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DYNAMIC STABILITY OF SPACE VEHICLES

Structural Dynamics Model Testing

by J. W. Wissmann

Prepared by
GENERAL DYNAMICS CORPORATION
San Diego, Calif.

for

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FOREWORD

This report supplements the series of volumes listed below, all of which were prepared under Contract NAS8-11486. The reports of the series are intended to illustrate methods used to determine parameters required for the design and analysis of flight control systems of space vehicles. Below is a list of the reports of the series previously published.

Volume I	Lateral Vibration Modes
Volume II	Determination of Longitudinal Vibration Modes
Volume III	Torsional Vibration Modes
Volume IV	Full Scale Testing for Flight Control Parameters
Volume V	Impedence Testing for Flight Control Parameters
Volume VI	Full Scale Dynamic Testing for Mode Determination
Volume VII	The Dynamics of Liquids in Fixed and Moving Containers
Volume VII	Atmospheric Disturbances that Affect Flight Control Analysis
Volume IX	The Effect of Liftoff Dynamics on Launch Vehicle Stability and Control
Volume X	Exit Stability
Volume XI	Entry Disturbance and Control
Volume XII	Re-entry Vehicle Landing Ability and Control
Volume XIII	Aerodynamic Model Tests for Control Parameters Determination
Volume XIV	Testing for Booster Propellant Sloshing Parameters
Volume XV	Shell Dynamics with Special Applications to Control Problems

The work was conducted under the direction of Clyde D. Baker and George F. McDonough, Aero Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The General Dynamics Convair Program was conducted under the direction of David R. Lukens.

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LIST OF SYMBOLS

Symbols used in the fundamental theory are identified by their equation number of first occurrence. Indices are generally omitted in this list. Symbols which occur only in the Appendix are not listed here.

a	exponent matrix (3-2); acceleration; bending deflection
A	cross section; bending mode shape
b	exponent matrix (3-4); shear deflection
B	shear mode shape
c	number of basic dimensions (3-2)
d	basic dimension (3-2)
E	modulus of elasticity
f( )	function (3-3)
F	force
g	damping coefficient
g( )	function (3-3)
G	shear modulus
h	liquid depth
I	section moment of inertia
K	shear constant
ℓ	length; subscript indicating "liquid"
L	basic dimension length
m	number of dimensional variables (3-1); mass per unit length
(m)	superscript indicating "model" (3-10)
M	basic dimension mass
n	number of dimensionless variables (3-3); material constant for damping
N_M	Mach number
N_R	Reynolds number
p	pressure
(p)	superscript indicating "prototype" (3-10)

P	basic dimension force
r	rank of exponent matrix a (3-7); tank radius
s	number of dimensionless variables which can be scaled exactly (3-30)
S_α	static mass imbalance about elastic axis per unit length
t	time
T	basic dimension time
u	displacement
v	velocity
w	total lateral deflection; subscript indicating "wall"
x	dimensional variable (3-1); orthogonal Cartesian coordinate; subscript indicating "component in x-direction"
x_o	elastic axis location
y	orthogonal Cartesian coordinate; subscript indicating "component in y-direction"
z	orthogonal Cartesian coordinate; subscript indicating "component in z-direction"
α	variation parameter (3-33); dimensionless bending deflection
β	dimensionless shear deflection
δ	basic scale factor (3-17)
Δ	increment (3-36)
ϵ	characteristic length of surface roughness
η	dimensionless coordinate; dynamic viscosity of liquid
μ	rotational inertia per unit length
$\bar{\mu}$	coefficient of friction
ν	Poisson's ratio
ξ	derived scale factor (3-18)
π	dimensionless variable (3-3)
ρ	mass density
σ	normal stress
τ	shear stress; characteristic time; relaxation time
φ ( )	function (3-32)

$\psi(\ )$	function (3-34)
ω	circular frequency
ω_α	circular torsional frequency
ω_b	circular bending frequency
[ ]	dimensional brackets (3-2)
{ }	matrix brackets
$\sum_i$	sum over all permissible values of index i
$\prod_i$	product over all permissible values of index i

1/INTRODUCTION

Model testing is probably one of the oldest tools devised by man to gain insight into the behavior of physical systems. In fact, it may be the only rational means to assess the influence of the variation of some parameter on a hitherto unknown phenomenon. For instance, to formulate Hooke's law of linear elasticity, it was necessary to construct model tests with geometrically similar models to evaluate the influence of the cross section, thus leading to the concept of stress.

Tests must be constructed whenever a relevant physical phenomenon cannot be formulated analytically due to a lack of knowledge about it, or, whenever the analytical representation (mathematical model) becomes too complicated, costly, or unreliable. Before proceeding with tests, however, a thorough evaluation of the merits of tests versus analysis should be made. Tests may look deceivingly simple but may run into enormous cost. On the other hand, a simple test may easily replace a tedious analysis. Very often test and analysis are carried out in parallel. This is done especially in the aerospace industry to assure validity of the results.

The question of tests versus analysis should be seen in the proper perspective. The recurring costs of tests may be very high, whereas routine analysis is usually much cheaper. Therefore, every effort should be made to develop analytical tools. On the other hand, more extensive tests are necessary to develop these capabilities and will continue to be needed to verify analytical predictions. Only a systematic integration of tests and analyses will satisfy both requirements: accuracy of prediction and cost-effectiveness.

This monograph deals with the use of models, not with simulation. Models in this sense are devices which employ the same physical phenomena that govern the behavior of the hardware under investigation. In contrast to this, simulation uses different phenomena. Well-known examples are electrical analog simulations, supersonic flow simulations using liquid surface waves, or the membrane analogy of torsional shear-stress patterns.

Once it has been decided that tests will be conducted, the question arises as to whether to build a model or to test the full-scale hardware. Some of the factors that influence this decision are cost, available test facilities, size of the test item, and the time at which the results should be available. The latter consideration, especially, is strongly in favor of models, if the test results are needed for the final design of the hardware; generally, models can be built much faster than their full-size counterparts. Another interesting argument for models is this: using several models which exhibit a variation of some important parameter, an insight into the behavior of the physical phenomenon in response to this variation is achieved. Sometimes this is the only way which allows an intelligent interpretation of the test results.

The testing of models and of full-scale hardware are, in principle, not different from each other. The differences are of a technical nature and in the area of interest, i.e., for launch vehicle control, they occur mainly in aerodynamic testing. Consequently, the bulk of this monograph will be concerned with the laws and difficulties of valid modeling. How to plan and conduct the actual tests is a different subject, and is adequately covered in other monographs of this series, e.g., Reference 9.

In connection with the control dynamics requirements of launch vehicles, there are four basic areas which call for testing of models:

a. Structural dynamics

b. Liquid fuel sloshing

c. Aerodynamics

d. Thermodynamics

This monograph is concerned with the first item in the list. The second item is treated in another monograph of this series (Reference 10).

2/STATE OF THE ART

As mentioned in the introduction, model testing has an ancient tradition. However, only at the end of the 19th century, through the works of Lord Rayleigh, Reynolds, et. al., was model testing put on a scientific basis. In 1914 Buckingham of the National Bureau of Standards published his famous "π-theorem." This theorem introduced dimensional analysis as the basis for all model work, and it still remains to the present day. In subsequent years Bridgman[2] and especially Langhaar[1] put the π-theorem on a sound mathematical basis and started to exploit its powers and pitfalls. Since then nothing fundamentally new has appeared in the theory of models. In recent years several investigators[6,7] working in areas previously untouched by modeling have formalized some interesting means of combining dimensional analysis with a partial insight into a given problem. This leads to immensely practical "distorted" models in which only the relevant unknown parameters need to be scaled; it also opens the door to the investigation of so-far "unscalable" problems.

The classical area of model testing has been in fluid and gas dynamics, with complete geometrical similarity between model and full-scale hardware. The reason for this was that flow patterns can be accurately calculated only in exceptionally simple cases and, second, that the test facilities demanded small-scale models. The advent of aeroelastic flutter models in the 1940's brought the realization to the aircraft industry that complete dynamical similarity ("replica models") was impractical and would also lead to erroneous results because of model imperfections[12]. Dynamically distorted models which would scale only those modes of natural vibration which correspond with the lowest frequencies brought the desired results. Pure dynamic testing has been carried out widely on the full-scale product. However, the sheer size of modern launch vehicles such as Saturn again renders extremely attractive the use of models (Reference 13 to 19) to make the test item more tractable and less expensive. At present, very expensive replica models are in use; however, they are far less expensive than prototype hardware. With a better understanding of vehicle dynamics, it is to be expected that distorted models will be used eventually.

3/MODEL DESIGN CRITERIA

Scale modeling is more an art than a science, because a physical phenomenon requiring model work is either poorly understood or its application is very complex. In addition, the theory of models may put constraints on a particular model which cannot be fulfilled in practice. The situation need not be hopeless, however, if an investigator understands the theory and the possibilities it offers. Ideal modeling is the exception rather than the rule, and it occurs mostly in textbooks. In the following sections it will be attempted to outline the principles of realistic modeling. Most sections will be accompanied with illustrative examples.

3.1 DIMENSIONAL ANALYSIS

Two principles are involved in scale-model work:

a. The physical phenomena are independent of the units of measurement (invariant under coordinate transformation).

b. The same physical laws govern the behavior of the model and the full-scale article.

At least in classical mechanics, item a. assumes the role of an axiom; i.e., it must be fulfilled under all circumstances. From this point of view, a physical phenomenon must be described after a certain level of abstraction has been achieved. This approach uses the relevant dimensions as the problem coordinates, the units of which are purely coincidental and defined by convenience.

Theoretically, item b. is always true. In practice, however, "scale effects" play an important part. Between a model and a full-scale article a physical phenomenon may be important for one but not for the other. Neglecting the unimportant phenomenon in the full-scale tests, for instance, makes it appear as if the model were exhibiting different phenomena. The classical example for scale effects is small-scale modeling of liquid surface waves. The surface tension of the liquid is important for the model, not the full-scale application.

A physical variable, y, may depend on several independent variables, y_1, y_2, etc. An unknown relation

$$y = f_y (y_1, y_2, \cdots)$$

will exist which ties these quantities together.

An alternate way of writing this relationship is

$$f(x_1, x_2, \cdots, x_m) = 0 \tag{3-1}$$

which is more convenient for the following developments. One of the variables, say x_1, is the dependent one; all others must be independent of each other to assure that Equation (3-1) does exist.

There are several things that need to be said about Equation (3-1). As to the selection of variables: the investigator must be able to make at least an intelligent guess as to the identity of the important variables that enter into the problem. If there is any analytical formulation of the problem available, this will be of great help in identifying these variables. If not, intuition and experience are very important. In case of doubt, a variable should be included rather than left out. It is generally much simpler to discard an extraneous variable during the experiments than to account for a missing one which proves to be important[5]. Next come the dimensions: the variables of Equation (3-1) must be measured relative to a consistent set of basic dimensional units, i.e., units for length, time, mass, etc. If this is not done, conversion factors will distort the physical phenomena and render Equation (3-1) useless. Some consistent systems of dimensions are[1]:

a. cgs system (centimeter, gram, second) which uses the gram as the mass unit; the derived force unit is "dyne."

b. mks mass system (meter, kilogram, second) which uses the kilogram as the mass unit; the derived force unit is "Newton."

c. mks force system (meter, kilogram, second) which uses the kilogram as the force unit; there is no special name for the derived mass unit.

d. British Mass System (foot, pound, second) which uses the pound as the mass unit; the derived force unit is "poundal."

e. American Engineering System (foot, pound, second) which uses the pound as the force unit; the derived mass unit is "slug."

In American aerospace technology, the basic unit for length is the inch rather than the foot. For dimensional analysis it is of no consequence which system of measurements is adopted as long as it is consistent. In fact, an investigator can make up his own system and use it to advantage. For instance, in a problem requiring the dimensions of length and time an alternate system of dimensions would be velocity and time or velocity and acceleration.

Another useful feature in dimensional analysis is the fact that several basic dimensions may be of the same type. For instance, different units of length may be employed parallel to the x-, y-, z-axes of a spatial coordinate system, thus obtaining three rather than one basic unit of length. This procedure is permissible only if the dimensions of all variables can be separated in this way.

Given a set of basic dimensions $[d_1]$, $[d_2]$, $\cdots$, $[d_c]$, where c is the total number of dimensions employed, the dimensions of the variables, x, of Equation (3-1) are given through

$$[x_p] = \prod_i [d_i]^{a_{ip}} \tag{3-2}$$

$$(i = 1, 2, \cdots, c; p = 1, 2, \cdots, m)$$

Square brackets are used here to express the fact that the dimensions are involved, not the numerical quantities. For instance, x_p would be the magnitude of the p^{th} variable x in Equation (3-1), whereas $[x_p]$ is the corresponding dimension. The notation $\prod_i$ indicates that the product is to be taken over the term to the right of it for i=1 times that for i=2, etc. The exponents a_{ip} form a c-by-m matrix of numerical values which is characteristic for the variable dimensions relative to the basic dimensions chosen (see example in Section 3.1.1). The matrix "a" is fully known.

The Buckingham "π-theorem" states that a dimensional relation of the form (3-1) can be transformed into nondimensional form, such that

$$g(\pi_1, \pi_2, \cdots, \pi_n) = 0 \tag{3-3}$$

The letter g is used here to denote that this is not the same function which occurs in (3-1). The nondimensional variable π_1 may assume the role of the dependent variable formerly occupied by x_1. The variables π_2, π_3, etc. are necessarily independent of each other. The number n of the dimensionless variables is smaller than the number m of dimensional variables in (3-1).

The value of relation (3-3) for scale-model work lies in the fact that it is nondimensional. This means that it applies equally well to a large-scale representation of a physical phenomenon as it does to a small-scale version, because the only difference between the two lies in the units of the dimensions. In other words, model and full-scale "prototype" have an equation of the type (3-3) in common. Therefore, conclusions drawn from experimental scale-model work in nondimensional form can be applied directly to the dimensionless variables of the full-scale prototype.

The dimensionless variables, π, of (3-3) must be some function of the dimensional variables, x, in (3-1). For the purpose of dimensional analysis it will be sufficient to assume that the nondimensional variables are obtained as products of powers of the dimensional variables (these could also be sums, etc. over variables of equal dimension).

$$\pi_q = \prod_p (x_p)^{b_{pq}} \tag{3-4}$$

$$(p = 1, 2, \cdots, m; q = 1, 2, \cdots, n)$$

The exponents b_{pq} form an m-by-n matrix of numerical values and are so far unknown. Equation (3-4) relates the numerical values of the variables. A similar equation relates the dimensions themselves. Since the variables π are dimensionless by definition, i.e., $[\pi_q] = [0]$, this relation is

$$[0] = \prod_p [x_p]^{b_{pq}} \qquad (p = 1,2,\cdots,m;\ q = 1,2,\cdots,n) \tag{3-5}$$

Substitution of (3-2) into (3-5) yields

$$[0] = \prod_p \left(\prod_i [d_i]^{a_{ip}} \right)^{b_{pq}}$$

Multiplying the expressions forming the base is equivalent to adding the exponents; it follows that

$$[0] = \prod_i [d_i]^{\sum_p a_{ip} b_{pq}}$$

The basic dimensions $[d_1]$, $[d_2]$, etc. are independent of each other. Therefore, each individual exponent must be zero to generate the dimensionless quantities [0].

$$\sum_p a_{ip} b_{pq} = 0 \tag{3-6}$$

$$(i = 1, 2, \cdots, c;\ p = 1, 2, \cdots, m;$$
$$q = 1, 2, \cdots, n)$$

This is the condition that is needed to calculate the exponent matrix which, according to Equation (3-4), transforms the dimensional variables x into the nondimensional variables π. The theory of simultaneous linear, homogeneous equations[20] gives exact criteria about how these solutions can be obtained. Generally, when only a few basic dimensions are used, the solutions can be found by simple arithmetic operations (see example in Section 3.1.1).

Equation (3-6) describes c simultaneous linear equations, where c is the number of basic dimensions used in the problem. There are a total of

$$n = m - r \tag{3-7}$$

linearly independent solutions, i.e., columns of the matrix b which determine the number n of dimensionless variables π. In this equation m is the number of dimensional variables x, and r is the rank of the matrix a. Usually, r is equal to c; however, there are exceptional cases where r may be smaller[1].

Equation (3-6) does not provide a unique solution for the matrix b. In fact, there is an infinite number of valid solutions. There is usually a relatively large finite set of solutions when only the simple powers of the variables are considered. Watkins[21,22] has developed a computer program which will automatically compute this finite set of solutions and display the resulting sets of dimensionless variables π. Tradition and the ease of comparison with other model experiments make it advisable to use certain standard dimensionless variables (see Appendix A) such as Reynolds number, Mach number, etc. To simplify experimental control, it is a good rule to set up the dimensionless variables π in such a way that the influence of the dimensional variables x which can be varied is nicely separated. How to achieve this has been outlined by Langhaar[1] (see also example in Section 3.1.1).

The existence of n linearly independent, nontrivial solutions in (3-6) asserts the validity of the Buckingham π-theorem (3-3). It should be noted that this proof is relatively straightforward as compared with the proofs given in the textbooks.

The entire procedure of dimensional analysis can now be described as follows: The dimensional variables x are collected in (3-1); the exponent matrix a is found in (3-2); Equations (3-6) and (3-7) permit the selection of some suitable exponent matrix b; using Equation (3-4), the nondimensional variables, π, are computed; these variables enter into relation (3-3), which is suitable for scale-model work.

3.1.1 EXAMPLE: LATERAL VIBRATIONS OF A LAUNCH VEHICLE. It is intended to perform model tests on the lateral vibrations of a long, slender launch vehicle. For this purpose it will be sufficient to treat the vehicle as a beam which exhibits bending and shear deformations[18].

The number of basic dimensions to be encountered is c = 3. These are, using a system of dimensions based on the force unit,

$$[d_1] = [L]; \text{ Length}$$

$$[d_2] = [P]; \text{ Force}$$

$$[d_3] = [T]; \text{ Time}$$

The variables which enter this problem are:

Variable	Symbol	Dimension	Dependency
Circular frequency	$x_1 = \omega$	$[T^{-1}]$	Dependent
Length of vehicle	$x_2 = \ell$	$[L]$	Independent
Mass per unit length	$x_3 = m$	$[L^{-2}\,PT^2]$	Independent

Cross section	$x_4 = A$	$[L^2]$	Independent
Section moment of inertia	$x_5 = I$	$[L^4]$	Independent
Modulus of elasticity	$x_6 = E$	$[L^{-2} P]$	Independent
Shear modulus	$x_7 = G$	$[L^{-2} P]$	Independent
Shear constant	$x_8 = K$	—	Independent

This establishes Equation (3-1) with m = 8 as

$$f(\omega, \ell, m, A, I, E, G, K) = 0$$

With the information given above, the exponent matrix a of Equation (3-2) can be written:

$$\{a\} = \begin{array}{c} \\ [L] \\ [P] \\ [T] \end{array} \begin{array}{cccccccc} \underline{\omega} & \underline{\ell} & \underline{m} & \underline{A} & \underline{I} & \underline{E} & \underline{G} & \underline{K} \\ 0 & 1 & -2 & 2 & 4 & -2 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{array}$$

The rank of this matrix is equal to the size of the largest nonvanishing determinant that can be constructed out of its columns.

The first three columns yield a 3-by-3 nonvanishing determinant. Therefore, r = 3, and with Equation (3-7)

$$n = 8 - 3 = 5$$

is the number of dimensionless variables π.

Instead of using Equation (3-6), now, to compute the exponent matrix b, the dimensionless variables will be written by inspection in the form of Equation (3-4). Equation (3-6) is used afterwards to check the validity of the operation. This procedure is advisable for small numbers of basic dimensions.

The five dimensionless variables are selected in the following manner:

$$\pi_1 = \frac{m\,\omega^2}{E}, \quad \pi_2 = \frac{I}{\ell^4}, \quad \pi_3 = \frac{\sqrt{\frac{I}{A}}}{\ell}, \quad \pi_4 = \frac{G}{E}, \quad \pi_5 = K$$

There is nothing unique about this selection. Other variables such as $\frac{I}{A^2}$, $\frac{E}{KG}$, etc. could have been used as well. Since K is already a dimensionless quantity, it can (but need not) be used directly as a nondimensional variable. From Equation (3-4) the assumed exponent matrix b can be deduced.

$$\{b\} = \begin{array}{c} \\ \omega \\ \ell \\ m \\ A \\ I \\ E \\ G \\ K \end{array} \begin{array}{c} \begin{array}{ccccc} \underline{\pi_1} & \underline{\pi_2} & \underline{\pi_3} & \underline{\pi_4} & \underline{\pi_5} \end{array} \\ \left\{\begin{array}{ccccc} 2 & 0 & 0 & 0 & 0 \\ 0 & -4 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right\} \end{array}$$

The matrices a and b are multiplied together to check whether condition (3-6) is fulfilled.

$$\{a\} \cdot \{b\} = \begin{array}{c} \\ [L] \\ [P] \\ [T] \end{array} \begin{array}{c} \begin{array}{ccccc} \underline{\pi_1} & \underline{\pi_2} & \underline{\pi_3} & \underline{\pi_4} & \underline{\pi_5} \end{array} \\ \left\{\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right\} \end{array}$$

This is indeed the case. However, one more check must be made to assure the validity of the dimensionless set of variables π. It is conceivable that a set could have been chosen that is linearly inter-dependent, i.e., that one variable π could be expressed as a function of the others. This cannot be the case if a nonvanishing n-by-n (here 5-by-5) determinant can be constructed out of the rows of the matrix b. Selecting rows 3, 4, 5, 7, 8 of b:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2} \neq 0$$

This relationship proves that the set of dimensionless variables π is valid.

The dimensionless statement of the problem (3-3) is now

$$g\left(\frac{m\,\omega^2}{E}, \frac{I}{\ell^4}, \frac{\sqrt{\frac{I}{A}}}{\ell}, \frac{G}{E}, K\right) = 0 \tag{3-8}$$

or, when isolating the dependent variable, ω

$$\omega = \sqrt{\frac{E}{m}} \cdot g_1\left(\frac{I}{\ell^4}, \frac{\sqrt{\frac{I}{A}}}{\ell}, \frac{G}{E}, K\right) \tag{3-9}$$

3.2 SIMILITUDE AND SCALING LAWS

So far, only a single physical configuration has been considered. At this point two separate ones will be introduced, one for the full-scale or "prototype," the other for a scale "model." Quantities relating to these representations will be labeled with the superscripts (p) and (m) respectively.

It should be clear from the development in Section 3.1 that the dimensionless mathematical representation of Equation (3-3) is valid for both model and prototype since it is independent of the choice of dimensional units. Therefore, two equations can be written

$$\left.\begin{aligned} g\left(\pi_1^{(m)}, \pi_2^{(m)}, \dots, \pi_n^{(m)}\right) &= 0 \\ g\left(\pi_1^{(p)}, \pi_2^{(p)}, \dots, \pi_n^{(p)}\right) &= 0 \end{aligned}\right\} \tag{3-10}$$

and the function g is exactly the same for both cases. Taking π_1 as the dependent variable as before, Equations (3-10) may be written as

$$\left.\begin{aligned} \pi_1^{(m)} &= g_1\left(\pi_2^{(m)}, \pi_3^{(m)}, \dots, \pi_n^{(m)}\right) \\ \pi_1^{(p)} &= g_1\left(\pi_2^{(p)}, \pi_3^{(p)}, \dots, \pi_n^{(p)}\right) \end{aligned}\right\} \tag{3-11}$$

The essence of scale-modeling consists of "scaling" the dimensionless variables π_2, $\pi_3 \cdots$ in such a way that they are equal for model and prototype. This results in the similitude requirements[5]

$$\pi_q^{(m)} = \pi_q^{(p)} ; \qquad (q = 2, 3, \cdots, n) \tag{3-12}$$

How these are to be fulfilled is demonstrated in the example (3.2.1) and in a more formal way in Section 3.3. When (3-12) is substituted into the second equation (3-11), a relation between the independent variables of the model and the dependent variable of the prototype results.

$$\pi_1^{(p)} = g_1\left(\pi_2^{(m)}, \pi_3^{(m)}, \cdots, \pi_n^{(m)}\right) \tag{3-13}$$

This shows that a property of the prototype can be determined completely from properties of the model.

Comparison of Equation (3-13) with the first equation (3-11) leads to the scaling law[5]

$$\pi_1^{(m)} = \pi_1^{(p)} \tag{3-14}$$

Obviously, scaling law (3-14) and similitude requirements (3-12) can be combined in one equation.

$$\pi_q^{(m)} = \pi_q^{(p)} ; \qquad (q = 1, 2, \cdots, n) \tag{3-15}$$

From a mathematical point of view this relation could have been inferred directly. However, for the physical interpretation, the separation into a dependent variable, π_1, and the independent ones, $\pi_2, \pi_3, \cdots, \pi_n$, is more rewarding.

3.2.1 EXAMPLE: LATERAL VIBRATIONS OF A LAUNCH VEHICLE. The dimensionless representation of the example chosen in Section 3.1.1 was (3-8)

$$g\left(\frac{m\omega^2}{E}, \frac{I}{\ell^4}, \frac{\sqrt{\frac{I}{A}}}{\ell}, \frac{G}{E}, K\right) = 0$$

The similitude requirements (3-12) are

$$\left(\frac{I}{\ell^4}\right)^{(m)} = \left(\frac{I}{\ell^4}\right)^{(p)}$$

$$\left(\frac{\sqrt{\frac{I}{A}}}{\ell}\right)^{(m)} = \left(\frac{\sqrt{\frac{I}{A}}}{\ell}\right)^{(p)}$$

$$\left(\frac{G}{E}\right)^{(m)} = \left(\frac{G}{E}\right)^{(p)}$$

$$K^{(m)} = K^{(p)}$$

The equation (3-13) can nowbe written as

$$\omega^{(p)} = \sqrt{\left(\frac{E}{m}\right)^{(p)}}\; g_1\left(\frac{I}{\ell^4}, \frac{\sqrt{\frac{I}{A}}}{\ell}, \frac{G}{E}, K\right)^{(m)}$$

This equation may be interpreted as if in the model the mass "m" had no influence whatsoever. That this is not so is revealed by the scaling law (3-14) which demands that

$$\left(\frac{m\,\omega^2}{E}\right)^{(m)} = \left(\frac{m\,\omega^2}{E}\right)^{(p)}$$

Once the physical implications are understood, it is much easier to proceed as follows: write the scaling law (3-14), which leads to

$$\omega^{(p)} = \sqrt{\frac{\left(\frac{E}{m}\right)^{(p)}}{\left(\frac{E}{m}\right)^{(m)}}}\;\omega^{(m)}$$

Then scale the model according to the similitude requirements (3-12) (performed above), and perform the experiment according to the first equation (3-11);

$$\omega^{(m)} = \sqrt{\left(\frac{E}{m}\right)^{(m)}}\; g_1\left(\frac{I}{\ell^4}, \frac{\sqrt{\frac{I}{A}}}{\ell}, \frac{G}{E}, K\right)^{(m)}$$

i.e., measure the frequency of the model.

3.3 SCALE FACTORS

For many engineers working with models, the development in Sections 3.1 and 3.2 may appear to be of purely academic interest. There is, in fact, seemingly a totally different approach which yields all the criteria necessary for a single model test in a much more straightforward manner. However, this approach by itself works only for complete similitude of the model. As soon as distortions and scale effects become of importance, the more general approach of Sections 3.1 and 3.2 must be consulted. Nevertheless, a calculation of the scale factors as outlined below will be sufficient for most routine model work.

The starting point is again the knowledge of the dependent variable, x_1, and the independent variables, x_2, x_3, $\cdots$, x_m, which enter into the problem area of interest. These variables will generally possess nonvanishing basic dimensions $[d_1]$, $[d_2]$, $\cdots$, $[d_c]$ such as length, force, time, etc. The relation of the dimensions (3-2)

$$[x_p] = \prod_i [d_i]^{a_{ip}} ; \qquad (i = 1, 2, \cdots, c; p = 1, 2, \cdots, m) \tag{3-16}$$

is rewritten here for convenience. It establishes the c-by-n exponent matrix, a.

The basic assumption underlying the derivation of scale factors is that the problem under investigation is insensitive to a change in scale of the basic dimensions. This assumption is equivalent to the Buckingham π-theorem stated in Equation (3-3), and its validity is assured by the existence of an exponent matrix, b, which fulfills Equation (3-6). The computation of scale factors therefore becomes just a simplification of the Buckingham π-theorem for the general class of engineering structural problems.

Since the problem is insensitive to changes in the units of the "c" basic dimensions, arbitrary basic scale factors for these units can be defined.

$$\delta_i = \frac{d_i^{(p)}}{d_i^{(m)}} ; \qquad (i = 1, 2, \cdots, c) \tag{3-17}$$

The superscripts (p) and (m) indicate "prototype" and "model" as before. The lack of square brackets around the d_i indicates that units are compared, not the dimensions per se.

A different type of scale factor is defined to compare the "m" variables, x_p, which enter into the problem. These derived scale factors are

$$\xi_p = \frac{x_p^{(p)}}{x_p^{(m)}}; \qquad (p = 1, 2, \cdots, m) \tag{3-18}$$

Equation (3-16) relates dimensions. The same type of equation can be written to relate the quantities associated with these variables (e.g., a velocity of 2 ft/sec is the same as one of 6 ft/3 sec).

$$x_p = \prod_i (d_i)^{a_{ip}}; \qquad (i = 1, 2, \cdots, c; \ p = 1, 2, \cdots, m) \tag{3-19}$$

Equation (3-19) is substituted into (3-18)

$$\xi_p = \frac{\prod_i \left(d_i^{(p)}\right)^{a_{ip}}}{\prod_i \left(d_i^{(m)}\right)^{a_{ip}}} = \prod_i \left(\frac{d_i^{(p)}}{d_i^{(m)}}\right)^{a_{ip}}$$

and with (3-17) as follows

$$\xi_p = \prod_i (\delta_i)^{a_{ip}}; \qquad (i = 1, 2, \cdots, c; \ p = 1, 2, \cdots, m) \tag{3-20}$$

Comparison of Equation (3-19) and (3-20) shows that the scale factors ξ and δ are related in the same way as the quantities "x" and "d" which they are scaling.

The general procedure to be followed is extremely simple: given the significant variables, x, of a problem and the relation of their dimensions [x] to the basic dimensions [d] as expressed in Equation (3-16), basic scale factors, δ, for the basic dimensions are chosen, and the derived scale factors, ξ, for the variables, x, are computed from Equation (3-20). In practice, however, the procedure may not be quite so straightforward. A basic scale factor, δ, for the length, for instance, may be chosen. The necessity to make model and prototype out of the same material defines some derived scale factor, ξ. Application of Equation (3-20) will show whether this is compatible with the decisions made so far, etc. The example in Section 3.3.1 should demonstrate this point clearly.

The traditional notation for scale factors is "λ" for the length, "τ" for time, "χ" for forces, and "σ_E, σ_V, σ_f" for elastic modulus, velocity, frequency, etc.[23] This tradition has been broken here to produce a clear distinction between basic scale factors, δ, and the derived ones, ξ. This allowed the general equation (3-20) to be written which replaces whole sets of equations found in the literature[1]. There is logically a very important distinction between the scale factors "δ" and "ξ": the former can be chosen at will, whereas the latter are subject to the constraint expressed by Equation (3-20). In this connection it is good to remember that the basic dimensions of a problem as defined in Section 3.1 need not necessarily be chosen as length, time, etc., but may very well be velocity, acceleration, etc.

It is easily explained why the dimensionless variables, π, of Section 3.1 do not make their appearance here in the development of the scale factors. As already mentioned at the beginning of this section, complete sets of scale factors can be derived only if complete similitude is maintained. Under these circumstances, all dimensionless variables are constant according to Equation (3-15). This, clearly, makes it unnecessary for them to be even formulated (e.g., the example in Section 3.3.1). Only when similitude requirements are violated are the dimensionless variables a necessary ingredient of the theory. In most practical cases, the theory of the dimensionless variables and that of the scale factors are applied in parallel.

3.3.1 EXAMPLE: LATERAL VIBRATIONS OF A LAUNCH VEHICLE. The basic dimensions, d, the dimensional variables, x, and the exponent matrix, b, are given in example 3.3.1. Equation (3-17) defines the basic scale factors

$$\delta_L = \frac{L^{(p)}}{L^{(m)}} \; ; \quad \delta_P = \frac{P^{(p)}}{P^{(m)}} \; ; \quad \delta_T = \frac{T^{(p)}}{T^{(m)}} \tag{3-21}$$

The derived scale factors are defined in (3-18) and are computed from (3-20).

$$\left.\begin{aligned}
\xi_\omega &= \frac{\omega^{(p)}}{\omega^{(m)}} = \delta_T^{-1} \\
\xi_\ell &= \frac{\ell^{(p)}}{\ell^{(m)}} = \delta_L \\
\xi_m &= \frac{m^{(p)}}{m^{(m)}} = \delta_L^{-2}\,\delta_P\,\delta_T^{2} \\
\xi_A &= \frac{A^{(p)}}{A^{(m)}} = \delta_L^{2}
\end{aligned}\right\} \tag{3-22}$$

$$\left.\begin{aligned}
\xi_I &= \frac{I^{(p)}}{I^{(m)}} = \delta_L^{\,4} \\
\xi_E &= \frac{E^{(p)}}{E^{(m)}} = \delta_L^{\,-2}\,\delta_P \\
\xi_G &= \frac{G^{(p)}}{G^{(m)}} = \delta_L^{\,-2}\,\delta_P \\
\xi_K &= 1
\end{aligned}\right\}\quad \begin{matrix}(3\text{-}21)\\ (\text{Contd})\end{matrix}$$

Based on these scale factors two different ways of scaling will be discussed.

3.3.1.1 Replica Model Using Identical Material. A length scale factor of 5 is chosen in (3-21).

$$\delta_L = 5 \text{ (first decision)}$$

With Equations (3-22) it follows that

$$\xi_\omega = \delta_T^{\,-1}$$

$$\xi_L = 5$$

$$\xi_m = \frac{1}{5^2}\,\delta_P\,\delta_T^{\,2}$$

$$\xi_A = 5^2$$

$$\xi_I = 5^4$$

$$\xi_E = \frac{1}{5^2}\,\delta_P$$

$$\xi_G = \frac{1}{5^2}\,\delta_P$$

$$\xi_K = 1$$

The scale factors to the right are not yet fully defined. Making the prototype and the model from the same material implies that

$$\xi_E = 1 \quad \text{(second decision)}$$

With this it follows that

$$\delta_P = 5^2 \xi_E = 5^2$$

$$\xi_\omega = \delta_T^{-1}$$

$$\xi_m = \delta_T^2$$

$$\xi_G = 1$$

One more scale factor must be selected. Assuming that the mass per unit length of the model is determined by the geometrical dimensions, i.e., that the mass per unit length decreases with the square of the length, it follows that

$$\xi_m = 5^2 \quad \text{(third decision)}$$

From which

$$\delta_T = \sqrt{\xi_m} = 5$$

$$\xi_\omega = \frac{1}{5}$$

This relation determines all eleven scale factors. This set of scale factors has been used in the Saturn one-fifth scale tests[13].

3.3.1.2 Model With Identical Frequency. Rather than scaling the mass with the geometry, as has been done in the example above, it is possible to postulate, for instance, that model and prototype should exhibit the same frequency of vibration. The point of departure from the example above is

$$\xi_\omega = 1 \quad \text{(third decision)}$$

and then

$$\delta_T = 1$$

$$\xi_m = 1$$

This result demands that model and prototype possess the same mass-per-unit-length properties. This may not necessarily be a practical way, but it is a possibility that exists. The model is not "distorted;" it conserves complete similitude for the variables chosen for the representation.

3.3.2 KINEMATIC SIMILITUDE. Kinematics involves two basic dimensions: length and time. Taking three-dimensional physical space and allowing different scale factors for all three directions, the exponent matrix, a, of Equation (3-16) becomes:

$$\begin{array}{ccc} & & \text{Variables} \\ & & \begin{array}{ccccccccc} \underline{u_x} & \underline{u_y} & \underline{u_z} & \underline{v_x} & \underline{v_y} & \underline{v_z} & \underline{a_x} & \underline{a_y} & \underline{a_z} \end{array} \\ \{a\} = & \begin{array}{c} [L_x] \\ [L_y] \\ [L_z] \\ [T] \end{array} & \left\{ \begin{array}{ccccccccc} 1 & & & 1 & & & 1 & & \\ & 1 & & & 1 & & & 1 & \\ & & 1 & & & 1 & & & 1 \\ & & & -1 & -1 & -1 & -2 & -2 & -2 \end{array} \right\} \\ & \text{Basic dimensions} & \end{array}$$

The variable names are: "u" for displacements, "v" for velocities, and "a" for accelerations, each subscripted with the corresponding directions. The basic scale factors are defined by Equation (3-17).

$$\delta_x = \frac{x^{(p)}}{x^{(m)}} \qquad \delta_y = \frac{y^{(p)}}{y^{(m)}}$$

$$\delta_z = \frac{z^{(p)}}{z^{(m)}} \qquad \delta_T = \frac{t^{(p)}}{t^{(m)}}$$

The basic scale factors or their equivalent may be assigned arbitrarily. The derived scale factors are, according to (3-20):

$$\left.\begin{aligned} &\xi_{u_x} = \delta_x \, ; \quad \xi_{u_y} = \delta_y \, ; \quad \xi_{u_z} = \delta_z \\ &\xi_{v_x} = \delta_x \, \delta_T^{-1} \, ; \quad \xi_{v_y} = \delta_y \, \delta_T^{-1} \, ; \quad \xi_{v_z} = \delta_z \, \delta_T^{-1} \end{aligned}\right\} \quad (3\text{-}23)$$

$$\xi_{a_x} = \delta_x \delta_T^{-2} \; ; \quad \xi_{a_y} = \delta_y \delta_T^{-2} \; ; \quad \xi_{a_z} = \delta_z \delta_T^{-2} \qquad \begin{array}{c}(3\text{-}23)\\ (\text{Contd})\end{array}$$

Alternate ways of writing the velocity and acceleration scale factors are:

$$\left.\begin{aligned} \xi_{v_x} &= \xi_{u_x} \delta_T^{-1} \text{; etc.} \\ \xi_{a_x} &= \xi_{v_x} \delta_T^{-1} = \xi_{v_x}^{2} \xi_{u_x}^{-1} \text{; etc.} \end{aligned}\right\} \quad (3\text{-}24)$$

If the basic length scale factors are made equal, i.e.;

$$\delta_L = \frac{x^{(p)}}{x^{(m)}} = \frac{y^{(p)}}{y^{(m)}} = \frac{z^{(p)}}{z^{(m)}}$$

then

$$\left.\begin{aligned} \xi_u &= \delta_L \\ \xi_v &= \delta_L \delta_T^{-1} = \xi_u \delta_T^{-1} \\ \xi_a &= \delta_L \delta_T^{-2} = \xi_v \delta_T^{-1} = \xi_v^{2} \xi_u^{-1} \end{aligned}\right\} \quad (3\text{-}25)$$

3.3.3 DYNAMIC SIMILITUDE. Dynamics involves three basic dimensions: length, time, and mass. Taking again three-dimensional physical space and allowing different scale factors for all three directions, the exponent matrix, a, of Equation (3-16) becomes:

$$\{a\} = \begin{array}{c} \\ \\ [L_x] \\ [L_y] \\ [L_z] \\ [T] \\ [M] \\ \text{Basic dimensions} \end{array} \begin{array}{c} \text{Variables} \\ \begin{array}{ccccccccccccc} \underline{u_x} & \underline{u_y} & \underline{u_z} & \underline{v_x} & \underline{v_y} & \underline{v_z} & \underline{a_x} & \underline{a_y} & \underline{a_z} & \underline{F_x} & \underline{F_y} & \underline{F_z} & \underline{m} \end{array} \\ \left\{\begin{array}{ccccccccccccc} 1 & & & 1 & & & 1 & & & 1 & & & \\ & 1 & & & 1 & & & 1 & & & 1 & & \\ & & 1 & & & 1 & & & 1 & & & 1 & \\ & & & -1 & -1 & -1 & -2 & -2 & -2 & -2 & -2 & -2 & \\ & & & & & & & & & 1 & 1 & 1 & 1 \end{array}\right\} \\ \\ \end{array}$$

The variable names are "u" for displacements, "v" for velocities, "a" for accelerations, "F" for forces, "m" for masses. The basic scale factors of Equation (3-17) are:

$$\delta_x = \frac{x^{(p)}}{x^{(m)}} \; ; \quad \delta_y = \frac{y^{(p)}}{y^{(m)}} \; ; \quad \delta_z = \frac{z^{(p)}}{z^{(m)}}$$

$$\delta_T = \frac{t^{(p)}}{t^{(m)}} ; \quad \delta_M = \frac{m^{(p)}}{m^{(m)}}$$

These basic scale factors or their equivalent may be assigned arbitrarily. The derived scale factors of Equation (3-20) are as follows:

$$\left.\begin{array}{lll}
\xi_{u_x} = \delta_x \; ; & \xi_{u_y} = \delta_y \; ; & \xi_{u_z} = \delta_z \\
\xi_{v_x} = \delta_x \, \delta_T^{-1} \; ; & \xi_{v_y} = \delta_y \, \delta_T^{-1} \; ; & \xi_{v_z} = \delta_z \, \delta_T^{-1} \\
\xi_{a_x} = \delta_x \, \delta_T^{-2} \; ; & \xi_{a_y} = \delta_y \, \delta_T^{-2} \; ; & \delta_{a_z} = \delta_z \, \delta_T^{-2} \\
\xi_{F_x} = \delta_x \, \delta_T^{-2} \, \delta_M \; ; & \xi_{F_y} = \delta_y \, \delta_T^{-2} \, \delta_M \; ; & \xi_{F_z} = \delta_z \, \delta_T^{-2} \, \delta_M \\
\xi_m = \delta_M & &
\end{array}\right\} \quad (3\text{-}26)$$

Some alternate ways of writing the velocity, acceleration, and force scale factors are:

$$\left.\begin{array}{ll}
\xi_{v_x} = \xi_{u_x} \, \delta_T^{-1} & ; \text{ etc.} \\
\xi_{a_x} = \xi_{v_x} \, \delta_T^{-1} = \xi_{v_x}^{\;2} \, \xi_{u_x}^{-1} & ; \text{ etc.} \\
\xi_{F_x} = \xi_{a_x} \, \delta_M & ; \text{ etc.} \\
\xi_m = \xi_{F_x} \, \xi_{a_x}^{-1} & ; \text{ etc.}
\end{array}\right\} \quad (3\text{-}27)$$

If the basic length scale factors are equal, i.e., if

$$\delta_L = \frac{x^{(p)}}{x^{(m)}} = \frac{y^{(p)}}{y^{(m)}} = \frac{z^{(p)}}{z^{(m)}}$$

it follows that

$$\left.\begin{aligned}
\xi_u &= \delta_L \\
\xi_v &= \delta_L \delta_T^{-1} = \xi_u \delta_T^{-1} \\
\xi_a &= \delta_L \delta_T^{-2} = \xi_v \delta_T^{-1} = \xi_v^{2} \xi_u^{-1} \\
\xi_F &= \delta_L \delta_T^{-2} \delta_M = \xi_a \delta_M \\
\xi_m &= \delta_M = \xi_F \xi_a^{-1}
\end{aligned}\right\} \quad (3\text{-}28)$$

If, in addition to this, the mass is scaled with the geometry, i.e., if it is proportional to the volume,

$$\delta_M = \delta_L^{3}$$

and the derived force and mass scale factors become

$$\left.\begin{aligned}
\xi_F &= \delta_L^{4} \delta_T^{-2} = \xi_a \delta_L^{3} \\
\xi_m &= \delta_L^{3}
\end{aligned}\right\} \quad (3\text{-}29)$$

The dynamic similitude requirements contain the kinematic requirements completely. This can be verified by comparing the results obtained in Section 3.3.2 with those obtained here.

3.4 DISTORTED MODELS

It is more often the rule rather than the exception that a scale model does not fulfill all similitude requirements expressed in Equation (3-12). This implies that the scaling law (3-14) does not hold. However, valid scale modeling is still possible, provided that an experimental, empirical, or theoretical relation can be supplied between the dependent variable and each one of the independent ones for which true scaling is not achieved. As a matter of fact, distorted scale models may prove to be much more practical than true scale models. A good example for this is the use of airplane flutter models[12]: from experience it is known that only the lower frequencies of vibration have any significant influence. Building models which scale only these lower frequencies results in much better experiments than expensive replica models which represent the entire frequency spectrum, but which are much less accurate at the low frequency end.

To discuss the laws governing distorted models it is most advantageous to bring the mathematical representation into nondimensional form as outlined in Sections 3.1 and 3.2. With π_1 as the dependent variable, the representation for model and prototype is given in Equations (3-11). It will be assumed that it is possible to scale the model for the variables $\pi_2, \pi_3, \cdots, \pi_s$, but not for $\pi_{s+1}, \pi_{s+2}, \cdots \pi_n$; i.e.,

$$\left.\begin{aligned} \pi_q^{(m)} &= \pi_q^{(p)} ; && (q = 2, 3, \cdots, s) \\ \pi_q^{(m)} &\neq \pi_q^{(p)} && (q = s+1, s+2, \cdots, n) \end{aligned}\right\} \tag{3-30}$$

These relations replace the complete similitude requirements (3-12). Selecting the first s-1 variables as those that can be scaled does not impair the generality of the results since the arrangement of variables is arbitrary anyhow.

With (3-30) it follows for model and prototype that

$$\left.\begin{aligned} \pi_1^{(m)} &= g_1\left(\pi_2^{(p)}, \pi_3^{(p)}, \cdots, \pi_s^{(p)}, \pi_{s+1}^{(m)}, \cdots, \pi_n^{(m)}\right) \\ \pi_1^{(p)} &= g_1\left(\pi_2^{(p)}, \pi_3^{(p)}, \cdots, \pi_n^{(p)}\right) \end{aligned}\right\} \tag{3-31}$$

These equations replace Equation (3-13) and instead of the scaling law (3-14),

$$\pi_1^{(m)} \neq \pi_1^{(p)}$$

The first equation (3-31) describes the model experiment. In order for this to be of any use for the investigation of the prototype, a relation

$$\pi_1^{(p)} = \varphi\left(\pi_1^{(m)}\right) \tag{3-32}$$

is needed.

3.4.1 TESTS OF SEVERAL MODELS. To assess the influence of the unscaled variables $\pi_{s+1}^{(m)} \cdots \pi_n^{(m)}$ in the first equation (3-31), several models may have to be built for which the scaled variables π_2 to π_s are kept constant, i.e., the first equation (3-30) remains valid. However, the unscaled variables are varied over as wide a range as possible or necessary. This variation will be described by factors "α" so that

$$\pi_q = \alpha_q \pi_q^{(p)} ; \qquad (q = s+1, s+2, \cdots, n) \tag{3-33}$$

where the non-superscripted π is a model variable. The first equation (3-31) becomes

$$\pi_1 = g_1\left(\pi_2^{(p)}, \pi_3^{(p)}, \cdots, \pi_s^{(p)}, \alpha_{s+1}\pi_{s+1}^{(p)}, \alpha_{s+2}\pi_{s+2}^{(p)}, \cdots, \alpha_n\pi_n^{(p)}\right)$$

All superscripted variables are constant prototype parameters. Lumping these constants with a model function, ψ , allows this equation to be written

$$\pi_1 = \psi\left(\alpha_{s+1}, \alpha_{s+2}, \cdots, \alpha_n\right) \tag{3-34}$$

Variation of the factors α, i.e., conducting the experiment with the different models, allows an (n-s+1) - dimensional plot to be drawn. Extrapolation or interpolation will yield the value of the function ψ at the point which represents the prototype:

$$\pi_1^{(p)} = \psi\ (1,\ 1,\ \cdots,\ 1) \tag{3-35}$$

The general procedure that has been outlined here can be modified in many ways to accommodate particular problems. Effective modeling is the art of making the best use of all the knowledge about a given system so that valid results are obtained with a minimum of cost and effort. Sections 3.4.2 and 3.4.3 will further elaborate on this point.

3.4.2 MODEL VARIATION. This method differs from the general procedure of Section 3.4.1 insofar as it uses one particular model configuration as the starting point and employs variations to define the model law in the vicinity of this model[5]. In other words, the method derives the model law from the experimental (and theoretical) knowledge of a function and its derivations at one particular point. It is founded on the premise that a function can be fully defined by its properties at one point. Obviously, this need not be the case for non-steady functions or when the influence of some variable is significant only in a certain range. However, the approach is definitely useful for applications with "well-behaved" variables or when model and prototype are not too far separated.

The foregoing argument suggests a model law that is expandable in a Taylor's series. With the definition of the increments

$$\Delta\pi_q = \pi_q^{(p)} - \pi_q^{(m)}\ ; \qquad (q = s+1, s+2,\ \cdots,\ n) \tag{3-36}$$

and with the incomplete similitude requirements expressed in the first equation (3-30), the second equation (3-31) can be written as

$$\pi_1^{(p)} = g_1\left[\pi_2^{(m)}, \pi_3^{(m)}, \cdots, \pi_s^{(m)}, \left(\pi^{(m)} + \Delta\pi\right)_{s+1}, \left(\pi^{(m)} + \Delta\pi\right)_{s+2} \cdots \left(\pi^{(m)} + \Delta\pi\right)_n\right]$$

The Taylor expansion is

$$\pi_1^{(p)} = \pi_1^{(m)} + \frac{1}{1!} \sum_{q=s+1}^{n} \frac{\partial \pi_1^{(m)}}{\partial \pi_q^{(m)}} \Delta\pi_q$$

$$+ \frac{1}{2!} \sum_{p=s+1}^{n} \sum_{q=s+1}^{n} \frac{\partial^2 \pi_1^{(m)}}{\partial \pi_p^{(m)} \partial \pi_q^{(m)}} \Delta\pi_p \Delta\pi_q + \cdots \tag{3-37}$$

The superscripts in the derivatives indicate that the derivatives are to be evaluated for the model configuration. Each derivative must be found through at least one additional experiment or theoretical relationship. The number of derivatives in Equation (3-37) increases rapidly with the number of unscaled variables, π_q, and the order of the derivatives to be taken. Therefore, a careful evaluation of the number of tests required is definitely indicated.

3.4.3 EXAMPLE: LATERAL VIBRATION OF A LAUNCH VEHICLE; ROTATIONAL INERTIA CORRECTION OF THE SCALING LAW. This example will demonstrate how the analytical formulation of a problem can be used to advantage to design an experiment, to reduce the number of independent variables, and to account for model distortions.

The rotational inertia usually has a not too pronounced effect on the frequencies of natural vibration of a launch vehicle. In the example of Section 3.1.1 this influence has been neglected, and, therefore, it would be a mere coincidence if the model possessed the properly scaled rotational inertia. For the purpose of this example it shall be required to evaluate the influence of this distortion.

The list of dimensionless variables given in Section 3.1.1 will be augmented by

$$\pi_6 = \frac{\mu}{m\,\ell^2} \tag{3-38}$$

where μ is the rotational inertia per unit length and has the dimension $[PT^2]$. This variable will not be the same (supposedly) for model and prototype. Instead, with Equation (3-33) it is to be expected that

$$\alpha_6 = \frac{\pi_6^{(m)}}{\pi_6^{(p)}} \neq 1 \tag{3-39}$$

The differential equations governing the free vibrations of an elastic beam which, according to Section 3.1.1 resembles that of a laterally excited launch vehicle, are given as[24]

$$w = a + b \tag{3-40}$$

$$\frac{\partial^2}{\partial y^2}\left(E\,I\,\frac{\partial^2 a}{\partial y^2}\right) - \frac{\partial}{\partial y}\left(\mu\,\frac{\partial^3 a}{\partial y\,\partial t^2}\right) + m\,\frac{\partial^2 w}{\partial t^2} = 0 \tag{3-41}$$

$$K\,A\,G\,\frac{\partial b}{\partial y} + \frac{\partial}{\partial y}\left(E\,I\,\frac{\partial^2 a}{\partial y^2}\right) - \mu\,\frac{\partial^3 a}{\partial y\,\partial t^2} = 0 \tag{3-42}$$

The symbols are the same as those introduced in Section 3.1.1. In addition, w, a, and b are the total, the bending, and the shear deflections. The coordinate along the axis is y, and t is the time. The deflections and their coordinates are rendered dimensionless by

$$\alpha = \frac{a}{\ell}\,; \qquad \beta = \frac{b}{\ell}\,; \qquad \eta = \frac{y}{\ell} \tag{3-43}$$

The form of the solution will be

$$\left.\begin{aligned} \alpha &= A_i \cos \omega_i\, t \\ & \qquad\qquad (i = 1,\ 2,\ \cdots\cdots) \\ \beta &= B_i \cos \omega_i\, t \end{aligned}\right\} \tag{3-44}$$

which assumes that bending and shear deformations are in phase with each other. The index i identifies the mode, and A_i and B_i are the corresponding dimensionless mode shapes.

Equations (3-43) and (3-44) are substituted into (3-41) and (3-42). With the definition of the variables π, with ℓ and ω_i as constants, and $\partial t/\partial\eta = 0$:

$$\frac{1}{E}\,\frac{\partial^2}{\partial\eta^2}\left(E\,\pi_2\,\frac{\partial^2 A_i}{\partial\eta^2}\right) + \frac{\pi_{1,i}}{m}\,\frac{\partial}{\partial\eta}\left(m\,\pi_6\,\frac{\partial A_i}{\partial\eta}\right) - \pi_{1,i}\,(A_i + B_i) = 0 \tag{3-45}$$

$$\pi_2 \pi_{(3,4,5)} \frac{\partial B_i}{\partial \eta} + \frac{1}{E} \frac{\partial}{\partial \eta} \left(E \pi_2 \frac{\partial^2 A_i}{\partial \eta^2} \right) + \pi_{1,i} \pi_6 \frac{\partial A_i}{\partial \eta} = 0 \tag{3-46}$$

where

$$\pi_{(3,4,5)} = \frac{\pi_4 \pi_5}{\pi_3^2} = \frac{\ell^2 KAG}{EI} \tag{3-47}$$

The significance of the simplification (3-47) is that the variables π_3, π_4, and π_5 can be lumped into one variable for scaling purposes. This is typical for a simplification of the scaling laws, gained from insight into the analytical representation. However, the two surviving dimensional variables, m and E, in (3-45) and (3-46) must still be accounted for separately. This is the case through variable $\pi_{1,i}$.

Equations (3-45) and (3-46) are valid for prototype and model. Not all of the variables are different, though. The expressions (3-30) are for this case

$$\left.\begin{aligned} &\pi_q^{(m)} = \pi_q^{(p)} \quad ; \qquad (q = 2,\ 3,\ 4,\ 5) \\ &\pi_{1,i}^{(m)} \neq \pi_{1,i}^{(p)} \\ &\pi_6^{(m)} \neq \pi_6^{(p)} \end{aligned}\right\} \tag{3-48}$$

Another requirement, not expressed in terms of the original variables, is that

$$\eta^{(m)} = \eta^{(p)} \tag{3-49}$$

This demands that the units of length measurement in prototype and model be identical, a condition that is usually overlooked but seldom violated. To simplify matters, one assumption will be made: the mode shapes of prototype and model are similar; i.e., their derivatives with respect to the dimensionless length coordinate, η, are identical. This assumption violates the theory because it makes Equation (3-45) incompatible with (3-46). However, it is felt that this is a minor offense as long as the rotational mass of the model is not too far off-scale. The final results, however, must be interpreted with this assumption in mind.

The assumption of the similarity of mode shapes allows one to use Equation (3-46) alone and also to drop the mode identifying index, i. With (3-48) and (3-49),

$$\pi_2 \pi_{(3,4,5)} \frac{\partial B}{\partial \eta} + \frac{1}{E^{(m)}} \frac{\partial}{\partial \eta} \left(E^{(m)} \pi_2 \frac{\partial^2 A}{\partial \eta} \right) + \pi_1^{(m)} \pi_6^{(m)} \frac{\partial A}{\partial \eta} = 0 \qquad (3\text{-}50)$$

$$\pi_2 \pi_{(3,4,5)} \frac{\partial B}{\partial \eta} + \frac{1}{E^{(p)}} \frac{\partial}{\partial \eta} \left(E^{(p)} \pi_2 \frac{\partial^2 A}{\partial \eta} \right) + \pi_1^{(p)} \pi_6^{(p)} \frac{\partial A}{\partial \eta} = 0 \qquad (3\text{-}51)$$

To make use of these equations, it is necessary to realize that in the original model design of Section 3.1.1 it was not sufficient to scale the vehicle properties in some lumped fashion. Indeed, it was tacitly assumed that the variables and their distribution over the length were properly scaled. A more formal approach should have included the derivatives with respect to the length in the list of independent variables.

With the condition put forward in Section 3.3.1.1 that the model and the prototype be built from the same material, Equations (3-50) and (3-51) lead to the condition

$$\pi_1^{(m)} \pi_6^{(m)} \frac{\partial A}{\partial \eta} - \pi_1^{(p)} \pi_6^{(p)} \frac{\partial A}{\partial \eta} = 0$$

or, with Equation (3-39)

$$\pi_1^{(p)} = \alpha_6 \, \pi_1^{(m)} \qquad (3\text{-}52)$$

The variable $\pi_1^{(m)}$ is determined from model tests. To find the prototype variable, $\pi_1^{(p)}$, the result from the model test must be multiplied by the correction factor, α_6. For perfect scaling of the rotational inertia, $\alpha_6 = 1$, and no correction is necessary, as is to be expected. Equation (3-52) is the desired relation (3-32) for the particular application treated here.

Taking the scale factors used in example 3.3.1.1, the relation between the frequencies becomes

$$\omega_1^{(p)} = \sqrt{\alpha_6} \; \bar{\omega}_1^{(p)} \qquad (3\text{-}53)$$

where

$$\bar{\omega}_1^{(p)} = \frac{1}{5} \, \omega_1^{(m)}$$

$$\alpha_6 = 5^4 \, \frac{\mu^{(m)}}{\mu^{(p)}}$$

3.4.4 SCALE EFFECTS. Scale effects represent a special case of model distortions and they are treated in much the same way. Scale effects occur when, due to the different dimensions of model and prototype, a physical phenomenon gains importance for one but not for the other. The classical example is provided in fluid dynamics by models which possess a free liquid surface, e.g., models of waterways. In the full-scale application, the surface tension has no appreciable influence; however, for the model with its much smaller geometric dimensions, surface tension becomes important. Since it is extremely impractical for these cases to include the surface tension in the scaling law, no direct comparison of the full-scale and small-scale applications can be made. However, empirical and theoretical corrections can be applied as outlined in Section 3.4.

4/RECOMMENDED PRACTICES

A test program consists of the following major parts:

a. Definition of the problem.

b. Justification of the use of models.

c. Assessment of the model design parameters.

d. Model engineering.

e. Test setup.

f. Test procedure.

g. Data evaluation and interpretation.

This is the most logical sequence of events; however, practical considerations will usually demand some modifications. This section will elaborate on the various points listed as far as they are typical for model work. This excludes the test setup, procedure, and evaluation of the structural dynamic models because it is felt that these points are adequately covered in other monographs of this series, e.g., Reference 9.

4.1 STRUCTURAL DYNAMIC MODELS

At the present time, structural models are used to determine the vibration characteristics, frequency, mode, and damping of complete launch vehicles with their payloads (References 13 to 19). Component testing, separation dynamics, and other equally important aspects see very little, if any, systematic model testing. It is probably realistic to assume that structural model testing will find more applications in the future.

4.1.1 JUSTIFICATION OF MODEL TESTS. Structural dynamic model tests seem to have been the exception rather than the rule for launch vehicle development. So far, full-scale testing has been employed in the majority of development programs. However, increased demands on efficiency and reliability may very well reverse this situation[18]. An excellent argument for the use of models is made by Jaszlics et al.[19] which centers on the versatility of the scaled experiment.

Tests are indicated whenever no adequate analysis capability exists and, in addition, when an independent verification of analytical results is desired. These tests should be conducted on models whenever:

a. Information is needed before the availability of full-scale test hardware, or full-scale hardware is unavailable for testing the complete vehicle.

b. It is cheaper to test models.

c. Various test conditions will be encountered.

d. No other test facilities are available.

4.1.2 ENGINEERING OF MODELS. It is the foremost requirement of a model that it exactly duplicate all important features of the full-scale hardware. Any deviations from this rule should be kept to a minimum, or, their influence should be sufficiently well understood that the test results can be corrected accordingly. Especially for structural replica models, tolerances and joints must be scaled carefully. This requirement may very well determine the smallest useful scale to which a model can be built[18, 19].

On the other hand, models should not be over-elaborate with respect to unimportant properties. For instance, non-structural parts can be omitted and accounted for by dead weights. The best way to reach a good compromise between over-complication and over-simplification is to define the model parameters as outlined in Section 3. With this and any other available information, simplifications can be judged against some valid criteria.

4.1.3 MODEL PARAMETERS. It has been mentioned in Section 3.1 that for one and the same problem many different sets of dimensionless variables (model parameters) may be chosen. This choice should be dictated mainly by the convenience with which these parameters can be controlled in the experiment. However, for the sake of comparison with previous experiments it is also a good idea to use parameters which have found prior application. The following list of parameters is taken from Runyan, Morgan, and Mixson[18]. A list of more general model parameters is given in Appendix A.

Problem Area	Dimensionless Parameters
Lateral vibrations	$\frac{m\omega^2}{E}, \frac{I}{\ell^4}, \frac{I}{KA\ell^2}, \frac{G}{E}, \frac{\sqrt{\frac{I}{A}}}{\ell}$
Longitudinal vibrations	$\frac{m\omega^2}{E}, \frac{A}{\ell^2}$
Local vibrations	$\frac{\rho_w \omega^2 \ell^2}{E}, \frac{t}{r}, \frac{h}{r}, \frac{r}{\ell}, \frac{G}{E}, \frac{P}{E}, \frac{\rho_\ell}{\rho_w}$
Sloshing (rigid tank)	$\frac{\rho_\ell r^2}{\eta r}, \frac{h}{r}, \frac{a\tau^2}{r}$

Problem Area	Dimensionless Parameters
Buffetting (rigid vehicle)	$\frac{\ell \omega}{v}, \frac{r}{\ell}, \frac{\epsilon}{r}, \frac{P}{\rho v^2}, N_M, N_R$
Flutter	$\frac{\ell \omega}{v}, \frac{m}{\rho \ell^2}, \frac{S_\alpha}{m\ell}, \frac{\mu}{m \ell^2}, \frac{\omega}{\omega_\alpha}, \frac{\omega_h}{\omega_\alpha}, \frac{x_0}{\ell}, N_M$
Ground winds	$\frac{\ell \omega}{v}, \frac{r}{\ell}, \frac{\epsilon}{r}, \frac{m}{\rho \ell^2}, \frac{m \omega^2}{E}, \frac{I}{\ell^4}, \omega\tau, N_R$

The symbols used here are in agreement with those used before in the examples; additional symbols are given in the list of symbols.

Reference 18 also provides a discussion of these problem areas.

Vibrations: Lateral and longitudinal vibrations can be scaled in the same model as long as a vehicle can be considered to be primarily a beam. Only the first two dimensionless parameters apply if shear deformation is neglected. It is probably better to employ separate models for scaling local and overall vibrations because of the many parameters which would enter the problem simultaneously.

Sloshing: The three parameters shown contain the Reynolds and Froude numbers. Elastic walls will increase the number of model parameters substantially. Difficulties are encountered in scaling the gravitational field. For replica models built from the same material, the bending and shell frequency increases linearly with the length scale factor, whereas the slosh frequency increases only with the square root of this scale factor.

Buffetting: The correct scaling of the surface roughness, ϵ, is critical. To include the effects of the vehicle elasticity, at least the lateral vibration characteristics must be scaled in addition.

Flutter: Modeling techniques developed for aircraft apply for launch vehicles, too.

Ground Winds: Important variables are local geometry and the surface roughness. The lateral vibration characteristics must be scaled correctly. For very large vehicles, the dimensions of available wind tunnels may make it difficult to maintain the correct Reynolds number and to stay below an acceptable Mach number.

4.1.4 SCALING LAW FOR ELASTIC STRUCTURES. The differential equation of three-dimensional, linearized theory of isotropic elasticity[25] is employed to define the variables which enter the modeling relations of elastic structures[23].

$$\frac{E}{2(1+\nu)(1-2\nu)}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)$$

$$+ \frac{E}{2(1+\nu)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(u_x, u_y, u_z\right)$$

$$- \rho\left(\frac{\partial^2 u_x}{\partial t^2}, \frac{\partial^2 u_y}{\partial t^2}, \frac{\partial^2 u_z}{\partial t^2}\right) = 0 \tag{4-1}$$

This equation is written for an orthogonal Cartesian reference frame (x, y, z). The displacements are u_x, u_y, u_z, and it is assumed that only inertial forces are acting.

Equation (4-1) must be valid for prototype and model; i.e., it must be insensitive to a change in scale. With the definition of the basic and derived scale factors in Equations (3-17) and (3-18), Equation (4-1) can be transformed into a different scale. Assuming that the basic length scale factor (Section 3.3), is the same for all spatial directions, i.e.,

$$\delta_L = \frac{x^{(p)}}{x^{(m)}} = \frac{y^{(p)}}{y^{(m)}} = \frac{z^{(p)}}{z^{(m)}}$$

Equation (4-1) may be written as

$$\frac{\xi_E E}{2(1+\xi_\nu \nu)(1-2\xi_\nu \nu)} \frac{1}{\delta_L}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)$$

$$+ \frac{\xi_E E}{2(1+\xi_\nu \nu)} \frac{1}{\delta_L^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\delta_L\left(u_x, u_y, u_z\right)$$

$$- \xi_\rho \rho \frac{\delta_L}{\delta_T^2}\left(\frac{\partial^2 u_x}{\partial t^2}, \frac{\partial^2 u_y}{\partial t^2}, \frac{\partial^2 u_z}{\partial t^2}\right) = 0$$

This equation does, indeed, agree with (4-1) if

$$\left.\begin{aligned} \xi_\nu &= \frac{\nu^{(p)}}{\nu^{(m)}} = 1 \\ \frac{\xi_E\,\delta_T^{\,2}}{\xi_\rho\,\delta_L^{\,2}} &= \frac{E^{(p)}\,t^{(p)^2}\,\rho^{(m)}\,\ell^{(m)^2}}{E^{(m)}\,t^{(m)^2}\,\rho^{(p)}\,\ell^{(p)^2}} = 1 \end{aligned}\right\} \quad (4\text{-}2)$$

The first condition (4-2) demands that Poisson's ratio for model and prototype are identical. The second condition allows a wide choice of scaling laws to be selected since three (this is the number of basic dimensions) scale factors can be selected arbitrarily. The equations of Section 3.3 will be helpful in applying this result.

4.1.5 SCALING LAW FOR DAMPING. It is very difficult to formulate rational design criteria which will furnish proper model damping characteristics. However, in many cases, model damping is not very important: it may be sufficient to run experiments on modes and frequencies alone. These results, when applied to the prototype, are combined with empirical damping coefficients to supply approximate response data.

To scale dry friction (Coulomb damping), it is necessary that the relation

$$\tau = \bar{\mu}\,\sigma \cdot \qquad (4\text{-}3)$$

remain constant[23], where τ and σ are the shear and normal stress in a joint, and $\bar{\mu}$ is the coefficient of friction. For material damping[23,26] a complex elastic modulus $E(1+ig)$ can be used, where the damping coefficient

$$g = \left(\frac{\sigma}{\sigma_o}\right)^n \qquad (4\text{-}4)$$

applies reasonably well for steel and brass. Material constants are "σ_o" and "n", and "σ" is the stress amplitude. For aluminum, a somewhat better correlation is obtained by

$$g = \left(\frac{\sigma}{\sigma_o}\right)^n \frac{\omega\tau}{1+\omega^2\tau^2} \qquad (4\text{-}5)$$

where "τ" is an additional material constant which expresses a relaxation time for the equalization of temperature.

At present, exact prediction of the damping characteristics is very difficult. Further information may be gathered from References 23 and 26.

4.1.6 SCALING LAW FOR SHELLS. Various sets of differential equations are available for the analysis of shell structures. These equations can be used to advantage to identify the variables that enter the problem and need to be scaled. An example of this is given by Morgan[27] who uses Fluegge's equations for an orthotropic layered shell. The procedure to be followed is either that of identifying the dimensionless variables, π, as outlined in Sections 3.1 and 3.2 or the use of scale factors as defined in Section 3.3. In practice, both approaches should complement each other. A systematic example of the former approach is given in Reference 27, whereas the latter approach is analogous to that of Section 4.1.4.

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APPENDIX A

SOME COMMONLY USED MODEL PARAMETERS

This list of dimensionless variables has been compiled from References 1, 28 and 29. To avoid confusion, all these variables have been identified by N and are distinguished by their subscripts. In a few cases, discrepancies in the definitions exist which are, however, limited to the powers of the dimensionless variables.

The characteristic quantities used are given in the following table. The dimensions are given with respect to a length, force, time, temperature, i.e., L, P, T, θ system.

QUANTITY	SYMBOL	DIMENSION
Length	ℓ	[L]
Force	F	[P]
Time	t	[T]
Temperature	ϑ	[θ]
Modulus of Elasticity	E	[L^{-2} P]
Velocity	v	[L T^{-1}]
Mass density	ρ	[L^{-4} PT^2]
Pressure	p	[L^{-2} P]
Dynamic viscosity	η	[L^{-2} PT]
Kinematic viscosity	ν	[L^2 T^{-1}]
Velocity of sound	c	[L T^{-1}]
Liquid surface tension	σ	[L^{-1} P]
Acceleration of gravity	g	[L T^{-2}]
Coefficient of temperature conduction	a_θ	[L^2 T^{-1}]
Coefficient of heat conduction	k	[P T^{-1} θ^{-1}]
Heat transfer coefficient	h	[L^{-1} PT^{-1} θ^{-1}]

a. Hooke's number (static elasticity)

$$N_{Ho} = \frac{F}{E\,\ell^2}$$

b. Newton's number (dynamics)

$$N_{Ne} = \frac{F}{\rho\, v^2\, \ell^2}$$

This subject is treated in Section 3.3.3.

c. Pressure coefficient (fluid dynamics)

$$N_P = \frac{p}{\rho\, v^2}$$

This number follows from b. with $p = \frac{F}{\ell^2}$.

d. Cauchy's number (dynamic elasticity)

$$N_C = \sqrt{\frac{N_{Ho}}{N_{Ne}}} = v \sqrt{\frac{\rho}{E}}$$

Another definition[29] of the same number is

$$N_C' = N_C^2$$

e. Mach number (gas dynamics)

$$N_M = \frac{v}{c}$$

This number follows from d. with $c = \sqrt{\frac{E}{\rho}}$.

f. Weber's number (fluid dynamics with surface tension)

$$N_W = \frac{v^2 \rho \ell}{\sigma}$$

which follows from d. with $\sigma \approx E\ell$.

g. Reynolds' number (viscous fluid dynamics)

$$N_R = \frac{\rho\, v\ell}{\mu} = \frac{v\ell}{\nu}$$

h. Froude's number (fluid dynamics with gravitation)

$$N_F = \frac{v}{\sqrt{g\ell}}$$

Another definition[1] of the same number is

$$N_F' = N_F^2$$

i. Fourier's number (heat flow)

$$N_{Fo} = \frac{\ell^2}{t a_\theta}$$

The same number is also defined as[29]

$$N_{Fo}' = \frac{1}{N_{Fo}}$$

j. Péclet's number (heat flow in liquid and gas)

$$N_{Pe} = \frac{v\ell}{a_\theta}$$

This number is obtained from i. with $v = \frac{\ell}{t}$

k. Prandtl's number (heat convection)

$$N_{Pr} = \frac{N_{Pe}}{N_R} = \frac{\nu}{a_\theta}$$

l. Nusselt's number (heat transfer)

$$N_{Nu} = \frac{h\ell}{k}$$