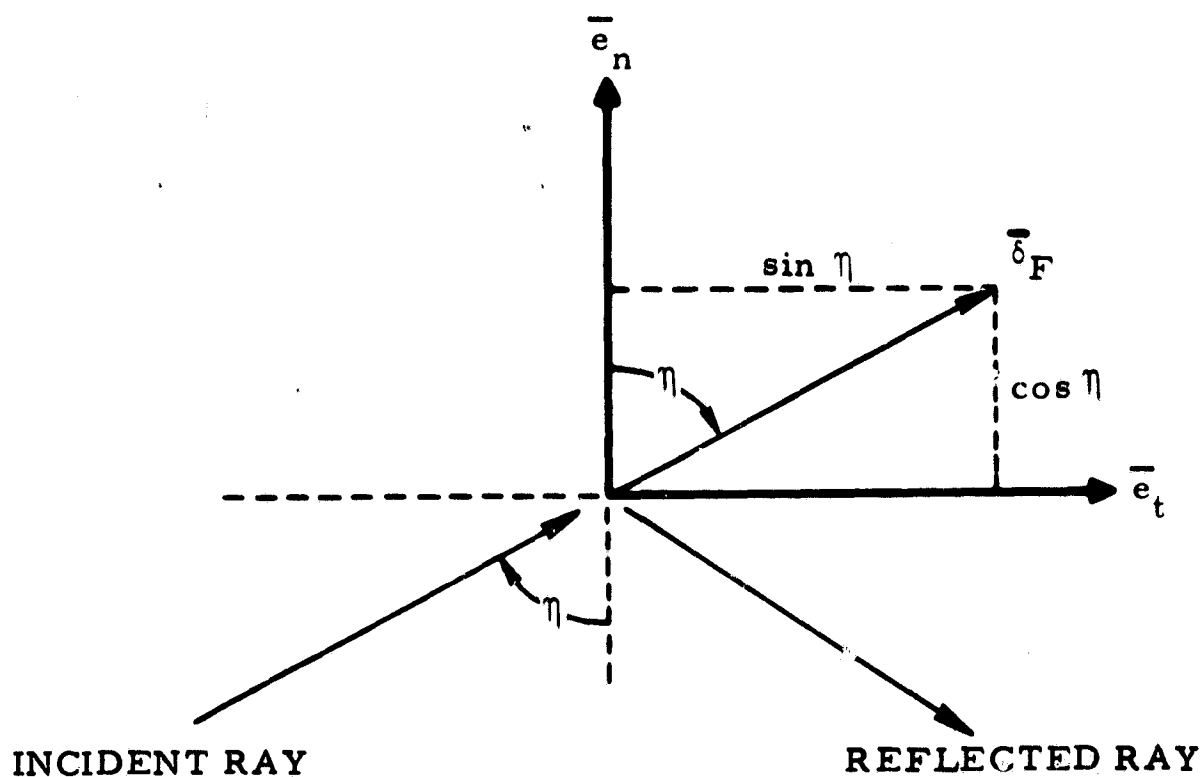


Figure 13.2-1. Surface Coordinates

$$d\bar{F}_t = V_{Fn} (1 - \nu_T) \sin \eta \, dA \, \bar{e}_t \quad (13.2-5)$$

$$d\bar{F}_m = 0 \quad (13.2-6)$$

where

ν_N = normal reflectivity coefficient for solar forces and normal momentum exchange coefficient for aerodynamic forces

ν_T = tangential reflectivity coefficient for solar forces and tangential momentum exchange coefficient for aerodynamic forces

$$V_{Fn} = V_F |\cos \eta|$$

$$V_F = V_S = .94 \times 10^{-7} \text{ \#/ft}^2 \text{ for solar forces}$$

$$V_F = V_A = 1/2 \zeta V_A^2 = \text{dynamic pressure in \#/ft}^2 \text{ for aerodynamic forces.}$$

Then

$$d\bar{F}_n = V_F (1 + \nu_N) |\cos \eta| \cos \eta dA \bar{e}_n \quad (13.2-7)$$

$$d\bar{F}_t = V_F (1 - \nu_T) |\cos \eta| \sin \eta dA \bar{e}_t \quad (13.2-8)$$

using Equations 13.2-3, Equation 13.2-8 becomes

$$d\bar{F}_t = V_F (1 - \nu_T) |\cos \eta| dA \bar{\delta}_F - V_F (1 - \nu_T) |\cos \eta| dA \cos \eta \bar{e}_n \quad (13.2-9)$$

The total force on the element dA is $d\bar{F}$ given by

$$d\bar{F} = d\bar{F}_n + d\bar{F}_t + d\bar{F}_m$$

$$d\bar{F} = V_F |\cos \eta| dA \left((1 - \nu_T) \bar{\delta}_F + (\nu_N + \nu_T) \cos \eta \bar{e}_n \right) \quad (13.2-10)$$

If $\bar{\delta}_F$ and $\bar{e}_n$ are defined in the same coordinate system, the force Equation 13.2-10 may be written in matrix notation as

$$|d\bar{F}| = V_F |\cos \eta| dA \left\{ (1 - \nu_T) |\bar{\delta}_F| + (\nu_N + \nu_T) \cos \eta |e_n| \right\} \quad (13.2-11)$$

Let dA be an element of area of a surface enclosing a volume and let $\bar{e}_n$ be an inward directed unit normal to the surface element. From Figure 13.2-2 it may be seen that the flow does not reach an area for which $\cos \eta$ is negative or an area which is shaded. Therefore when integrating to obtain the total force, the integration must not be carried out over these areas. Introducing the restriction that $\cos \eta$ must be positive Equations 13.2-10 and 13.2-11 become,

$$d\bar{F} = V_F \left((1 - \nu_T) \cos \eta \bar{\delta}_F + (\nu_N + \nu_T) \cos^2 \eta \bar{e}_n \right) dA \quad (13.2-12)$$

$$|d\bar{F}| = V_F \left\{ (1 - \nu_T) \cos \eta |\bar{\delta}_F| + (\nu_N + \nu_T) \cos^2 \eta |e_n| \right\} dA$$

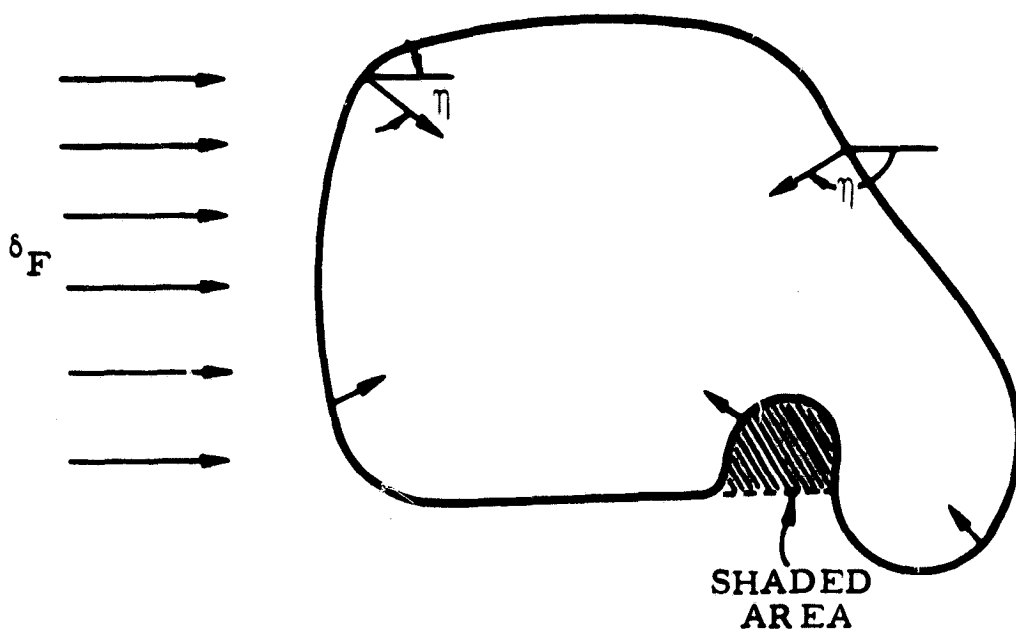


Figure 13.2-2. Body Self Shading

Equations 13.2-12 represent the general force equations for solar and aerodynamic forces. It should be noted that the total force on an element dA is the sum of two non-orthogonal forces, one along the flow vector and one along the inward directed normal to the surface. The following factors must be considered when using these equations for obtaining total forces:

1. $\cos \eta = \left| \delta_F \right|_T \left| e_n \right|$
2. $\cos \eta$ must be positive $-\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2}$
3. $\left| dF \right|$ are zero if area is shaded.

13.3 FORCES AND MOMENTS ON A FLAT PLATE

The flat plate area attached at the end of boom k of auxiliary body i is defined by the unit vector $\bar{A}_{Fik}$ and the scalar area A_{Fik} . For example the third boom on body two would have area A_{F23} . In the case of body 1 which may have up to four shapes attached, the same symbols apply except the subscript k refers to the shape number. For example if the third shape placed upon body 1 is a flat plate, its area is defined by A_{F13} .

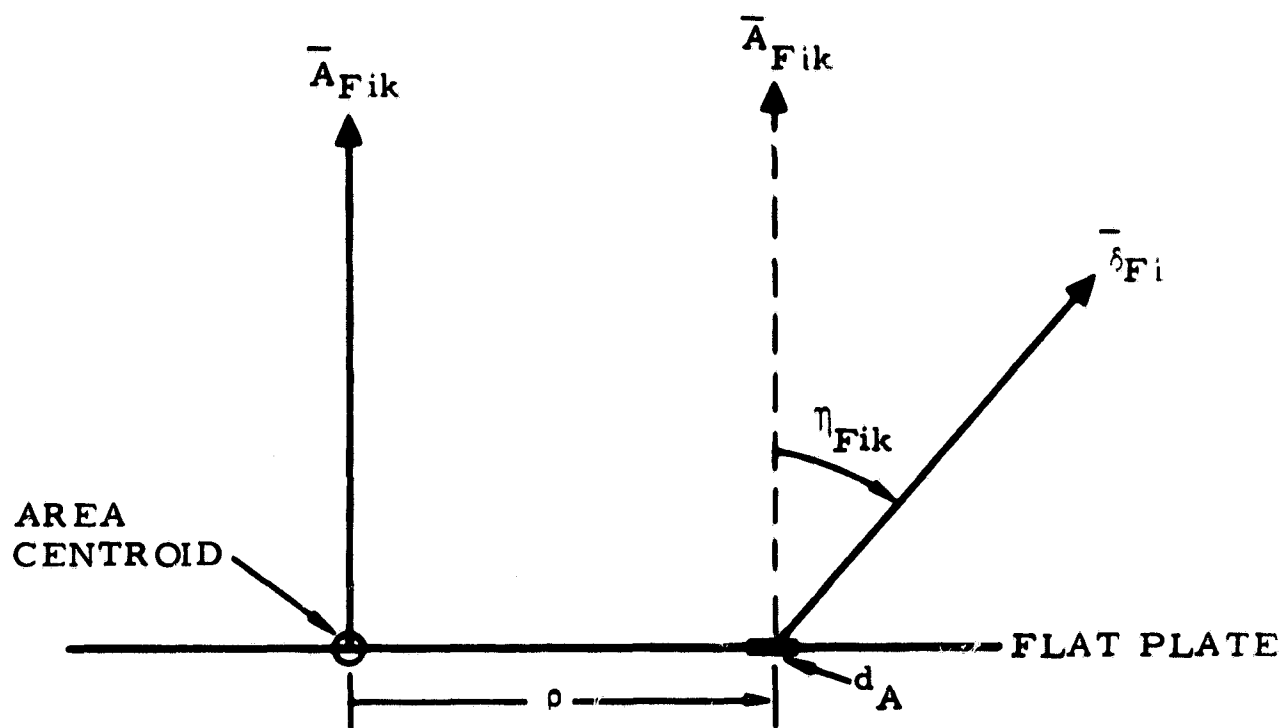


Figure 13.3-1. Forces on a Flat Plate

Referring to Figure 13.3-1, the angle η_{Fik} is constant and $\bar{e}_n$ is equal to $\bar{A}_{Fik}$ for every element of area dA . Since the area of the plate does not enclose a volume and $\bar{e}_n$ is not an inward directed normal, the total force on the plate is found by integrating Equation 13.2-11.

$$\begin{aligned} |P_{Fik}| = V_F A_{Fik} |\cos \eta_{Fik}| \left\{ (1 - \nu_{TFik}) |\delta_{Fi}| \right. \\ \left. + (\nu_{NFik} + \nu_{TFik}) \cos \eta_{Fik} |A_{Fik}| \right\} \end{aligned} \quad (13.3-1)$$

where

$$\cos \eta_{Fik} = \frac{|\delta_{Fi}|_T |A_{Fik}|}{|A_{Fik}|} \quad (13.3-2)$$

$|A_{Fik}|$ is found from Equation 13.1-10

$|P_{Fik}|$ = components of force at centroid of the plate

when $|A_{Fik}|$ and $|\delta_{Fi}|$ are in body i coordinates, the force components $|P_{Fik}|$ are in body i coordinates. It should be noted that Equation 13.3-1 will give the total force when the flow is on either side of the plate. When

the flow is against the under side of the plate in Figure 13.3-1 $\cos \eta_{Fik}$ is positive and when the flow is against the top of the plate $\cos \eta_{Fik}$ is negative. Either side of the plate can be shaded by setting $|P_{Fik}|$ to zero in accordance with the sign of $\cos \eta_{Fik}$. For the general expressions for a single flat plate neither side will be shaded.

The moment of the forces about the centroid of the plate area is

$$|dM_{Fik}| = \rho |dP_{Fik}| = \rho dA \text{ (constant)}$$

Therefore

$$|M_{Fik}| = 0 \quad (13.3-3)$$

13.4 FORCES AND MOMENTS ON A SPHERE

The spherical shape attached at the end of boom k of auxiliary body i is defined by the scalar radius R_{Sik} . For body 1 this parameter is R_{S1k} where the subscript k denotes the k^{th} shape of body 1. The element of area dA and the inward directed unit normal $\bar{e}_n$ will be defined in a right hand orthogonal coordinate system established in the sphere such that $\bar{y}_S = -\bar{\delta}_F$, Figure 13.4-1. Then in the $\bar{Q}_S$ system

$$|\delta_{FS}| = - \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad (13.4-1)$$

$$|R| = R \begin{Bmatrix} \sin \theta \cos \phi \\ \cos \theta \\ \sin \theta \sin \phi \end{Bmatrix} \quad (13.4-2)$$

The inward directed normal to the surface is

$$\bar{e}_n = -\frac{\bar{R}}{R}$$

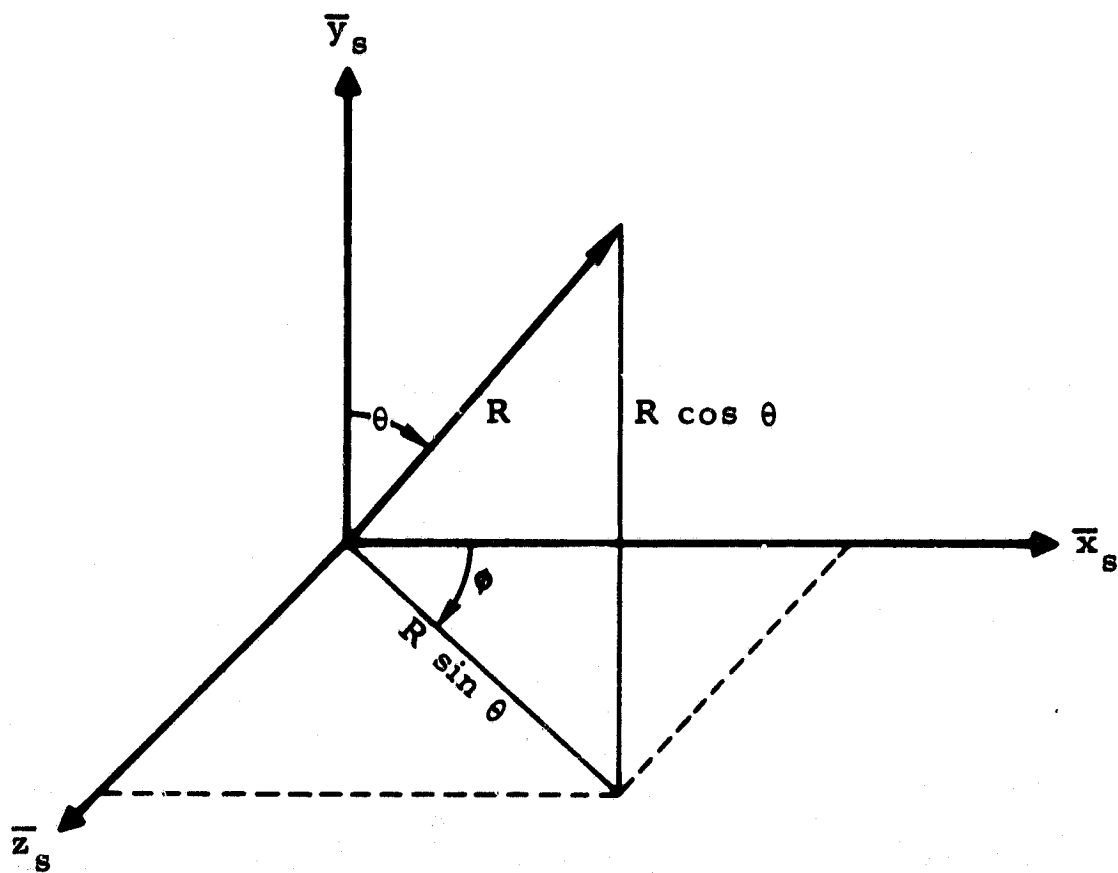
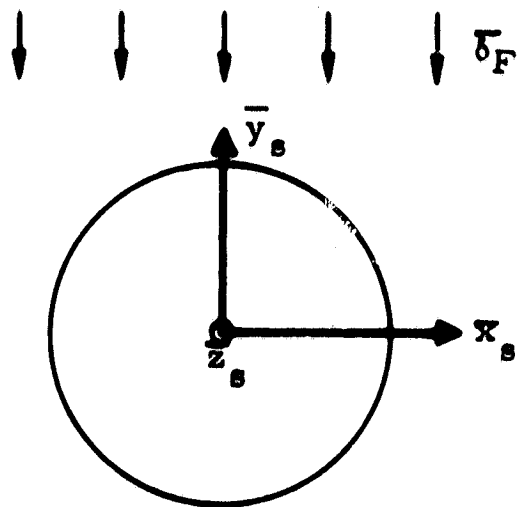


Figure 13.4-1. Coordinator for Cylindrical Shapes

$$|e_n| = - \begin{Bmatrix} \sin \theta \cos \phi \\ \cos \theta \\ \sin \theta \sin \phi \end{Bmatrix} \quad (13.4-3)$$

Then

$$\cos \eta = - \begin{Bmatrix} \sin \theta \cos \phi \\ \cos \theta \\ \sin \theta \sin \phi \end{Bmatrix}_T \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix} = \cos \theta \quad (13.4-4)$$

$\cos \eta$ is positive if $0 \leq \theta \leq \frac{\pi}{2}$

$$dA = R^2 \sin \theta \, d\theta \, d\phi \quad (13.4-5)$$

Then from Equation 13.2-12

$$\begin{aligned} |dP_{Sik}| &= - R_{Sik}^2 V_F (1 - \nu_T) \sin \theta \cos \theta \, d\theta \, d\phi \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \\ &\quad - R_{Sik}^2 V_F (\nu_N + \nu_T) \sin \theta \cos^2 \theta \begin{Bmatrix} \sin \theta \cos \phi \\ \cos \theta \\ \sin \theta \sin \phi \end{Bmatrix} d\theta \, d\phi \end{aligned} \quad (13.4-6)$$

Integrating with respect to $d\phi$ from 0 to 2π

$$\begin{aligned} |dP_{Sik}| &= - 2\pi R_{Sik}^2 V_F \left((1 - \nu_{TSik}) \sin \theta \cos \theta \, d\theta \right. \\ &\quad \left. + (\nu_{NSik} + \nu_{TSik}) \sin \theta \cos^3 \theta \, d\theta \right) \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \end{aligned}$$

Integrating with respect to $d\theta$ from 0 to $\frac{\pi}{2}$,

$$|P_{Sik}| = \pi R_{Sik}^2 V_F \left(1 + \frac{1}{2} \nu_{NSik} - \frac{1}{2} \nu_{TSik}\right) |\delta_{Fi}| \quad (13.4-7)$$

If δ_{Fi} is in body i coordinates P_{Sik} is also in body i coordinates. For solar forces $\nu_N = \nu_T$ and the reflectivity is arbitrary and no solar reflectivity need be input for a sphere.

The moment of the force on element dA about the centroid of the spheres is given by

$$d\bar{M}_{Fik} = \bar{R} \times d\bar{P}_{Sik}$$

$$|dM_{Sik}| = |J_R| |dP_{Sik}|$$

Substituting Equations 13.4-2 and 13.4-6 and integrating gives

$$|M_{Sik}| = 0 \quad (13.4-8)$$

13.5 FORCES AND MOMENTS ON AN ELEMENT OF A CYLINDER

The force, and the resulting torque, due to an element of area of a cylinder will be required for both the body shapes and for the booms. Therefore the equations for the differential forces and torques will be derived in this section. The element of area dA and the inward directed unit normal, $\bar{e}_n$ will be defined in a flow oriented coordinate system established in the cylindrical volume such that $\bar{y}_c$ lies along the axis of the element, Figure 13.5-1. From Section 13.1

$$\cos \eta_c = |\delta_F|_T |y_c| \quad (13.5-1)$$

$$|x_c| = \frac{|\delta_F| - \cos \eta_c |y_c|}{\sin \eta_c} \quad (13.5-2)$$

$$|z_c| = \frac{|J_{\delta F}| |y_c|}{\sin \eta_c} \quad (13.5-3)$$

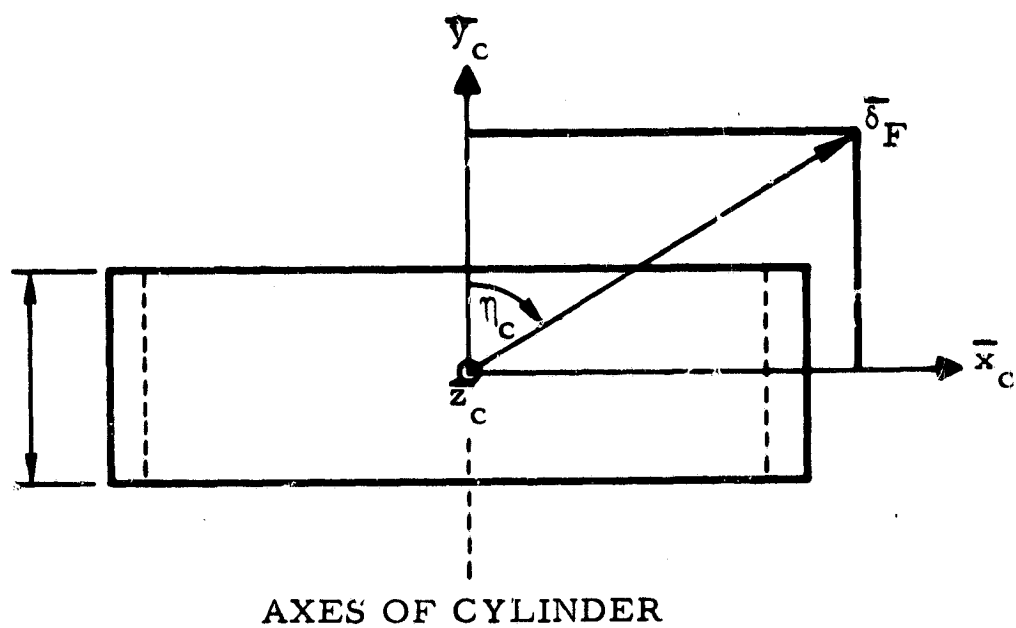


Figure 13.5-1. Coordinate for Cylindrical Element

From Figure 13.5-1

$$\left\{ \delta_F \right\} = \begin{Bmatrix} \sin \eta_c \\ \cos \eta_c \\ 0 \end{Bmatrix} \quad (13.5-4)$$

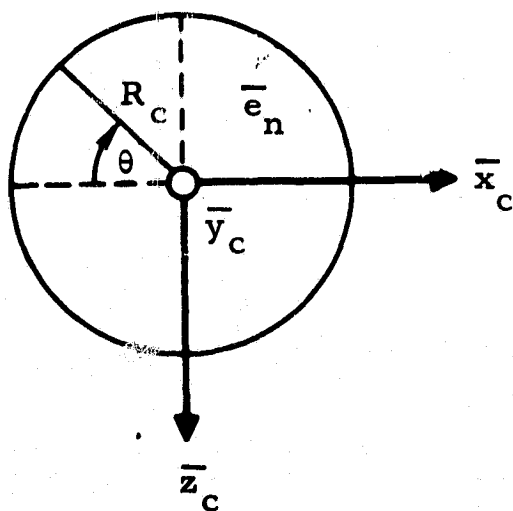


Figure 13.5-2. Coordinate for a Cylindrical Element

$$\left\{ R_c \right\} = - R_c \begin{Bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{Bmatrix} \quad (13.5-5)$$

The inward directed normal to the surface $\bar{e}_n$ is

$$\bar{e}_n = - \frac{\bar{R}_c}{R_c}$$

$$\left\{ e_n \right\} = \begin{Bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{Bmatrix} \quad (13.5-6)$$

$$dA = R d\theta dh \quad (13.5-7)$$

Note that η_c is the angle between the unit flow vector $\bar{\delta}_F$ and the axis of the cylindrical element and is constant for all area elements dA of the cylindrical element. Let η_e be the local angle between the unit flow vector and the normal $\bar{e}_n$ to the area dA . Then

$$\cos \eta_e = \left\{ e_n \right\}_T \left\{ \delta_F \right\}$$

$$\cos \eta_e = \begin{Bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{Bmatrix}_T \begin{Bmatrix} \sin\eta_c \\ \cos\eta_c \\ 0 \end{Bmatrix} = \sin\eta_c \cos\theta \quad (13.5-8)$$

$\cos\eta_e$ is positive if

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Assume that the inside of the cylinder is shaded. Then from Equation 13.2-12 the force on the element of area dA is

$$\begin{aligned} |dP|_F = R_c V_F (1 - \nu_T) \begin{Bmatrix} \sin \eta_c \\ \cos \eta_c \\ 0 \end{Bmatrix} \sin \eta_c \cos \theta d\theta dh \\ + R_c V_F (\nu_N + \nu_T) \sin^2 \eta_c \cos^2 \theta d\theta dh \begin{Bmatrix} \cos \theta \\ 0 \\ \sin \theta \end{Bmatrix} \end{aligned} \quad (13.5-9)$$

Integrating $d\theta$ from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$

$$\begin{aligned} |dP|_F = 2 R_c V_F (1 - \nu_T) \begin{Bmatrix} \sin^2 \eta_c \\ \sin \eta_c \cos \eta_c \\ 0 \end{Bmatrix} dh \\ + R_c V_F (\nu_N + \nu_T) \sin^2 \eta_c \begin{Bmatrix} \frac{4}{3} \\ 0 \\ 0 \end{Bmatrix} dh \end{aligned} \quad (13.5-10)$$

Since the flow vector $\bar{\delta}_{Fi}$ and the unit normal $\bar{e}_n$ were both defined in the flow oriented system given by Equations 13.5-1, 13.5-2, and 13.5-3, the force components $|dP|_F$ are also in these coordinates (denoted by the subscript F). From Equation 13.1-12 the force components in body i coordinates are given by

$$|dP| = [\eta_c]_T |dP|_F$$

From Equation 13.5-10 and letting $D_c = 2R_c = \text{diameter of cylinder}$

$$\begin{aligned} |dP| = & D_c V_F (1 - \nu_T) \sin \eta_c [\eta_c]_T \begin{Bmatrix} \sin \eta_c \\ \cos \eta_c \\ 0 \end{Bmatrix} dh \\ & + \frac{2}{3} D_c V_F (\nu_N + \nu_T) \sin \eta_c [\eta_c]_T \begin{Bmatrix} \sin \eta_c \\ 0 \\ 0 \end{Bmatrix} dh \end{aligned}$$

Substituting Equations 13.5-1, 13.5-2, and 13.5-3 into Equation 13.1-13

$$\sin \eta_c [\eta_c]_T = \left[\left[|\delta_{Fi}| - \cos \eta_c |y_c| \right] \left[|y_c| \sin \eta_c \right] \left[[J_{\delta Fi}] |y_c| \right] \right] \quad (13.5-11)$$

$$\sin \eta_c [\eta_c]_T \begin{Bmatrix} \sin \eta_c \\ \cos \eta_c \\ 0 \end{Bmatrix} = |\delta_{Fi}| \sin \eta_c$$

$$\sin \eta_c [\eta_c]_T \begin{Bmatrix} \sin \eta_c \\ 0 \\ 0 \end{Bmatrix} = |\delta_{Fi}| \sin \eta_c - \sin \eta_c \cos \eta_c |y_c|$$

The force components in body coordinates then become

$$\begin{aligned} |dP| = & D_c V_F (1 - \nu_T) \sin \eta_c |\delta_{Fi}| dh \\ & + \frac{2}{3} D_c V_F (\nu_N + \nu_T) \sin \eta_c \left[|\delta_{Fi}| - \cos \eta_c |y_c| \right] dh \end{aligned} \quad (13.5-12)$$

Equation 13.5-12 is the general expression for the force on a cylindrical element of length dh in body coordinates where $\bar{y}_c$ defines the axis of the element and $\bar{\delta}_{Fi}$ defines the flow direction both in body coordinates. The moment of the force $d\bar{P}$ about the centroid of the cylindrical element is

$$d\bar{M} = \bar{R}_c \cdot d\bar{P}$$

and in flow oriented coordinates

$$[dM]_F = [J_{Rc}] [dP]_F$$

$$[J_{Rc}] = R_c \begin{bmatrix} 0 & \sin \theta & 0 \\ -\sin \theta & 0 & \cos \theta \\ 0 & -\cos \theta & 0 \end{bmatrix} \quad \text{from 13.5-5}$$

Using Equation 13.5-9

$$[dM]_F = R_c^2 V_F dh (1 - \nu_T) \begin{bmatrix} (0)(\sin \theta \cos \theta d\theta)(0) \\ (-\sin \theta \cos \theta d\theta)(0)(\cos^2 \theta d\theta) \\ (0)(-\cos^2 \theta d\theta)(0) \end{bmatrix}$$

$$\cdot \begin{Bmatrix} \sin^2 \eta_c \\ \sin \eta_c \cos \eta_c \\ 0 \end{Bmatrix}$$

Integrating $d\theta$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$[dM]_F = -\frac{\pi}{2} R_c^2 V_F dh (1 - \nu_T) \sin \eta_c \cos \eta_c \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad (13.5-13)$$

where $|dM|_F$ are the components of the moments about the centroid of the cylindrical element in flow oriented coordinates. From Equation 13.1-12 the moment components in body i coordinates are

$$|dM| = [\eta_c]_T |dM|_F$$

Using Equation 13.5-11 and 13.5-13

$$|dM| = \frac{\pi D_c^2}{8} V_F dh (1 - \nu_T) \cos \eta_c [J_{yc}] |\delta_{Fi}| \quad (13.5-14)$$

Equation 13.5-14 is the general expression for the moment on a cylindrical element of length dh in body coordinates, where $\bar{y}_c$ defines the axis of the element and δ_{Fi} defines the flow direction both in body coordinates.

13.6 FORCES AND MOMENTS ON A CYLINDRICAL SHAPE

The cylindrical shape attached at the end of boom k of auxiliary body i is defined by the unit vector $\bar{A}_{cik}$ and the scalars L_{cik} and R_{cik} . For body 1 the subscript k denotes the k^{th} shape. The unit vector $\bar{A}_{cik}$ lies along the axis of the cylinder and L_{cik} and R_{cik} are the length and radius of the cylinder in feet.

The total force on the side of the cylinder is obtained by integrating Equation 13.5-12 over the length L_{cik}

$$\begin{aligned} |P_{cik}|_{side} &= D_{cik} L_{cik} V_F (1 - \nu_{Tcik}) \sin \eta_{cik} |\delta_{Fi}| \\ &+ \frac{2}{3} D_{cik} L_{cik} V_F (\nu_{Ncik} + \nu_{Tcik}) \\ &\cdot \sin \eta_{cik} \left\{ |\delta_{Fi}| - \cos \eta_{cik} |A_{cik}| \right\} \end{aligned} \quad (13.6-1)$$

The total moment about the centroid due to forces on the side is obtained from Equation 13.5-14

$$|M_{cik}|_{side} = \frac{\pi}{8} D_{cik}^2 L_{cik} V_F (1 - \nu_{Tcik}) \cos \eta_{cik} \left\{ [J_{Acik}] |\delta_{Fi}| \right\} \quad (13.6-2)$$

where

1. $D_{cik} = 2 R_{cik}$
2. $|A_{cik}|$ is obtained from Equation 13.1-10
3. The inside of the cylinder is assumed to be shaded
4. $\cos \eta_{cik} = \frac{|\delta_{Fi}|_T |A_{cik}|}{|P_{cik}|}$
5. $\sin \eta_{cik} = +\sqrt{1 - \cos^2 \eta_{cik}}$

The ends of the cylinder may be considered as flat plates of area $1/4 \pi D_{cik}^2$. Since the flow can reach only one end of the cylinder at any instant and the inside surface of each end is shaded, the two ends can be replaced by a single two sided plate. From Equation 13.3-1 the force components on this plate are

$$|P_{cik}|_{ends} = \frac{1}{4} \pi D_{cik}^2 V_F \cos \eta_{cik} \left\{ (1 - v'_{Tcik}) |\delta_{Fi}| + (v'_{Ncik} + v'_{Tcik}) \cos \eta_{ik} |A_{cik}| \right\} \quad (13.6-3)$$

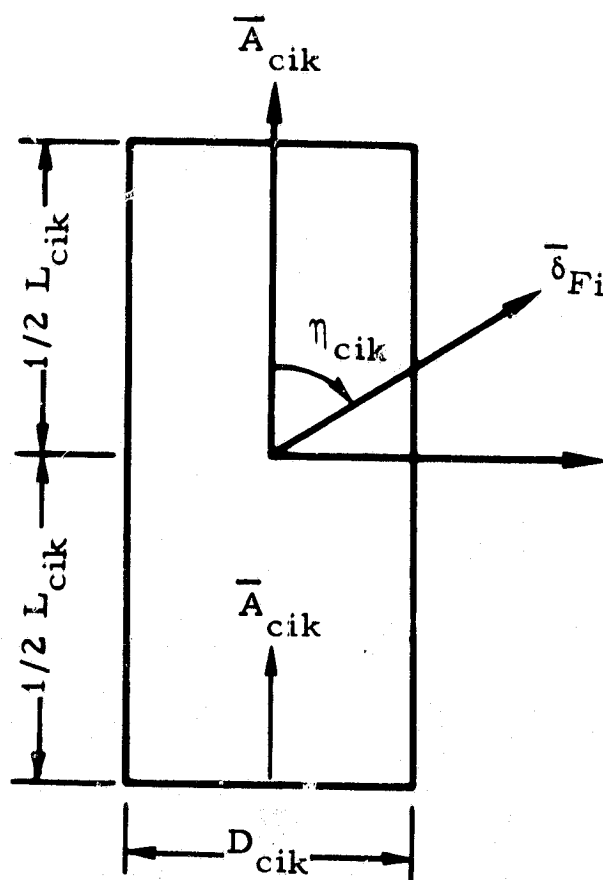


Figure 13.6-1. Definition of Cylindrical Shapes

where the primes are used to allow different surface conditions on the ends and sides of the cylinder. From Figure 13.6-1 the components $|P_{cik}|_{ends}$ are applied at the centroid of the lower end when $\cos \eta_{cik}$ is positive and at the centroid of the upper end when $\cos \eta_{cik}$ is negative. The vector location of the applied force is $-1/2 L_{cik} \bar{A}_{cik}$ when $\cos \eta_{cik}$ is positive and $1/2 L_{cik} \bar{A}_{cik}$ when $\cos \eta_{cik}$ is negative. From Equation 13.3-3 the moment about the centroid of the end is zero, the moment about the centroid of the cylinder due to forces on the ends is

$$\bar{M}_{cik} = -\frac{1}{2} L_{cik} \bar{A}_{cik} \times P_{cik}$$

or

$$|M_{cik}|_{ends} = -(\text{sign } \cos \eta_{cik}) \frac{1}{2} L_{cik} [J_{Acik}] |P_{cik}|_{ends}$$

Substituting for $|P_{cik}|_{ends}$

$$\begin{aligned} |M_{cik}|_{ends} = & -\frac{1}{8} \pi D_{cik}^2 L_{cik} V_F (\text{sign } \cos \eta_{cik}) |\cos \eta_{cik}| \\ & \cdot [J_{Acik}] \left\{ (1 - \nu'_{Tcik}) |\delta_{Fi}| + (\nu'_{Ncik} + \nu'_{Tcik}) \right. \\ & \cdot \cos \eta_{cik} |A_{cik}| \left. \right\} \end{aligned}$$

However

$$(\text{sign } \cos \eta_{cik}) |\cos \eta_{cik}| = \cos \eta_{cik}$$

$$[J_{Acik}] |A_{cik}| = 0$$

Therefore,

$$|M_{cik}|_{ends} = -\frac{1}{8} \pi D_{cik}^2 L_{cik} V_F \cos \eta_{cik} (1 - \nu'_{Tcik}) \left\{ [J_{Acik}] |\delta_{Fi}| \right\} \quad (13.6-4)$$

The total force and moment on the cylinder may now be written as

$$\begin{aligned}
 |P_{cik}| &= |P_{cik}|_{side} + |P_{cik}|_{ends} \\
 |P_{cik}| &= D_{cik} L_{cik} V_F (1 - \nu_{Tcik}) \sin \eta_{cik} |\delta_{Fi}| \\
 &\quad + \frac{2}{3} D_{cik} L_{cik} V_F (\nu_{Ncik} + \nu_{Tcik}) \sin \eta_{cik} \\
 &\quad \left\{ |\delta_{Fi}| - \cos \eta_{cik} |A_{cik}| \right\} + \frac{1}{4} \pi D_{cik}^2 V_F |\cos \eta_{cik}| \\
 &\quad \left\{ (1 - \nu'_{Tcik}) |\delta_{Fi}| + (\nu'_{Ncik} + \nu'_{Tcik}) \cos \eta_{cik} |A_{cik}| \right\}
 \end{aligned}
 \tag{13.6-5}$$

$$\begin{aligned}
 |M_{cik}| &= |M_{cik}|_{side} + |M_{cik}|_{ends} \\
 |M_{cik}| &= \frac{\pi}{8} D_{cik}^2 L_{cik} V_F (\nu_{Tcik} - \nu'_{Tcik}) \cos \eta_{cik} \\
 &\quad \left\{ [J_{Acik}] |\delta_{Fi}| \right\}
 \end{aligned}
 \tag{13.6-6}$$

It should be noted that for a cylinder of uniform reflectivity the net moment is zero for any position of the flow vector.

13.7 FORCES AND MOMENTS ON A RECTANGULAR PARALLELEPIPED

The rectangular parallelepiped attached at the end of boom k of auxiliary body i is defined by three unit vectors normal to any three non-parallel sides and the length, width and depth. The order of the dimensions W_p , L_p , and D_p is arbitrary as long as the vectors are defined consistently. The two opposite sides normal to a given vector can be treated in the same manner as the ends of a cylinder. Since the flow can reach only one side of a parallel pair at any instant and the inside surfaces are shaded, the two sides can be replaced by a single two sided plate. From Equation 13.3-1 the force components on this plate are

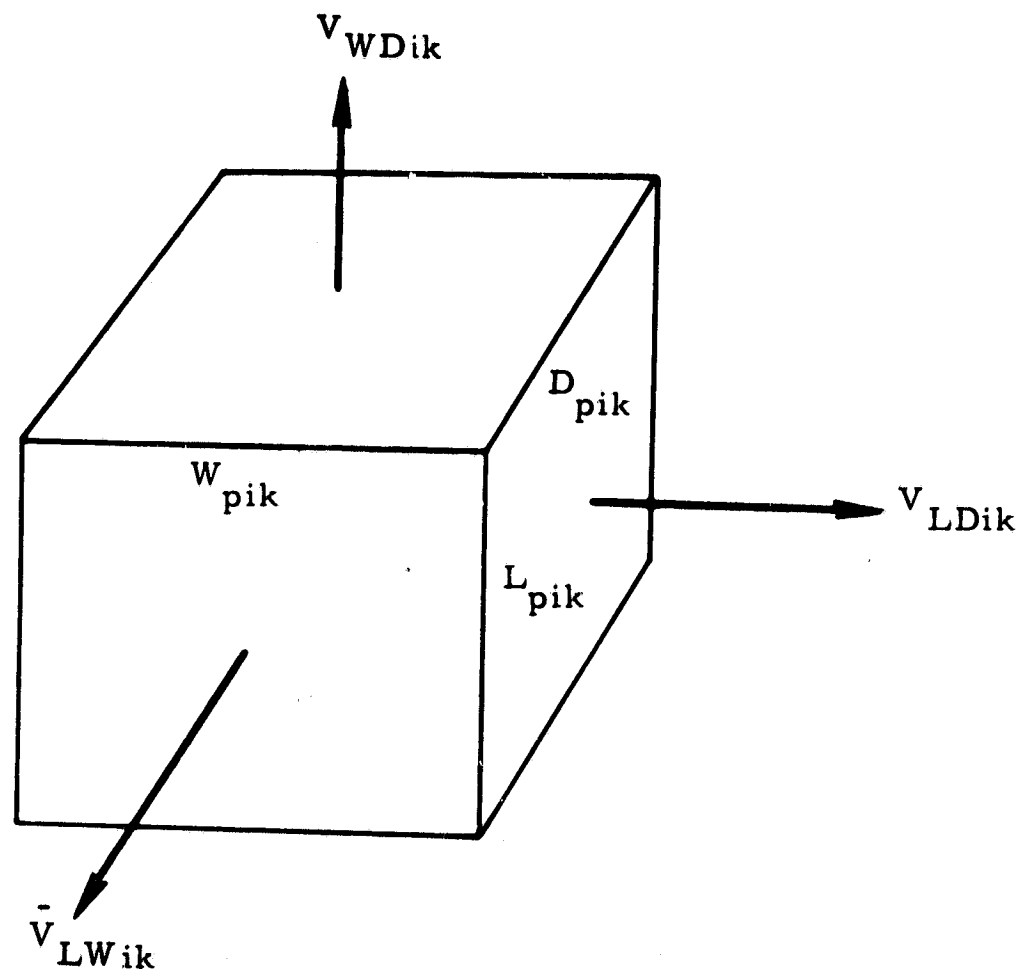


Figure 13.7-1 Definition of Parallelipiped

$$|P_{pik}|_{WD} = W_{pik} D_{pik} V_F |\cos \eta_{WDik}|$$

$$\left\{ (1 - \nu_{TWDik}) |\delta_{Fi}| + (\nu_{NWDik} + \nu_{TWDik}) \cos \eta_{WDik} |V_{WDik}| \right\}$$

(13.7-1)

where

$$\cos \eta_{WDik} = |\delta_{Fi}|_T |V_{WDik}|$$

$$|V_{WDik}| = |\tau_{ik}|_T |\alpha_{ik}|_T |\tau_{ik}| |V'_{WDik}| \text{ from Equation 13.1-10}$$

From Equation 13.6-4 the moment of the centroid of the parallelepiped is

$$|M_{pik}|_{WD} = -\frac{1}{2} W_{pik} D_{pik} L_{pik} V_F \cos \eta_{WDik} (1 - \nu_{TWDik}) [J_{VWDik}] |\delta F_i| \quad (13.7-2)$$

The forces and moments due to pairs of sides LD and LW are found in the same manner and the total force at the centroid of the parallelepiped is

$$|P_{pik}| = |P_{pik}|_{WD} + |P_{pik}|_{LD} + |P_{pik}|_{LW} \quad (13.7-3)$$

and the total moment is

$$|M_{pik}| = |M_{pik}|_{WD} + |M_{pik}|_{LD} + |M_{pik}|_{LW} \quad (13.7-4)$$

13.8 FORCES AND MOMENTS ON A DEFLECTED BOOM ELEMENT

The k^{th} boom of auxiliary body i is defined by the vector ${}_O \bar{R}_{ik}$ along the undeflected position. Vector ${}_O \bar{R}_{ik}$ is defined in body i coordinates with magnitude R equal to the length of the boom in feet. The cross-section of the boom is assumed to be circular with diameter D_{ik} . A flow oriented coordinate system is established with its origin at the base of the boom and its $\bar{y}_{Bik}$ axis along ${}_O \bar{R}_{ik}$. The solar oriented system is then identical to the solar oriented system established for the derivation of the boom bending equations. For boom computations the angle η is replaced by the angle λ . Since the equations are long and involved, the subscript ik will be dropped in the derivation and reintroduced into the final equation. In boom flow oriented coordinates, Figure 13.8-1.

$$|\delta F| = \begin{Bmatrix} \sin \lambda \\ \cos \lambda \\ 0 \end{Bmatrix} \quad (13.8-1)$$

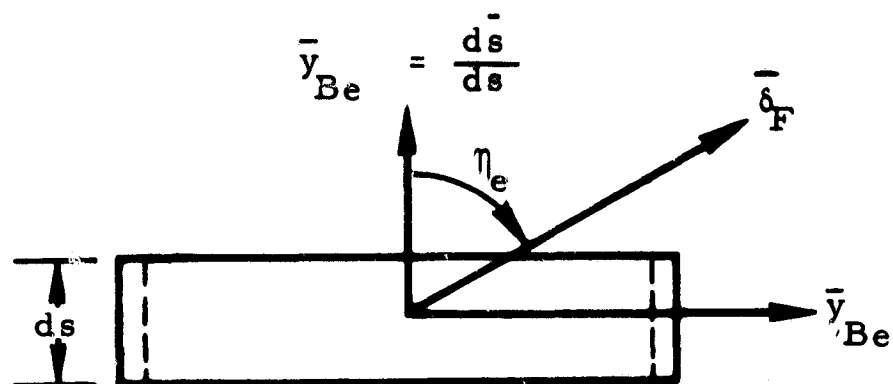


Figure 13.8-2 Boom Element

$$\cos \eta_e = \bar{\delta}_F \cdot \bar{y}_{Be} = \frac{1}{ds} \begin{Bmatrix} \sin \lambda \\ \cos \lambda \\ 0 \end{Bmatrix}_T \begin{Bmatrix} dx_B \\ dy_B \\ dz_B \end{Bmatrix}$$

$$\cos \eta_e = \frac{dx_B}{ds} \sin \lambda + \frac{dy_B}{ds} \cos \lambda$$

For small deflections

$$\frac{dy_B}{ds} \approx 1.0$$

$$\frac{dx_B}{ds} \approx \frac{dx_B}{dy_B} \equiv x'_B$$

$$\frac{dz_B}{ds} \approx \frac{dz_B}{dy_B} = z'_B$$

$$1 \pm (x'_B)^2 \approx 1 \pm (z'_B)^2 \approx 1.0$$

$$\text{Then } \cos \eta_e = \cos \lambda + x'_B \sin \lambda$$

(13.8-4)

and for no deflection $\cos \eta_e \neq \cos \lambda$

$$\sin^2 \eta_e = 1 - \cos^2 \eta_e = (\sin \lambda - x'_B \cos \lambda)^2$$

$$\sin \eta_e = \sin \lambda - x'_B \cos \lambda \quad (13.8-5)$$

and for no deflection $\sin \eta_e = \sin \lambda$

From Equation 13.8-3

$$(y_{Be}) \approx \begin{Bmatrix} x'_B \\ 1 \\ z'_B \end{Bmatrix} \quad (13.8-6)$$

The force components on the boom element in the element flow oriented coordinate system $\bar{x}_{Be}$, $\bar{y}_{Be}$, and $\bar{z}_{Be}$ may be obtained from Equation 13.5-10.

$$\begin{aligned} |dp|_e &= DV_F (1 - \nu_T) \begin{Bmatrix} \sin^2 \eta_e \\ \sin \eta_e \cos \eta_e \\ 0 \end{Bmatrix} dy_B \\ &+ \frac{D}{2} V_F (\nu_N + \nu_T) \sin^2 \eta_e \begin{Bmatrix} \frac{4}{3} \\ 0 \\ 0 \end{Bmatrix} dy_B \end{aligned} \quad (13.8-7)$$

Let the transformation matrix from the boom flow oriented system to the element flow oriented system be $[\eta_e]$, then the force components on the boom element in the boom flow oriented coordinate system $\bar{x}_B$, $\bar{y}_B$, and $\bar{z}_B$ is

$$|dp|_B = [\eta_e]_T |dp|_e$$

Then from Equation 13.8-7

$$\begin{aligned} \left| dp \right|_B = DV_F (1 - \nu_T) \sin \eta_e \left[\eta_e \right]_T \begin{Bmatrix} \sin \eta_e \\ \cos \eta_e \\ 0 \end{Bmatrix} dy_B \\ + \frac{2D}{3} V_F (\nu_N + \nu_T) \sin^2 \eta_e \left[\eta_e \right] \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} dy_B \quad (13.8-8) \end{aligned}$$

Since $\left| dp \right|_B$ is in boom-flow oriented coordinates Equation 13.8-8 may be integrated over the length of the boom to get the total force. However, the variables $\sin \eta_e$, $\cos \eta_e$, and $\left[\eta_e \right]_T$ must first be put in the boom parameters x'_B , z'_B , and λ . The expressions for $\sin \eta_e$ and $\cos \eta_e$ are given by Equations 13.8-4, and 13.8-5 while

$$\left[\eta_e \right]_T = \begin{bmatrix} \left| x_{Be} \right| & \left| y_{Be} \right| & \left| z_{Be} \right| \end{bmatrix} \quad (13.8-9)$$

From Figure 13.8-2

$$\bar{\delta}_F = \sin \eta_e \bar{x}_{Be} + \cos \eta_e \bar{y}_{Be}$$

and

$$\left| x_{Be} \right| = \frac{1}{\sin \eta_e} \left[\left| \delta_F \right| - \cos \eta_e \left| y_{Be} \right| \right]$$

From Figure 13.8-1, Equations 13.8-4, 13.8-6

$$|x_{Be}| = \frac{1}{\sin \eta_e} \begin{Bmatrix} \sin \lambda - x'_B \cos \lambda \\ -x'_B \sin \lambda \\ -z'_B (\cos \lambda + x'_B \sin \lambda) \end{Bmatrix} \quad (13.8-10)$$

From Equations 13.8-5 and 13.8-6

$$|y_{Be}| = \frac{1}{\sin \eta_e} \begin{Bmatrix} x'_B \sin \eta_e \\ \sin \eta_e \\ z'_B \sin \eta_e \end{Bmatrix} = \frac{1}{\sin \eta_e} \begin{Bmatrix} x'_B \sin \lambda - (x'_B)^2 \cos \lambda \\ \sin \lambda - x'_B \cos \lambda \\ z'_B (\sin \lambda - x'_B \cos \lambda) \end{Bmatrix} \quad (13.8-11)$$

$$|z_{Be}| = -[J_{yBe}] |x_{Be}|$$

where

$$[J_{yBe}] = \begin{bmatrix} 0 & -z'_B & 1 \\ z'_B & 0 & -x'_B \\ -1 & x'_B & 0 \end{bmatrix}$$

from which

$$|z_{Be}| = \frac{1}{\sin \eta_e} \begin{Bmatrix} z'_B \cos \lambda \\ -z'_B \sin \lambda \\ \sin \lambda - x'_B \cos \lambda \end{Bmatrix} \quad (13.8-12)$$

From Equations 13.8-9, 13.8-10, 13.8-11, and 13.8-12

$$\begin{aligned} \sin \eta_e [\eta_e]_T &= \begin{bmatrix} \sin \lambda & 0 & 0 \\ 0 & \sin \lambda & 0 \\ 0 & 0 & \sin \lambda \end{bmatrix} + \begin{bmatrix} (-x'_B \cos \lambda)(x'_B \sin \lambda)(z'_B \cos \lambda) \\ (-x'_B \sin \lambda)(-x'_B \cos \lambda)(-z'_B \sin \lambda) \\ (-z'_B \cos \lambda)(z'_B \sin \lambda)(-x'_B \cos \lambda) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & (-x'_B)^2 \cos \lambda & 0 \\ 0 & 0 & 0 \\ (-x'_B z'_B \sin \lambda) & (-x'_B z'_B \cos \lambda) & 0 \end{bmatrix} \end{aligned} \quad (13.8-13)$$

From Equations 13.8-4, 13.8-5, and 13.8-13

$$\sin \eta_e [\eta_e]_T \begin{Bmatrix} \sin \eta_e \\ \cos \eta_e \\ 0 \end{Bmatrix} = \begin{Bmatrix} \sin^2 \lambda - x'_B \sin \lambda \cos \lambda \\ \sin \lambda \cos \lambda - x'_B \cos^2 \lambda \\ 0 \end{Bmatrix} \quad (13.8-14)$$

$$\begin{aligned} \sin^2 \eta_e [\eta_e]_T \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} &= \begin{Bmatrix} \sin^2 \lambda \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -2x'_B \sin \lambda \cos \lambda \\ -x'_B \sin^2 \lambda \\ -z'_B \sin \lambda \cos \lambda \end{Bmatrix} \begin{Bmatrix} (x'_B)^2 \cos^2 \lambda \\ (x'_B)^2 \sin \lambda \cos \lambda \\ x'_B z'_B (\cos^2 \lambda - \sin^2 \lambda) \end{Bmatrix} \\ &\quad (13.8-15) \end{aligned}$$

Stustituting into Equations 13.8-8

$$\begin{aligned}
 |dp|_B = D_A V_F (1 - \nu_T) dy_B & \begin{Bmatrix} \sin^2 \lambda - x'_B \sin \lambda \cos \lambda \\ \sin \lambda \cos \lambda - x'_B \cos^2 \lambda \\ 0 \end{Bmatrix} \\
 + \frac{2D}{3} V_F (\nu_N + \nu_T) dy_B & \begin{Bmatrix} \sin^2 \lambda - 2x'_B \sin \lambda \cos \lambda + (x'_B)^2 \cos^2 \lambda \\ -x'_B \sin^2 \lambda + (x'_B)^2 \sin \lambda \cos \lambda \\ x'_B z'_B (\cos^2 \lambda - \sin^2 \lambda) - z'_B \sin \lambda \cos \lambda \end{Bmatrix}
 \end{aligned}
 \tag{13.8-16}$$

Equation 13.8-16 is the general expression for the force components on an element of a boom in boom coordinates.

Let $|dM_o|$ equal the components of the moment about the centroid of the cylindrical element. From Equation 13.5-13 the components in element-flow oriented coordinates are

$$|dM_o|_e = -\frac{\pi D^2}{8} V_F (1 - \nu_T) dy_B \sin \eta_e \cos \eta_e \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

and

$$|dM_o|_B = [\eta_e]_T |dM_o|_e = -\frac{\pi D^2}{8} V_F (1 - \nu_T) dy_B \sin \eta_e \cos \eta_e [\eta_e]_T \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}
 \tag{13.8-17}$$

From Equation 13.8-4, and 13.8-13

$$\sin \eta_e \cos \eta_e [\eta_e]_T \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} z'_B \cos^2 \lambda & x'_B z'_B \sin \lambda \cos \lambda \\ -z'_B \sin \lambda \cos \lambda & -x'_B z'_B \sin^2 \lambda \\ \sin \lambda \cos \lambda & -x'_B (\cos^2 \lambda - \sin^2 \lambda) \end{Bmatrix}$$

Substituting into Equation 13.8-17

$$\left| dM_o \right|_B = -\frac{\pi D^2}{8} V_F (1 - \nu_T) \begin{Bmatrix} z'_B \cos^2 \lambda + x'_B z'_B \sin \lambda \cos \lambda \\ -z'_B \sin \lambda \cos \lambda - x'_B z'_B \sin^2 \lambda \\ \sin \lambda \cos \lambda - x'_B (\cos^2 \lambda - \sin^2 \lambda) \end{Bmatrix} dy_B$$

(13.8-18)

Equation 13.8-18 is the general expression for the moment components about the centroid for an element of a boom in boom coordinates.

The moment at the base of the boom resulting from the forces on the the boom element is given by

$$\left| dM \right|_B = \left| dM_o \right|_B + \left[J_{\rho B} \right] \left| dp \right|_B$$

where $\bar{\rho}_B$ is the vector from the base of the boom to the centroid of the boom element.

$$\left| \rho_B \right| = \begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix}$$

Then

$$\left| dM \right|_B = \left| dM \right|_B + \begin{bmatrix} 0 & -z_B & y_B \\ z_B & 0 & -x_B \\ -y_B & x_B & 0 \end{bmatrix} \left| dp \right|_B \quad (13.8-19)$$

Substituting Equation 13.8-16

$$\begin{aligned} \left| dM \right|_B = \left| dM_o \right|_B & DV_F (1 - v_T) dy_B \begin{bmatrix} -z_B \sin \lambda \cos \lambda + x'_B z_B \cos^2 \lambda \\ z_B \sin^2 \lambda - x'_B z_B \sin \lambda \cos \lambda \\ -y_B \sin^2 \lambda - x'_B y_B \sin \lambda \cos \lambda + x_B \sin \lambda \cos \lambda - x_B x'_B \cos^2 \lambda \end{bmatrix} \\ & + \frac{2}{3} DV_F (v_N + v_T) dy_B \begin{bmatrix} +z_B x'_B \sin^2 \lambda - y_B z'_B \sin \lambda \cos \lambda - z_B (x'_B)^2 \sin \lambda \cos \lambda + y_B x'_B z'_B (\cos^2 \lambda - \sin^2 \lambda) \\ z_B \sin^2 \lambda - 2z_B x'_B \sin \lambda \cos \lambda + x_B z'_B \sin \lambda \cos \lambda + z_B (x'_B)^2 \cos^2 \lambda - x_B x'_B z'_B (\cos^2 \lambda - \sin^2 \lambda) \\ -y_B \sin^2 \lambda + 2y_B x'_B \sin \lambda \cos \lambda - x_B x'_B \sin^2 \lambda - y_B (x'_B)^2 \cos^2 \lambda + x_B (x'_B)^2 \sin \lambda \cos \lambda \end{bmatrix} \end{aligned} \quad (13.8-20)$$

Equation 13.8-20 is the general expression for the moment components about the base of the boom for an element of a boom in boom coordinates.

13.9 SOLAR FORCES AND MOMENTS ON A DEFLECTED BOOM

The expressions for the force and moment due to a particle flow impinging upon an element of a deflected boom element are derived in Section 13.8. The force at the centroid of the element is given by Equation 13.8-16 and the resulting moment at the base of the boom is given by Equation 13.8-20. The force and moment components are with respect to a flow oriented coordinate system $\bar{x}_B, \bar{y}_B, \bar{z}_B$ (see Figure 13.8-1), and are expressed in terms of the element location x_B, y_B, z_B expressed in this system. In the case of solar force and torques the components are defined with respect to a solar oriented system and the boom deflection with respect to this system are derived in Section 14.0.

From Equations 14.5-1 and 14.5-2

$$x_B = k_e C_B \frac{y_B^2}{R} \sin \lambda \quad (13.9-1)$$

$$z_B = k_e C'_B \sin \sigma \frac{y_B^2}{R} \sin \lambda \quad (13.9-2)$$

differentiating with respect to y_B

$$x'_B = 2k_e C_B \frac{y_B}{R} \sin \lambda \quad (13.9-3)$$

$$z'_B = 2k_e C'_B \sin \sigma \frac{y_B}{R} \sin \lambda \quad (13.9-4)$$

Substituting Equations 13.9-1, 13.9-2, 13.9-3, and 13.9-4 into Equations 13.8-16 and 13.8-20 and integrating dy_B from 0 to R Equations 13.9-5 and 13.9-6 of Table 13.9-1 are obtained. Equation 13.9-5 gives the components of the total solar force on a thermally bent boom in solar oriented coordinates.

Equation 13.9-6 gives the components of the total solar moment at the base of a thermally bent boom in solar oriented coordinates. The corresponding force and moment components in body i coordinates are given by Equation 13.9-7, and 13.9-8 where $[\tau_{ik}]$ is the transformation from body i coordinates to boom coordinates. The matrix $[\tau_{ik}]$ has already been computed in connection with thermal bending disturbances, Equation 14.2-27. In programming Equations 13.9-5 and 13.9-6 the second and third order terms in C_B and C'_B have been neglected.

13.10 TOTAL SOLAR FORCES AND TORQUES

Let the general expressions for the solar force and solar torque about the centroid of the k^{th} shape element of body i be $\bar{P}_{Nik}$ and $\bar{M}_{Nik}$. The subscript N will change to indicate the type of shape considered: F for flat plate, S for sphere, C for cylinder, and P for parallelepiped. In

Table 13.9-1. Solar Forces and Moments on a Deflected Boom

$$\{P_B\}_B = V_S DR^2 (1-\nu_T) \left\{ \begin{array}{c} \sin^2 \lambda \\ \sin \lambda \cos \lambda \\ 0 \end{array} \right\} - k_E \left\{ \begin{array}{c} C_B \sin^2 \lambda \cos \lambda \\ C_B \sin \lambda \cos^2 \lambda \\ 0 \end{array} \right\} + \frac{2}{3} V_S DR^2 (\nu_T + \nu_N) \left\{ \begin{array}{c} \sin^2 \lambda \\ 0 \\ 0 \end{array} \right\} - k_E \left\{ \begin{array}{c} 2 C_B \sin^2 \lambda \cos \lambda \\ C_B \sin^3 \lambda \\ C_B' \sin \sigma \sin^2 \lambda \cos \lambda \end{array} \right\} + \frac{4}{3} k_E \left\{ \begin{array}{c} C_B C_B' \sin^2 \lambda \cos^2 \lambda \\ C_B C_B' \sin^3 \lambda \cos \lambda \\ C_B C_B' \sin \sigma (\sin^2 \lambda \cos^2 \lambda - \sin^4 \lambda) \end{array} \right\} \quad (13.9-5)$$

$$\{M_B\}_B = V_S DR^2 (1-\nu_T) \left\{ \begin{array}{c} 0 \\ 0 \\ \cos^2 \lambda \end{array} \right\} + \frac{1}{3} k_E \left\{ \begin{array}{c} -C_B' \sin \sigma \sin^2 \lambda \cos \lambda \\ C_B' \sin \sigma \sin^3 \lambda \\ -C_B \sin^2 \lambda \cos \lambda \end{array} \right\} + \frac{1}{2} k_E \left\{ \begin{array}{c} C_B C_B' \sin \sigma \sin^2 \lambda \cos^2 \lambda \\ -C_B C_B' \sin \sigma \sin^3 \lambda \cos \lambda \\ -C_B C_B' \sin^2 \lambda \cos^2 \lambda \end{array} \right\} \\ + \frac{2}{3} V_S DR^2 (\nu_T + \nu_N) \left\{ \begin{array}{c} 0 \\ 0 \\ \sin^2 \lambda \end{array} \right\} - \frac{1}{2} k_E \left\{ \begin{array}{c} -2 C_B' \sin \sigma \sin^2 \lambda \cos \lambda \\ C_B' \sin \sigma \sin^3 \lambda \\ 4 C_B \sin^2 \lambda \cos \lambda \end{array} \right\} + \frac{1}{2} k_E \left\{ \begin{array}{c} 2 C_B C_B' \sin \sigma (\sin^2 \lambda \cos^2 \lambda - \sin^4 \lambda) \\ -C_B C_B' \sin \sigma \sin^3 \lambda \cos \lambda \\ -C_B C_B' \sin^4 \lambda - 2 C_B C_B' \sin^2 \lambda \cos^2 \lambda \end{array} \right\} + \frac{4}{5} k_E \left\{ \begin{array}{c} -C_B C_B C_B' \sin \sigma \sin^4 \lambda \cos \lambda \\ C_B C_B C_B' \sin \sigma \sin^5 \lambda \\ C_B C_B C_B' \sin^4 \lambda \cos \lambda \end{array} \right\} \\ - \frac{\pi D^2}{8} R V_S (1-\nu_T) \left\{ \begin{array}{c} 0 \\ 0 \\ \sin \lambda \cos \lambda \end{array} \right\} + k_E \left\{ \begin{array}{c} C_B' \sin \sigma \sin \lambda \cos^2 \lambda \\ -C_B' \sin \sigma \sin^2 \lambda \cos \lambda \\ -C_B' (\sin \lambda \cos^2 \lambda - \sin^3 \lambda) \end{array} \right\} + \frac{4}{3} k_E \left\{ \begin{array}{c} C_B C_B' \sin \sigma \sin^2 \lambda \cos \lambda \\ -C_B C_B' \sin \sigma \sin^4 \lambda \\ 0 \end{array} \right\} \quad (13.9-6)$$

The subscripts i k have been omitted from all parameters except V_S and k_E in Equations 13.9-5 and 13.9-6

$$\{P_{Bib}\} = \{r_{ik}\}_T \{P_{B_{ik}B}\} \quad (13.9-7)$$

$$\{M_{Bib}\} = \{r_{ik}\}_T \{M_{B_{ik}B}\} \quad (13.9-8)$$

addition let $\bar{C}_{Plk}$ be the vector location of the center of pressure of the k^{th} shape of body 1 in body 1 coordinates. Then for body 1

$$|F_{S11}| = \sum_k |P_{N1k}| \quad 13.10-1)$$

$$|T_{S11}| = \sum_k \left\{ |M_{N1k}| + [J_{CP1k}] |P_{N1k}| \right\} \quad 13.10-2)$$

and for auxiliary body i

$$|F_{Sii}| = \sum_k \left\{ |P_{Nik}| + |P_{Bik}| \right\}$$

$$|T_{Sii}| = \sum_k \left\{ |M_{Nik}| + [J_{Rik}] |P_{Nik}| \right\} \quad 13.10-4)$$

13.11 AERODYNAMIC FORCES AND TORQUES

The present version of the program, Version C, does not include the capability for computing aerodynamic forces and torques. However, provisions have been made for the future incorporation of this capability. Aerodynamic force and torque components have been programmed into the dynamic equations of motion, and a dummy aerodynamic subroutine is provided which presently sets these forces and torques to zero. Future expansion of the program will consist of replacing the dummy routine by one which computes aerodynamic forces and torques using the composite equations already presented for solar disturbance.

14.0 BOOM DEFLECTION EQUATIONS

It is assumed that each of the rods supporting the masses m_{ik} of the auxiliary bodies may change shape due to thermal bending. All of the forces and torques associated with thermal bending are internal and there is no net change in the momentum of the system, however, the bodies will change their relative positions. The torques are reflected into the dynamic equations in a number of ways. Consider Equation 4.1-1

$$\begin{aligned} \{T_{Eii}\} = & M_i [J_{vii}] \{G_{nii}\} + [I_{in}] \{\dot{\omega}_{ii}\} + [J_{\omega ii}] [I_{iH}] \{\omega_{ii}\} \\ & + [I_{iH}] \{\omega_{ii}\} + [J_{\omega ii}] \{H_{ii}\} + \{H_{ii}^{\cdot}\} \end{aligned}$$

If thermal bending changes the body inertia matrix by $[\Delta I_{iH}]$, the torques $[\Delta I_{iH}] \{\dot{\omega}_{ii}\}$ and $[J_{\omega ii}] [\Delta I_{iH}] \{\omega_{ii}\}$ are created. Since $[I_{iH}]$ is due only to thermal bending, the torque $[I_{iH}] \{\dot{\omega}_{ii}\}$ is a thermal bending torque. The change of body i momentum and the rate of change of momentum with respect to the body axes are partly due to thermal bending and the torques $[J_{\omega ii}] \{H_{Rii}\}$ and $\{H_{Rii}^{\cdot}\}$ are created. The center of mass location $\{v_{ii}\}$ together with the linear acceleration $\{G_{nii}\}$ are affected by thermal bending. Therefore, all of the inertia torque terms at the right hand side of Equation 4.1-1 are affected. In addition the inertia change is reflected into the gravity torques and the center of mass movement is reflected into the hinge reaction forces. If $\{\Delta R_{ik}\}$ are the components of the deflection of mass m_{ik} due to thermal bending, all of the above effects may be expressed in terms of the components $\{\Delta R_{ik}\}$, $\{\Delta \dot{R}_{ik}\}$, and $\{\Delta \ddot{R}_{ik}\}$. Expressions for these components will be derived in this section.

14.1 NORMALIZED DEFLECTION EQUATION

Consider a single rod or boom deflecting in the $\bar{x}_B, \bar{y}_B$ plane due to thermal heating, where the $\bar{x}_B, \bar{y}_B$ plane is the plane containing the undeflected boom position $\bar{y}_B$ and the unit sun vector $\bar{\delta}_s$, Figure 14.1-1.

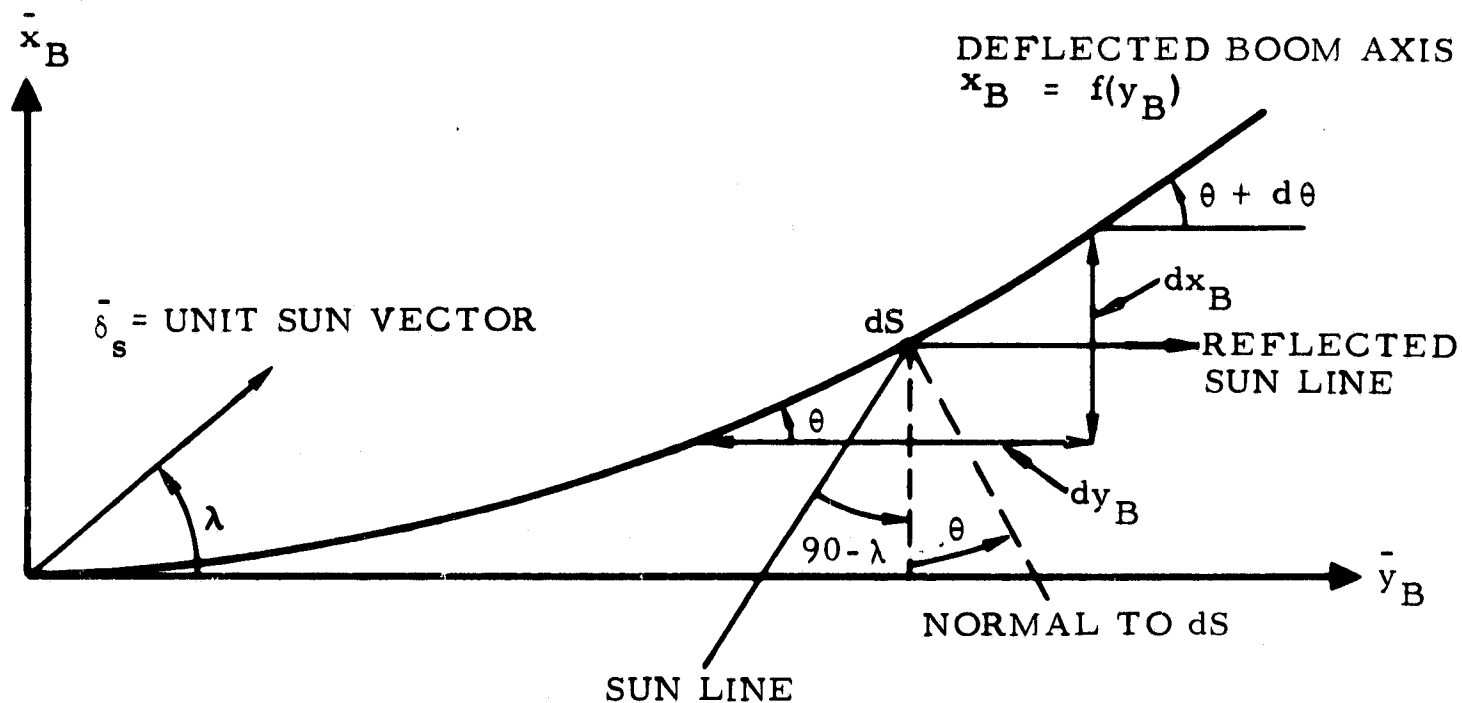


Figure 14.1-1. Deflected Boom Parameters

The heating of element dS is proportional to $(1-\nu) \cos (90-\lambda+\theta)$, where ν is the reflectivity constant for the boom and $(90-\lambda+\theta)$ is the angle between the sun vector and the normal to the element dS . The change in direction of the boom in length dS is $d\theta$ where $d\theta$ is proportional to the heat absorbed by the element.

Then

$$d\theta = \beta_B (1-\nu) [\sin \lambda \cos \theta dS - \cos \lambda \sin \theta dS] \quad (14.1-1)$$

where β_B = radians per feet for total absorption and a normal sun vector.
From Figure 14.0-1

$$dS \cos \theta = dy_B$$

$$dS \sin \theta = dx_B$$

$$\theta = \tan^{-1} \frac{dx_B}{dy_B}$$

then

$$\frac{d\theta}{dy_B} = \beta_B (1-\nu) \left[\sin \lambda - \cos \lambda \frac{dx_B}{dy_B} \right]$$

but

$$\frac{d\theta}{dy_B} = \frac{\frac{dx_B^2}{dy_B^2}}{1 + \left(\frac{dx_B}{dy_B}\right)^2}$$

and

$$\frac{dx_B^2}{dy_B^2} = \left[1 + \left(\frac{dx_B}{dy_B}\right)^2\right] \beta_B (1-\nu) \left[\sin\lambda - \cos\lambda \frac{dx_B}{dy_B}\right] \quad (14.1-2)$$

assume that $1 + \left(\frac{dx_B}{dy_B}\right)^2 \approx 1$ then

$$\frac{dx_B^2}{dy_B^2} = \beta_B (1-\nu) \left[\sin\lambda - \cos\lambda \frac{dx_B}{dy_B}\right] \quad (14.1-3)$$

Let

$$\left. \begin{aligned} P_1 &= \beta_B (1-\nu) \sin \lambda \\ P_2 &= \beta_B (1-\nu) \cos \lambda \end{aligned} \right\} \text{constants for time } t \quad (14.1-4)$$

Then

$$\frac{d^2 x_B}{dy_B^2} = P_1 - P_2 \frac{dx_B}{dy_B}$$

Integrating

$$\frac{dx_B}{dy_B} = P_1 y_B - P_2 x_B \quad (14.1-5)$$

The solution of this differential equation is

$$x_B = \frac{P_1}{P_2^2} \left[e^{-P_2 y_B} - 1 \right] + \frac{P_1}{P_2} y_B \quad (14.1-6)$$

Substituting for P_1 and P_2

$$x_B = \frac{\sin \lambda}{\cos \lambda} \left[\frac{e^{-\beta_B y_B (1-\nu) \cos \lambda} - 1}{\beta_B (1-\nu) \cos \lambda} + y_B \right]$$

Let

$$y_B = k_b L$$

$$x_B = m L$$

where L is the length of the boom and $0 \leq k_b \leq 1$, the normalized deflection equation is then

$$m = \frac{\sin \lambda}{\cos \lambda} \left[\frac{e^{-[\beta_B L (1-\nu) \cos \lambda] k_b} - 1}{\beta_B L (1-\nu) \cos \lambda} + k_b \right]$$

$\beta_B L (1-\nu) = 2C_B$ where C_B is a non-dimensional coefficient defining the material and length of boom.

Then

$$m = 2C_B \sin \lambda \left[\frac{e^{-(2C_B \cos \lambda) k} - 1}{(2C_B \cos \lambda)^2} + \frac{k}{2C_B \cos \lambda} \right] \quad (14.1-7)$$

Consider the expansion of e^{-ak}

$$e^{-ak} = 1 - ak + \frac{a^2 k^2}{2!} - \frac{a^3 k^3}{3!} + \frac{a^4 k^4}{4!} - \frac{a^5 k^5}{5!} + \dots$$

Then

$$\left[\frac{e^{-ak} - 1}{a^2} + \frac{k}{a} \right] = \left[\frac{k^2}{2} - \frac{a k^3}{6} + \frac{a^2 k^4}{24} - \frac{a^3 k^5}{120} + \dots \right]$$

Letting $a = 2C_B \cos \lambda$ Equation 14.0-7 becomes

$$m = 2C_B \sin \lambda \left[\frac{k^2}{2} - \frac{k^3 (2C_B \cos \lambda)}{6} + \frac{k^4 (2C_B \cos \lambda)^2}{24} - \frac{k^5 (2C_B \cos \lambda)^3}{120} + \dots \right] \quad (14.1-8)$$

The nominal value of the coefficient C_B is approximately .01, therefore, the maximum values of the terms in the series expansion are

$$[.5 - .003 + .000066 - .0000003 + \dots]$$

and neglecting all but the first term will introduce an error of less than 1/2%. Then the normalized deflection equation becomes

$$m = C_B k_b^2 \sin \lambda \quad (14.1-9)$$

The deflection curve is a parabola, the magnitude of m is proportional to the sine of the angle between the sun vector and the nominal position of the boom. Consider the maximum deflection which occurs when the sun vector is normal to the nominal boom position, $\lambda = 90$ degrees.

$$m_o = C_B k_b^2 \quad (14.1-10)$$

The maximum deflection curve is shown in Figure 14.1-2

It may be seen from Figure 14.1-2 that when $k = 1.0$, y_B equals L and the length of the curve, S , is greater than L . The approximate length of the arc of a parabola is given by

$$S = \sqrt{y_B^2 + \frac{4 x_B^2}{3}}$$

$$\frac{S}{L} = \sqrt{k_b^2 + \frac{4 m^2}{3}}$$

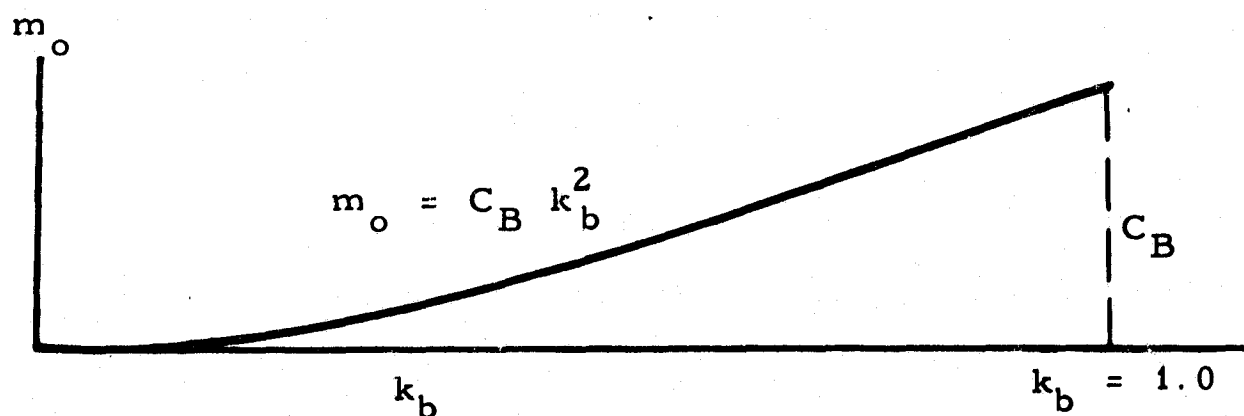


Figure 14.1-2. Normalized Boom Deflection

$$\frac{S}{L} = \sqrt{k^2 + \frac{4}{3} C_B^2 k^4}$$

Then for $k = 1$

$$\left(\frac{S}{L}\right)' = \sqrt{1 + \frac{4}{3} C_B^2}$$

and for $C_B = .01$

$$\left(\frac{S}{L}\right)' = 1.0001$$

Therefore, it is assumed that C_B is the maximum deflection of the end of the boom and that the end of the boom deflects normal to the nominal position. The normalized deflection equation is

$$m = C_B k_b^2 \sin \lambda \quad (14.1-11)$$

The slope of the deflected boom is given by

$$\frac{dm}{dk_b} = 2C_B k_b \sin \lambda \quad (14.1-12)$$

where $2C_B \sin \lambda$ is the slope at the end of the boom and C_B is the maximum normalized deflection at the end of the boom.

14.2 THERMAL BENDING DEFLECTIONS

The plane containing the nominal position vector of mass m_{ik} , $\bar{R}_{ik}$, and the instantaneous unit sun vector $\bar{\delta}_s$ is defined as the instantaneous bending plane for mass m_{ik} . It is assumed that the total deflection need not lie in the instantaneous bending plane and at any instant of time may be expressed as the sum of an in-plane component in the bending plane and an out-of-plane component normal to the bending plane. Both components are taken to be normal to the nominal position vector of the rod, $\bar{R}_{ik}$, Figure 14.2-1.

Then

$$\Delta \bar{R}_{ik} = (\Delta \bar{R}_{ik})_x + (\Delta \bar{R}_{ik})_z \quad (14.2-1)$$

and the instantaneous vector position of mass m_{ik} is given by

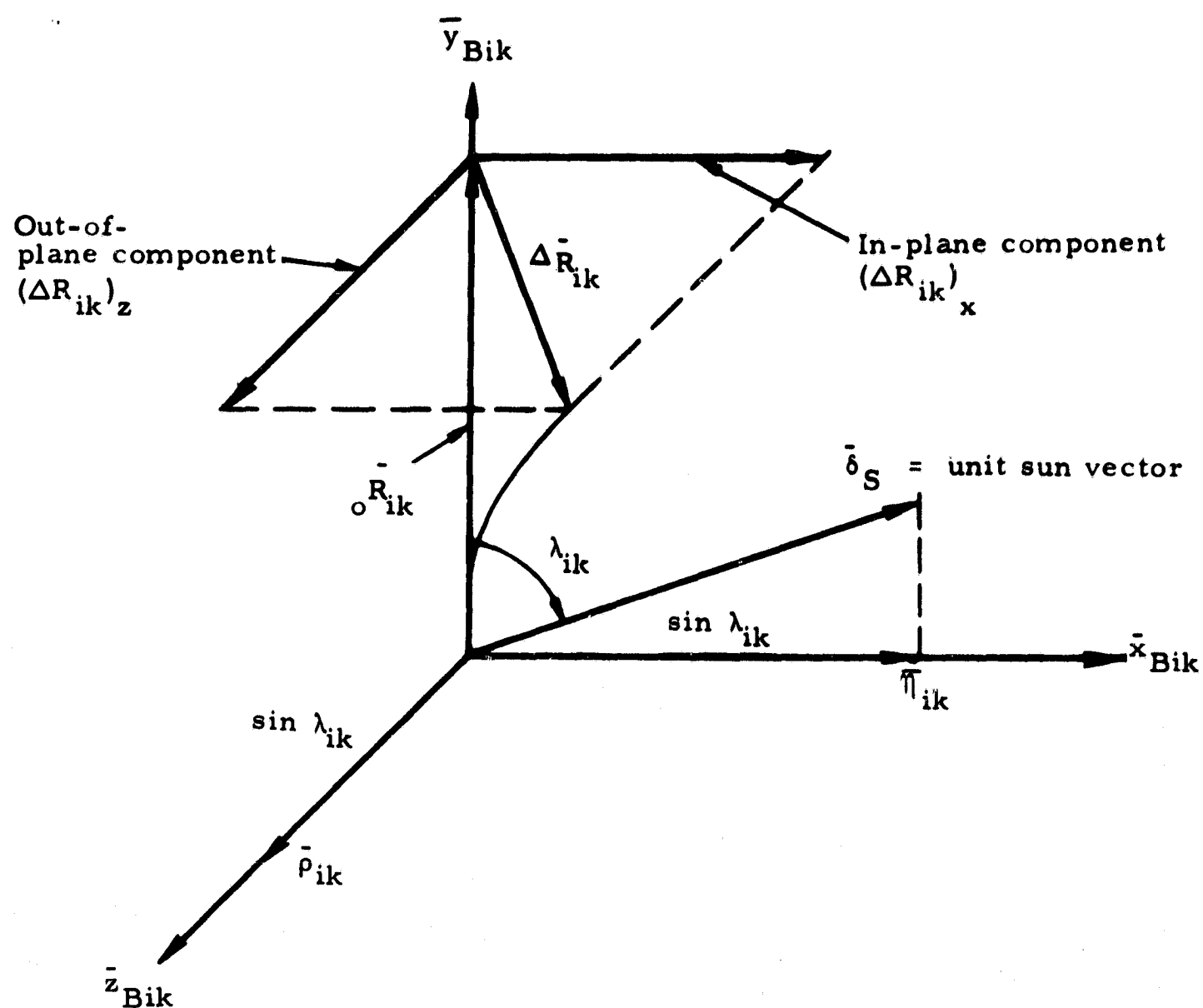


Figure 14.2-1 Tip Mass Displacement

$$\bar{R}_{ik} = {}_o\bar{R}_{ik} + \Delta\bar{R}_{ik} \quad (14.2-2)$$

and when expressed as components in body i coordinates

$$\{R_{ik}\} = \{{}_oR_{ik}\} + \{\Delta R_{ik}\} \quad (14.2-3)$$

where subscripts denoting the coordinate system are not used since the $\bar{R}$ vectors will always be defined in the corresponding body i coordinates. It should also be noted the components $\{{}_oR_{ik}\}$ are input constants.

The angle λ_{ik} is the angle between the nominal position vector and the unit sun vector. The vectors $\bar{\eta}_{ik}$ and $\bar{\rho}_{ik}$ are normal to the nominal position vector. η_{ik} is in the instantaneous bending plane with the magnitude $\sin\lambda_{ik}$ and is, therefore, the component of the unit sun vector normal to the nominal position vector. $\bar{\rho}_{ik}$ is normal to the instantaneous bending plane and also has the magnitude of $\sin\lambda_{ik}$.

From Equation 14.1-11, the magnitude of the in-plane deflection at the end of the rod is

$$\left| (\Delta\bar{R}_{ik})_x \right| = R_{ik} C_{Bik} \sin \lambda_{ik}$$

where R_{ik} is the length of rod ik and k_b is unity.

Since $(\Delta\bar{R}_{ik})_x$ is the direction of η_{ik} which has the magnitude $\sin\lambda_{ik}$, the in-plane bending component is given by

$$(\Delta\bar{R}_{ik})_x = R_{ik} C_{Bik} \bar{\eta}_{ik} \quad (14.2-4)$$

The magnitude of the out-of-plane deflection is assumed to be a function of the relative positions of the instantaneous bending plane and a reference bending plane fixed in body i. The reference bending plane is defined as the body fixed bending plane for which the out-of-plane bending deflection is zero. Since the reference bending plane for mass m_{ik} must contain the nominal position vector, ${}_o\bar{R}_{ik}$, its position is defined by ${}_o\bar{R}_{ik}$ and any other vector $\bar{F}_{ik}$ in the plane, not parallel to ${}_o\bar{R}_{ik}$. Let $\bar{\rho}_{ik}$ be a unit vector along the component of $\bar{F}_{ik}$ in the $\bar{\rho}_{ik} - \bar{\eta}_{ik}$ plane normal to ${}_o\bar{R}_{ik}$,
Figure 12.2-2

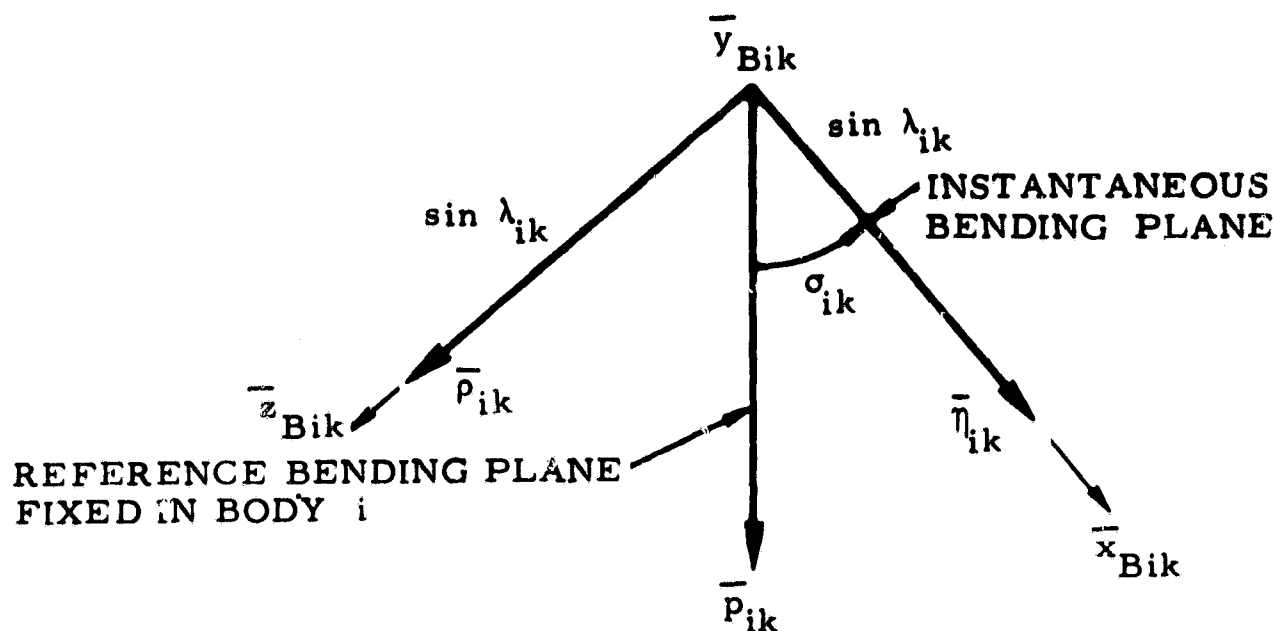


Figure 14.2-2. Thermal Bending Planes

Let the maximum magnitude of the out-of-plane bending be defined by Equation 14.1-11 and let the magnitude vary as the sine of σ_{ik} . Where σ_{ik} is the relative position of the instantaneous bending plane with respect to the reference bending plane measured by right-hand rotation about $\bar{R}_{ik}$.

From Equation 14.1-11 the magnitude of the out-of-plane deflection is

$$\left| (\Delta \bar{R}_{ik})_z \right| = R_{ik} C'_{Bik} \sin \lambda_{ik} \sin \sigma_{ik}$$

where C'_{Bik} is the out-of-plane bending coefficient. The direction of the out-of-plane component is along $\bar{p}_{ik}$ which has the magnitude $\sin \lambda_{ik}$, the out-of-plane bending component is given by

$$(\Delta \bar{R}_{ik})_z = R_{ik} C'_{Bik} \sin \sigma_{ik} \bar{p}_{ik} \quad (14.2-5)$$

From Equation 12.2-1 the total bending deflection may be written as

$$\Delta \bar{R}_{ik} = R_{ik} (C_{Bik} \bar{\eta}_{ik} + C'_{Bik} \sin \sigma_{ik} \bar{p}_{ik}) \quad (14.2-6)$$

and

$$\{\Delta R_{ik}\} = R_{ik} (C_{Bik} \{\eta_{ik}\} + C'_{Bik} \sin \sigma_{ik} \{p_{ik}\}) \quad (14.2-7)$$

where C_{Bik} and C'_{Bik} are in-plane and out-of-plane normalized thermal deflection coefficients for rod ik. The length of the rod ik is

$$R_{ik} = \left| {}_o\bar{R}_{ik} \right| = \text{length of rod ik}$$

$$R_{ik}^2 = \{ {}_oR_{ik} \}_T \{ {}_oR_{ik} \} \quad (14.2-8)$$

where $\{ {}_oR_{ik} \}$ are the input components of ${}_o\bar{R}_{ik}$. Therefore, R_{ik} is a constant computed from input data.

From Figure 14.2-1

$$\bar{y}_{Bik} = \frac{{}_o\bar{R}_{ik}}{R_{ik}} \quad (14.2-9)$$

Then in body coordinates

$$\{ y_{Bik} \} = 1/R_{ik} \{ {}_oR_{ik} \} \quad (14.2-10)$$

Therefore, $\{ y_{Bik} \}$ is a constant matrix computed from input data and defines the unit vector $\bar{y}_{Bik}$ in the $\{ \bar{Q}_i \}$ coordinate system. Define the input vector $\bar{F}_{ik}$ which defines the reference bending plane as

$$\bar{F}_{ik} = \{ F_{ik} \}_T \{ \bar{Q}_i \}$$

where $\bar{F}_{ik}$ is not parallel to ${}_o\bar{R}_{ik}$ and $\{ F_{ik} \}$ are input components of $\bar{F}_{ik}$ in the $\{ \bar{Q}_i \}$ coordinate system, see Figure 14.2-3.

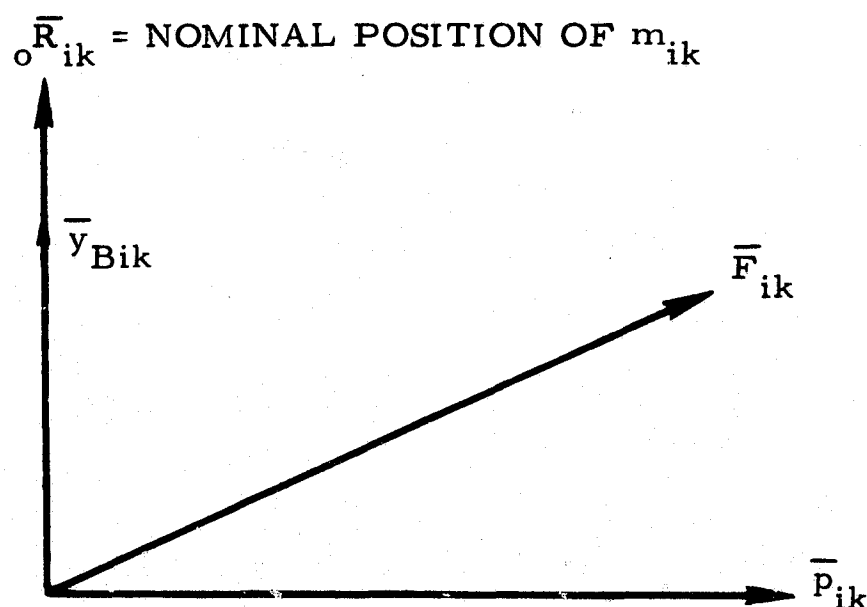


Figure 14.2-3. Location of Reference Bending Plane

Let $\bar{f}_{ik}$ = component of $\bar{F}_{ik}$ normal to $\bar{y}_{Bik}$

Then

$$\bar{f}_{ik} = -\bar{y}_{Bik} \times (\bar{y}_{Bik} \times \bar{F}_{ik})$$

and

$$\{f_{ik}\} = -[J_{yBik}] [J_{yBik}] \{F_{ik}\} \quad (14.2-11)$$

The magnitude of $\bar{f}_{ik}$ may be found from

$$f_{ik}^2 = \{f_{ik}\}_T \{f_{ik}\} \quad (14.2-12)$$

and then

$$\{p_{ik}\} = \frac{1}{f_{ik}} \{f_{ik}\} \quad (14.2-13)$$

$\{y_{Bik}\}$ and $\{p_{ik}\}$ are constant matrices computed from input data and represent the components in the $\{\bar{Q}_i\}$ coordinate system for two normal unit vectors.

The components of the unit sun vector in the $\{\bar{Q}_i\}$ coordinate system, $\{\delta_{Si}\}$, have been defined by Equation 10.1-3. From Figure 14.2-1

$$\bar{p}_{ik} = \bar{\delta}_S \times \bar{y}_{Bik} \quad (14.2-14)$$

and

$$\bar{\eta}_{ik} = \bar{y}_{Bik} \times (\bar{\delta}_S \times \bar{y}_{Bik}) \quad (14.2-15)$$

From which

$$\{p_{ik}\} = [J_{\delta Si}] \{y_{Bik}\} = -[J_{yBik}] \{\delta_{Si}\} \quad (14.2-16)$$

$$\{\eta_{ik}\} = [J_{yBik}] [J_{\delta Si}] \{y_{Bik}\} = -[J_{yBik}] [J_{yBik}] \{\delta_{Si}\} \quad (14.2-17)$$

Substituting Equations 14.2-16 and 14.2-17 into Equation 14.2-7, the deflection components of mass m_{ik} becomes

$$\{\Delta R_{ik}\} = -R_{ik} \left[C_{Bik} [J_{yBik}] [J_{yBik}] + C'_{Bik} \sin \sigma_{ik} [J_{yBik}] \right] \{\delta_{Si}\} \quad (14.2-18)$$

The expressions for $\sin \sigma_{ik}$ and $\cos \sigma_{ik}$ are obtained from the components of the unit vectors $\bar{x}_{Bik}$ and $\bar{z}_{Bik}$ in body i coordinates. From Figure 12.2-1 and the fact that $\bar{\delta}_S$ and $\bar{y}_{Bik}$ are unit vectors

$$\cos \lambda_{ik} = \bar{y}_{Bik} \cdot \bar{\delta}_S = \{y_{Bik}\}_T \{\delta_{Si}\} \quad (14.2-19)$$

and

$$\sin \lambda_{ik} = \sqrt{1 - \cos^2 \lambda_{ik}} \quad (14.2-20)$$

where the sign of $\sin \lambda_{ik}$ is always positive since $0 \leq \lambda_{ik} \leq \pi$. Since

$$\bar{x}_{Bik} = \frac{\bar{\eta}_{ik}}{\sin \lambda_{ik}}$$

and

$$\bar{z}_{Bik} = \frac{\bar{\rho}_{ik}}{\sin \lambda_{ik}},$$

Equations 14.2-16 and 14.2-17 give

$$\{x_{Bik}\} = \frac{\{\eta_{ik}\}}{\sin \lambda_{ik}} = - \frac{[J_{yBik}] [J_{yBik}] \{\delta_{Si}\}}{\sin \lambda_{ik}} \quad (14.2-21)$$

and

$$\{z_{Bik}\} = \frac{\{\rho_{ik}\}}{\sin \lambda_{ik}} = - \frac{[J_{yBik}] \{\delta_{Si}\}}{\sin \lambda_{ik}} \quad (14.2-22)$$

where $[J_{yBik}]$ and $[J_{yBik}] [J_{yBik}]$ are constant matrices. Also from Figure 14.2-2

$$\sin \sigma_{ik} = \bar{\rho}_{ik} \cdot \bar{z}_{Bik} = \{\rho_{ik}\}_T \{z_{Bik}\} \quad (14.2-23)$$

$$\cos \sigma_{ik} = \bar{y}_{ik} \cdot \bar{x}_{Bik} = \{p_{ik}\}_T \{x_{Bik}\} \quad (14.2-24)$$

Let the transformation from the body i axes to the boom axes for the k^{th} boom of body i be $[\tau_{ik}]$ such that

$$\{\bar{Q}_{Bik}\} = [\tau_{ik}] \{\bar{Q}_i\} \quad (14.2-25)$$

Thus from Equations 14.2-21, 14.2-10, and 14.2-22

$$[\tau_{ik}] = [\{x_{Bik}\} \{y_{Bik}\} \{z_{Bik}\}] \quad (14.2-26)$$

$$[\tau_{ik}] = \left[\left| -\frac{[J_{yBik}][J_{yBik}]\{\delta_{Si}\}}{\sin \lambda_{ik}} \right| \left| \frac{\{R_{ik}\}}{R_{ik}} \right| \left| -\frac{[J_{yBik}]\{\delta_{Si}\}}{\sin \lambda_{ik}} \right| \right] \quad (14.2-27)$$

14.3 ECLIPSE TRANSITION

The in-plane and out-of-plane thermal bending coefficients C_{Bik} and C'_{Bik} are nominally constant at their input values, however, they are assumed to vary linearly with time in Equation 14.2-18 in order to represent the transition at the entrance to and exit from eclipse. Normally, in sunlight, boom k of auxiliary body i is bent and in motion with respect to the body i coordinate system, and C_{Bik} and C'_{Bik} are constant at their nominal values with both $\dot{C}_{Bik}$ and $\dot{C}'_{Bik}$ equal to zero. When the spacecraft enters eclipse the boom will return to its undeflected position, ${}_o\bar{R}_{ik}$, during the short period of time t_E , after which the deflection and motion remains zero until exit from eclipse. At exit the boom deflects to its normal deflected position in time t_E . The boom motion during these transition periods is represented in the program by allowing the bending coefficients to reduce linearly to zero during time t_E at entrance to eclipse and to increase linearly from zero to their normal input values in time t_E at exit from eclipse.

The time rates of change of the bending coefficients are given by

$$\left. \begin{aligned} \dot{C}_{Bik} &= \pm \frac{C_{Bik}(\text{nominal})}{60 t_E} \\ \dot{C}'_{Bik} &= \pm \frac{C'_{Bik}(\text{nominal})}{60 t_E} \end{aligned} \right\} \begin{array}{l} \text{in eclipse transition where the positive} \\ \text{sign is used for exit and the negative} \\ \text{sign for entrance} \end{array}$$

$$\left. \begin{array}{l} C_{Bik} = 0 \\ \dot{C}'_{Bik} = 0 \end{array} \right\} \text{ not in eclipse transition}$$

The parameter p_E has been defined in Section 10.4 such that

$$p_E = - \frac{1}{60 t_E} \quad \text{when in entrance eclipse transition}$$

$$p_E = + \frac{1}{60 t_E} \quad \text{when in exit eclipse transition}$$

$$p_E = 0 \quad \text{when not in eclipse transition}$$

Therefore

$$\dot{C}_{Bik} = p_E C_{Bik} \text{ (nominal)} \quad (14.3-1)$$

$$\dot{C}'_{Bik} = p_E C'_{Bik} \text{ (nominal)} \quad (14.3-2)$$

In addition, the parameter k_E has also been defined in Section 10.4 such that

$$k_E = 1.0 \text{ for no eclipse and no transition}$$

$$k_E = 1 - \frac{\Delta t}{60 t_E} = (1 + p_E \Delta t) \text{ for entrance transition}$$

$$k_E = \frac{\Delta t}{60 t_E} = p_E \Delta t \text{ for exit transition}$$

$$k_E = 0 \text{ for eclipse and no transition}$$

where Δt is the time since start of transition,

$$C_{Bik} = k_E C_{Bik} \text{ (nominal)} \quad (14.3-3)$$

$$C'_{Bik} = k_E C'_{Bik} \text{ (nominal)} \quad (14.3-4)$$

and

$$\dot{k}_E = p_E \quad (14.3-5)$$

The values of p_E and k_E are set for each integration step by the eclipse subroutine and are held constant for the next step. Therefore if C_{Bik} and C'_{Bik} are multiplied by k_E in the thermal bending equations the effects of eclipse and eclipse transition are incorporated. The bending coefficients are reduced in a stair step manner during transition requiring the use of small time steps. The thermal bending Equation 12.2-18, modified to include eclipse effects, becomes

$$\{\Delta R_{ik}\} = -k_E R_{ik} \left[C_{Bik} [J_{yBik}] [J_{yBik}] + C'_{Bik} \sin \sigma_{ik} [J_{yBik}] \right] \{\delta_{Si}\} \quad (14.3-6)$$

Let

$$[\beta_{ik}] = -R_{ik} \left[C_{Bik} [J_{yBik}] [J_{yBik}] + C'_{Bik} \sin \sigma_{ik} [J_{yBik}] \right] \quad (14.3-7)$$

where all terms in $[\beta_{ik}]$ are constants except $\sin \sigma_{ik}$

Then

$$\{\Delta R_{ik}\} = k_E [\beta_{ik}] \{\delta_{Si}\} \quad (14.3-8)$$

14.4 THERMAL BENDING RATES AND ACCELERATIONS

Since k_E is p_E the rate of change of $\{\Delta R_{ik}\}$ with respect to the body i coordinate system is

$$\{\dot{\Delta R}_{ik}\} = p_E [\beta_{ik}] \{\delta_{Si}\} + k_E [\dot{\beta}_{ik}] \{\delta_{Si}\} + k_E [\beta_{ik}] \{\dot{\delta}_{Si}\}$$

From Equation 10.1-4

$$\{\dot{\delta}_{Si}\} = -[J_{\omega ii}] \{\delta_{Si}\}.$$

Then

$$\{\dot{\Delta R}_{ik}\} = \left[p_E [\beta_{ik}] + k_E [\dot{\beta}_{ik}] - k_E [\beta_{ik}] [J_{\omega ii}] \right] \{\delta_{Si}\} \quad (14.4-1)$$

Differentiating Equation 14.4-1 and substituting for $\{\dot{\delta}_{Si}\}$ and $\{\ddot{\delta}_{Si}\}$ from Equations 10.1-4 and 10.1-5 gives

$$\begin{aligned} \{\Delta \ddot{R}_{ik}\} = & 2p_E \left[[\dot{\beta}_{ik}] - [\beta_{ik}] [J_{\omega ii}] \right] \{\delta_{Si}\} - k_E \left[2[\dot{\beta}_{ik}] - [\beta_{ik}] [J_{\omega ii}] \right] [J_{\omega ii}] \{\delta_{Si}\} \\ & + k_E \left[[\dot{\beta}_{ik}] - [\beta_{ik}] [J_{\omega ii}] \right] \{\delta_{Si}\} \end{aligned} \quad (14.4-2)$$

The first term in Equation 12.4-2 is zero except when in eclipse transition, the second term is a function of position and velocity, and the third term is a function of the unknown accelerations of the system. A portion of the third term must then be placed on the unknown side of the dynamic equations. Since the terms in $[\beta_{ik}]$ are all constants except $\sin \sigma_{ik}$, $[\dot{\beta}_{ik}]$ and $[\ddot{\beta}_{ik}]$ result from the assumed variation in the magnitude of the out-of-plane bending deflection, and will contain $\dot{\sigma}_{ik}$ and $\ddot{\sigma}_{ik}$. In order to further expand the bending equations $\dot{\sigma}_{ik}$ and $\ddot{\sigma}_{ik}$ must be expressed in terms of $\{\omega_{ii}\}$ and $\{\dot{\omega}_{ii}\}$.

From Figures 14.2-1 and 14.2-2 it may be seen that $\dot{\sigma}_{ik}$ and $\ddot{\sigma}_{ik}$ are the rotational rate and acceleration $\bar{\eta}_{ik}$ and, therefore, $\bar{\delta}_S$ and $\bar{y}_{Bik}$, as seen from the $\{\bar{Q}_i\}$ coordinate system. The components of the velocity of the tip of the unit sun vector in the $\{\bar{Q}_i\}$ coordinate system are $\{\dot{\delta}_{Si}\}$ and the components of this velocity normal to $\bar{\delta}_S$ are $\bar{\eta}_{ik}$ in the $\bar{x}_{Bik} - \bar{z}_{Bik}$ plane are $\{\dot{\delta}_{Si}\}_T \{z_{Bik}\}$, see Figure 14.4-1.

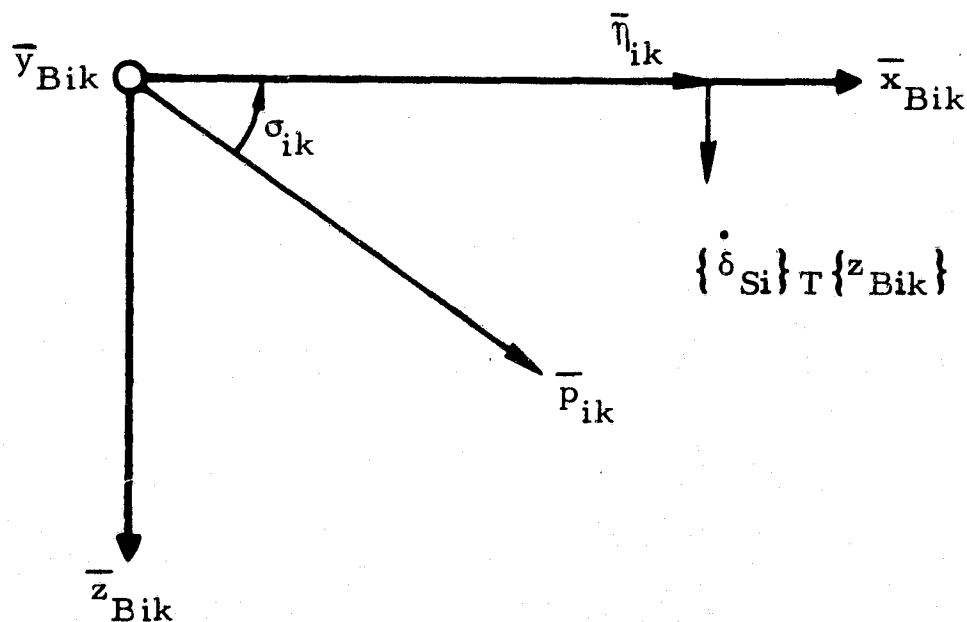


Figure 14.4-1. Motion of Bending Plane

The rotational velocity of $\bar{\eta}_{ik}$ about $\bar{y}_{Bik}$ is then

$$\dot{\sigma}_{ik} = - \frac{\{\delta_{Si}\}_T \{z_{Bik}\}}{\sin \lambda_{ik}} \quad (14.4-3)$$

From Equation 10.1-4

$$\{\delta_{Si}\} = - [J_{\omega ii}] \{\delta_{Si}\}$$

and

$$\dot{\sigma}_{ik} = \frac{\{[J_{\omega ii}] \{\delta_{Si}\}\}_T \{z_{Bik}\}}{\sin \lambda_{ik}} \quad (14.4-4)$$

From the fact that the cross and dot may be interchanged in a triple vector product or by direct expansion

$$\dot{\sigma}_{ik} = \left\{ \frac{[J_{\delta Si}] \{z_{Bik}\}}{\sin \lambda_{ik}} \right\}_T \{\omega_{ii}\} \quad (14.4-5)$$

Likewise the acceleration of the tip of the unit sun vector is $\{\ddot{\delta}_{Si}\}$ and the components of acceleration normal to $\bar{\delta}_S$ and $\bar{\eta}_{ik}$ in the $\bar{x}_{Bik} - \bar{z}_{Bik}$ plane are $\{\ddot{\delta}_{Si}\}_T \{z_{Bik}\}$ and

$$\ddot{\sigma}_{ik} = - \frac{\{\ddot{\delta}_{Si}\}_T \{z_{Bik}\}}{\sin \lambda_{ik}} \quad (14.4-6)$$

From Equation 10.1-5

$$\{\ddot{\delta}_{Si}\} = [J_{\omega ii}] [J_{\omega ii}] \{\delta_{Si}\} - [J_{\dot{\omega} ii}] \{\delta_{Si}\}$$

and

$$\ddot{\sigma}_{ik} = - \frac{\{[J_{\omega ii}] [J_{\omega ii}] \{\delta_{Si}\}\}_T \{z_{Bik}\}}{\sin \lambda_{ik}} + \left\{ \frac{[J_{\delta Si}] \{z_{Bik}\}}{\sin \lambda_{ik}} \right\}_T \{\dot{\omega}_{ii}\} \quad (14.4-7)$$

In order to simplify the equations, matrix expressions which are not functions of the unknown independent variables will be grouped and assigned new symbols. Where possible the grouping and symbols will correspond to those used in the digital program.

Let

$$\{\Delta_{ik}\} = \frac{[J_{\delta Si}] \{z_{Bik}\}}{\sin \lambda_{ik}} \quad (14.4-8)$$

$${}_o\Delta_{ik} = - \frac{\left\{ [J_{\omega ii}] [J_{\omega ii}] \{\delta_{Si}\} \right\}_T \{z_{Bik}\}}{\sin \lambda_{ik}} (\text{scalar}) \quad (14.4-9)$$

From Equations 12.4-5 and 12.4-7

$$\dot{\sigma}_{ik} = \{\Delta_{ik}\}_T \{\omega_{ii}\} \quad (14.4-10)$$

$$\ddot{\sigma}_{ik} = {}_o\Delta_{ik} + \{\Delta_{ik}\}_T \{\dot{\omega}_{ii}\} \quad (14.4-11)$$

Returning to Equation 14.3-7

$$[\beta_{ik}] = - R_{ik} \left[C_{Bik} [J_{yBik}] [J_{yBik}] + C'_{Bik} \sin \sigma_{ik} [J_{yBik}] \right]$$

Let

$$[\dot{\beta}_{ik}] = [\gamma_{ik}] = - R_{ik} C'_{Bik} \dot{\sigma}_{ik} \cos \sigma_{ik} [J_{yBik}] \quad (14.4-12)$$

then

$$[\ddot{\beta}_{ik}] = - R_{ik} C'_{Bik} \ddot{\sigma}_{ik} \cos \sigma_{ik} [J_{yBik}] + R_{ik} C'_{Bik} \dot{\sigma}_{ik}^2 \sin \sigma_{ik} [J_{yBik}] \quad (14.4-13)$$

Let

$$[\rho_{ik}] = R_{ik} C'_{Bik} \dot{\sigma}_{ik}^2 \sin \sigma_{ik} [J_{yBik}] \quad (14.4-14)$$

then

$$[\dot{\beta}_{ik}] = [\rho_{ik}] - R_{ik} C'_{Bik} \ddot{\sigma}_{ik} \cos \sigma_{ik} [J_{yBik}] \quad (14.4-15)$$

Substituting for $\ddot{\sigma}_{ik}$ from Equation (12.4-11)

$$\begin{aligned} [\ddot{\beta}_{ik}] &= [\rho_{ik}] - R_{ik} C'_{Bik} ({}_o\Delta_{ik}) \cos \sigma_{ik} [J_{yBik}] \\ &\quad - R_{ik} C'_{Bik} \cos \sigma_{ik} \{\Delta_{ik}\}_T \{\dot{\omega}_{ii}\} [J_{yBik}] \end{aligned} \quad (14.4-16)$$

Let

$$[{}_o\gamma_{ik}] = - R_{ik} C'_{Bik} ({}_o\Delta_{ik}) \cos \sigma_{ik} [J_{yBik}] \quad (14.4-17)$$

where $[{}_o\gamma_{ik}]$ is the same as $[\gamma_{ik}]$ except $({}_o\Delta_{ik})$ replaces $(\dot{\sigma}_{ik})$ then

$$[\ddot{\beta}_{ik}] = [\rho_{ik}] + [\gamma_{ik}] - R_{ik} C'_{Bik} \cos \sigma_{ik} \{\Delta_{ik}\}_T \{\dot{\omega}_{ii}\} [J_{yBik}] \quad (14.4-18)$$

Substituting for $[\beta_{ik}]$ and $[\ddot{\beta}_{ik}]$ in Equation 14.4-2

$$\begin{aligned} \{\Delta \ddot{R}_{ik}\} = & \left[2p_E \left[[\gamma_{ik}] - [\beta_{ik}] [J_{\omega ii}] \right] + k_E \left[[\rho_{ik}] + [\beta_{ik}] [J_{\omega ii}] \right. \right. \\ & \left. \left. - 2 [\gamma_{ik}] \right] [J_{\omega ii}] + [\gamma_{ik}] \right] \{\delta_{Si}\} - k_E [\beta_{ik}] [J_{\omega ii}] \{\delta_{Si}\} \\ & - k_E R_{ik} C'_{Bik} \cos \sigma_{ik} \{\Delta_{ik}\}_T \{\dot{\omega}_{ii}\} [J_{yBik}] \{\delta_{Si}\} \end{aligned}$$

By interchanging cross vector products

$$\begin{aligned} [\beta_{ik}] [J_{\omega ii}] \{\delta_{Si}\} &= - [\beta_{ik}] [J_{\delta Si}] \{\dot{\omega}_{ii}\} \\ \{\Delta_{ik}\}_T \{\dot{\omega}_{ii}\} [J_{yBik}] \{\delta_{Si}\} &= - \left[[J_{\delta Si}] \{y_{Bik}\} \right] \{\Delta_{ik}\}_T \{\dot{\omega}_{ii}\} \end{aligned}$$

Let

$$[\Delta_{ik}] = \left[[J_{\delta Si}] \{y_{Bik}\} \right] \{\Delta_{ik}\}_T \quad (14.4-19)$$

Then

$$\{\Delta_{ik}\}_T \{\dot{\omega}_{ii}\} [J_{yBik}] \{\delta_{Si}\} = - [\Delta_{ik}] \{\dot{\omega}_{ii}\}$$

and

$$\begin{aligned} \{\Delta \ddot{R}_{ik}\} = & \left[2p_E \left[[\gamma_{ik}] - [\beta_{ik}] [J_{\omega ii}] \right] + k_E \left[[\rho_{ik}] + [\beta_{ik}] [J_{\omega ii}] \right. \right. \\ & \left. \left. - 2 [\gamma_{ik}] \right] [J_{\omega ii}] + [\gamma_{ik}] \right] \{\delta_{Si}\} + k_E [\beta_{ik}] [J_{\delta Si}] \\ & + R_{ik} C'_{Bik} \cos \sigma_{ik} [\Delta_{ik}] \{\dot{\omega}_{ii}\} \end{aligned}$$

From Equation 10.5-4 the term $\{\ddot{R}_{ik}\}$ has been defined as

$$\{\ddot{R}_{ik}\} = \{\lambda_{ik}\} + [\lambda_{ik}] \{\dot{\omega}_{ii}\}$$

Therefore,

$$\begin{aligned} \{\lambda_{ik}\} = & \left[2p_E \left[[\gamma_{ik}] - [\beta_{ik}] [J_{\omega ii}] \right] + k_E \left[[\rho_{ik}] + [\beta_{ik}] [J_{\omega ii}] \right. \right. \\ & \left. \left. - 2 [\gamma_{ik}] \right] [J_{\omega ii}] + [\gamma_{ik}] \right] \{\delta_{Si}\} \end{aligned} \quad (14.4-20)$$

$$[\lambda_{ik}] = k_E \left[[\beta_{ik}] [J_{\delta Si}] + R_{ik} C'_{Bik} \cos \sigma_{ik} [\Delta_{ik}] \right] \quad (14.4-21)$$

14.5 TIP MASS COORDINATES

In determining disturbances on the spacecraft, the solar radiation and aerodynamic forces and torques will be considered for bodies attached at the end of the booms. As a result it will be necessary to establish a coordinate system at the end of each boom and determine its rotation due to thermal bending. Referring to Figure 12.2-1, the $\bar{x}_{Bik}$, $\bar{y}_{Bik}$, and $\bar{z}_{Bik}$ system will be translated to the end of the boom and designated as the $\bar{x}'_{Bik}$, $\bar{y}'_{Bik}$, and $\bar{z}'_{Bik}$ system. It is assumed that this $\{\bar{Q}'_{Bik}\}$ system is fixed to, and rotates with, the end of the boom, see Figure 14.5-1.

Introducing the eclipse transition factor k_E into Equation 14.1-11 gives

$$x_B = k_E C_B \frac{y_B^2}{R} \sin \lambda \quad (14.5-1)$$

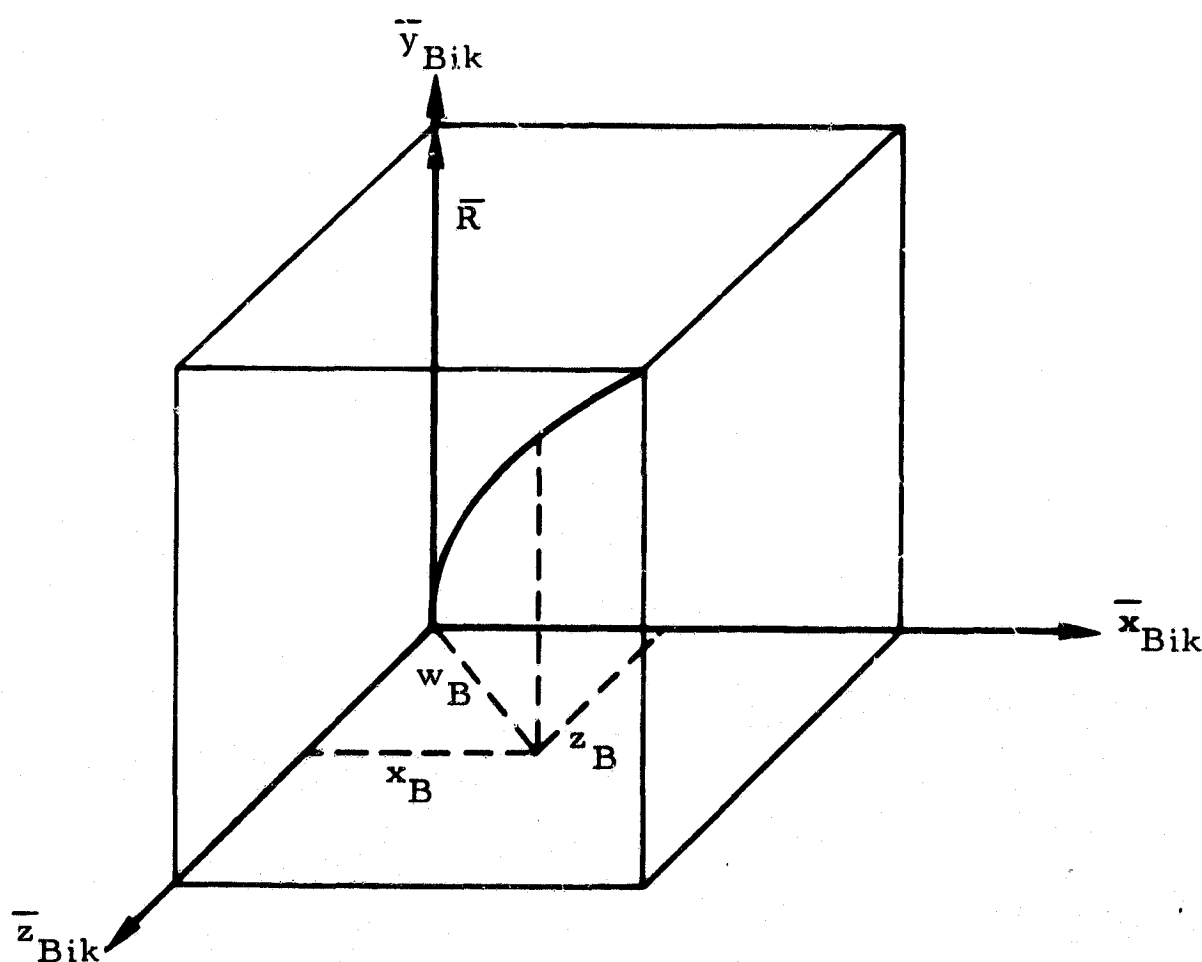


Figure 14.5-1. Tip Mass Coordinates

$$z_B = k_E C'_B \sin \sigma \frac{y_B^2}{R} \sin \lambda \quad (14.5-2)$$

where $C'_B \sin \sigma$ is the effective out of plane bending coefficient. In the true bending plane the deflection is w_B where

$$w_B = k_E \left[C_B^2 + (C'_B \sin \sigma)^2 \right]^{1/2} \frac{y_B^2}{R} \sin \lambda$$

Let

$$C_{B \text{ eff}} = \left[C_B^2 + (C'_B \sin \sigma)^2 \right]^{1/2} \quad (14.5-3)$$

Then

$$w_B = k_E C_{B \text{ eff}} \frac{y_B^2}{R} \sin \lambda \quad (14.5-4)$$

$$\frac{dw_B}{dy_B} = 2k_E C_{B \text{ eff}} \frac{y_B}{R} \sin \lambda$$

Let the slope at the end of the boom equal the sine of the angle of rotation of the end of the boom in the true bending plane

$$\sin \theta_B = 2C_{B \text{ eff}} \sin \lambda \quad (14.5-5)$$

assume $\cos \theta_B = 1.0$.

The rotation occurs about a unit vector $\bar{\phi}_B$ normal to the true bending plane and fixed in body i.

$$\phi_x = \frac{z_B}{w_B} = \frac{C'_B \sin \sigma}{C_{B \text{ eff}}}$$

$$\phi_y = 0$$

$$\phi_z = \frac{x_B}{w_B} = -\frac{C_B}{C_{B \text{ eff}}}$$

Then

$$\{\phi_{Bi}\} = \frac{1}{C_{B \text{ eff}}} \begin{Bmatrix} C'_B \sin \sigma \\ 0 \\ -C_B \end{Bmatrix} \quad (14.5-6)$$

From Equations 4-49 of Reference 1, the transformation matrix for fixed axis rotation gives

$$[\alpha] = [U] + [J_{\phi_{Bi}}] [J_{\phi_{Bi}}] (1 - \cos \theta_B) - [J_{\phi_B}] \sin \theta_B \quad (14.5-7)$$

Using 14.5-5 and 14.5-6 the transformation matrix for the k^{th} boom of the i^{th} body is

$$[\alpha_{ik}] = [U] + 2k_E \begin{Bmatrix} J \begin{Bmatrix} -C'_{Bik} \sin \sigma_{ik} \\ 0 \\ C_{Bik} \end{Bmatrix} \end{Bmatrix} \sin \lambda_{ik} \quad (14.5-8)$$

and

$$\{\bar{Q}'_{Bik}\} = [\alpha_{ik}] \{\bar{\theta}_{Bik}\} \quad (14.5-9)$$

15. MAGNETIC TORQUES

The magnetic torque on body i is given by the expression

$$\bar{T}_{Mi} = K_M (\bar{N}_i \times \bar{B}_{Mi}) \quad (15.0-1)$$

where $\bar{B}_{Mi}$ is the magnetic field vector at body i , $\bar{N}_i$ is the magnetic moment of body i , and K_M is a constant. Expressing the vectors in body coordinates the matrix equation for the magnetic torque components is

$$\{T_{Mii}\} = K_M [J_{Nii}] \{B_{mii}\} \quad (15.0-2)$$

In Section 11.3 the magnetic field components at the center of mass of the spacecraft were expressed in body i coordinates. Equation 11.3-9 gives the field components $\{B_{mii}\}$ in gauss for $1 \leq i \leq N$. The body i magnetic moment components $\{N_{ii}\}$ used in this program are in ampere-turn-ft², therefore,

$$K_M = 6.86 \times 10^{-6} \quad (15.0-3)$$

and the torque components $\{T_{Mii}\}$ are in #-ft.

It will be noted that provisions are made in the digital program to input four magnetic moments for each body of the spacecraft. In the present version of the program only the first magnetic moment is used in the computation of magnetic torques. The additional input was incorporated for future expansion of the program. Magnetic moments for each boom will be used in connection with the tip mass coordinates defined in Section 14.5 to provide for the simulation of magnetic viscous dampers at the end of booms.

16. MAGNETIC HYSTERESIS DAMPER

Provisions have been made to introduce a magnetic hysteresis damper at each of the auxiliary body gimbal axes, Equation 7.2-1. An expression will now be derived for the damper torque about the x gimbal axis between body i and the main body. The torque equation will then be generalized to apply to the y and z gimbal axes and the results will be expressed in the desired matrix form.

16.1 BASIC DAMPER TORQUE EQUATION

The three hysteresis damper torques at the gimballed hinge between body i and body 1 have been defined as $\{T_{Bi}\}$, Equation 7.2-1, where

$$\{T_{Bi}\} = \begin{Bmatrix} T_{Bix} \\ T_{Biy} \\ T_{Biz} \end{Bmatrix} \quad (16.1-1)$$

and T_{Bix} is the $\bar{x}_g$ axis torque. The corresponding gimbal deflection and velocity are θ_{ix} and $\dot{\theta}_{ix}$, Equation 7.1-1.

The damper torque T_{Bix} at any instant of time is a function of the past history of the motion about the $\bar{x}_g$ gimbal axis. Assume that the angular velocity about the $\bar{x}_g$ axis, $\dot{\theta}_{ix}$ changes sign when the gimbal deflection is θ_{ixsc} and the damper torque is T_{Bixsc} , see Figure 16.1-1. The change in damper torque during the period after sign change, ΔT_{ix} , is a function of the displacement from the point of sign change, $\Delta\theta_i$, until the velocity again changes sign or the damper saturates.

$$\text{Let} \quad \Delta\theta_i = |(\theta_{ixsc} - \theta_{ix})| \quad (16.1-2)$$

The value of ΔT_{ix} is computed from

$$\Delta T_{ix} = (\text{sign } \dot{\theta}_{ix}) \left[H_{1ix} (\Delta\theta_{ix}) + H_{2ix} (\Delta\theta_{ix})^2 + H_{3ix} (\Delta\theta_{ix})^3 + H_{4ix} (\Delta\theta_{ix})^4 \right] \quad (16.1-3)$$

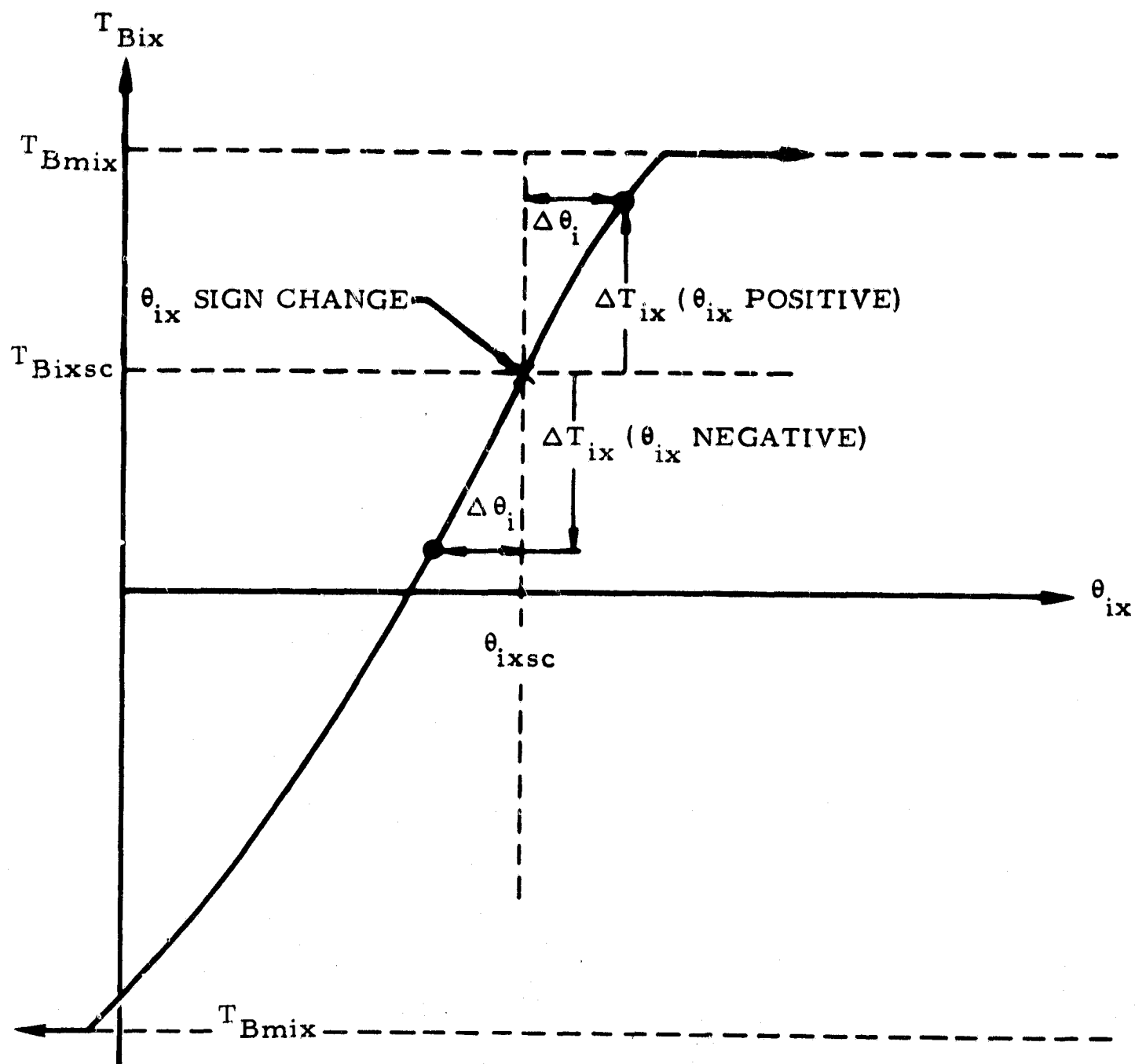


Figure 16.1-1. Hysteresis Damper Torque Vs Deflection

The damper torque equation for the $\bar{x}_g$ gimbal axis of body i is then

$$T_{Bix} = T_{Bixsc} + \Delta T_{ix} \quad |T_{Bix}| \leq T_{Bmix} \quad (16.1-4)$$

where T_{Bmix} is the saturation level of the damper. When the magnitude of T_{Bix} is found to be greater than T_{Bmix} it is set equal to T_{Bmix} and remains at this value until θ_{ix} changes sign. During this period the value of $\Delta\theta_{ix}$ and ΔT_{ix} need not be computed.

Expanding the damper torque equation to include all three gimbal axes of body i, let

$$\{T_{Bmi}\} = \begin{Bmatrix} T_{Bmix} \\ T_{Bmiy} \\ T_{Bmiz} \end{Bmatrix} \quad (16.1-5)$$

$$\{T_{Bisc}\} = \begin{Bmatrix} T_{Bixsc} \\ T_{Biysc} \\ T_{Bizsc} \end{Bmatrix} \quad (16.1-6)$$

$$\{\Delta\theta_i\} = \begin{Bmatrix} \Delta\theta_{ix} \\ \Delta\theta_{iy} \\ \Delta\theta_{iz} \end{Bmatrix} \quad (16.1-7)$$

$$\{\Delta T_i\} = \begin{Bmatrix} \Delta T_{ix} \\ \Delta T_{iy} \\ \Delta T_{iz} \end{Bmatrix} \quad (16.1-8)$$

$$\{H_{li}\} = \begin{Bmatrix} H_{lix} \\ H_{liy} \\ H_{liz} \end{Bmatrix} \quad [\text{etc. for } \{H_{2i}\}, \{H_{3i}\}, \text{ and } \{H_{4i}\}] \quad (16.1-9)$$

Then

$$\{T_{Bi}\} = \{T_{Bisc}\} + \{\Delta T_i\} \quad (16.1-10)$$

where

$$|T_{Bix}| \leq T_{Bmix}$$

$$|T_{Biy}| \leq T_{Bmiy}$$

$$|T_{Biz}| \leq T_{Bmiz}$$

and the elements of $\{\Delta\theta_i\}$ and $\{\Delta T_i\}$ are computed from expression corresponding to Equations 16.1-2 and 16.1-3.

16.2 MAJOR LOOP OPERATION

For large angle operation of the hysteresis damper, the damper saturates for both positive and negative torques. The resulting torque deflection curve is shown in Figure 16.2-1.

On the torque-deflection plot the coordinates of a given sign change, T_{Bixsc} and θ_{ixsc} , define the point at which a given curve segment starts. This point will be denoted as the anchor point for that curved segment. Any curved segment with an anchor point on either the upper or lower saturation level is considered a portion of a major loop. Conversely any curved segment with an anchor point not at a saturation level is considered a portion of a minor loop. The saturation levels are considered as a portion of a major loop, therefore, the entire loop shown in Figure 16.2-1 is a major loop. The single exception to the above definitions is the initial curve at the start of the program which is considered as a portion of a major loop independent of the location of its anchor point.

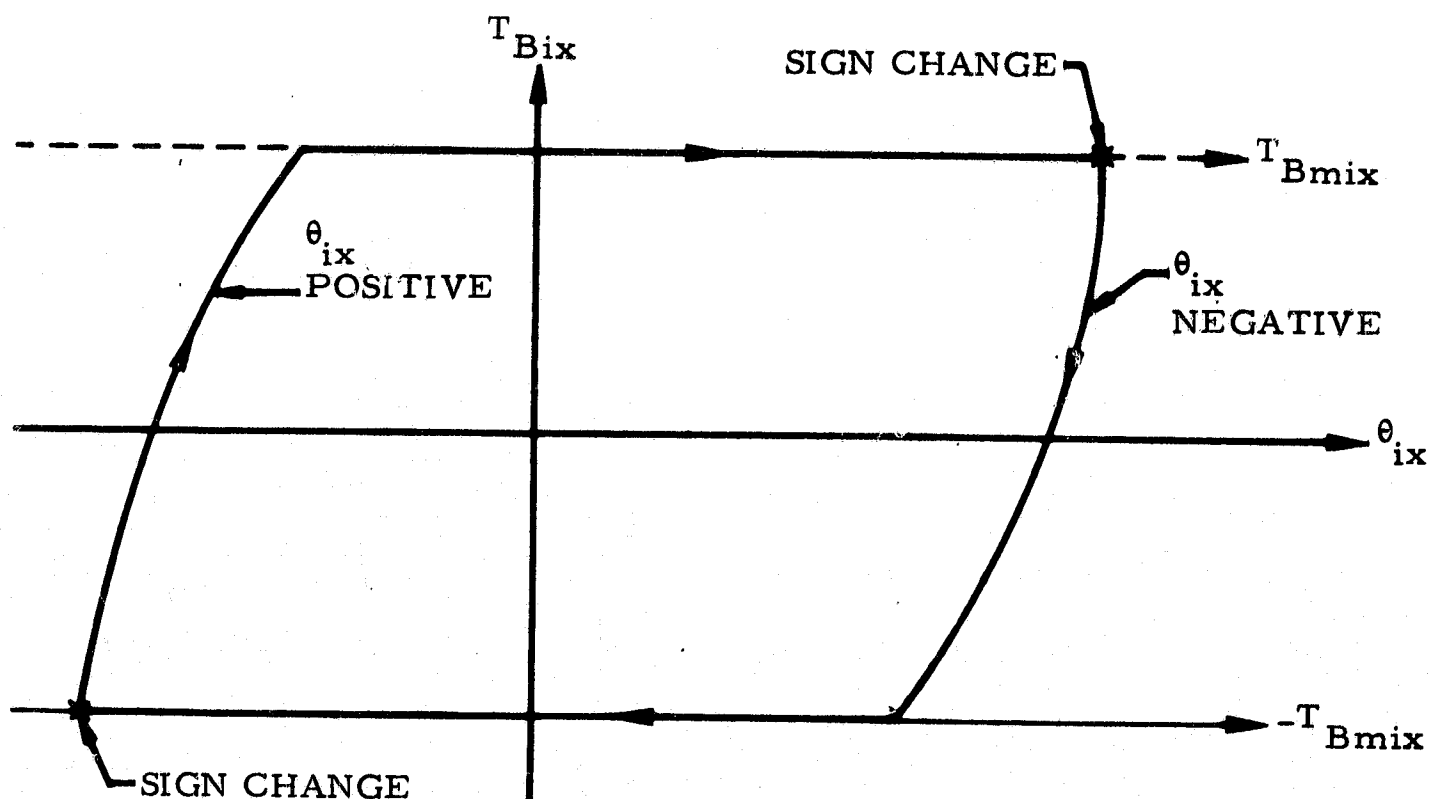


Figure 16.2-1. Hysteresis Damper Major Loop Operator

It should be noted that the hysteresis damper torques $\{T_{Bi}\}$ are added to the spring and viscous damper torques in Equation 7.2-1. Therefore, a positive damper torque is a positive torque transmitted from body i to the main body and is a retarding torque on body i . The energy absorbed by the damper during a cycle of operation is equal to the area inside the loop of Figure 16.2-1. For a more detailed description of the damper operation see the program user manual Reference 3.

16.3 MINOR LOOP OPERATION

For small angle operation of the damper, the damper does not saturate and the torque deflection curve is as shown in Figure 16.2-2. Consider the minor loop shown on the right side of the figure, assume that $\dot{\theta}_{ix}$ is negative and the torque deflection relationship is moving downward from the upper anchor point. If $\dot{\theta}_{ix}$ changes sign to positive before the lower saturation level is reached, the lower minor loop anchor point is created and a new upward curve is created. Then if the sign of $\dot{\theta}_{ix}$ does not change the upward curve must pass through the original upper anchor point.

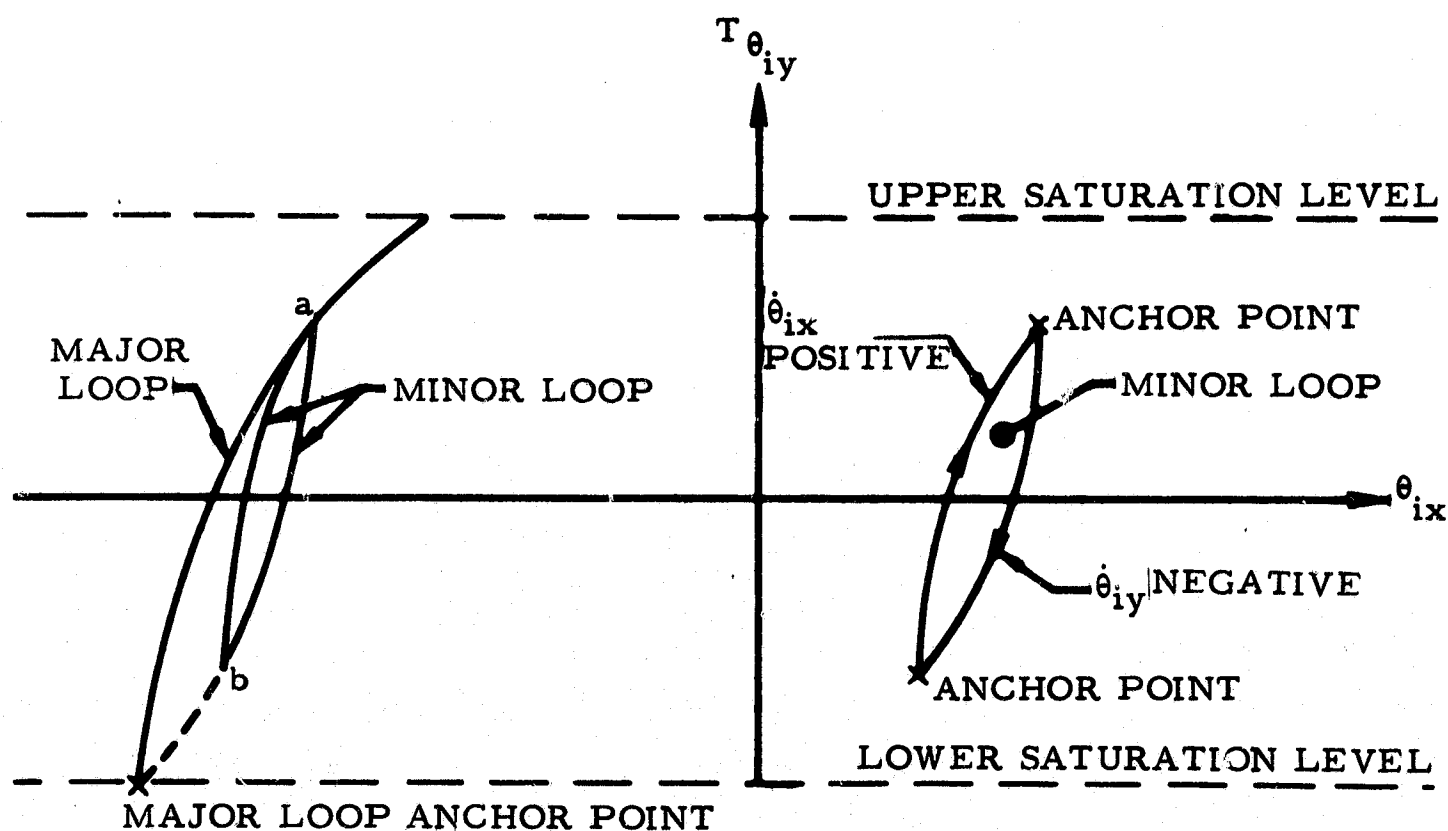


Figure 16.2-2. Hysteresis Damper Minor Loop Operation

Consider the minor loop shown on the left of Figure 16.2-2, assume that the torque deflection relationship is shown by the major loop with an anchor point on the lower saturation level. If the sign of $\dot{\theta}_{iy}$ changes to negative at point a, the relationship is downward along the minor loop with an anchor point at a. If no sign change is encountered the minor loop will saturate at the original major loop anchor point, however, if a sign change in $\dot{\theta}_{ix}$ is encountered at point b the relationship moves upward from minor loop anchor point b and will again pass through point a. It has been shown by both experiment and damper theory that the minor loop returning to point a will not cross the original major loop and the torque deflection relationship returns to the major loop.

The above discussion is an attempt to explain the following basic operating principles of the hysteresis damper:

- a) any minor loop can cross or return to a previous minor loop only at the anchor point of the previous loop
- b) any series of minor loops generated from a major loop can return to the major loop only at its anchor point or the original point from which it left the major loop. Upon returning to the major loop, the relationship reverts to the major loop as if it had never left the loop.

16.4 SHAPED HYSTERESIS DAMPERS

It has been shown that a trade off exists between the optimum damper parameters for steady state operation and the optimum damper parameters for acquisition. For steady state operation, when the deflection amplitude is small, a relatively low saturation level is required, while for large angle acquisition a much higher saturation level provides optimum performance. In order to eliminate this trade off in performance, hysteresis dampers are being manufactured with tapered vanes such that the saturation level is a function of the deflection angle. A typical shaped saturation curve is shown in Figure 16.4-1. The saturation level is minimum at θ_{ix} equal to zero and increase with a value of approximately 40 times the minimum value at a deflection of one radian.

The operation of the shaped damper is rather complicated, however, the damper torques can be arrived at in a relatively simple

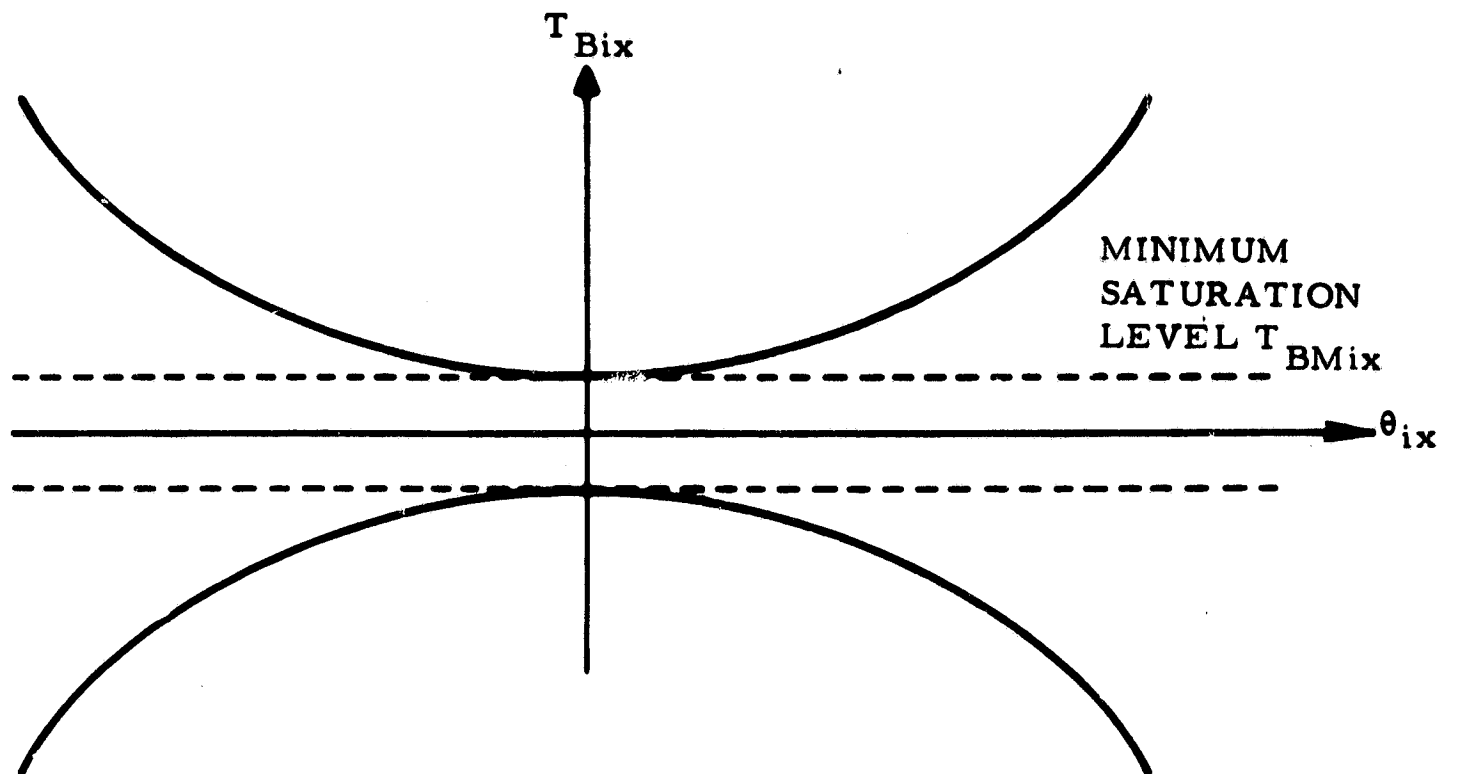


Figure 16.4-1. Shaped Damper Characteristics

manner. The damper coefficients in Equation 16.1-3, H_{1ix} through H_{4ix} are linear functions of the damper saturation level. As a result the damper torque for a given θ_{ix} may be computed on the basis of a constant saturation level, equal to the minimum value, and the resulting torques multiplied by the corresponding ratio of the actual saturation level to the minimum saturation level.

17. BOOM DEPLOYMENT

In Section 10.5 the parameters variations resulting from changes in shape of the auxiliary bodies are defined. It was noted that these variations may result from either thermal boom bending or boom deployment but not both. Expressions for the parameter variations in terms of thermal bending are given in Section 14.0 and corresponding expressions in terms of boom deployment will be presented in this section. The parameters required at any instant of time are: $\{\Delta R_{ik}\}$, $\{\Delta \dot{R}_{ik}\}$, $\{\lambda_{ik}\}$, and $[\lambda_{ik}]$.

It should be remembered that in the derivation of the dynamic equations the mass of the auxiliary bodies is concentrated in the tip masses m_{ik} , and the distributed mass of the booms has been neglected. As noted in the User's Handbook, Reference 3, the error created by this assumption can be compensated for in the case of an extended boom. If the mass of the boom is not small in comparison to the tip mass, the body inertias and the body momentum due to thermal bending is corrected by adding one third of the boom mass to the tip mass. The mass ratios are then corrected by placing the remaining two thirds of the boom mass at the hinge point on a zero length boom. In the case of boom deployment the boom mass is changing and the amount of mass added at the tip and at the hinge is a function of the boom length. Since the program does not provide for variations in the mass distribution as a function of time, the effects of boom deployment will be approximated by neglecting the distributed mass of the boom.

It is assumed that the k^{th} boom of body i starts at some initial extension defined by ${}_0\bar{R}_{ik}$ and extends along $\bar{R}_{ik}$ to a length L_{Dik} . Extension will start at some time interval t_{oB} after the start of the program, see Figure 17.0-1. The values of m_{ik} and $\{{}_0R_{ik}\}$ are part of the input

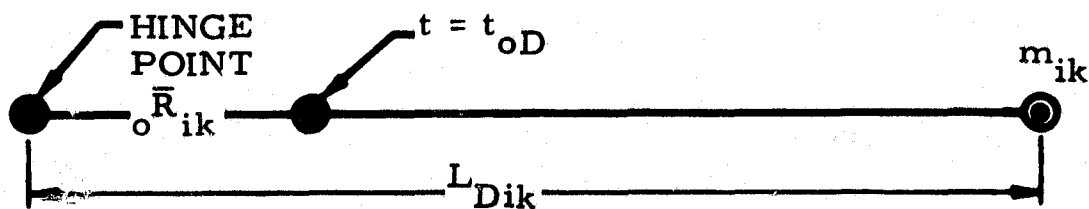


Figure 17.0-1. Initial and Final Deployed Positions

parameters defining auxiliary body i . The direction of deployment is along the unit vector $\bar{y}_{Bik}$ where the components $\{y_{Bik}\}$ are computed as in the case of thermal bending using Equation 14.2-10. It should be noted that the initial position components $\{R_{ik}\}$ cannot be zero but must have some finite value in order to define $\bar{y}_{Bik}$.

It is assumed that a given boom ik deploys at a constant initial acceleration for a time t_{Dik} until the velocity reaches V_{Dik} . The extension will then occur at a constant velocity of V_{Dik} and decelerate at a constant deceleration for the period t_{Dik} reaching the final length L_{Dik} . The additional inputs required for each boom are:

- a) t_{oDik} = deployment time
- b) t_{Dik} = acceleration and deceleration time
- c) V_{Dik} = maximum deployment velocity
- d) L_{Dik} = deployed length

The five phases of deployment together with the acceleration, velocity, and displacement during each phase are shown in Figure 17.0-2. From the figure and simple kinematics it can be seen that

$$A_{Dik} = \frac{V_{Dik}}{t_{Dik}} \quad (17.0-1)$$

$$l_3 = l_1 = \frac{1}{2} V_{Dik} t_{Dik} \quad (17.0-2)$$

$$l_2 = L_{Dik} - R_{ik} - 2 l_1 \quad (17.0-3)$$

$$t_{mik} = \frac{l_2}{V_{Dik}} \quad (17.0-4)$$

where A_{Dik} , l_1 , l_2 , and t_{mik} are constants computed at the start of the program. The magnitudes of the acceleration, velocity, and change in displacement from R_{ik} during each phase of deployment are given in Table 17.0-1.

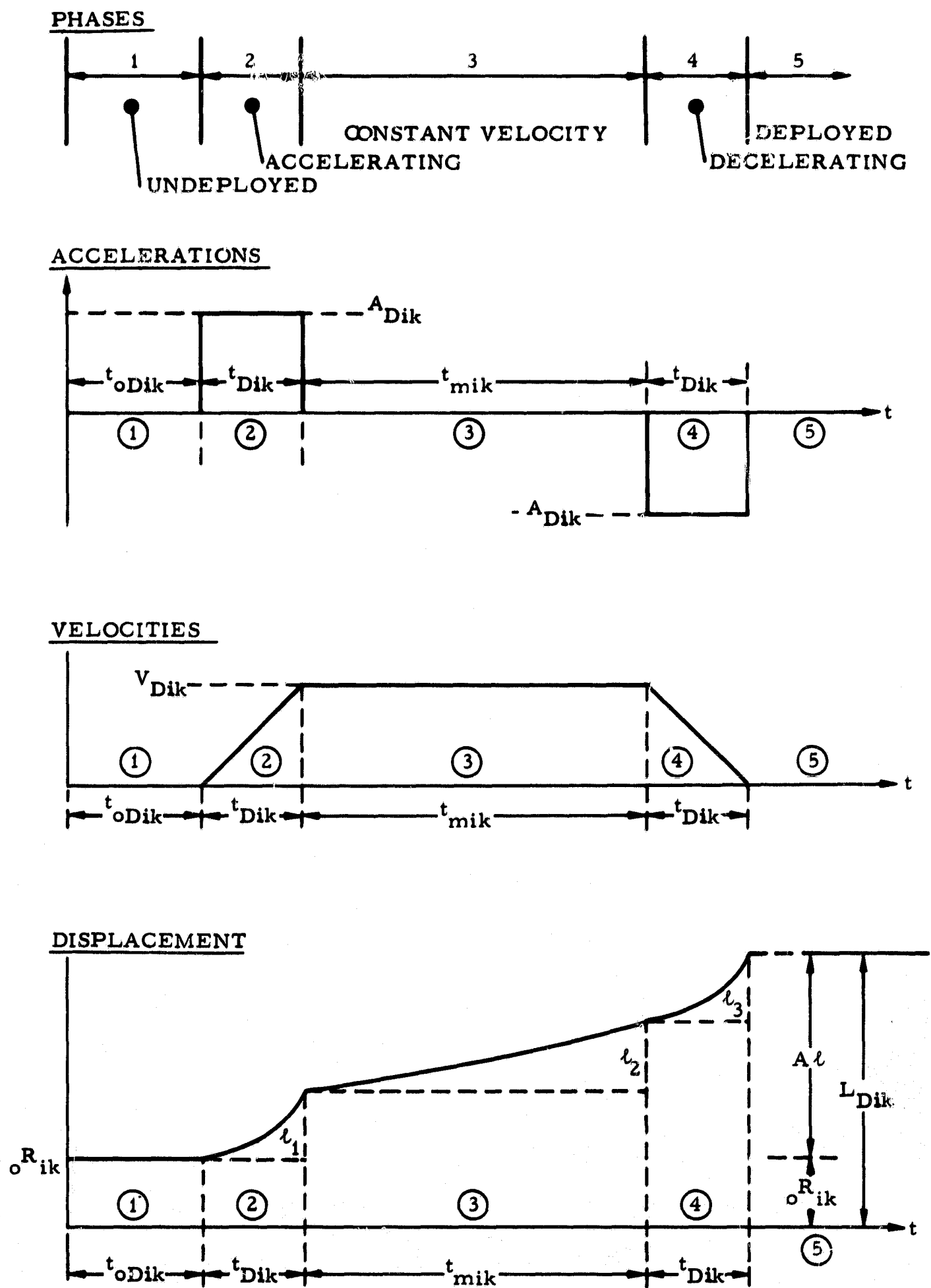


Figure 17.0-2. Phases of Boom Deployment

Table 17.0-1. Boom Deployment Acceleration, Velocities, and Displacements

Phase	Acceleration	Velocity	Relative Displacement
1	0	0	0
2	A_{Dik}	$A_{Dik} (t-t_{oDik})$	$+\frac{1}{2} A_{Dik} (t-t_{oDik})^2$
3	0	V_{Dik}	$+l_1 + V_{Dik} (t-t_{oDik} - t_{Dik})$
4	$-A_{Dik}$	$V_{Dik} - A_{Dik} (t-t_{oDik} - t_{Dik} - t_{mik})$	$+l_1 + l_2 + \frac{1}{2} A_{Dik} (t-t_{oDik} - t_{Dik} - t_{mik})^2$
5	0	0	$L_{Dik} - R_{ik}$

The required parameters in terms of boom deployment are

$$\{\Delta R_{ik}\} = (\text{Relative Displacement}) \{y_{Bik}\} \quad (17.0-5)$$

$$\{\dot{\Delta R}_{ik}\} = (\text{Velocity}) \{y_{Bik}\} \quad (17.0-6)$$

$$\{\lambda_{ik}\} = (\text{Acceleration}) \{y_{Bik}\} \quad (17.0-7)$$

$$[\lambda_{ik}] = 0 \quad (17.0-8)$$

The total time for boom deployment is relatively short, therefore, a large number of small integration steps may be taken and still maintain reasonable length runs. It is proposed that the time of each phase of the deployment be divided into a large number of constant steps.

REFERENCES

1. Palmer, J. L., "A Derivation of Basic Dynamic Equations Using Matrix Notation" TRW Systems - 9427-600-3RU000, 26 January 1965
2. Jones, A. W., "Computer Program Descriptions of the Generalized Spacecraft Simulation" TRW Systems 06464-6008-T000, 28 February 1967
3. Jones, A. W. and Palmer, J. L., "User's Handbook for the Generalized Spacecraft Simulation", TRW Systems 06464-6007-T000, 28 February 1967
4. McElvain, R. J., "Solar Radiation Torques Acting on an Earth-Sun Oriented Satellite", STL/TR-60-0000-04164, June 1960
5. Palmer, J. L., "Interim Technical Report Dynamic Equations for the Generalized Spacecraft Simulation", TRW Systems 8427-6004-R7000, 1 February 1965

APPENDIX A. PROGRAMMED DYNAMIC EQUATIONS

It was noted in Section 4.0 of the body of this report, that the dynamic equations presented in Section 12.0 are not in the format desired for digital programming. The equations presented contain both the gimbal deflections as independent variables and the locked gimbal reaction torques as dependent variables. Although two methods of handling the elimination of degrees of freedom were discussed, the problem was not resolved and the reformulation of the equations was left to those responsible for the digital programming (see acknowledgement). The final programmed dynamic equations as established by the Mathematical Analysis and Programming Departments are presented in this appendix for reference. The material presented does not stand alone but refers back to the body of the report, and is presented as a bridge between the dynamic equations derived in the body of the report and the equations programmed for the computer.

The final dynamic equations presented in the body of the report, Equations 12.3-4 and 12.3-5, are

$$[L_{11}] \{\dot{\Omega}_{11}\} + \sum_{j=2}^N [L_j] \{\dot{\Omega}_{jj}\} = \{q_{11}\} + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} \{T_{sgi}\} + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} \{T_{yi}\} \quad (A-1)$$

$$[L_{ii}] \{\dot{\Omega}_{ii}\} = \sum_{j=2}^N [L_{ij}] \{\dot{\Omega}_{jj}\} = \{q_{ii}\} - [G_i]_T^{-1} \{T_{sgi}\} - [G_i]_T^{-1} \{T_{yi}\} \quad (A-2)$$

Equation A-1 is the matrix moment equation for the main spacecraft body, body 1, and Equation A-2 is the matrix moment equation for the i^{th} auxiliary body. The torque components $\{T_{sgi}\}$ are the body i gimbal torques for gimbals with degrees of freedom while the torque components $\{T_{yi}\}$ are the body i gimbal torques for gimbals without degrees of freedom. Therefore, for a given gimbal axis one of the corresponding elements in

either $\{T_{sg_i}\}$ or $\{T_{yi}\}$ must be zero. If the non-zero element occurs in $\{T_{sg_i}\}$ it is a driving torque which is a function of position and velocity and it should be added to $\{q_{11}\}$ or $\{q_{ii}\}$. If the non-zero element occurs in $\{T_{yi}\}$ it is a dependent variable and should be eliminated between equations. Premultiplying Equation (A-2) by $[G_i]_T$

$$[G_i]_T [L_{i1}] \{\dot{\Omega}_{11}\} + [G_i]_T \sum_{j=2}^N [L_{ij}] \{\dot{\Omega}_{jj}\} = [G_i]_T \{q_{ii}\} - \{T_{sg_i}\} - \{T_{yi}\} \quad (A-3)$$

It should be noted that Equation A-2 is written in terms of torques about the body i reference axes $\{\bar{Q}_1\}$ and since $[G_i]_T$ is the transformation matrix transforming body i reference torques to gimbal torques, reference Equation 7.2-2, Equation (A-3) is the body i moment equation in terms of body i gimbal torques. From Equation 12.3-9

$$\{\dot{\Omega}_{jj}\} = [G_j] \{\dot{\theta}_j\} + [\dot{G}_j] \{\theta_i\} \quad (A-4)$$

Substituting Equation A-4 into A-3 the body i equation in terms of gimbal torques and gimbal accelerations is

$$[G_i]_T [L_{i1}] \{\dot{\Omega}_{11}\} + [G_i]_T \sum_{j=2}^N [L_{ij}] [G_j] \{\dot{\theta}_j\} = \{q_{ig}\} - \{T_{sg_i}\} - \{T_{yi}\} \quad (A-5)$$

where

$$\{q_{ig}\} = [G_i]_T \left\{ \{q_{ii}\} - \sum_{j=2}^N [L_{ij}] [G_j] \{\dot{\theta}_j\} \right\} \quad (A-6)$$

Define the matrix $[P_{yi}]$ such that for body i

$$[P_{yi}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for no degrees of freedom}$$

$$\begin{aligned}
[P_{yi}] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} && \text{for one degree of freedom} \\
[P_{yi}] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} && \text{for two degrees of freedom} \\
[P_{yi}] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && \text{for three degrees of freedom}
\end{aligned} \tag{A-7}$$

Note that the restriction has already been made in Section 4.0 that the first degree of freedom is about the x gimbal axis and the second degree of freedom is about the y gimbal axis.

Then from Equation A-7

$$\begin{aligned}
[U-P_{yi}] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && \text{for no degrees of freedom} \\
[U-P_{yi}] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && \text{for one degree of freedom} \\
[U-P_{yi}] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} && \text{for two degrees of freedom} \\
[U-P_{yi}] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} && \text{for three degrees of freedom}
\end{aligned} \tag{A-8}$$

Consider Equation A-5 which represents the torques about the three gimbal axes of body i including those with degrees of freedom and those without degrees of freedom. If Equation A-5 is premultiplied by $[P_{yi}]$ all of

the elements representing gimbal axes without degrees of freedom have been set to zero. If Equation A-5 is premultiplied by $[U-P_{yi}]$ all of the elements representing gimbal axes with degrees of freedom have been set to zero. Therefore

$$(A-5) = [P_{yi}] (A-5) + [U-P_{yi}] (A-5)$$

Premultiplying (A-5) by $[P_{yi}]$

$$[P_{yi}] [G_i]_T [L_{i1}] \{\dot{\Omega}_{11}\} + [P_{yi}] [G_i]_T \sum_{j=2}^N [L_{ij}] [G_j] \{\dot{\theta}_j\} = [P_{yi}] \{q_{ig}\} - \{T_{sgi}\} \quad (A-9)$$

Premultiplying (A-5) by $[U-P_{yi}]$

$$[U-P_{yi}] [G_i]_T [L_{i1}] \{\dot{\Omega}_{11}\} + [U-P_{yi}] [G_i]_T \sum_{j=2}^N [L_{ij}] [G_j] \{\dot{\theta}_j\} = [U-P_{yi}] \{q_{ig}\} - \{T_{yi}\} \quad (A-10)$$

Note

$$[P_{yi}] \{T_{sgi}\} = \{T_{sgi}\}$$

$$[U-P_{yi}] \{T_{yi}\} = \{T_{yi}\}$$

Equation A-9 is the body i matrix moment equation with the dependent variables $\{T_{yi}\}$ eliminated. Since the elements representing locked gimbals are zero, there will be a row of zeros in the system matrix for each degree of freedom eliminated. In order to prevent the system matrix from being singular a value of unity must be placed on the diagonal in each of the zero rows. This is accomplished by adding the term $[U-P_{yi}]$

$\sum_{j=2}^N \delta_{ij} \{\dot{\theta}_j\}$ to Equation A-9. The matrix moment equation for body i may

now be written as

$$\begin{aligned}
& [P_{yi}] [G_i]_T [L_{i1}] \{\dot{\Omega}_{11}\} + \sum_{j=2}^N \left[[P_{yi}] [L_{ij}] [G_j] + [U-P_{yi}] \delta_{ij} \right] \{\ddot{\theta}_{jj}\} \\
& = [P_{yi}] \{q_{ig}\} - \{T_{sgi}\} \quad (A-11)
\end{aligned}$$

Equation A-10 is an auxiliary equation defining the dependent variables $\{T_{yi}\}$. Premultiplying A-10 by $[D_i]_T [G_i]_T^{-1}$ and summing over all bodies

$$\begin{aligned}
& \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} [U-P_{yi}] [G_i]_T [L_{i1}] \{\dot{\Omega}_{11}\} \\
& + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} [U-P_{yi}] [G_i]_T \sum_{j=2}^N [L_{ij}] [G_j] \{\ddot{\theta}_{jj}\} \quad (A-12) \\
& = \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} [U-P_{yi}] \{q_{ig}\} - \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} \{T_{yi}\}
\end{aligned}$$

If Equation A-12 is added to the main body Equation A-1 the dependent variables $\{T_{yi}\}$ are eliminated from all equations. Substituting Equation A-4 into A-1 and adding A-12 the main body equation becomes

$$\begin{aligned}
& \left[[L_{11}] + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} [U-P_{yi}] [G_i]_T [L_{i1}] \right] \{\dot{\Omega}_{11}\} \\
& + \sum_{j=2}^N \left[[L_{1j}] + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} [U-P_{yi}] [G_i]_T [L_{ij}] \right] [G_j] \{\ddot{\theta}_{jj}\} \quad (A-13) \\
& = \{q_{11}\} - \sum_{j=2}^N [L_{1j}] [G_j] \{\dot{\theta}_{jj}\} + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} [U-P_{yi}] \{q_{ig}\} \\
& + \sum_{i=2}^N [D_i]_T [G_i]_T^{-1} \{T_{sgi}\}
\end{aligned}$$

Rewriting the main body Equation A-13 in the form of Equation 4.0-6

$$[M_{11}] \{\dot{\Omega}_{11}\} + \sum_{j=2}^N [M_{1j}] \{\ddot{\theta}_{jj}\} = \{b_1\} \quad (A-14)$$

where

$$[M_{11}] = [L_{11}] + \sum_{i=2}^N \left[[D_i]_T [G_i]^{-1} [U-P_{yi}] [G_i]_T [L_{i1}] \right] \quad (A-15)$$

$$[M_{1j}] = \left[[L_{1j}] + \sum_{i=2}^N [D_i]_T [G_i]^{-1} [U-P_{yi}] [G_i]_T [L_{ij}] \right] [G_j] \quad (A-16)$$

$$\begin{aligned} \{b_1\} = \{q_{11}\} - \sum_{j=2}^N [L_{1j}] [\dot{G}_j] \{\dot{\theta}_{jj}\} + \sum_{i=2}^N [D_i]_T [G_i]^{-1} [U-P_{yi}] \{q_{ig}\} \\ + \sum_{i=2}^N [D_i]_T [G_i]^{-1} \{T_{sgi}\} \end{aligned} \quad (A-17)$$

Rewriting the auxiliary body i Equation A-11 in the form of Equation 4.0-6

$$[M_{i1}] \{\dot{\Omega}_{11}\} + [M_{ij}] \{\ddot{\theta}_{jj}\} = \{b_i\} \quad (A-18)$$

where

$$[M_{i1}] = [P_{yi}] [G_i]_T [L_{i1}] \quad (A-19)$$

$$[M_{ij}] = [P_{yi}] [G_i]_T [L_{ij}] [G_j] + [U-P_{yi}] \hat{\delta}_{ij} \quad (A-20)$$

$$\{b_i\} = [P_{yi}] \{q_{ig}\} - \{T_{sgi}\} \quad (A-21)$$

Eliminating $\{T_{sgi}\}$ between A-17 and A-21

$$\{b_1\} = \{q_{11}\} - \sum_{j=2}^N [L_{1j}] [\dot{G}_j] \{\dot{\theta}_{jj}\} + \sum_{i=2}^N [D_i]_T [G_i]^{-1} \left\{ \{q_{ig}\} - \{b_i\} \right\} \quad (A-22)$$

LIST OF SYMBOLS

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
COORDINATE SYSTEMS		
3.1	$\{\bar{Q}_I\}$	Inertial reference system at center of mass of the earth
3.2	$\{\bar{Q}_M\}$	Orbital reference system at center of mass of spacecraft
3.2	$\{\bar{Q}_R\}$	Orbital reference system at center of mass of body 1
3.3	$\{\bar{Q}_1\}$	Main body reference system at center of mass of main body
3.4	$\{\bar{Q}_{ni}\}$	Body i reference system fixed in body 1 at body i attach point ($2 \leq i \leq N$). Defines nominal position of body i.
3.5	$\{\bar{Q}_i\}$	Body i reference system at body i attach point ($2 \leq i \leq N$).
9.2	$\{\bar{Q}_\beta\}$	Intermediate systems in the transformation of $\bar{Q}_I$ to $\bar{Q}_M$ by rotations β , ξ , α , and a change in sign
9.2	$\{\bar{Q}_o\}$	
9.2	$\{\bar{Q}_\alpha\}$	
10.2	$\{\bar{Q}_G\}$	Earth fixed system, $\bar{y}_G$ is polar, and $\bar{z}_G$ is at the prime meridian
10.2	$\{\bar{Q}_m\}$	Earth fixed magnetic system, $\bar{y}_m$ at magnetic pole
14.2	$\{\bar{Q}_{Bik}\}$	Solar oriented reference system for k^{th} boom of body i
14.5	$\{\bar{Q}'_{Bik}\}$	Reference system for k^{th} boom of body i, located at tip of boom and rotates as boom bends.
13.1	$\{\bar{Q}_{Nik}\}$	Flow oriented system for k^{th} shape element of body i
13.2	$\{\bar{Q}_e\}$	Flow oriented system for area element dA

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
COORDINATE SYSTEMS (Continued)		
13.4	$\{\bar{Q}_S\}$	Reference system used in the integration of forces and moments on a sphere
13.5	$\{\bar{Q}_C\}$	Reference system used in the integration of forces and moments on an element of a cylinder
13.8	$\{\bar{Q}_{Be}\}$	Reference system used in determining the forces and moments on a boom element

COORDINATE TRANSFORMATION MATRICES

3.3	$[A]$	Transforms $\{\bar{Q}_R\}$ to $\{\bar{Q}_1\}$ and defines the position of body 1 with respect to the orbital reference system
3.4	$[B_i]$	Transforms $\{\bar{Q}_1\}$ to $\{\bar{Q}_{ni}\}$ and defines the position of the body i reference system in body 1 (constant)
3.5	$[C_i]$	Transforms $\{\bar{Q}_{ni}\}$ to $\{\bar{Q}_i\}$ and defines the position of body i with respect to $\{\bar{Q}_{ni}\}$
3.5	$[D_i] = [C_i][B_i]$	Transforms $\{Q_1\}$ to $\{Q_i\}$ and defines the position of body i with respect to body 1
3.2	$[E]$	Transforms $\{\bar{Q}_I\}$ to $\{\bar{Q}_M\}$ and $\{Q_R\}$; defines position of orbital reference systems with respect to the inertial system
9.2	$[E_{\beta I}]$	Intermediate transformations in the transformation $[E]$ representing rotations β , ξ , and α and a change in signs
9.2	$[E_{o\beta}]$	
9.2	$[E_{\alpha o}]$	
9.2	$[E_{M\alpha}]$	
9.2	$[E_{Mo}]$	
11.3	$[F_i] = [D_i][A]$	Transforms $\{\bar{Q}_R\}$ to $\{\bar{Q}_i\}$ and defines the position of body i with respect to the orbital reference system

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
COORDINATE TRANSFORMATION MATRICES Continued)		
7.2	$[K_i]$	Transforms body i gimbal displacements to gimbal spring torques
7.2	$[P_i]$	Transforms body i gimbal velocities to gimbal viscous damping torques
14.5	$[\alpha_{ik}]$	Transforms the reference system for the k^{th} boom of body i to the rotated system at the tip of the boom
14.2	$[\tau_{ik}]$	Transforms the body i system $\{\bar{Q}_i\}$ to the boom reference system $\{\bar{Q}_{Bik}\}$
10.2	$[\Delta]$	Constant matrix which transforms the earth fixed axes through the prime meridian $\{\bar{Q}_G\}$ to the earth fixed magnetic axes $\{Q_m\}$.
10.2	$[\phi]$	Transforms $\{\bar{Q}_I\}$ to $\{\bar{Q}_G\}$ and defines the earth fixed axes $\{\bar{Q}_G\}$ with respect to the inertial system
13.1	$[\eta_{Nik}]$	Transforms the body i axes $\{\bar{Q}_i\}$ to the flow oriented axes associated with the shape of the k^{th} element of body i $\{\bar{Q}_{Nik}\}$

INERTIA MATRICES Slug-ft²)

2.1	$[I_1]$	Inertia matrix of body 1 with respect to the $\{\bar{Q}_1\}$ system
10.6	$[I_i]$	Inertia matrix of body i with respect to the $\{\bar{Q}_i\}$ system translated to the center of the mass for body i
4.1	$[I_{iH}]$	Inertia matrix of body i with respect to the $\{\bar{Q}_i\}$ system at the body i hinge point
10.6	$[I_{ik}]$	Inertia matrix of tip mass m_{ik} with respect to the $\{\bar{Q}_i\}$ system translated to the tip of boom k

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
SPECIAL SQUARE MATRICES		
7.1	$[G_i]$	Transforms body i gimbal rotational velocities to body i relative velocities
7.2	$[G_i]^{-1}_T$	Transforms body i gimbal torques to torques about the body axes $\{\bar{Q}_i\}$
14.3	$[\beta_{ik}]$	Transforms components of sun vector in body i coordinates into tip mass deflections due to thermal bending
14.4	$[\gamma_{ik}] = [\beta_{ik}]$	
1.4	$[\rho_{ik}]$	Grouping of terms in expression for $[\beta_{ik}]$
14.4	$[\gamma_{ik}]$	
14.4	$[\Delta_{ik}]$	
10.5	$[\lambda_{ik}]$	Transforms body i inertial accelerations into components of $\{\Delta \ddot{R}_{ik}\}$ which are not independent of accelerations.
12.1	$[R_{11}] = \sum_{i=2}^N \frac{M_i}{M_T} [D_i]_T [\beta_i'] [D_i]$	
10.6	$[\beta_i'] = \sum_k \frac{m_{ik}}{M_i} [\lambda_{ik}]$	
10.6	$[W_i] =$	Transforms body i inertial accelerations into components of $\{H_{Rii}\}$ which are not independent of accelerations.
12.0	$[L]$	3N x 3N System matrix for dynamic equations in terms of torques and accelerations about body axes.
4.0	$[M]$	3N x 3N System matrix for dynamic equations in terms of torques and accelerations about gimbal axes.

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
VECTORS		
13.1	$\bar{A}_{Nik}$	A unit vector defining the orientation of the k^{th} shape in body i ($1 \leq 2 \leq N$). The subscript N changes to define the shape, F for fiat plate, S for sphere, C for cylinder, and P for parallele
10.3	$\bar{B}_m$	Magnetic field vector at center of mass of spacecraft, magnitude in gauss
6.1	$\bar{F}_{Ai}$	Aerodynamic force on body (pounds).
6.1	$\bar{F}_{Cl}$	Control force on body 1 (pounds)
6.1	$\bar{F}_{Ei}$	Net external force on body i at its reference point (pounds)
6.1	$\bar{F}_{gi}$	Gravity force on body i (pounds)
8.0	$\bar{F}_{gm}$	Total gravity force on spacecraft (pounds)
14.2	$\bar{F}_{ik}$	Input vector to define reference bending plane for out of plane thermal bending (N.D.)
14.2	$\bar{f}_{ik}$	Component of $\bar{F}_{ik}$ normal to boom i
6.1	$\bar{F}_{Ri}$	Internal reaction force between body i and main body (pounds)
6.1	$\bar{F}_{Si}$	Solar force on body i (pounds)
5.4	$\bar{G}_i$	Inertial acceleration of body i center of mass (ft. per sec. ²)
5.4	$\bar{G}_{ni}$	Inertial acceleration of body i reference point (ft. per sec. ²)
5.4	$\bar{G}_M$	Inertial acceleration of center of mass of total spacecraft (ft. per sec. ²)
5.4	$\bar{G}_R$	Inertial acceleration of center of mass of main body (ft. per sec. ²)
8.0	$\bar{G}_{Mo}$	Inertial acceleration of total center of mass used to define orbit
5.4	$\bar{H}_i$	Angular momentum of body i due to internal motion (ft. -lb. -sec.)

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
VECTORS (Continued)		
5.4	$\bar{H}_M$	Acceleration of total center of mass with respect to main body center of mass (ft. per sec. ²)
5.4	$\bar{H}_{pi}$	Acceleration of body i center of mass with respect to main body center of mass (ft. per sec. ²)
12.1	$A \bar{H}_{pi}$	That portion of $\bar{H}_{pi}$ containing unknown acceleration of the system
12.1	$N \bar{H}_{pi}$	That portion of $\bar{H}_{pi}$ not containing unknown acceleration of the system
6.2	$\bar{H}_{Ri}$	Angular momentum of body i due to thermal bending (ft. -lb. -sec.)
3.4	$\bar{H}_{ui}$	Acceleration of body i reference point with respect to the main body center of mass (ft. per sec. ²)
5.4	$\bar{H}_{vi}$	Acceleration of body i center of mass with respect to body coordinate (ft. per sec. ²)
6.2	$\bar{H}_{wi}$	Angular momentum of body i due to reaction wheels (ft. -lb. -sec.)
13.0	$\bar{M}_{Nik}$	General expression for the solar torque about the centroid of the k th shape of body i ($1 \leq i \leq N$). The subscript N changes to define the shape (see $\bar{A}_{Nik}$). Torques are in ft-lbs.
13.3	$\bar{M}_{Bik}$	Total solar movement due to boom k of body i ($2 \leq i \leq N$). Torque in ft-lbs. about the body i reference point.
15.0	$\bar{N}_i$	Magnetic moment of body i in ampere-turns -ft ²
5.4	$\bar{p}_i$	Vector position of body i center of mass with respect to main body center of mass in feet
14.2	$\bar{p}_{ik}$	Unit vector along $\bar{f}_{ik}$

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
VECTORS (Continued)		
13.0	$\bar{P}_{Nik}$	General expression for the solar force on the k^{th} shape of body i ($1 \leq i \leq H$). The subscript N changes to define the shape (see A_{Nik}). Forces are in pounds.
13.3	$\bar{P}_{Bik}$	Total solar force on boom k of body i ($2 \leq i \leq N$)
2.1	$\bar{R}_{ik}$	Vector position of mass m_{ik} in body coordinates. Magnitude in feet.
10.5	${}_o\bar{R}_{ik}$	Nominal input value of $\bar{R}_{ik}$ before changes in shape, magnitude in feet
5.4	$\bar{R}_M$	Vector position of total spacecraft center of mass with respect to the main body center of mass (feet).
11.1	$\bar{R}_{Mi}$	Vector position of body i center of mass with respect to total spacecraft center of mass (feet)
.4	$\bar{S}_i$	Vector position of the center of mass of body i with respect to the center of mass of the earth (nautical miles)
3.2	$\bar{S}_M$	Vector position of the center of mass of total spacecraft with respect to the center of mass of the earth (nautical miles)
5.4	$\bar{S}_{ni}$	Vector position of the body i reference point with respect to the center of mass of the earth (nautical miles)
5.4	$\bar{S}_R$	Vector position of the main body center of mass with respect to the center of mass of the earth (nautical miles)
6.1	$\bar{T}_{Ai}$	Aerodynamic torque on body i about the body i reference point (lb. -ft.)
6.1	$\bar{T}_{Cl}$	Control torque on body l about the center of mass of body l (lb. -ft.)
6.1	$\bar{T}_{Ei}$	Net external torque about the body i reference point (lb. -ft.)

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
VECTORS (Continued)		
6.1	$\bar{T}_{gi}$	Gravity torque on body i about the body i center of mass (lb. -ft.)
6.1	$\bar{T}_{Hi}$	Torque transmitted from body i to body 1 (lb. -ft.)
6.1	$\bar{T}_{MI}$	Magnetic couple on body i (lb. -ft.)
6.1	$\bar{T}_{Si}$	Solar torque on body i about the body i reference point (lb. -ft.)
10.6	$\bar{T}_{Ti}$	That portion of the thermal bending torque which is due to the accelerations of mass m_{ik} and is independent of unknown accelerations of the system.
2.1	$\bar{u}_i$	Vector position of body i reference point with respect to body 1 center of mass (feet)
3.5	$\bar{v}_i$	Vector position of body i center of mass with respect to body i reference point (feet)
10.6	$\bar{v}_i$	Nominal value of $\bar{v}_i$ for no change in shape of body i
10.3	$\bar{\alpha}$	Unit vector along $\bar{S}_M$
13.1	$\bar{\delta}_F$	Unit vector in direction of particle flow
10.1	$\bar{\delta}_S$	Unit sun vector
13.1	$\bar{\delta}_A$	Unit vector along negative velocity vector
11.1	$\bar{\zeta}$	Vector position of mass element dm with respect to body i center of mass (feet)
14.2	$\bar{\eta}_{ik}$	Component of sun vector along $\bar{x}_{Bik}$
14.2	$\bar{\rho}_{ik}$	Vector along $\bar{z}_{Bik}$ with magnitude of η_{ik}
14.5	$\bar{\phi}_B$	Axis of rotation due to thermal bending
3.3	$\bar{\omega}_1$	Inertial rotational velocity of body 1 in radians per sec

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
VECTORS (Continued)		
3.1	$\bar{\omega}_M$	Inertial rotational velocity of orbital reference system in radians per sec.
3.2	$\bar{\omega}_R$	Equal to $\bar{\omega}_M$
3.4	$\bar{\omega}_{ni}$	Inertial rotational velocity of body i reference system equal to $\bar{\omega}_1$
3.5	$\bar{\omega}_i$	Inertial rotational velocity of body i in radians per sec.
3.2	$\bar{\Omega}_R$	Relative rotational velocity of orbital reference system with respect to the inertial system, equal to $\bar{\omega}_R$.
5.2	$\bar{\Omega}_1$	Relative rotational velocity of body 1 with respect to the orbital reference system in radians per sec.
5.2	$\bar{\Omega}_i$	Relative rotational velocity of body i with respect to body 1 in radians per sec.

GIMBAL AXES PARAMETERS

7.3	$\{T_{Bi}\}$	Hysteresis damper torques at body i gimbal axes in lb. -ft.
16.1	$\{T_{Bmi}\}$	Hysteresis damper saturation torque for body i gimbal axes in lb. -ft.
16.1	$\{T_{BiSC}\}$	Hysteresis damper torques at last corresponding sign change in $\{\theta_i\}$ (lb. -ft.).
7.2	$\{T_{gi}\}$	Net gimbal torques at body i gimbal axes (lb. -ft.)
7.3	$\{T_{Li}\}$	Motor torques on body i gimbal axes due to drive motors (lb. -ft.).
7.3	$\{T_{Ni}\}$	Stop torques at body i gimbal axes (lb. -ft.)
7.2	$\{T_{sgi}\}$	Reaction torques at body i gimbal axes with degrees of freedom (lb. -ft.)

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
GIMBAL AXES PARAMETERS (Continued)		
7.3	$\{T_{yi}\}$	Reaction torques at body i gimbal axes without degrees of freedom
16.1	$\{\Delta\theta_i\}$	Relative gimbal displacements since last sign change in corresponding gimbal velocities (radians)
16.1	$\{\Delta T_i\}$	Change in hysteresis damper torques since change in corresponding gimbal velocities (lb. -ft.)
7.1	$\{\theta_i\}$	Body i gimbal displacements from nominal axes (radians)
16.1	$\{\theta_{isc}\}$	Gimbal displacements at last sign change in corresponding gimbal velocities (radians)

SCALARS AND CONSTANTS

8.0	a	Semi-major axis of orbit in nautical miles
10.4	a_E	Distance from center of mass of spacecraft to earth's shadow in nautical miles
13.3	A_{Fik}	Area of flat plate k of body i (ft ²)
17.0	A_{Bik}	Acceleration magnitude due to boom deployment for boom ik (ft per sec ²)
14.1	C_{Bik}	Non-dimensional in-plane thermal bending coefficient
14.1	C'_{Bik}	Non-dimensional out-of-plane thermal bending coefficient
13.6	D_{Cik}	Diameter of cylindrical shape k of body i (feet)
13.9	D_{Bik}	Diameter of k th boom of body i (feet)
10.1	D_{AE}	Number of days after autumnal equinox
13.7	D_{pik}	Depth of k th paralleliped of body i (feet)

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
SCALARS AND CONSTANTS (Continued)		
9.3	g_i	Acceleration of gravity at center of mass of spacecraft (ft per sec ²)
	$\left. \begin{array}{l} \{H_{1i}\} \\ \{H_{2i}\} \\ \{H_{3i}\} \\ \{H_{4i}\} \end{array} \right\}$	Hysteresis damper coefficients for body i
16.1		
2.1	i	General subscript denoting body i
2.1	k	General subscript denoting the k th boom or the k th shape of body i
9.0	K_N	Number of feet in a nautical mile
15.0	K_M	Conversion factor from ampere-turns-ft ² -gauss to foot-pounds
10.4	k_E	Ratio of instantaneous thermal bending coefficient to the nominal value
17.0	L_{Bik}	Deployed length of boom ik (feet)
13.6	L_{Cik}	Length of cylindrical shape k of body i (feet)
13.7	L_{pik}	Length of parallelepiped k of body i (feet)
10.3	M_e	Magnetic field constant
2.1	M_i	Mass of body i (slugs)
2.1	m_{ik}	Mass of element k of body i (slugs)
5.4	M_T	Total mass of spacecraft (slugs)
1.0	N	Number of bodies
8.0	P	Orbital period in hours
10.4	p_E	Time rate of change of parameter k_E

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
SCALARS AND CONSTANTS (Continued)		
13.6	R_{Cik}	Radius of cylindrical shape k of body i (feet)
13.4	R_{Sik}	Radius of spherical shape k of body i (feet)
9.0	R_e	Radius of Earth in nautical miles
10.4	t_E	Time of eclipse transition in minutes
10.2	t_g	GMT
17.0	t_{DBik}	Initial time of deployment for boom ik (minutes)
17.0	t_{Dik}	Acceleration time during deployment of boom ik (minutes)
17.0	V_{Dik}	Deployment velocity of boom ik (feet per sec)
13.7	W_{pik}	Width of k^{th} parallelepiped of body i (feet)
9.2	α	Orbit angle from line of nodes (radius)
8.0	α_p	Orbit angle from perigee (radians)
10.4	α_o	Orbit angle for closest approach to earth's shadow
10.4	α_e	Orbit angle at which eclipse computations are started
9.2	β	Angle between autumnal equinox and ascending line of node measured in the equatorial plane
8.0	γ	Gravitational constant
8.0	ϵ	Eccentricity of orbit
13.1	η_{ik}	Angle between flow vector and the orientation vector for the k^{th} shape of body i
13.8	λ_{ik}	Angle between flow vector and the nominal position of boom ik

<u>Section</u>	<u>Symbol</u>	<u>Definition</u>
SCALARS AND CONSTANTS (Continued)		
13.2	ν_T	Tangential reflectivity coefficient or momentum exchange coefficient
13.2	ν_N	Normal reflectivity coefficient or momentum exchange coefficient
9.2	ξ	Orbit inclination with respect to equatorial plane
10.1	θ_S	Position of sun measured from autumnal equinox in the ecliptic plane
10.1	θ'_S	Position of sun measured from autumnal equinox in the equatorial plane
14.5	θ_{Bik}	Rotation of tip of boom ik due to thermal expansion
10.4	σ_E	Angle between orbital radius vector and sun vector
14.4	σ_{ik}	Angle between bending plane and reference bending plane of boom ik
10.2	ϕ_G	Angle between prime meridian and autumnal equinox
8.0	ψ	Orbit angle from perigee
8.0	ω_o	Equivalent circular orbit rate