DISTRIBUTION OF ELEVATIONS ON A CRATERED PLANETARY SURFACE

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ABSTRACT

The distribution of elevations on a cratered planetary surface is computed from a theoretical model which assumes the surface is altered only by the formation of crater bowls, rims, and ejecta blankets. The broad inverse power law distribution of crater diameters induces a very broad distribution of elevations with (asymptotic) inverse power law tails which, in some cases, can be explicitly expressed. Typical surface elevations grow at least as fast as the age of the surface, although the volume of fragmental material grows more slowly than the age of the surface. Estimates by Oberbeck and Quaide (1967) of the distribution of the thickness of the fragmental surface layer in Oceanus Procellarum are consistent with the theory.

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1. INTRODUCTION AND SUMMARY

Much of the roughness of cratered planetary surfaces such as on the Moon and Mars is in the form of crater bowls, rims, and material such as dust blankets, blocks and boulders ejected from craters. We will compute the distribution of elevations on a cratered surface, considering only the contributions of crater bowls, rims and smooth ejecta blankets. We assume that craters are distributed at random on an initially plane surface, with crater shape and size distribution appropriate to the meteoroidal impact hypothesis. (We do not believe that the impact hypothesis is necessarily correct, but it is the only hypothesis with computable consequences.) The surface elevations have a "moving average" Poisson point process representation, with crater shape as the weight function. In some particular cases, it is possible to compute the distribution of surface elevations explicitly.

The distribution of surface elevations usually has the form of a heavy-tailed inverse power law. The typical surface elevation (mean value, where it exists) increases at least as fast as the age of the surface. The elevations increase faster than the age of the surface only if large craters are sufficiently frequent. This fact notwithstanding, the rate of production of fragmental material must decrease with time.

Some of the model predictions are verified by Oberbeck and Quaide's estimate (1967) of the distribution of thickness of the layer of fragmental material on Oceanus Procellarum.

2. CRATER MODEL

Some physical justification for the model functions assumed here is given elsewhere (Marcus, 1968). At present we will simply collect the necessary data. We assume that crater bowls are paraboloidal in shape. The initial rim-to-floor depth $H(x)$ of the crater bowl is a power function of crater rim diameter x (see Figure 3),

$$H(x) = C_0 x^{\delta} \quad (C_0 > 0 \quad , \quad \delta > 0) \tag{1}$$

For craters of diameters up to about 5 to 15 kilometers, we seem to have $\delta = 1$ and $C_0 = 0.25$ approximately. The initial rim height $R(x)$ of a crater of diameter x (see Fig. 3) is the maximum height above the pre-crater mean local surface, and is also assumed to be a power function

$$R(x) = R_0 x^h \qquad (R_0 > 0 \quad , \quad h > 0) \qquad (2)$$

with $h = 1$ and $R_0 = 0.085$ approximately for, roughly, 10 meters $< x <$ 20 kilometers. The constant R_0 is rather poorly determined. The value of R_0 above is taken from lunar craters, but for terrestrial explosion craters we have only $R_0 = 0.055$. Lunar and terrestrial craters agree more closely on the value of C_0.

The exterior rim of an impact or explosion crater is somewhat irregular in shape, often being described as "hummocky". We follow the suggestion of Carlson and Roberts (1963) that the thickness of the fine material ejected from the crater can be described by a power law. The thickness $\zeta_B(x,r)$ of a layer formed at a distance r from the center of a crater of diameter x is

$$\zeta_B(x,r) = R(x)(2r/x)^{-k} \qquad (r > x/2 \quad , \quad k > 0) \qquad (3)$$

For large explosion craters we seem to have $k = 4$ approximately ($3 \leq k \leq 5$ in almost all cases). We use the term "ejecta blanket" rather loosely, since the exterior rim near the crater wall also includes the uplifted surface underneath the fragmental material.

Upon combining (1), (2), and (3), we obtain a description of the elevation profile $\zeta(x,r)$ of a crater of diameter x at a distance r from its center:

$$\zeta(x,r) = H(x)\left[(2r/x)^2 - 1\right] + R(x)$$

$$= C_0 x^\delta \left[(2r/x)^2 - 1\right] + R_0 x^h$$

$$\text{if} \quad 0 \leq r < x/2 \qquad (4)$$

$$\zeta(x,r) = R_0 x^h (x/2r)^k \qquad \text{if } x/2 < r \tag{5}$$

We believe this description is usefully accurate.

The craters are assumed to be distributed randomly over the surface, presumably (but not necessarily) a result of meteoroidal impacts. The average number of craters of diameter x per unit area per unit diameter interval which have formed on the surface is given by an expected number density $\xi(x)$. Since we are concerned here only with mare surfaces, we may assume that $\xi(x) = F\,p(x)$, where $p(x)$ is the probability density function of new-born craters of diameter x, $0 \leq x_0 < x < x_m \leq \infty$, and F is the cumulative mean number of craters of diameter x_0 to x_m formed per unit area during the lifetime of the surface (Marcus, 1966). The probability density usually assumed is the inverse power law

$$p(x) = \frac{\gamma\, x_0^{\gamma}}{1 - (x_0/x_m)^{\gamma}} \frac{1}{x^{\gamma+1}} \quad \text{if} \quad x_0 < x < x_m$$

$$= 0 \text{ otherwise} \qquad (\gamma > 0) \tag{6}$$

The relevant value of γ is rather uncertain. For postmare craters γ probably lies between 2.6 and 3.4, and may increase slowly with decreasing x.

Because the value of γ for postmare primary impact craters smaller than 1 or 2 km diameter is rather large (between 2.6 and 3.4) it may be possible to ignore, to a considerable extent, the contribution to surface roughness from possible secondary impact craters. The diameter distribution for secondary craters from a given primary is, like (6), an inverse power law over a large range of diameters. But the population index of secondary craters (corresponding to γ in (6) is some number ω which is in general different from γ. The value of ω is approximately 3 to 3.5 (Marcus, 1966). Walker (1967) has suggested $\omega = 3.34$ or $\omega = 3.56$.

If γ is larger than ω, then at all diameters of interest here most of the craters will be primary impact craters; if ω is larger than γ, then secondary craters will predominate. Because γ is comparable to ω, if not actually larger, we can no longer accept the statements by Marcus (1966) or Walker (1967) that most lunar craters of 50 to 500 meters diameter are of secondary impact or internal origin. Recent unpublished studies by D. Gault (personal communication, April 12, 1968) also suggest that, except for statistically rare clusters of secondary craters, most small craters on the lunar maria are of primary impact origin. Furthermore, crater morphology does not uniquely point to the origin of a crater. The "soft" morphology which some authors believe indicative of secondary impact or internal origin can also appear in old primary craters which have suffered from slumping and micrometeor erosion (Ross, 1968).

3. TOTAL DEPTH EXCAVATED BY CRATERS

We may compute the total depth $Z_C(\underset{\sim}{R})$ excavated by craters which cover the point $\underset{\sim}{R}$ by assuming a linear superposition (Marcus, 1968).

$$Z_C(\underset{\sim}{R}) = \int \zeta_C(x,r)\,dN(x,\underset{\sim}{R} + \underset{\sim}{r}) \tag{7}$$

where $\zeta_C(x,r)$ is the elevation decrease due to the formation of a crater of diameter x at a distance r = (length of $\underset{\sim}{r}$) from $\underset{\sim}{R}$. The random variable $dN(x,\underset{\sim}{R} + \underset{\sim}{r})$ is the number of craters of diameter x to x + dx formed in the small region $d(\underset{\sim}{R} + \underset{\sim}{r})$ surrounding $\underset{\sim}{R} + \underset{\sim}{r}$. Assuming that the secondary crater contribution to surface roughness is negligible, we may justifiably suppose that $dN(x,\underset{\sim}{R} + \underset{\sim}{r})$ is a Poisson point process with mean value $\xi(x)dx\, d(\underset{\sim}{R} + \underset{\sim}{r})$, where $\xi(x)$ is the expected number density of craters of diameter x. With the representation (7) we find the $Z_C(\underset{\sim}{R})$ has an "infinitely divisible" distribution law. The probability density function of elevations $p_C(z)$ has a characteristic function

$$\begin{aligned}\phi_C(u) &= \int_{-\infty}^{\infty} e^{iuz}\, p_C(z)\,dz \\ &= \exp\left(\int_{x_0}^{x_m} \xi(x)\,dx \int_0^{\infty} 2\pi r \left[e^{iu\zeta_C(x,r)} - 1\right] dr\right)\end{aligned} \tag{8}$$

Our model functions from (4) and (6) are

$$\zeta_C(x,r) = 0 \quad \text{if} \quad r > x/2$$

$$= C_0 x^\delta \left[(2r/x)^2 - 1\right] + R_0 x^h \quad \text{if} \quad r < x/2 \tag{9}$$

and

$$\xi(x) = \frac{\gamma\ F\ x_0{}^\gamma}{1 - (x_0/x_m)^\gamma} \frac{1}{x^{\gamma+1}} \quad \text{if} \quad x_0 < x < x_m$$

$$= 0 \quad \text{otherwise} \tag{10}$$

After straightforward reductions, (8), (9), and (10) imply

$$\phi_C(u) = \exp\Bigg(\frac{\pi\gamma F\ x_0{}^\gamma}{4[1 - (x_0/x_m)^\gamma]} \int_{x_0}^{x_m} \frac{dx}{x^{(\gamma-2)+1}} \frac{1}{iuC_0 x^\delta} \cdot \left\{\left[e^{iuR_0x^h} - 1 - iuR_0x^h\right] - \left[e^{iu(R_0x^h - C_0x^\delta)} - 1 - iu(R_0x^h - C_0x^\delta)\right]\right\}\Bigg) \tag{11}$$

The calculations are greatly simplified if we can allow $x_m \to \infty$, $x_0 \to 0$. The first is possible only if $\gamma > 2$ which is physically probable.

In order to permit $x_0 \to 0$, we require

$$\gamma < 2 + h \quad , \quad \gamma < 2 + \delta \tag{12}$$

$$\lim_{x_0 \to 0} F\ x_0{}^\gamma = C \quad , \quad \text{constant} \tag{13}$$

Let us for the moment accept (12) and (13).

The calculations are further simplified by assuming that

$$h = \delta \tag{14}$$

That crater rims and bowls should scale in the same manner is plausible for craters of up to about 5 to 15 kilometers diameter.

With the transformations

$$y = R_0 x^h \quad , \quad y = (C_0 - R_0) x^h$$

and (12), (13), and (14), we obtain from (11)

$$\phi_C(u) = \exp\left(\frac{\pi\gamma C}{4hC_0 iu}\left\{\frac{1}{R_0^{\alpha+1}} \int_0^\infty \frac{dy}{y^{\alpha+2}}\left[e^{iuy} - 1 - iuy\right] - \frac{1}{(C_0-R_0)^{\alpha+1}} \int_0^\infty \frac{dy}{y^{\alpha+2}}\left[e^{-iuy} - 1 + iuy\right]\right\}\right) \tag{15}$$

where

$$\alpha = (\gamma - 2)/h \quad , \quad 0 < \alpha < 1 \tag{16}$$

(we assume $C_0 > R_0$). For evaluation of the integrals appearing in (15), see Gnedenko and Kolmogorov (1954). We obtain after some reductions

$$\phi_C(u) = \exp\left(-\lambda_C |u|^\alpha \left[1 - i\beta_C \operatorname{sgn}(u) \tan(\pi\alpha/2)\right]\right) \tag{17}$$

where

$$\alpha = (\gamma - 2)/h$$

$$\beta_C = \left[R_0^{\alpha+1} - (C_0-R_0)^{\alpha+1}\right] \Big/ \left[R_0^{\alpha+1} + (C_0-R_0)^{\alpha+1}\right] \tag{18}$$

$$\operatorname{sgn}(u) = +1 \text{ if } u>0, \ \operatorname{sgn}(u) = -1 \text{ if } u<0$$

$$\lambda_C = \frac{\pi\gamma C}{4hC_0}\left[R_0^{\ \alpha+1} + (C_0-R_0)^{\ \alpha+1}\right]\frac{\Gamma(1-\alpha)}{\alpha(1+\alpha)}\cos(\pi\alpha/2) \tag{19}$$

The parameters satisfy

$$0 < \alpha < 1$$

$$-1 < \beta_C < 1$$

$$\lambda_C > 0$$

The characteristic function $\phi_C(u)$ in (17) is of known type, that of a stable distribution law (Gnedenko and Kolmogorov, 1954). Unfortunately, it is not possible to obtain explicitly the probability density function corresponding to $\phi_C(u)$ except for a few specific cases. The only case for general β_C is $\alpha = 1/2$, when (Zolotarev, 1954)

$$p_C(z) = \text{Real part of}\left\{\frac{\omega}{\pi z}\left[\sqrt{\pi}\, e^{-\omega^2} - 2iw(\omega)\right]\right\} \tag{20}$$

where

$$\alpha = 1/2 \quad , \quad \lambda_C = 1$$

$$z > 0$$

$$\omega = [(1-\beta_C) - i(1+\beta_C)]/2(2x)^{1/2} \tag{21}$$

$$w(\omega) = e^{-\omega^2}\int_0^{\omega} \exp(V^2)dV \tag{22}$$

This formula is not too useful except for $\beta_C = 0$, when we can express $p_C(z)$ in terms of Fresnel integrals. Some explicit results for $\beta_C = 1$ are listed in the next section, but are not relevant here.

Some asymptotic results are available (Skorohod, 1954) which are of interest. Assuming $\lambda_C = 1$, $0 < \alpha < 1$ and $-1 < \beta_C < 1$, we have

$$p_C(z) = \frac{\Gamma(1+\alpha)}{\pi z^{1+\alpha}}\left[1 + \beta_C^2 \tan^2(\pi\alpha/2)\right]^{1/2}\left(\sin \frac{\pi\alpha}{2} + \arctan\left(\beta_C \tan \frac{\pi\alpha}{2}\right)\right)$$

$$(z \to \infty) \qquad (23)$$

$$p_C(z) = \frac{\Gamma(1+\alpha)}{\pi(-z)^{1+\alpha}}\left[1 + \beta_C^2 \tan^2(\pi\alpha/2)\right]^{1/2} \sin\left(\frac{\pi\alpha}{2} - \arctan\left(\beta_C \tan \frac{\pi\alpha}{2}\right)\right)$$

$$(z \to -\infty) \qquad (24)$$

The distribution of Z_C is thus a heavy-tailed inverse power law for large $|z|$, skewed toward positive or negative values of z respectively, according as $\beta_C > 0$ $(C_0 < 2R_0)$ or $\beta_C < 0$ $(C_0 > 2R_0)$.

We cannot permit $x_0 \to 0$ when $\alpha \geq 1$. However, if $\alpha > 1$, we can compute the mean value $E\{Z_C\}$ due to craters (this does not exist if $\alpha \leq 1$). We find from (11) that for $x_m \to \infty$,

$$E\{Z_C\} = -i \frac{d}{du} \phi_C(u)\Big|_{u=0} = \frac{\pi\gamma F}{4} x_0^2\left[\frac{2R_0 \; x_0{}^h}{\gamma-2-h} - \frac{C_0 \; x_0{}^\delta}{\gamma-2-\delta}\right] \qquad (25)$$

where

$$\gamma > 2 + h \quad , \quad \gamma > 2 + \delta \qquad (26)$$

If also

$$h = \delta$$

then

$$E\{Z_C\} = \frac{\pi\gamma F\ x_0^{2+h}}{4(\gamma-2-h)} [2R_0 - C_0] \tag{27}$$

The uncratered part of the surface has $Z_C = 0$ with probability e^{-A}, where

$$A = \int_{x_0}^{x_m} \pi x^2\ \xi(x)dx/4 = \frac{\pi\gamma F\ x_0{}^{\gamma}}{(\gamma-2)[1-(x_0/x_m)^{\gamma}]} \left[\frac{1}{x_0^{\gamma-2}} - \frac{1}{x_m^{\gamma-2}}\right] \tag{28}$$

assuming (10). Since $\gamma > 2$, we can let $x_m \to \infty$. But we must then have $A \to \infty$ as $x_0 \to 0$. Thus, under the conditions under which (17) was derived $(0 < \alpha < 1)$, with probability one the surface is covered by craters completely.

In these calculations we have ignored a possibly important difficulty. We have implicitly assumed that whenever a crater was formed at $\underset{\sim}{R}+\underset{\sim}{r}$, it changed the surface elevation at $\underset{\sim}{R}$ by the same amount, whatever the elevation difference between $\underset{\sim}{R}$ and $\underset{\sim}{R}+\underset{\sim}{r}$ at the time the crater is formed. The validity of this approximation depends to some extent on the value of α. Preliminary studies (Marcus, 1968) of the covariance function on a cratered surface show that if $0 \leq \alpha \leq 1$, the surface elevations are significantly correlated (correlation coefficient 0.5) at distances up to about 20% of the diameter of the largest crater affecting the roughness of the region, but this distance falls to about 3% of the largest crater diameter for $\alpha = 2$. Thus the elevation difference will probably be small if $\alpha \leq 1$, but may be appreciable if $\alpha > 1$.

4. TOTAL HEIGHT OF EJECTA BLANKETS

Our usual assumption of linear superposition is more justified for the building up of a fragmental surface layer by accumulation of ejecta blankets than it is for the addition of crater bowls. With the same assumptions as in (7), we represent the elevation increase $Z_B(\underset{\sim}{R})$ at point $\underset{\sim}{R}$ due to ejecta blanket formation by

$$Z_B(\underset{\sim}{R}) = \int \zeta_B(x,r)dN(x,\underset{\sim}{R}+\underset{\sim}{r}) \tag{29}$$

where $\zeta_B(x,r)$ is the blanket thickness from a crater of diameter x formed a distance r = (length of $\underset{\sim}{r}$) away from $\underset{\sim}{R}$. We denote the probability density of Z_B by $p_B(Z)$, and the characteristic function corresponding to $p_B(Z)$ by

$$\phi_B(u) = \int_{-\infty}^{\infty} e^{iuz}\, p_B(z)dx = \exp\left(\int_{x_0}^{x_m} \xi(x)dx \int_0^{\infty} 2\pi r\left[e^{iu\zeta_B(x,r)} - 1\right]dr\right) \tag{30}$$

We use $\xi(x)$ defined by (10) and

$$\begin{aligned} \zeta_B(x,r) &= 0 && \text{if} \quad r < x/2 \\ &= R_0 x^h (x/2r)^k && \text{if} \quad r > x/2 \end{aligned} \tag{5}$$

We obtain from (30), after some reductions,

$$\phi_B(u) = \exp\left(\frac{\pi\gamma F\ x_0{}^{\gamma}\ R_0^{2/k}}{2[1-(x_0/x_m)^{\gamma}]} \int_{x_0}^{x_m} \frac{dx}{x^{(\gamma-2-2h/k)+1}} \int_0^{R_0x^h} \frac{[e^{iuy}-1]dy}{y^{2/k+1}}\right) \tag{31}$$

or, if $\gamma \neq 2 + 2h/k$,

$$\phi_B(u) = \exp\left(\frac{\pi\gamma F\ x_0{}^{\gamma}\ R_0^{2/k}}{2(\gamma-2-2h/k)\left[1 - \left(\frac{x_0}{x_m}\right)^{\gamma}\right]}\right.$$

$$\cdot\left[\int_{R_0x_0{}^h}^{R_0x_m{}^h} \frac{[e^{iuy} - 1]dy}{y^{2/k+1}}\left\{\frac{R_0{}^{(\gamma-2)/h-2/k}}{y^{(\gamma-2)/h-2/k}} - \frac{1}{x_m{}^{\gamma-2-2h/k}}\right\}\right.$$

$$\left.\left. + \int_0^{R_0x_0{}^h} \frac{[e^{iuy} - 1]dy}{y^{2/k+1}}\left\{\frac{1}{x_0{}^{\gamma-2-2h/k}} - \frac{1}{x_m{}^{\gamma-2-2h/k}}\right\}\right]\right) \tag{32}$$

This result is not of great use as it stands. Appreciable simplifications are possible if we can take $x_m \to \infty$, $x_0 \to 0$. To achieve the first, we require

$$\gamma > 2 + 2h/k \quad , \quad k > 2 \tag{33}$$

from which we obtain

$$\phi_B(u) = \exp\left(\frac{\pi\gamma F\ x_0{}^{\gamma}\ R_0{}^{(\gamma-2)/h}}{2(\gamma-2-2h/k)}\int_{R_0x_0{}^h}^{\infty} \frac{[e^{iuy} - 1]dy}{y^{(\gamma-2)/h+1}}\right.$$

$$\left. + \frac{\pi\gamma F\ x_0{}^{2+(2h/k)}R_0{}^{2/k}}{2(\gamma-2-2h/k)}\int_0^{R_0x_0{}^h} \frac{[e^{iuy} - 1]dy}{y^{2/k+1}}\right) \tag{34}$$

In order to assume $x_0 \to 0$ we require also

$$\gamma < 2 + h \tag{35}$$

Since we may validly assume

$$F = C/x_0^{\gamma}$$

for some constant C, (34) and (35) imply

$$\lim_{\substack{x_0 \to 0 \\ x_m \to \infty}} \phi_B(u) = \exp\left(-\lambda_B |u|^{\alpha}\left[1 - i\ \mathrm{sgn}(u)\tan(\pi\alpha/2)\right]\right) \tag{36}$$

where from (33) and (35)

$$2/k < \alpha = (\gamma-2)/h < 1$$

$$\lambda_B = \frac{\pi\gamma h\ R_0^{(\gamma-2)/h}\ C}{2(\gamma-2)(\gamma-2-2h/k)}\ \Gamma\left(1 - \frac{\gamma-2}{h}\right)\cos\left(\pi\left(\frac{\gamma-2}{h}\right)\right) \tag{37}$$

(see, e.g., Gnedenko and Kolmogorov (1954) for evaluation of the integral in (34)).

The limiting characteristic function (36) can be inverted explicitly only if α = 1/3, 1/2, 2/3. Assuming we have a depth scale on which $\lambda_B = 1$, the probability density $p_B(Z)$ corresponding to (36), i.e., for which

$$p_B(Z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}\ \phi_B(u)du$$

is, for z > 0, (Zolotarev, 1954)

$$p_B(Z) = \frac{1}{\pi\ 3^{1/4}}\left(\frac{2}{3z}\right)^{3/2}\ K_{1/3}\left(3^{1/4}\ \frac{4}{9}\left(\frac{2}{3z}\right)^{1/2}\right)$$

$$\text{for} \quad \alpha = 1/3 \tag{38}$$

$$p_B(Z) = \frac{1}{(2\pi)^{1/2}} \frac{1}{z^{3/2}} e^{-1/2z} \qquad \text{for} \qquad \alpha = 1/2 \tag{39}$$

$$p_B(Z) = (3/\pi)^{1/2} \frac{1}{z} e^{-16/27z^2} W_{1/2,1/6}(32/27z^2)$$
$$\text{for} \qquad \alpha = 2/3 \tag{40}$$

and $p_B(Z) = 0$ for $z \leq 0$, all α. $K_a(x)$ is a MacDonald function and $W_{a,b}(x)$ a Whittaker function. The density (39) is shown in Fig. 1. However, we know that for any α, $0 < \alpha < 1$, $p_B(Z)$ has, for large Z, the character of an inverse power law with index α. Explicitly (Skorohod, 1954)

$$p_B(Z) = \frac{2\Gamma(1+\alpha)\ \sin(\pi\alpha/2)}{\pi\ Z^{1+\alpha}} + O\left(\frac{1}{Z^{1+2\alpha}}\right) \tag{41}$$

for large Z, and $\lambda_B = 1$.

The heavy inverse power law tail (41) occurs only because we assume $x_m \to \infty$. This means that a given point on the surface has a finite (if small) chance of being covered by an ejecta blanket of any thickness whatever, no matter how large. There must then be some (though not many) places on the surface which have been greatly built up by crater formation. We identify these with the exterior rims of the largest craters.

If $\gamma \geq 2 + h$, we cannot permit $x_0 \to 0$ without changing the model. In this case we could fix some size x_1 as the smallest diameter for which $\zeta_B(x,r)$ is defined by (3). For craters smaller than x_1, we would redefine $\zeta_B(x,r)$ to take into account the rapid relative decrease in crater rim height and relative thinning out of the ejecta blanket with decreasing crater diameter. The model functions are at present too poorly defined to make these computations worthwhile; a reasonable choice for x_1, however, is about 5 to 10 meters (based on unpublished data of H. J. Moore).

On the other hand, if $\gamma > 2 + h$ then Z_B has a finite mean value $E\{Z_B\}$. (This is not true if $\gamma \leq 2 + h$.) We obtain from (32) with $x_0 > 0$,

$$E\{Z_B\} = -i \frac{d}{du} \phi_B(u) \Big|_{u=0} = \frac{\pi\gamma k \ R_0}{2(k-2)(\gamma-2-h)} F \ x_0^{\ 2+h}$$

$$(\gamma > 2 + h) \qquad (42)$$

A test of the reasonability of the model is whether or not the "typical" total blanket thickness is of a reasonable numerical size, where the "typical" thickness is $\lambda_B^{\ 1/\alpha}$ if $\alpha<1$ and $E\{Z_B\}$ if $\alpha>1$. We consider a typical lightly cratered mare, defined by

$2.6<\gamma<3.4$

$F = 0.2$ craters per square meter larger than one meter in diameter

$h = 1$

$k = 4$

$R_o = 0.085$

The results are shown in Figure 2. Except for the unavoidable (in the model) singularities at $\gamma = 2.5 = 2+2h/k$ and $\gamma = 3 = 2+h$, the blanket thickness is numerically plausible.

For $2.67<\gamma<2.97$, the typical blanket thickness varies from 2.6 to 7 meters, which is reasonable (at $\gamma = 2.6$ we obtain 17.4 meters, which is rather large). For $\gamma>3$ we must specify the *minimum* diameter x_o at which craters can effectively rework the surface. Since almost certainly 1 cm$<x_o<$10 m, the average blanket thickness is almost certainly between 12 cm and 7 m for $3.03<\gamma<3.5$, and is more likely between 37 cm and 4 m. The model is, therefore, not grossly inaccurate.

From the curves in Figure 1, it is evident that positive stable laws with $\alpha<1$ are strongly concentrated around their peak or "mode". We note that the mode $M_o(\alpha)$ increases rapidly with increasing α. Thus, another measure of the the central location of the distribution of Z_B is

$$M_o(\alpha)\lambda_B^{1/\alpha}$$

This is also plotted in Figure 2. The results are again reasonable, ranging from 4 to 10 meters for $2.6<\alpha<2.9$.

5. TOTAL SURFACE ELEVATION

The surface elevation at $\underset{\sim}{R}$, $Z(\underset{\sim}{R})$, may be represented as the sum of all the cratering events which have affected the point $\underset{\sim}{R}$

$$Z(\underset{\sim}{R}) = Z_B(\underset{\sim}{R}) + Z_C(\underset{\sim}{R}) = \int \zeta(x,r)dN(x,\underset{\sim}{R}+\underset{\sim}{r}) \tag{43}$$

where $\zeta(x,r) = \zeta_B(x,r) + \zeta_C(x,r)$, as in (4). One easily verifies that the random variables Z_B and Z_C are statistically independent, since they pick up only the events for which $r > x/2$ or $r < x/2$, respectively. The characteristic function of Z is then simply

$$\phi_Z(u) = \phi_B(u)\ \phi_C(u) \tag{44}$$

In the probable event that crater rims and bowls scale similarly so that $h = \delta$, the distribution of Z is readily described in some detail. If $2/k < \alpha = (\gamma-2)/h < 1$, then allowing $x_0 \to 0$ and $x_m \to \infty$ yields, from (17) and (36)

$$\phi_Z(u) = \exp\left(-\lambda|u|^{\alpha}\ [1 - i\beta\ \mathrm{sgn}(u)\tan(\pi\alpha/2)]\right) \tag{45}$$

where

$$\begin{aligned} \lambda &= \lambda_B + \lambda_C \\ \beta &= (\lambda_B + \beta_C\lambda_C)/(\lambda_B + \lambda_C) \end{aligned} \tag{46}$$

This is also a stable distribution law, and all the remarks made about Z_C for $0 < \alpha < 1$ apply equally well to Z, with λ replacing λ_C and β replacing β_C. Some preliminary estimated elevation densities (Figure 5) by Rowan and McCauley (1966) and Marcus (1967) show a heavy-tailed and positively skewed shape like the densities shown in Figure 1.

In the event that $\alpha > 1$, the mean value of Z is just (assuming $h = \delta$)

$$E\{Z\} = \frac{\pi\gamma F\ x_0^{\ 2+h}}{4(\gamma-2-h)} \left[\frac{4(k-1)}{(k-2)} R_0 - C_0\right] \tag{47}$$

from (27) and (42). This is positive if and only if $\frac{4(k-1)}{(k-2)} R_0 > C_0$; but using the data of Section 2, with $k = 4$, $R_0 = 0.085$ and $C_0 = 0.25$, we have indeed $\frac{4\cdot 3}{2}$ (0.085) = 0.51 > 0.25, so that the <u>average</u> surface elevation is increasing with time! In fact, E{Z} is positive if $R_0 > C_0/6 = 0.0417$.

6. TEMPORAL BEHAVIOR OF SURFACE ELEVATION

We have seen that the scale of surface relief is characterized by the parameter λ in the case that $\alpha < 1$ (46) or by the mean value E{Z} in the case $\alpha > 1$ (47). Both the quantities λ and E{Z} are proportional to the cumulative crater flux (F or C). However, the cumulative flux is roughly proportional to the age τ of the surface since the flux rate has probably been only slowly varying with time, if not actually constant. As usual, we must now distinguish the cases $\alpha < 1$ and $\alpha > 1$.

If $\alpha < 1$, the quantity $\lambda^{1/\alpha}$ (whose dimension is length) characterizes the extent to which the distribution of Z is spread out. Since λ is proportional to the age τ of the surface, the dispersion is proportional to $\tau^{1/\alpha}$. The location of the center of the distribution is characterized by $\beta\ \lambda^{1/\alpha}$ (the unimodality of the stable distributions has not been established, but is probable). Using the data of Section 2, $h = \delta = 1$, $C_0 = 0.25$, we find that for $R_0 = 0.085$, β decreases from 1.0 to 0.51 as γ increases from 2.5 to 3.0 (α increases from 0.5 to 1.0). For $R_0 = 0.050$, β decreases from 1.0 to 0.44 as γ increases from 2.5 to 3.0. The relatively large positive value of β establishes the tendency of the surface to grow upward relatively rapidly. This appears to be an inherent part of the model, not due to a volume defect, since elementary calculations show that the ratio of volume of the crater rim (both interior and exterior) to volume of the true crater increases from 0.56 when $R_0 = 0.085$ to 1.00 when $R_0 = 0.0625$, and to 1.46 when $R_0 = 0.050$.

We note further that since $\alpha < 1$, the elevation increases as $\tau^{1/\alpha}$, faster than the increasing age of the surface. The reason for this peculiar behavior is that the older a portion of the surface, the greater its chance of being covered by a really thick blanket of ejecta or riding up on a really large rim. The conditions for this are that R_0/C_0 be sufficiently large, and that large craters are sufficiently frequent ($\gamma < 2 + h$ and $x_m \to \infty$).

In the case $\alpha > 1$ we face a different situation. As (47) shows, the average value of the surface elevation is proportional to the age of the surface. The dispersion of the distribution is again of the order of $\tau^{1/\alpha}$ with $\alpha = (\gamma-2)/h > 1$; thus, the distribution of Z becomes relatively more concentrated around its mean value with increasing age of the surface. The tendency of the surface to grow upwards is even more marked in this case than for $\alpha < 1$, because of growing relative concentration of Z around its increasing mean value $E\{Z\}$.

We note that this analysis has relatively little to do with the rate of increase of the volume of fragmental material on the surface. We consider this problem next.

7. RATE OF PRODUCTION OF FRAGMENTAL MATERIAL

We previously identified the fragmental material produced by a crater with the material in the exterior rim, ignoring both the fragmental material within the crater (fall-back and brecciated material) and the uplifted solid substrate (if any) under the exterior rim. It did not matter then whether the fragmental material ejected from a crater had been freshly excavated or whether it was old fragmental material being reworked. This distinction is essential, as has been noted by others (Orrok (1964), Meloy and Faust (1965)) who have estimated the amount of fragmental material produced by impacts during lunar history. We make (as did they) the assumption that crater shape is the same whatever the nature of the medium in which the crater is formed. We also now assume $h = \delta = 1$.

A paraboloidal crater of diameter x across the rim crest has diameter $x(1 - R_0/C_0)^{1/2}$ at the point at which $\zeta_C(x,r) = 0$. Let $V(x,z)$ be the volume of material excavated by a crater of rim diameter x whose center is on a point at which the thickness of the fragmental layer is z. We assume that the thickness of the fragmental layer varies sufficiently slowly with the distance that the fragmental surface and its cohesive substrate are locally flat, approximately. Then (see Figure 3)

$$V(x,z) = \frac{\pi}{8} x^3 \frac{(C_0-R_0)^2}{C_0} [1 - z/x(C_0-R_0)]^2$$

$$\text{if} \quad 0 \leq z \leq (C_0-R_0)x$$

$$V(x,z) = 0 \quad \text{if} \quad z > (C_0-R_0)x \tag{48}$$

Now let $\nu_F(t)$ be the volume of fragmental material produced per unit area per unit time at time t, and let $p_F(z;t)$ be the probability density function of the thickness of the fragmental surface at time t. Let $f(t) = \frac{d}{dt} F$ be the mean number of craters formed per unit area per unit time at time t. Then

$$\nu_F(t) = f(t) \int_{x_0}^{x_m} p(x)dx \int_0^{\infty} V(x,z)p_F(z;t)dz \tag{49}$$

In order to correctly compute $p_F(z;t)$ we must take into account the actual time sequence of the formation of craters and blankets at a point. This is not an altogether straightforward problem and we will defer considering it for the time being. It is evident that on a lightly cratered surface, most of the fragmental material has been produced by a few relatively rare large craters which themselves cover only a small area on the surface. Consequently, outside the (assumed rare) large craters, we can assume

$$p_F(z;t) = p_B(z;t) \tag{50}$$

although in fact the thickness Z_F of the fragmental layer is not greater than Z_B, and is usually less.

It is evident from (49) that $\nu_F(t)$ is a decreasing function of time, since the bulk of the probability in $p_F(z;t)$ moves in the direction of increasing z, thus decreasing $V(x,z)$, with increasing time t. Attempts to work out explicit examples, assuming (50), ran into difficulties. If $\alpha \leq 1$, most of the volume of fragmental material is contributed by large craters, consequently we cannot assume $x_m \to \infty$ as this leads to a power law density (41) for Z_B. If $\alpha > 1$ we can allow $x_m \to \infty$, but do not have any explicit results for $p_B(z)$.

The procedure used by Orrok (1964) and Meloy and Faust (1965) is to equate the average volume of material produced by a crater of diameter x at time t

$$\int_0^\infty V(x,z)\, p_F(z;t)dz$$

with the volume $V(x,\bar{z}(t))$ of material produced by the impact of a crater of diameter x into a layer of thickness

$$\bar{z}(t) = \int_0^t \nu_F(\tau)d\tau \tag{51}$$

$\bar{z}(t)$ is the thickness achieved by smoothing the whole volume of fragmental material over the whole surface. With additional assumptions, $\nu_F(t)$ can be found from this approach. The author believes that the more precise formulation (49) merits further study, in spite of its greater mathematical complexity. However, our usual concern is with the distribution of the <u>thickness</u> of the fragmental layer, which must necessarily take into account the reworking and redeposition of material by crater formation (perhaps by the methods of Section 4).

8. SELECTION EFFECTS IN THE OBERBECK AND QUAIDE METHOD

The distribution of the thickness Z_F of the fragmental surface layer in Oceanus Procellarum has been estimated by Oberbeck and Quaide (1967) from crater morphology. Laboratory simulations show that if a crater of diameter x is formed in a layer of fragmental material of thickness a_1x to a_2x, the crater morphology is:

(a) "Normal" if $a_1 = 0.236$ and $a_2 = \infty$.

(b) "Flat-bottom" if $a_1 = 0.159$ and $a_2 = 0.236$.

(c) "Central mound" if $a_1 = 0.108$ and $a_2 = 0.159$.

(d) "Concentric ring" if $a_2 = 0.108$.

The lower limit a_1 for concentric ring geometry may be on the order of 0.01 to 0.04, since the experiments extended only to a relative layer thickness of 0.055. It is clear that as $a_1 \to 0$ crater shape passes into an essentially normal or conical geometry.

In comparing the laboratory studies with the Moon, Oberbeck and Quaide used only "fresh" craters, although it is likely that the definition of "fresh" varies slightly with different diameters. Let $\xi_o(x)$ be the number density of fresh craters of diameter x. Denote by $(m)\xi_o(x)$ the expected number density of fresh craters of morphological type m. The selection effect occurs in the following way: A crater of size x will be of morphological type m only if it happens to form at a point at which $a_1(m)x < Z_F < a_2(m)$; otherwise it will have some other morphology. Pools of fragmental material of given thickness are distributed more or less randomly across the surface, as are the "fresh" craters, and of course they are quite independent of each other. We thus compute

$$(m)\xi_o(x) = \xi_o(x)\text{Prob}\{a_1(m)x < Z_F < a_2(m)x\}$$

$$\sum_m (m)\xi_o(x) = \xi_o(x) \tag{52}$$

If x is sufficiently large, so is layer thickness $a_1(m)x$. Our analysis of Section 4, especially (41), suggests that for large z, Z_B and presumably also Z_F have an inverse power law distribution with index α. Thus, approximately for some constant C',

$$\text{Prob}\{a_1(m)x < Z_F < a_2(m)x\} = C'\left[a_1^{-\alpha} - a_2^{-\alpha}\right]\frac{1}{x^\alpha} \tag{53}$$

provided $a_1(m)x$ is sufficiently large. However, (6) implies that for fresh craters

$$\xi_o(x) = C''/x^{\gamma+1} \tag{54}$$

for some small constant C". Thus, for $a_1(m)x$ sufficiently large

$$(m)\xi_o(x) = C'C''\left[a_1(m)^{-\alpha} - a_2(m)^{-\alpha}\right]\frac{1}{x^{\alpha+\gamma+1}} \tag{55}$$

or

(Number of craters per unit area, of type m,

with diameter x or larger) = $C'''(m)/x^{\alpha+\gamma}$ (56)

where

$$C'''(m) = \frac{C'C''}{\alpha+\gamma}\left[a_1(m)^{-\alpha} - a_2(m)^{-\alpha}\right]$$

In Figure 4 we compare the prediction (56) with data from Table 1 in Oberbeck and Quaide. The inverse power law (56) gives an adequate fit to the data for x in the range 40-100 meters, for "normal" and "flat-bottom" (including "central mound") craters, and for "concentric ring" craters larger than 70 meters. We may thus conclude that Z_F has an inverse power law distribution in Oceanus Procellarum, at least for $Z_F > 6$ meters. Furthermore, the slope on the cumulative number density is approximately

$$\alpha + \gamma = 4 \tag{57}$$

Recalling that $\alpha = (\gamma-2)/h$, and assuming h = 1, we have

$$\gamma = 3 \quad , \quad \alpha = 1 \tag{58}$$

which is certainly within the range of possible values of γ.

For sufficiently large craters or sufficiently small ones, the approximation (52) fails. Large craters are formed almost wholly in the cohesive substrate, and small craters are formed wholly in the fragmental layer, thus must show essentially "normal" morphology except for small differences corresponding to the differences in density and cohesion between substrate and fragmental material. Thus

$$(\text{NORMAL})\,\xi_o(x) = \xi_o(x) = C''/x^{\gamma+1}$$

for x very large or very small.

There are undoubtedly other size-dependent selection effects affecting the morphological classification. In view of the great promise of the Oberbeck and Quaide method, these effects merit further study.

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2015-AHM-kse A. H. Marcus

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CAPTIONS TO FIGURES

FIGURE 1. Possible theoretical distributions of total ejecta blanket thickness. p(Z) is the probability density of a positive stable law with index α, standardized to $\lambda_B = 1$.

FIGURE 2. Typical thickness of total ejecta blanket on a mare surface with initial crater population index $\gamma = 2+\alpha$. For $\alpha>1$ the average thickness $E\{Z_B\}$ is given as a function of the minimum diameter X_o of craters with significant blankets of ejecta. For $\alpha<1$ the scale parameter $\lambda_B^{1/\alpha}$ and peak $M_o(\alpha)\ \lambda_B^{1/\alpha}$ are given.

FIGURE 3. Craters formed in a fragmental layer (diagonal lines) may (right side) or may not (left side) excavate new material (crossed diagonal lines) from a cohesive substrate (horizontal lines).

FIGURE 4. Cumulative numbers of craters of given morphological type in Oceanus Procellarum, from Oberbeck and Quaide (1967).

FIGURE 5. Frequency histograms of elevations on lunar surfaces. (a)-(c) continental terrain (Rowan and McCauley, 1966). (d) Mare Cognitum (Marcus, 1967).

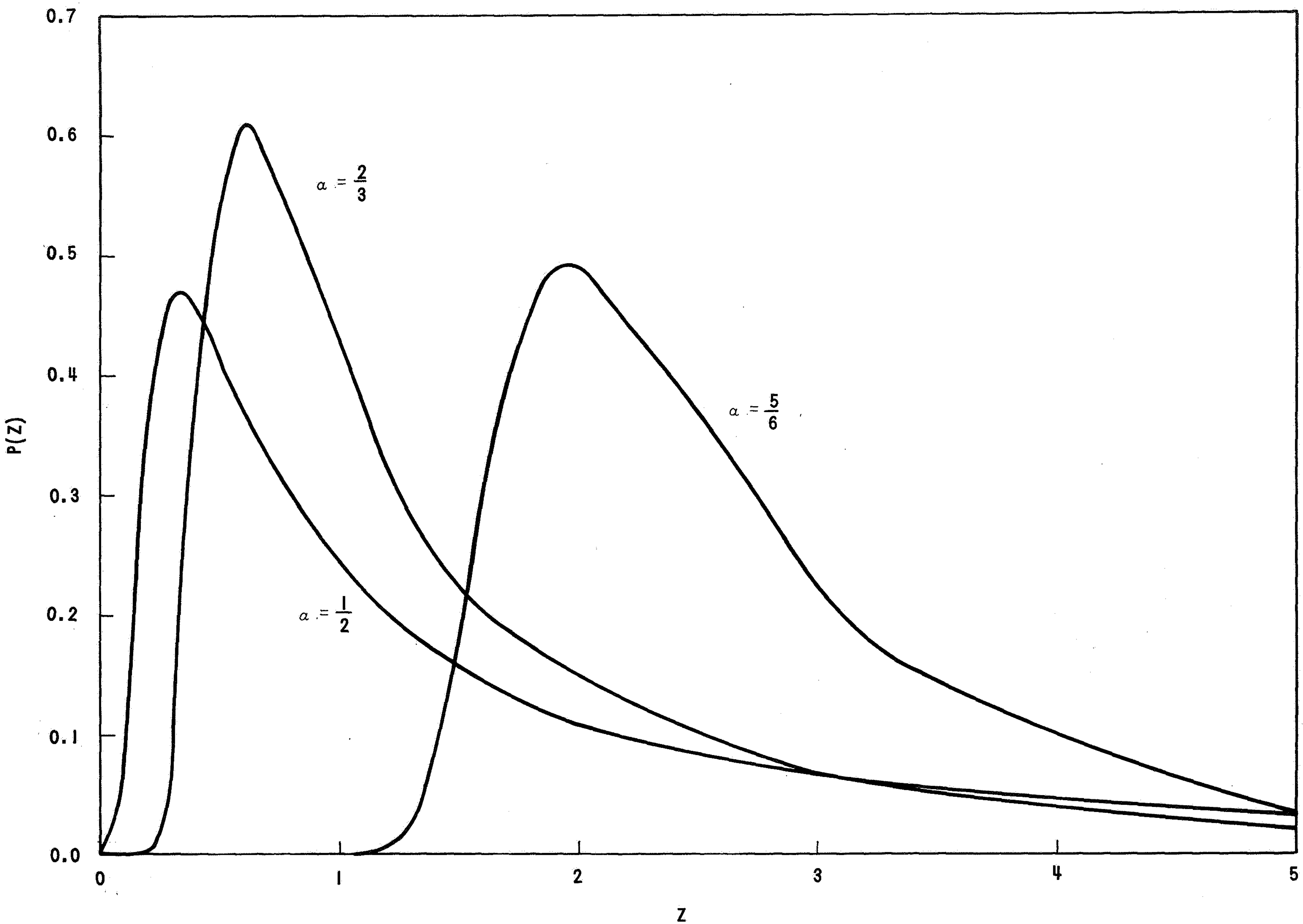
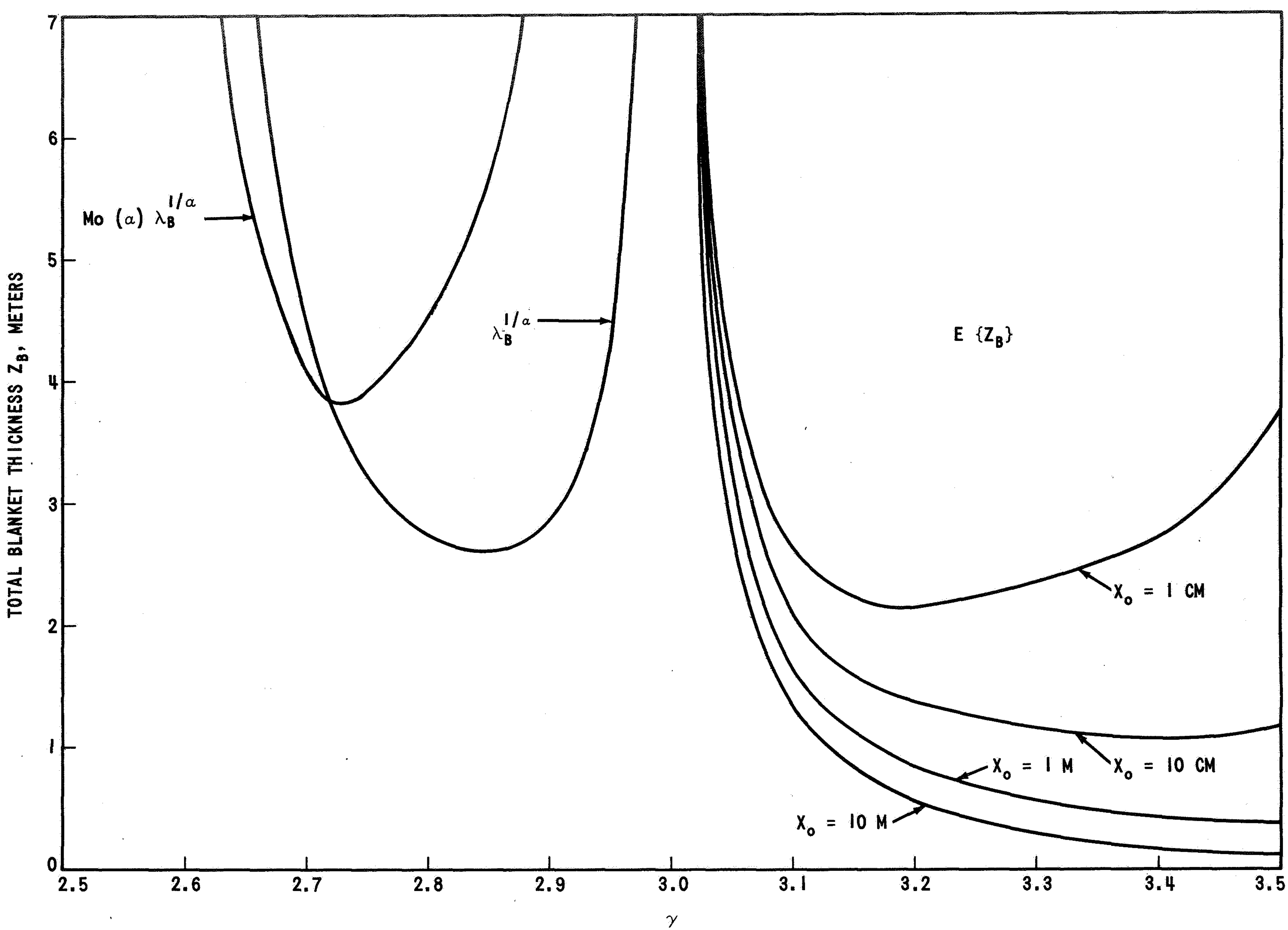


FIGURE 1 - PROBABILITY DENSITY P(Z) OF POSITIVE STABLE LAWS ($\lambda = 1.0$)



TOTAL BLANKET THICKNESS Z_B, METERS
7
6
5
4
3
2
1
0
2.5
2.6
2.7
2.8
2.9
3.0
3.1
3.2
3.3
3.4
3.5
γ
Mo (α) $\lambda_B^{1/\alpha}$
$\lambda_B^{1/\alpha}$
E {Z_B}
X_o = 1 CM
X_o = 10 CM
X_o = 1 M
X_o = 10 M

FIGURE 2

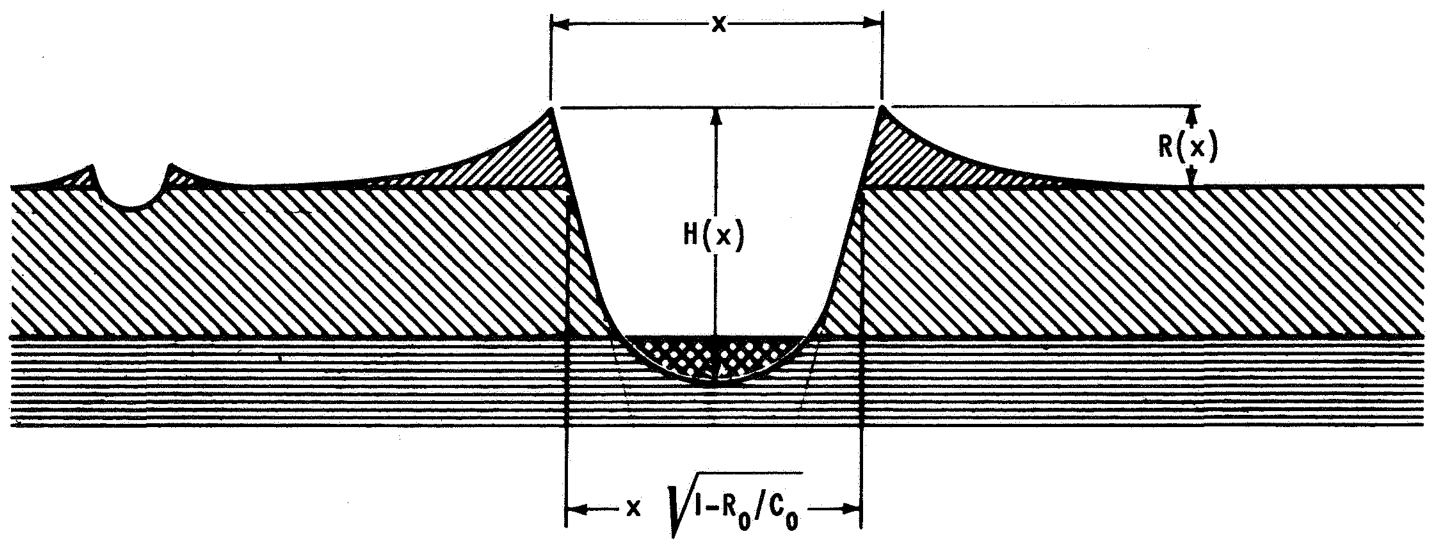


FIGURE 3

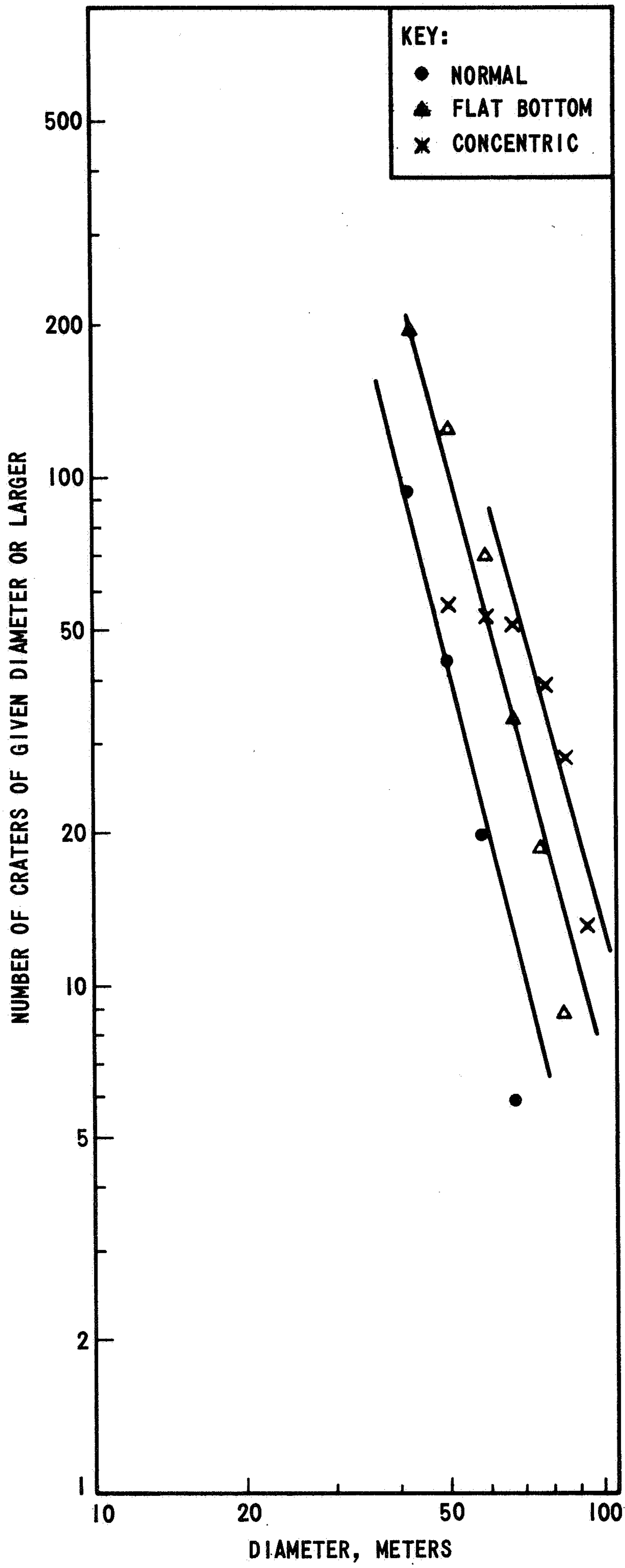

KEY:
NORMAL
FLAT BOTTOM
CONCENTRIC
500
200
100
50
20
10
5
2
1
10
20
50
100
NUMBER OF CRATERS OF GIVEN DIAMETER OR LARGER
DIAMETER, METERS

FIGURE 4

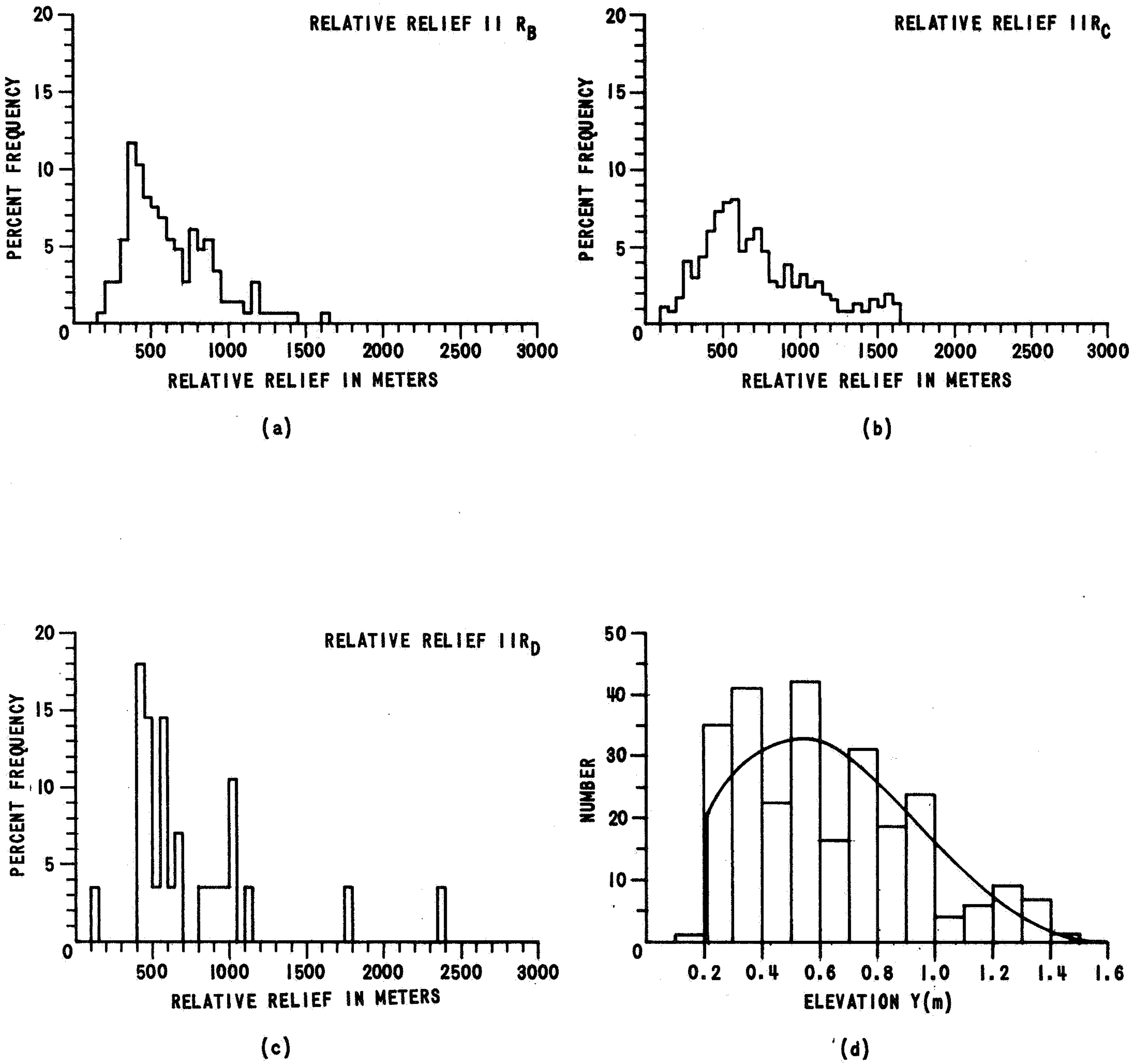


FIGURE 5 - DISTRIBUTIONS OF LUNAR ELEVATIONS

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