Report G-910557-11

A Survey of Impulsive Trajectories

Contract NAS8-21091
Final Report

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FOREWORD

This report summarizes the work performed under Contract NAS8-21091 (Impulsive Transfer Study). It is intended for use as a reference document on the subject of impulsive trajectories. The main body of the report consists of a survey of the state-of-the-art in this field. Also, an extensive bibliography of all papers, reports, and articles which deal with impulsive trajectories and related subjects has been included.

Before attempting to seek out information on a particular problem, the reader is encouraged to consult the introductory section in which the classification of subjects and definitions of important terms are explained. Once an understanding of the method of categorization has been gained, specific subjects can be identified readily from the breakdown by topics on page 8. Each of the sections is written independently so that a familiarity with material in the early sections is not a prerequisite to understanding later sections. A liberal referencing policy is used throughout the text to facilitate the pursuit of pertinent papers. Although the referencing system is basically chronological where more than one reference is listed, assignment of credit for the first solution of a given problem was not a concern in this study. Consequently individual authors are not generally singled out.

The authors gratefully acknowledge the guidance and support of Mr. Arthur Schwaniger and Mr. Rowland Burns of the NASA Marshall Space Flight Center in the conduct of this research study. We are also indebted to Mr. Theodore N. Edelbaum of Analytical Mechanics Associates who served as a consultant during the study and who contributed Appendix III, Singular Arcs.

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A Survey of Impulsive Trajectories

SUMMARY

An extensive survey of astrodynamics problems in which thrust periods are simulated by impulses has been made. This study was performed in three phases: (1) a literature search, (2) a categorization of problems and classification of papers into these categories, and (3) a review of papers and summary of known results on impulsive trajectories.

Basic objectives in this program were to perform a survey which would produce a coherent picture of the state-of-the-art in this field and to isolate problem areas in which future research should be applied. Impulsive trajectories were categorized into three major groupings: intercept, transfer, and rendezvous. Within these major categories, impulsive transfer topics were classified in a logical sequence, and detailed discussions of each topic are provided in this report.

CONCLUSIONS AND RECOMMENDATIONS

1. The subject of fixed-time trajectories, as applied to specific problems in intercept, transfer, and rendezvous, has received insufficient attention in the literature.

2. Optimal multi-impulse trajectories should be investigated for application in each of the major trajectory categories.

3. Optimal, time-fixed rendezvous is a subject for which the solution of basic problems would be beneficial to many priority space flight applications.

INTRODUCTION

The subject of impulsive trajectories has experienced a rapid growth which parallels the pace of advances in many fields of space technology. In fact, the accumulation of publications on this subject can be directly correlated with the launching of Sputnik I in 1957. The Bibliography on Impulsive Transfer which was compiled in this study contains 314 entries, fewer than 5% of which appeared before 1958. During the first five-year period after Sputnik, the literature on impulsive trajectories grew at an astronomical rate, and papers from this period comprise about 37% of the bibliography. The last five-year period has seen a continuation of that growth; 58% of the publications having appeared since 1963.

The subject of impulsive trajectories is unlike some other specialized study areas,however, in that it is not generally thought of as a separate discipline within space flight technology. Therefore, while the body of literature on the subject has reached a cumbersome state, no authoritative textbooks have appeared to organize and summarize the current state of knowledge.

For a newcomer to this field, it is difficult to arrive at a clear picture of what problems remain to be solved and what methods of approach have been successful in treating related problems. At the same time, even experienced researchers have difficulty keeping abreast of recent developments because of commitments to other fields of endeavor within space technology.

The primary goals of this study were: (1) to survey the field of impulsive trajectories, including both optimal and nonoptimal solutions, (2) to classify and describe known results in a form useful for reference purposes, and (3) to isolate problem areas which have received insufficient attention. Attainment of these goals serves to provide a foundation upon which future advances can be sought in a systematic way.

METHOD OF APPROACH

The program was divided into three tasks: (1) a literature search, (2) a classification of papers and categorization of problems, and (3) a review and summary of the results obtained in these papers.

Literature Search

The initial survey of the literature produced a list of well over 300 articles, papers, and reports on impulsive trajectories. This initial list, or unabridged bibliography, included all documents available in the following library reference sources:

Defense Documentation Center Technical Abstract Bulletins
NASA STAR Index
AIAA International Aerospace Abstracts
Engineering Index
Physics Abstracts
Electrical Engineering Abstracts
UA Library Catalog

Every item even vaguely related to impulsive trajectories was included in the unabridged file, and a copy of each document was requested by the UA Library Acquisitions Staff. Thus, this list served as a basic working file, the size and content of which varied during the study. The final result was the abridged bibliography consisting of 314 entries which appears in this report. In arriving at the final bibliography, many changes in the preliminary list were made during the course of the study. Discovery of additional sources for new papers, current contributions, translations of foreign papers, exclusion of nonapplicable papers in the unabridged list, and use of related surveys, all necessitated changes.

There have been a number of survey papers written on subjects related to impulsive trajectories. These are Entries 38, 53, 179, 180, 148, 195, and 198 as listed in the Bibliography of this report. Only two of these (195, 198) deal specifically with impulsive trajectories, and while both are recent, neither is available in English translation. Each of the surveys contributed additional references to the bibliography, but 195 and 198 can be singled out as especially useful because they include summaries of current knowledge in several important problem areas relevant to this study.

A few textbooks contain important information (66, 22,172, 253, 310) and a number of handbooks are available (80, 126, 127, 247, 304) from which data on specific problems can be obtained. Although the problems treated in these volumes are generally of the simplest type (textbook problems), each contains either an especially good treatment of a well known problem, a problem formulation not available in other publications, or a unique presentation of data. Reference is made to these documents under the appropriate categories in the survey.

The study was intended to be limited to impulsive trajectories, including: orbit transfer and rendezvous; minimum-fuel or minimum-time trajectories; optimum or nonoptimum trajectory modes; and approximate, analytical, and numerical methods. There are a number of papers which are related to impulsive trajectories, but only in a peripheral way. For example, many mission studies have been performed in which the impulsive approximation was used, but where the intent of the study was to obtain specific data such as payload requirements, launch opportunities, entry speed limits, etc. In these studies the use of impulses was incidental, and ordinarily no <u>new</u> results concerning <u>impulsive trajectories</u> were obtained. Other peripheral areas in which impulses are used, but only as a computational or conceptual convenience, include guidance and navigation studies and optimization techniques.

It was necessary to establish a criterion for including or excluding specific papers. The criterion used is as follows:

> Any paper not primarily concerned with the study of impulsive trajectories, and in which no new results are obtained, was excluded from the Bibliography.

A few papers were considered doubtful according to this definition, and they were included. Also, a number of papers on impulsive trajectories which appear in the Bibliography are not mentioned in the survey because they contain no new or significant results. Such papers are recorded in the Bibliography for the sake of completeness and to indicate that these papers were reviewed in the study.

Acquisition of papers was easy in the case of articles appearing in the open literature. But in some cases acquisition was difficult or impossible, and approximately 10% of the entries in the Bibliography were never obtained. Most of these were older papers which appeared as preprints in connection with technical meetings, foreign publications, theses, company reports, etc. Access to personal files of UARL employees and consultants was invaluable in obtaining some papers which other researchers may have difficulty in acquiring. For this reason a policy of multiple referencing was followed in the survey. In most cases at least one reference should be easily available. Furthermore, where a paper appears in more than one place, the most accessible reference is usually presented first, even though this sometimes upsets the otherwise chronological listing of papers by a particular author.

Foreign papers have been an important source of information in this study. However, it was felt that constant reference to untranslated papers would compromise the usefulness of the survey. Therefore, wherever possible, translations are referenced along with the original paper. Some of these tranlsations are difficult to obtain but several sources of translated papers are readily available. An example is the journal, Cosmic Research, which is a complete translation of the Russian journal, Kosmicheskie Issledovaniya.

It was apparent at the outset of the study that some important French papers (which were not translated at that time) would be of importance. Translations of these papers were undertaken early in the study, and these papers (193, 194, 199) will be available shortly after the publication of this report. A few other follow-up papers (195, 196, 198, 200, 201) are also referenced in the text of the survey, but have not yet appeared in translated form.

As an aid in ascertaining the degree of use made of each item in the Bibliography, Table I was prepared as a breakdown according to usage. If an item appears under the heading, Not Acquired, no copy of the paper was obtained. If it appears under the heading, Reviewed, But Not Referenced in Text, it is not referred to at all in the text of the survey. There are additional lists in Table I, one of which contains those foreign papers which were acquired, but not in English translations, and several more which are explained further on in this section.

Classification

The first problem which arose in attempting to classify and categorize the many papers on impulsive trajectories was the lack of universal definitions for some important words which are used frequently. In particular, it was found that the words terminal, transfer, and rendezvous are used inconsistently, and that definitions of some terms are required to allow a sensible classification. Therefore, the following definitions were made to describe types of trajectories and the boundary conditions which can be specified in particular problems.

Boundary Conditions

1. Free Orbit - radius, speed, and path angle specified as functions of true anomaly

2. Fixed Orbit - radius, speed, and path angle specified as functions of position in space

3. Terminal - specified radius and velocity vectors (a specific point on a specific orbit)

4. Subterminal - an incomplete terminal; i.e., one for which radius vector, velocity vector, or both are not completely specified

Trajectories

1. Intercept - starts from a prescribed or a partially prescribed initial condition (orbit, terminal, or subterminal), and ends at a partially prescribed final condition (subterminal); e.g., from a circular orbit to a specified radius and path angle, as in some disorbit problems

2. Transfer - starts from a prescribed initial motion and ends at a prescribed final motion; e.g., orbit-to-orbit, terminal-to-terminal

3. Rendezvous - starts from a prescribed initial motion and ends at a time-related prescribed final motion, e.g., satellite-to-satellite

"Prescribed" motion, as used in the above definitions, refers to the predictable motion which ensues if a terminal or orbit is specified. In the case of a subterminal the resulting motion cannot be predicted because position and/or velocity are not completely specified.

In the categorization of papers the above trajectory types were selected as the major categories. Thus Intercept, Transfer, and Rendezvous are taken up in that order as separate and distinct subjects in the survey. Subcategories were chosen somewhat differently within each of these major categories, the general objectives being to maximize the correlation among results and to achieve a reasonable balance by topics. Attainment of equal-length sections was not in itself an objective. Indeed, even in the major categories, Transfer is several times longer than either Intercept or Rendezvous.

The most common breakdown of categories was according to geometrical features such as coplanar or noncoplanar boundary conditions, intersecting or nonintersecting orbits, type of conic section, etc. Use of time constraints, such as time-fixed or time-free, as a major category was ruled out by the very small number of papers which considered such constraints. Therefore, all results described in the survey refer to time-open problems unless a time constraint is specifically mentioned. In a few cases (Rendezvous, Terminal-to-Terminal Transfer) time-fixed and time-open are used as minor categories. Everywhere else, the few examples of time-constrained problems are discussed separately within the text under the appropriate problem designations.

Review and Summary of Results

The task of reviewing and summarizing the results of over 300 technical papers is a difficult undertaking, just by virtue of the scope of the work. Each of the papers for which copies could be obtained (see Table I) was reviewed, and a short abstract of each paper was written to condense its important features into a manageable space. In some cases these reviews were only a few lines and in other cases they were several pages long, depending on the extent of new results obtained or the significance of the conclusions reached. The reviews contained at least a description of the subject and the results obtained and, where appropriate, the method of analysis and the intended application were also described.

These short reviews permitted classification of papers by categories. By selecting a category and leafing through the file, those papers whose subjects were appropriate to the category could be removed and grouped. Further subgroups could then be formed and, finally, papers treating identical problems could be isolated. The process was complicated by the fact that many papers deal with more than one specific problem, and therefore required special treatment.

There are a number of papers which are not referred to in the survey but which were considered important enough to be included in the Bibliography. These papers fall into a few problem areas which can be described as computational techniques, correction maneuvers, interplanetary applications, and terminal phase rendezvous. The papers in question are listed under these headings in Table I. The exclusion of these subject areas is not meant to downgrade their importance, but rather to underscore the fact they are distinct subjects in their own right and could not be adequately treated in a study of this length and scope. By consulting Table I, a hard core of useful papers on these subjects can be quickly extracted from the Bibliography and further references can be pursued in each area.

As an additional item of investigation in this program, effort was to have been devoted to the study of basic problems associated with space rescue and space station logistics. The rendezvous problems involved in these applications are complex, and solutions to them were not expected during the course of the program. However, the methods of analysis which may lead to solution of these problems are described in Appendix I and several recent contributions of significance in this regard were discovered during the program. These contributions are described in the section entitled Rendezvous.

Appendix II contains a discussion of the impulsive approximation. In Appendix III the subject of singular arcs is reviewed briefly from the standpoint of how such solutions relate to impulsive trajectories.

A complete breakdown by topics of all information contained in the survey is presented on the next page. In using this report the reader will find this topical summary useful in locating particular problem areas. A glossary of terms peculiar to the subject of impulsive trajectories is also provided at the end of this report.

INTERCEPT TRAJECTORIES

The most common intercept problem treated in the literature is disorbit, but there have been a few studies in which either no application was specifically mentioned, or else the application included intercept of a point farther from the focus than the initial condition. Since the analysis is not affected by whether the trajectory goes toward or away from the focus, general results hold for both cases. However, in those studies for which disorbit was the intended application, numerical data are provided only for inward paths.

BREAKDOWN BY TOPICS

- INTERCEPT TRAJECTORIES
  - General Intercept
    - Specified Radius
    - Specified Point
  - Disorbit
    - Coplanar
      - Prescribed Radius
      - Prescribed Radius and Entry Speed
      - Prescribed Radius and Entry Angle
      - Prescribed Radius, Entry Speed, and Entry Angle
    - Noncoplanar
- TRANSFER TRAJECTORIES
  - Free-Orientation Transfer
    - Orbit-to-Orbit
    - Orbit-to-Subterminal
    - Subterminal-to-Subterminal
  - Unbounded Orbit Transfer
    - Parabola-to-Parabola
    - Ellipse-to-Parabola
    - Hyperbola-to-Parabola
    - Circle-to-Hyperbola
      - Specified Energy
      - Nonintersecting Orbits
      - Specified Asymptote
    - Ellipse-to-Hyperbola
    - Hyperbola-to-Hyperbola
      - Unconstrained Approach
      - Constrained Approach
  - Circle-to-Circle Transfer
    - Coplanar Orbits
      - Two-Impulse
      - Three-Impulse
    - Noncoplanar Orbits
      - Two-Impulse
      - Three-Impulse
      - General Results
  - Circle-to-Ellipse Transfer
    - Coplanar Orbits
      - One-Impulse
      - Two-Impulse
      - N-Impulse
    - Noncoplanar Orbits
  - Coaxial Ellipse Transfer
    - Coplanar Orbits
      - Axes Aligned
      - Axes Opposed
        - Intersecting Orbits
        - Nonintersecting Orbits
        - Tangent Orbits
    - Noncoplanar Orbits
      - Axes Aligned
      - Axes Opposed
  - Special Cases
    - Congruent Orbits
      - Coplanar
      - Noncoplanar
    - Neighboring Orbits
      - Changing Orbit Elements
      - Circle-to-Circle
      - Small-Eccentricity Orbits
        - Coplanar
        - Noncoplanar
      - Multi-Impulse Transfer
    - Cotangential Transfer
      - Tangency Conditions
      - Nearly Tangent Orbits
      - Small-Eccentricity Orbits
    - Ascent
      - Coplanar
        - Ascent to Circular Orbit
        - Ascent to Elliptic Orbit
      - Noncoplanar
  - Terminal-to-Terminal Transfer
    - Time-Free
      - Coplanar Terminals
      - Noncoplanar Terminals
    - Time-Fixed
  - Ellipse-to-Ellipse Transfer
    - Coplanar
      - Useful Angle
      - Transfer Modes
      - Intersecting Orbits
      - Nonintersecting Orbits
    - Noncoplanar
- RENDEZVOUS TRAJECTORIES
  - Time-Open Rendezvous
  - Time-Limited Rendezvous
    - Circle-to-Circle
      - Coplanar
      - Noncoplanar
    - General Orbits
  - Time-Fixed Rendezvous
    - Neighboring Orbits
    - General Orbits

In view of the predominance of disorbit papers in the intercept category this section is divided into two parts, General Intercept and Disorbit. In both parts the discussion is categorized according to final conditions, i.e., intercept of a radius boundary, intercept of a point target, etc. Different initial conditions, consisting of orbits, terminals, and subterminals, are taken up successively under each category.

General Intercept

Almost all intercept problems have been analyzed under the assumption of a one-impulse mode. Unless otherwise noted, one-impulse intercept with time open is understood in the following.

Specified Radius

If the initial condition is a circle the obvious optimum intercept is a coplanar, Hohmann-type, 180-deg trajectory resulting in a grazing intercept of the target radius. In fact this same solution can be had for any initial condition if an initial coast to pericenter is used. If an initial coast is not used, the terminal-to-specified radius problem becomes considerably more complicated. It was shown in 263 that solution of this problem is equivalent to finding the minimum distance from a point to a conic section. The conic section is an ellipse if $r_2 > r_1$ and a hyperbola if $r_1 > r_2$ (11). A quartic equation must be solved to obtain the optimal solution (263) which is not generally characterized by tangential application of thrust (11).

Another problem, which has been treated in 58 and 82, is a close solar approach. The object is to approach the focus from an initially circular orbit, only the final radius being specified. A one-impulse maneuver would entail a tangential braking impulse to cause an elongated transfer ellipse with pericenter equal to the desired approach distance. It was shown in 58 that a two-impulse, bi-elliptic maneuver permits large reductions in ΔV at the expense of long transfer times. In the limit, escape to a parabola and return by an infinitesimal impulse is the optimal maneuver. An attempt to modify the finite two-impulse mode by minimizing time for fixed ΔV (82) did not result in appreciable improvement.

Specified Point

The most common intercept problem treated in the literature is interception of a specific point. If the initial condition is a circular orbit, specification of radius and central angle is equivalent to specifying the target point in the coplanar case. This problem has been studied in 22, 147, and 270. In 22, the important difference between the minimum-ΔV solution and the minimum-energy path was demonstrated. Only in the case of a 180-deg central angle are these solutions the same. For smaller angles the minimum-energy solution can result in a large increase in ΔV over the optimum (22). If the central angle exceeds 180 deg, an initial coast should be made until the 180-deg condition is reached (147).

If the target point and the initial circular orbit are coplanar there is a value for θ, the central angle, for each r_1/r_2, which divides elliptic paths from nonelliptic trajectories. The critical ranges of r_1/r_2 and θ within which elliptic solutions are possible are described in 270. A useful summary of coplanar circle-to-point intercepts is given in 270. These data are presented in Fig. 1 of this report. The configurations for which nonelliptical solutions occur are noted on the diagram. (It should be pointed out that the 180-deg solution can always be achieved for $\theta > 180$ deg by using an initial coast.) A complete summary of additional variables and their behavior with r_1/r_2 and θ can be found in 249.

Perhaps the most common intercept problem is that for which the initial condition is a terminal and the target is a point. This case has been considered in 272, 263, 249, 71, 11, and 270. It was shown in 263 that a geometrical analysis reduces the problem to that of finding the minimum distance from a point to a hyperbola. Extensions using similar geometrical reasoning were carried out in 249, 11, 270, and 40. In each case solution of a quartic equation is necessary, as shown in 84. Some extraneous solutions of this quartic result in "infinite" or "unrealistic" trajectories which reach infinite radii enroute. Rules for identifying optimal "realistic" solutions are given in 270.

The equations for optimal one-impulse intercept of a point target starting from an initial terminal are formulated in 272 and 249. Graphical data for various parameters, including flight path angle and ΔV, are provided for ranges of r_1/r_2 and θ in 249. An important study of a two-impulse intercept mode appears in 71. In this case a plane-change maneuver enroute to the target point overcomes the steep rise in one-impulse ΔV for central angles close to 180 deg. Although the maneuver described in 71 is not optimal, it improves the two-impulse solution drastically in this critical region.

Disorbit

Disorbit is categorized as an intercept problem since the initial condition is often prescribed as an orbit or terminal but the final condition is almost always a subterminal. The vast majority of studies have treated the problem of minimum-ΔV, time-open, one-impulse, coplanar disorbit from a circle, ellipse, or terminal, to a subterminal. The subterminal consists of a radius and a specified entry speed or entry angle, or both. The radius is that of the upper limit of the sensible atmosphere.

Some studies have considered multi-impulse disorbit and, in a few, entry angle or lateral range, rather than impulse, are extremized. However, since these studies are not numerous and since their results and conclusions are specialized in nature, they are not categorized separately in the discussion which follows.

The format used to summarize disorbit is similar to that used for General Intercept. Coplanar one-impulse disorbit is discussed first, and categorization is according to final conditions. Thus, disorbit to a prescribed radius with entry speed and angle open is considered first. Disorbit to a prescribed radius and entry velocity with entry angle open is considered next, and the last case is disorbit to a prescribed radius, velocity, and angle. Within each of these categories, different initial conditions consisting of circular or elliptical orbits or terminals are discussed separately and special considerations such as fixed or free central angle are taken up within each of these subcategories. The last category is that of noncoplanar disorbit.

Coplanar

Prescribed Radius

The problem of disorbit to a prescribed radius with no constraints on entry speed or angle has been considered in 186, 263, 273, 249, 75, 11, 270, and 40. If central angle is left open in this problem and the initial condition is an orbit, the optimum one-impulse solution is a 180-deg transfer to a "grazing" entry (zero entry angle) caused by a horizontal impulse at apogee. (It was shown in 40 that a two-impulse solution using parabolic arcs is always better than this minimum-energy solution if the radius ratio, r_1/r_2, is very large.)

If the initial condition is a circular orbit and the central angle is fixed, the results depicted in Fig. 1 apply (270). Additional data appear in 249.

If central angle is fixed, it was shown in 270 that for each direction of the initial velocity vector there is an optimal magnitude and for each magnitude a best direction. Whenever possible a grazing entry is used, but some boundary conditions preclude such a solution. The impulse is not generally tangential even when the disorbit point is apsidal, unless the initial velocity lies within a specific range (11).

If final entry angle is maximized for a fixed ΔV (steepest entry) from a circular orbit, two regimes of solution occur (75). A horizontal retro is optimum for small ΔV; for large ΔV, the retro angle for the steepest entry also yields minimum range. The large-ΔV, short-range solution was also obtained for the case of neighboring circular, or low-eccentricity orbits (186).

Prescribed Radius and Entry Speed

This problem has only been treated for the case of unprescribed central angle. Specification of both the final radius and entry velocity, but not entry angle, always leads to grazing entry as the minimum-impulse solution (40), whether the initial condition is an orbit or a terminal. If the initial orbit is specified (but not the disorbit point) a quintic equation results and two solution regimes occur. Below a certain entry speed, tangential disorbit from perigee is optimal. Conditions describing this limit are presented in 286 for initial circular orbits and in 40 for elliptical orbits. If the initial orbit is circular, the conditions describing this limit are functions of r_1/r_2 only.

If the initial condition is a terminal, closed-form solutions for optimal one-impulse disorbit can be derived (40). Grazing entry is again optimal regardless of entry speed, but tangential impulses are optimal only for entry speeds below a predictable value.

Prescribed Radius and Entry Angle

The optimum retrofire angle for minimum impulse is usually horizontal for one-impulse disorbit from a circular orbit to a given entry angle with central angle open. If $r_1/r_2 > 1.125$ the impulse is always horizontal, and it is below horizontal for $r_1/r_2 < 1.125$ (270 and 286). Minimizing impulse with respect to the retrofire angle, for a given entry angle, is equivalent to maximizing entry angle for a given impulse (18). For entry angles below a limiting value there exists a relative maximum of ΔV as well as a relative minimum (209). That is, there is a worst one-impulse disorbit altitude as well as a best altitude. (Disorbit altitude refers to the point of departure from the initial orbit). However, above this limiting entry angle, impulse decreases monotonically with increasing altitude. For small entry angles, γ_2, the minimum ΔV for given r_1/r_2 is shown in Fig. 2. The optimum thrust angle, ϕ, is indicated on the curves in the region where its value is different from zero.

When the initial orbit is noncircular, two solution regimes are again found (77), one for shallow entry angles and one for large angles. Horizontal impulses occur only if the disorbit point is at an apse, while for other disorbit points thrust angles are often large. Disorbit at apogee results in minimum impulse for shallow entry angles, although for large angles it is slightly better to disorbit either just before or just after apogee. In 40, conditions on eccentricity, major axis, and entry angle are presented to determine whether apogee disorbit is optimal.

In 266, the determination of the optimal disorbit location is reduced to simultaneous solution of two quartic polynomials. Conditions are defined (as in 40) to describe the configurations which lead to off-apogee disorbit, and it is shown that when the initial orbit has very low eccentricity, off-apogee disorbit is optimal. When this occurs there are two equal optima which are symmetric with respect to apogee. Characteristic velocity saving is up to 10%, compared with apogee disorbit, when off-apogee disorbit is optimal.

One further effect noted in 266 was that, if disorbit occurs prior to apogee on the initial orbit, the transfer ellipse is entered prior to apogee. Similarly, if disorbit occurs after apogee on the initial orbit, the ellipse is entered after apogee.

Prescribed Radius, Entry Speed, and Entry Angle

If both velocity and path angle are given at the final radius, with central angle open, the size and shape of the transfer orbit are predetermined, but its orientation is not. It is shown in 40 that a quintic equation must be solved to determine the optimum one-impulse disorbit from a given ellipse in this case. In general, the disorbit point is not apsidal, nor is the impulse horizontal. A two-impulse solution to this problem has been obtained in 114 for the case of disorbit from an ellipse. It is shown in 114 that a tangential impulse at apocenter is optimal for the first impulse. If the initial condition is described by a radius and velocity vector, the first impulse of an optimal two-impulse maneuver is tangential only if the initial velocity has no radial component (221). A closed-form solution to this problem appears in 221.

Noncoplanar

The only noncoplanar disorbit problems which appear in the literature are those with final radius prescribed but with entry speed and entry angle open. (The angle-fixed case is treated briefly in 275). Using concepts from spherical trigonometry, the entry or impact point is usually related to the disorbit point by two spherical arcs or angles. One of these is down-range angle measured in the plane of the initial orbit or terminal, and one is a lateral range measured normal to this plane.

If the initial condition is a circular orbit and down-range angle is left open, two simultaneous quartic equations must be solved (274). Even though a 180-deg down-range angle is optimal in the coplanar case, down-range angle quickly approaches limits of 90 deg and 270 deg if lateral range is increased from zero (74). Since these limits are approached rapidly as lateral range is increased, a 90-deg range angle is a good approximation to the optimum for most cases. Thrust angles are always small in the case where the down-range angle is open (74).

Only one quartic equation need be solved if the down-range angle is specified (274). If it is greater than 180 deg, a grazing entry results and an exact solution has been obtained (74). If down-range angle is less than 180 deg, an approximate solution is available (74), and if it is exactly 180 deg, the lateral range must be zero (74 and 270). It is interesting that if the radius ratio r_1/r_2 is near 1.0 and the lateral range is large, thrust is pointed outward (positive ϕ), resulting in a "lofted" disorbit trajectory (74).

The case of noncoplanar disorbit from an elliptic orbit with down-range angle open is considered in 275. For a specified lateral range the disorbit point appears to be at apogee and the down-range angle is 90 deg. The case where ΔV is specified and lateral range is maximized is also treated in 275.

Equations describing disorbit from a terminal to a noncoplanar impact point are derived in 8. Results are presented for some specific cases, and "footprints" for vehicles disorbiting from circular orbits are shown.

TRANSFER TRAJECTORIES

Free-Orientation Transfer

Early work on impulsive transfer (late 1950's and early 1960's) was devoted largely to the problem of free-orientation transfer. In the context used in this report, free orientation implies transfers between subterminals and orbits, since the orientation of a terminal is specified by definition. If an orbit is specified by its energy and angular momentum but not by argument of pericenter, its orientation is free. Transfer between two such orbits is a free-orientation transfer, and an optimal transfer of this type implies determining the optimum orientation angle as well as the number of impulses, impulse magnitudes, etc. By the same reasoning, a subterminal consisting of a radius and a velocity vector can be the initial or final condition of a free-orientation transfer.

The reader is referred to the definitions of terminals and subterminals stated earlier. Some references noted in this section use definitions which are different from those adhered to in this report.

Orbit-to-Orbit

Most of what is known about free-orientation orbit transfer was hypothesized or proved in 277 and 276. It was shown in 277 that the noncoplanar case can always be reduced to a planar problem because coplanar transfers are always more economical.

(Of course, physical constraints may preclude the zero-inclination solution in a real problem. Complete freedom of orientation is assumed unless otherwise noted.) In other words, the optimum inclination angle between orbits is always zero. Unless there is a restriction that the number of impulses cannot exceed one, the optimum orientation of the axes of elliptical orbits is always coaxial and aligned (277). It was stated in 193 that, for coplanar ellipses, ΔV increases monotonically with angle between pericenter directions. Thus transfer between free-orientation ellipses generally can be treated as transfer between coaxial, aligned orbits, a category which is taken up in a later section.

One-impulse transfers can be made only if the orbits intersect or are tangent for some orientation angle. Conditions for intersection of two orbits are described in 79, and a procedure for finding the optimum orbit orientation for one-impulse transfer appears in 277.

Optimal transfers using more than one impulse are always of the Hohmann type, with tangential, apsidal impulses. Although such transfers were shown to be only locally minimizing in 277, they were proved to be globally optimum in 20. In view of these results, one-impulse transfers should not be used if two or more impulses are permitted.

Since the optimum orbit orientation was shown to be coaxial, the results obtained for free-orientation orbit transfer coincide with those for coplanar, coaxial orbit transfer. Two-impulse, Hohmann-type transfers, and transfer through infinity both occur. Four- or more-impulse transfers are never optimal (277) and, in the two-impulse case, the apsides of the transfer ellipse coincide with the larger apocenter and opposing pericenter of the initial and final orbits (277, 113). The numerical results for these transfers are described in the section on coplanar, aligned, coaxial orbits.

It is interesting that in the cases where there are two unequal values of specific impulse (234), or where thrust is bounded, i.e., nonimpulsive (94), Hohmann-type solutions are still optimal for free-orientation transfer.

Radius-constrained transfers between free-orientation orbits have been studied in 123, 124, and 125. If upper and lower bounds are placed on the radius, the entire transfer (as well as the initial and final orbits) must take place inside an annulus. It was proved in 123 that if interior impulses occur in this problem, they must be tangential and apsidal. This result also holds if only the transfer orbits are radius-constrained (125). The basic results for the unconstrained problem were extended in 124 to cover radius constraints. Thus one-, two-, and three-impulse transfers occur. The two-impulse transfers connect one pericenter with the opposite apocenter. The three-impulse transfers connect pericenters and the intermediate impulse is always at the outer limit of the annulus.

Orbit-to-Subterminal

The optimum two-impulse transfer from an inner orbit to an outer coplanar subterminal consisting of a radial distance and a velocity vector is a transfer from pericenter of the inner orbit (114). The transfer orbit is tangent to the inner orbit at pericenter. Similarly, if the transfer is from an inner terminal to an outer orbit, the apocenter of the transfer orbit is tangent to the apocenter of the final orbit. It is not known whether a multi-impulse maneuver can improve on this two-impulse mode.

Subterminal-to-Subterminal

This case has been treated in 283, 219, 220, and 112. Of course, if the subterminals are apsidal, the previous results for orbit-to-orbit transfer apply (112). The general two-impulse case requires solution of a fourth-order system which was first done in 283 by an iterative numerical method. In 219, the problem was reduced to minimizing a function of two variables subject to a constraint and, in 220, it was further reduced to solution of two quadratics. This latter solution is therefore in closed form. The terminals considered in 219 and 220 are on orbits which cannot intersect for any orientation so that at least two impulses are always required.

Unbounded Orbit Transfer

There are a number of spaceflight problems which require the use of unbounded orbits, i.e., parabolic or hyperbolic orbits. Among these are: fast transfer between orbits, escape from orbit (or capture), and flyby without capture. In addition to such applications, parabolic orbits are useful in time-open transfer between ellipses and circles, as described in other sections of this report.

Parabola-to-Parabola

Since velocity on a parabolic orbit diminishes to zero as the radius becomes infinite, it is clear that infinity is the best place to transfer from one parabola to another and that such a transfer can be effected by two infinitesimally small impulses (199, 195, 121). Although the transfer between parabolas is an academic exercise, it should be pointed out that the principle of changing inclination and/or direction at great distance from the focus is important. The use of parabola-to-parabola transfer as an intermediate maneuver permits several of the problems which follow to be treated as essentially free-orientation problems.

Ellipse-to-Parabola

A tangential impulse at pericenter is the optimal mode of transfer from an ellipse to an escape parabola. (This is true even in the restricted thrust case (163).) If a particular parabolic orbit must be entered, two more impulses, both at infinity and of infinitesimal magnitude, permit transfer to any other parabolic orbit (199, 195, 121). Transfer from a parabola to an ellipse can, of course, be made by the same maneuvers in reverse sequence.

Hyperbola-to-Parabola

Transfer from a hyperbolic orbit to a parabolic orbit begins with an infinitesimal impulse at infinity which changes the direction of the initial hyperbolic speed so that a "grazing" passage occurs (199, 195). In other words, the closest possible approach to the focus should be made. At pericenter of this hyperbola, a finite tangential impulse results in a parabolic orbit which is followed to infinity. At infinity, two infinitely small impulses are used to transfer to the desired final parabolic orbit. If the orientation of the orbits is not prescribed, one of the impulses at infinity can be omitted (121). Also, if no limit on approach distance is prescribed, the approach hyperbola can be directed to the origin. In this limiting case, $\Delta V = 0$ for the entire maneuver.

Circle-to-Hyperbola

Specified Energy

Transfer between circular and hyperbolic orbits has received considerable attention because of its important application in escape and capture maneuvers. If only the energy of the hyperbolic orbit is specified (pericenter distance and axis orientation left open) the escape problem reduces to one of achieving a given hyperbolic excess speed, starting from an initial circle. Examples of one-, two-, and three-impulse escape maneuvers are shown in Fig. 3.

The two-impulse maneuver was first considered in 230. Later it was compared with one-impulse escape (155), where it was shown that for low values of V_∞ the single-impulse mode is superior. When V_∞ is equal to escape velocity at the circular orbit radius, the one- and two-impulse modes require equal ΔV's. It is interesting that for this condition ($V_\infty = V_{esc}$), the radius r_1 in Fig. 3b does not affect the calculation (61). For larger values of V_∞, the two-impulse mode is always superior to one-impulse escape.

A three-impulse escape, first presented in (58), can improve on either the one- or two-impulse maneuvers, depending on the radius r_i of the second impulse point in Fig. 3c, which should be chosen as large as possible. Some plots of $\Delta V/V_{c1}$ vs V_∞/V_{c1} are provided in Fig. 4 for various radius ratios, r_1/r_i and r_1/r_{ii}. Additional graphs of this type can be found in 58. The limiting three-impulse escape involves parabolic intermediate conditions, $r_i/r_1 = \infty$, and passage through the focus, $r_{ii}/r_1 = 0$. If the maneuver is constrained to the space outside the circular orbit, a single impulse is always the optimum coplanar escape mode.

Nonintersecting Orbits

Two-impulse transfer between a circle and a coplanar, nonintersecting hyperbola was studied in 217. This case is different from that just discussed in that both energy and pericenter distance of the hyperbola are specified, but orientation is still free. It was determined (217) that the first impulse should always be tangential and that the transfer orbit is a hyperbola entered at pericenter. The second impulse should be applied as far from the focus as possible. In the limit, the optimal two-impulse transfer is equivalent to a one-impulse transfer since the second impulse becomes infinitesimal.

Specified Asymptote

A hyperbolic asymptote consists of a hyperbolic excess speed, V_∞, with a prescribed direction. The line of action of this velocity vector is not fixed relative to the focus, but can be translated parallel to itself (pericenter of the hyperbolic orbit is unspecified). The plane of the circle need not contain the hyperbolic asymptote.

One-impulse transfer from a circle to a noncoplanar hyperbolic asymptote is discussed in detail in 22 for the case of escape from Earth orbit. It is shown that the best condition for transfer occurs when V_∞ lies in the plane of the circle and that launch should be delayed until this condition occurs. In this case the direction of V_∞ is assumed fixed relative to an Earth-centered corrdinate system, and the circular orbit plane can be rotated about the Earth's polar axis. However, for some orientations V_∞ never lies in the orbit plane, and in these cases the launch point should be chosen such that the out-of-plane component of the velocity impulse is as small as possible. The optimum impulse is horizontal if V_∞ lies in the circular orbit plane, and is usually close to horizontal (with a small downward component) in the noncoplanar case.

A two-impulse mode in which the second impulse is at infinity and the first is at the circular orbit radius was considered in 92. There is a critical angle (between V_∞ and the circular orbit plane) above which the two-impulse mode is superior to a one-impulse transfer. The larger the angle the greater is the benefit. The limiting case, for an angle of 90 deg, consists of transfer to a parabola followed by a 90-deg plane change and simultaneous acceleration to V_∞ at infinity.

It was shown in 294 that a three-impulse transfer is often better than either the one- or two-impulse solutions. The transfer described in 294 is not optimal since only apsidal impulses at prescribed radii are permitted. Nevertheless substantial ΔV savings are shown for the case of escape from a circular orbit about the moon to a specified noncoplanar hyperbolic asymptote.

In this three-impulse mode the first impulse is horizontal but may contain a plane change component. The second impulse is horizontal, at apocenter of the resulting ellipse, and consists entirely of a plane change. (The first and second coasting ellipses are congruent but inclined.) The third impulse is applied before pericenter of the second transfer ellipse, its position, magnitude and direction being optimized. Most of the plane change was found to occur at the second impulse unless the inclination angle is very small.

The absolute optimum transfer between a circle and a noncoplanar hyperbolic asymptote consists of: (1) transferring to a parabola by a tangential impulse, (2) changing to a second (inclined) parabola by two infinitesimal impulses at infinity, (3) returning on the second parabola to a "grazing" passage where, (4) another finite impulse establishes the desired escape hyperbola containing the prescribed asymptote. An escape maneuver similar to this was first presented in 32.

Ellipse-to-Hyperbola

If parabolic intermediate arcs are permitted, the limiting solutions described for the circle-to-hyperbola transfers also apply in the case of elliptic orbits (32, 195, 121). Injection from the elliptic orbit is always at pericenter.

The only studies of finite ellipse-to-hyperbola transfers in the literature appear in 11 and 312. In 11 the problem is formulated as a one-impulse transfer from a small-eccentricity ellipse to a coplanar hyperbolic asymptote. An approximate optimum impulsive solution is found, to first order in the elliptic orbit eccentricity e_1; ΔV for the maneuver is expressed as a function of e_1, V_∞, and the angle between V_∞ and the axis of the ellipse. The impulse is applied almost tangentially (in the circular case a tangential impulse is optimal) but it is not generally at pericenter. Location and direction of the impulse are determined and a considerable amount of data for various transfers is presented in 11.

In 312, one- and two-impulse maneuvers were compared for escape from various elliptical orbits to specified hyperbolic conditions. The one-impulse maneuver consists of a tangential impulse from the proper elliptic orbit true anomaly to achieve the desired hyperbolic conditions at infinity. The two-impulse maneuver consists of a tangential departure from the ellipse followed by an "adjustment" impulse at infinity to establish the hyperbola. For most end conditions, the one-impulse transfer is better, although the optimum two-impulse transfer is better for small hyperbolic excess speeds.

Hyperbola-to-Hyperbola

If only the asymptotes of the hyperbolic orbits are specified, the velocities V_{∞_1} and V_{∞_2} are known, as is the angle Γ between them. However, the pericenter radii of the orbits are left open. Thus, while the orientation of each orbit axis is fixed, the degree of "bending" during passage is not. The question of intersection of the orbits is not resolved until passage distances are specified. Actually, transfer between prescribed hyperbolic orbits can always be reduced to transfer between their asymptotes. This is accomplished by providing for two infinitesimal impulses at infinity which leave the velocity magnitudes unchanged but rotate the vectors slightly to adjust approach distances.

Unconstrained Approach

The case of unconstrained approach distance is a good starting point in this problem although it results in degenerate optimum solutions. Perhaps the most obvious case is that in which the hyperbolic excess velocities are equal, $V_{\infty_1} = V_{\infty_2}$. This transfer, referred to in 194 as the "free transfer", never requires propulsion. Any turning angle, $0 \leq \Gamma \leq 180$ deg, can be achieved by selecting the proper approach radius, a zero angle corresponding to infinite passage, $P = \infty$, and a 180-deg angle corresponding to passage through the focus, $P = 0$.

If the velocities are unequal, $V_{\infty_1} \neq V_{\infty_2}$, there is still a solution to the unconstrained problem which requires no propulsion. This transfer is a limiting case of the four-impulse transfer of 83. It consists of two impulses at the origin, where any change in velocity can be attained at negligible cost, and two impulses at infinity, where the change in direction requires negligible cost.

Constrained Approach

When a limit is placed on the approach distance, r_{min}, the transfers described above must be modified. A free transfer is still possible if the maximum turning angle compatible with r_{min} is large enough, $\Gamma \leq \Gamma_{max}$. If $\Gamma > \Gamma_{max}$, propulsion must be provided.

The one-impulse transfer between hyperbolic asymptotes was first considered in 45 and was subsequently studied in detail in 83. A summary of ΔV data for various values of Γ and ratios $V_{\infty_2}/V_{\infty_1}$ is provided in 83 in a series of summary graphs. The optimum one-impulse solutions are described in terms of the approach distance required to achieve the best one-impulse transfer, and the penalty incurred by non-optimal approach distances is indicated in the diagrams. These optimum solutions involve nontangential, nonapsidal impulses although it is shown that tangential impulses at pericenter are nearly optimal in all cases. An explicit solution to the one-impulse problem appears in 194.

A recent study of hyperbolic orbit transfer with constrained approach distance was described in 194. The transfers were categorized according to whether $\Gamma \gtreqless \Gamma_{max} = (\Gamma_1 + \Gamma_2)/2$, where Γ_1, Γ_2 are the total turning angles on the respective orbits. If $\Gamma = (\Gamma_1 + \Gamma_2)/2$, than a "grazing" one-impulse transfer is optimal. However, the cases where $\Gamma \gtrless \Gamma_1 + \Gamma_2$ involve several possible optimal modes of transfer, requiring up to six impulses.

When the required turning angle is less than that achievable without propulsion, $\Gamma < (\Gamma_1 + \Gamma_2)/2$, there are five possible modes of optimal transfer (194):

1. one-impulse nongrazing transfer with the impulse at a finite distance
2. two-impulse nongrazing transfer with one impulse at infinity and one at finite distance (the former is situated on the side of the smaller V_∞, and is an acceleration if $V_{\infty_1} < V_{\infty_2}$ and a brake if $V_{\infty_1} > V_{\infty_2}$)
3. one-impulse grazing transfer analogous to 1
4. two-impulse grazing transfer analogous to 2
5. six-impulse grazing transfer with two finite impulses at r_{min} and four infinitesimal impulses at ∞ (61).

Conditions on Γ, V_{∞_1}, and V_{∞_2} which define which transfer modes are candidates for the optimum are presented in 194 as series expansions in Γ.

When the required turning angle is too large, $\Gamma > (\Gamma_1 + \Gamma_2)/2$, there are six possible transfer modes, all grazing:

1. six-impulse transfer analogous to that described above (always optimal for Γ = 180 deg)
2. one-impulse transfer with the impulse at infinity
3. two-impulse transfer with both impulses at infinity, one on each side
4. one-impulse transfer with the impulse at finite distance
5. two-impulse transfer with one impulse at a finite distance and the other at infinity, but both on the same side
6. same as 5, but with impulses on the opposite side

Cotangential transfer between hyperbolic orbits was considered in 296, wherein it was shown that for nonintersecting orbits the transfer orbit is an ellipse, and for intersecting orbits the transfer orbit may be elliptical, parabolic, or hyperbolic.

Circle-to-Circle Transfer

The problem of transfer between circular orbits may be considered as a limiting case of transfer between free-orientation orbits because the major axis of a circle is undefined. For the same reason it is also a limiting case of coaxial orbit transfer.

More purely analytical effort has been devoted to circular orbit transfer than to any other impulsive transfer problem. The simple nature of circular orbit motion permits simplifications which naturally attract the analytically minded researcher. Many studies which purport to treat general elliptical orbit transfer provide results only when the orbits degenerate to circles.

Coplanar Orbits

Two-Impulse

Transfer between circular coplanar orbits was the first (109) and for many years the only orbital transfer problem to be studied. Hohmann concluded that the optimal two-impulse transfer between coplanar circles was by a transfer ellipse cotangential to the circles at its apses. To go from one circle to a larger one requires two accelerating impulses, and to get to a smaller circle requires two decelerating impulses. Until recently the optimality of the Hohmann transfer was a hypothesis, but now it has been established by a rigorous proof (20). Among all two-impulse, time-open transfers between coplanar circles, the Hohmann transfer is a global optimum. The possibility that Hohmann transfers may be optimal in force fields other than the inverse square field is considered in (12).

Magnitudes of the individual impulses can be expressed as simple functions of the radius ratio, r_2/r_1, which are plotted in Fig. 5 for radius ratios from 0.01 to 100.

It is apparent from Fig. 5 that transfers from a given circle to one k times as large ($k > 1.0$) are less costly than transfers to a correspondingly smaller circle. Thus, for example, it is more expensive to reach the origin than it is to escape. This is true regradless of the initial orbit radius because only the radius ratio affects the calculation.

An interesting feature of the Hohmann transfer to a larger circle ($k > 1.0$) is that ΔV reaches a maximum for k near 10.0. Although the first impulse increases monotonically with increasing k, the second reaches a maximum, then decreases to zero as k goes to infinity. Observation of this effect led to the discovery of the bi-elliptic transfer and the transfer "through infinity" which will be discussed further on.

A study of the coplanar circle-to-circle transfer was performed in (6) in terms of hodograph parameters. This study is of particular interest because it includes transfers with fixed central angles other than 180 deg. The data show that penalties of 10% or less in ΔV result when the transfer angle is as low as 160 deg. Also, the monotonic decrease in first-impulse magnitude and the interior maximum of the second impulse were confirmed when the fixed central angle is not 180 deg. These data, as well as additional information on the time required for each time-open transfer, are presented in (6).

In another study (143) it was found that for two-impulse, fixed-central-angle transfers between neighboring, circular orbits, the condition that the flight path angles on the transfer orbit are equal at the two impulse points results in a minimum ΔV. The flight path angles are always small and, for a 180-deg transfer, they are zero. Although the analysis in 143 was carried out for small orbit separations ($\Delta r/r_1 \ll 1$), numerical data are presented to show that the equal-angle solution results in a close correlation with the optimum even for moderately large separations.

Another approximate solution to this problem was performed in 290, in which equations for semi-latus rectum of the transfer ellipse and ΔV were derived as series expansions in $\Delta r/r_1$. It was shown in 290 that, for transfer angles less than 180 deg, the transfer arc lies entirely between the circular orbits, but that when the angle exceeds 180 deg the transfer arc goes beyond the outer circle, through apocenter, and returns. Comparison of the linear theory with exact results indicates good agreement up to radius ratios of about 1.5. Although the transfer ellipse semi-latus rectum cannot be represented accurately for small transfer angles, agreement with exact ΔV calculations is good for angles as small as 25 deg.

Transfer between coplanar circles was analyzed in 207 from the standpoint of reducing transfer time without imposing a large ΔV penalty. Three types of transfer orbits were considered: tangent to the inner circle, tangent to the outer circle, and intersecting both circles. It was found that for terminal orbit radius ratios of 3.0 or greater, transfer orbits tangent to the inner circle are nearly optimal and provide large reductions in transfer time relative to the Hohmann transfer without imposing large ΔV penalties. The proportionate time advantage increases with increasing radius ratio. If the radius ratio is less than 3.0, the optimum transfer ellipse intersects both circles.

The great majority of impulsive transfer papers are concerned with minimization of ΔV. The problem of minimum-time, two-impulse transfer between circular coplanar orbits with specified total ΔV has been considered in 176 and 293. The lowest ΔV is that of the Hohmann transfer which requires a specific transfer time. If transfer time is decreased from this value, a two-impulse transfer requires higher ΔV (176). If transfer time is increased, an initial (or final) coast should be used to achieve the Hohmann ΔV. Under the assumption that the transfer time is very small, it was found in 293 that approximately radial impulses result in a minimum-time transfer, and that the impulses should be almost equal in magnitude.

The case where both transfer time and central angle are fixed results in optimum transfers involving initial or final coasts (97). For transfer angles greater than 180 deg, there is a wedge-shaped region in the θ vs t plane for which the Hohmann solution can be achieved using terminal coasts.

Three-Impulse

A logical extension of the Hohmann transfer, the bi-elliptic transfer, is composed of two, cotangential, transfer ellipses with a common apocenter greater than the radius of either circular orbit (258, 107, 58). It was the first multi-impulse solution to improve on the two-impulse transfer (61). Furthermore, it was shown in 101 that four- or more-impulse cotangential transfers never improve on the bi-elliptic transfer.

Inspection of Fig. 5 for transfer to a larger circle ($r_2/r_1 > 1.0$) reveals that the Hohmann transfer is sometimes more costly of fuel than escape.

$$\frac{\Delta V_{esc}}{V_{c1}} = \sqrt{2} - 1$$

When the limiting case of transfer from the initial circle to a parabola and return on a second parabola tangent to the final circle was considered, it was discovered that this "bi-parabolic" transfer through infinity was superior to the Hohmann transfer for radius ratios greater than 11.93876. If, instead of parabolic intermediate conditions, the common apocenter of two, finite, cotangential, transfer ellipses is introduced, the maneuver is termed "bi-elliptic" and the ratio of this apocenter, r_i, to the initial orbit radius, r_1, is named the "conjunction ratio" (107). An exterior conjunction includes an intermediate apocenter greater than either circle and an interior conjunction involves an apocenter smaller than the larger circle. In Fig. 6, non-dimensional ΔV is plotted vs conjunction ratio for various terminal orbit radius ratios. The dashed curve represents Hohmann transfers ($r_2 = r_i$). Lines of constant radius ratio are shown solid for exterior conjunctions, and by long-short dashes for interior conjunctions.

If the terminal orbit radius ratio is known, Fig. 6 indicates what conjunction ratios will result in bi-elliptic transfers superior to the Hohmann for that configuration. Exterior conjunctions are always superior to interior conjunctions. Thus, each curve originates at a point on the Hohmann line and either decreases or increases from that ΔV value. If it decreases, bi-elliptic transfers result in lower

ΔV's. Using this criterion, Hoelker and Silber calculated the critical radius ratios for which bi-elliptic transfers may be better than Hohmann transfers. If $r_2/r_1 <$ 11.93876, Hohmann transfers are always superior. If $r_2/r_1 > 15.58172$, bi-elliptic transfers are always superior for conjunction ratios greater than 1.0. Between the two limiting values, the conjunction ratio must be steadily increased from 1.0, at $r_2/r_1 = 15.58172$, to ∞ at 11.93876, to make the bi-elliptic transfer equal to the Hohmann.

Noncoplanar Orbits

Two-Impulse

The first treatment of noncoplanar circular orbit transfer (115) was restricted to include only two impulses, both applied at apses. These assumptions required that the impulses also be nodal, and the optimizing parameter was the plane change split between the two impulses. It was found that most of the plane change should be made at the outer radius. The inner impulse never includes a plane change of more than 6.0 deg. Consequently the advantage of splitting the plane change between the two impulses is small (never more than 3.0% of inner circular speed) compared to making the entire change at the outer radius.

These results were verified and extended in 9, where it was assumed that the impulses are at the nodes. Although this assumption does not require that the projection of the transfer orbit be cotangential, it was shown in 9 that this projection should indeed be cotangential with the circular orbits at the apses. Analytical expressions for the plane change split were obtained by series expansion in 9.

Three-Impulse

A three-impulse transfer between noncoplanar circles represents a logical extension of the bi-elliptic transfer in the planar problem. The plane change is split up into three parts so that each impulse may include a plane change as well as a pericenter or apocenter change. The impulse is always circumferential, as in the coplanar transfer. Analyses of the noncoplanar problem appear in 241, 104, 17, and 251. (In addition, papers which deal with general coaxial orbit transfer usually include the circular orbit problem as a special case.) The first and third plane changes have been found to be small regardless of the orbit inclination and radius ratio. In 251, it is shown that these changes never exceed 5.3 deg. Therefore only a small penalty in ΔV is imposed if the entire plane change is made during the intermediate impulse (241, 104).

General Results

Only three possible transfer modes can be optimal for noncoplanar, circular orbit transfer: three-impulse, two-impulse (referred to as "Generalized Hohmann" - 195, 196, or "tilted Hohmann" - 104), and transfer through infinity. A detailed discussion of how the choice of the optimum mode can be made is given in 195. Extensive data on the impulse required and the plane change split appears in 104, and information on transfer time appears in 17. The three-impulse transfers referred to here have all impulses at finite radii, thus differentiating them from transfers through infinity which also have three-impulses but only two of which are at finite radii.

In Fig. 7, a complete summary of results for circle-to-circle transfer is presented in a single diagram. The axis parameters, radius ratio, and inclination angle between the terminal orbits are sufficient to determine the optimal transfers completely. The dashed lines are contours of constant total $\Delta V/V_{c_1}$, and the long-short dashed lines in the three-impulse region are drawn for constant values of the intermediate apocenter to initial radius ratio, or conjunction ratio. A distinct region of this parameter space is seen to be occupied by each transfer mode. Data describing the boundaries between the regions were taken from 196 and the data from which the curves were drawn were taken from 104 and 17.

Much of what is known about circle-to-circle transfer can be found in this diagram. Consider first the region of transfers through infinity. If orbit inclination exceeds 60.185 deg (195), transfers through infinity are optimal for all radius ratios. Similarly, if the radius ratio is less than 0.08376 (r_2/r_1 = 11.93876), transfers through infinity are optimal regardless of the inclination angle. The limiting values of ΔV are $\sqrt{2}-1$ when $r_1/r_2 = 0$, and $2(\sqrt{2}-1)$ when $r_1/r_2 = 1.0$. The former represents escape ($r_2 = \infty$), and the latter represents escape and return to the same radius. Since the entire plane change is made at infinity by an infinitely small impulse, ΔV is not a function of inclination angle in this region.

It is apparent that three-impulse transfers are optimal only for rather large inclination angles and/or for radius ratios close to 1.0 (241). The conjunction ratio varies from a value of 1.0 in the lower right-hand corner to ∞ on most of the boundary with transfers through infinity. Along the boundary with generalized Hohmann transfers, the intermediate apocenter is identical with the final orbit radius. Another interesting phenomenon which was pointed out in 104 is that, for inclinations greater than about 45 deg, the slope of the ΔV contours is always negative. Thus, for a given inclination, ΔV decreases as radius ratio decreases from 1.0. It is actually easier to get to a more distant orbit than to a nearby one!

The generalized Hohmann region occupies the domain of most practically interesting transfers. For radius ratios between 0.1 and 1.0 and inclinations of 10 deg or less, virtually all optimum transfers are of the Hohmann type. It is interesting that, at zero inclination, only generalized Hohmann and transfers through infinity are optimal, whereas, for $r_1/r_2 = 1.0$ and $i \neq 0$, only three-impulse and transfers through infinity can be optimal.

Three regions of this diagram have been studied extensively in the literature: (1) $i = 0$, (2) $r_1/r_2 = 1.0$, and (3) the region in the lower right corner where i is small and r_1/r_2 is near 1.0. The first of these was discussed in a preceding paragraph of this section. The equal-orbit case has been analyzed in 239, 295, and 291, with the primary results being that three-impulse transfer are always optimal for $i < 60.185$ deg and transfers through infinity are optimal for larger inclinations. At the boundary between transfers through infinity and three-impulse transfers (when $r_1/r_2 = 1.0$, $i = 60.185$ deg) two discrete solutions exist with equal ΔV (61). One is the transfer with $r_t/r_1 = \infty$. The other is the three-impulse transfer with $r_t/r_1 \simeq 10.0$. Transfer between neighboring orbits has been treated in 200, 81, and 62, and results for the special case of neighboring circular orbits are most easily deduced from the presentation in 62.

As a further point of interest, note that there is one point in Fig. 7 ($r_1/r_2 = 0.1505$, $\Delta i = 37.54$ deg) for which all three transfer modes yield identical values of ΔV (196).

Circle-to-Ellipse Transfer

Coplanar Orbits

There are three possible configurations of the orbits for coplanar circle-to-ellipse transfer. Two of these involve nonintersecting orbits: (a) ellipse entirely within circle and (b) circle entirely within ellipse. The third is: (c) intersecting orbits. In all cases tangential, apsidal impulses are used to effect optimal transfers (215, 237). (However if departure from (or arrival at) an apse of the ellipse is not possible for some reason, the transfer orbit should not be tangent at the ellipse (50); indeed in some cases it cannot be.)

One-Impulse

A one-impulse transfer can be made only if the orbits intersect or are tangent, and tangency can occur only at an apse of the ellipse. It has been shown that for interesecting orbits a one-impulse transfer is never more economical of ΔV than a two-impulse (Hohmann) transfer (237, 66). If the orbits are tangent, either at apocenter or pericenter, ΔV's required by one- and two-impulse transfers are equal (237, 113).

Therefore, a one-impulse transfer is never superior to a two-impulse transfer between a circle and a coplanar ellipse.

Two-Impulse

Two-impulse transfers are always of the Hohmann type so that, for each of the three configurations listed above, there are two candidate transfer orbits. In case (a) the optimum transfer orbit (Fig. 8a) connects the circle to pericenter of the ellipse (215, 50, 113). In case (b) the apocenter of the ellipse is used (240, 50, 259, et al).

A general rule describing the nature of the optimal transfer orbit for non-intersecting orbits was proposed in 113. The optimal transfer orbit always connects the higher apocenter and the lower pericenter. It is apparent that the impulses will both be forward (in support of the motion) to go to a larger orbit, and backward (opposing the motion) to go to a smaller orbit (237).

When the orbits intersect, case (c), the transfer connects the circle to the apocenter of the ellipse (113), as shown in Fig. 8c. In this case one of the impulses is forward and one is backward (237).

N-Impulse

A general treatment of coplanar circle-to-ellipse transfer is provided in 213, wherein necessary conditions for a minimum-ΔV transfer are applied and numerical results are expressed in a summary diagram of all such transfers. The optimality of tangential, apsidal impulses was confirmed in 213 and 299, and the transfers were found to include two types: two-impulse (Hohmann) transfers, and transfers through infinity.

A summary of all such transfers is provided in Fig. 9 which was taken from 299. In this diagram the origin represents the initial circular orbit and the coordinate axes are functions of the ratios of the final orbit pericenter and apocenter distances to the initial orbit radius which is assumed equal to 1.0.

In the shaded regions transfers through infinity are optimal and everywhere else a Hohmann transfer (of the appropriate type as discussed above) should be used. The arrows denote the proper sequence of impulses. For example, to transfer from the circle to an ellipse with $\sqrt{P_2} = 0.432$ and $\sqrt{A_2} = 0.452$ (this transfer is of type (a)), the first impulse results in a pericenter decrease to $\sqrt{P} = 0.432$, but no apocenter change. At $\sqrt{P} = 0.432$ an apocenter decrease is made to $\sqrt{A} = 0.452$, with no pericenter change. Thus two apsidal impulses, the first at apocenter and the second at pericenter, result in the proper transfer.

Noncoplanar Orbits

Unlike the noncoplanar circle-to-circle or coplanar circle-to-ellipse problems, noncoplanar circle-to-ellipse transfer is not a special case of coaxial orbit transfer because the line of nodes and the major axes of the orbits need not be coincident. This complication makes the noncoplanar circle-to-ellipse case considerably more difficult to treat analytically. Consequently, the problem has received less attention than coaxial transfer, and no completely general conclusions can be drawn concerning the nature of the solutions.

Two numerical studies of two-impulse transfers have been performed, each for a particular pair of terminal orbits at a prescribed orientation. The special case where the line of nodes is coincident with the latus rectum of the ellipse is considered in 56. Results are presented for a range of terminal ellipse eccentricities and orbit inclinations, with the circular orbit assumed to be entirely within the ellipse. The data indicate that transfer angles should not exceed 180 deg for time-open transfers. The transfer ellipse is entered just before its pericenter, and pericenter passage on this ellipse occurs from 0 to 90 deg before pericenter of the terminal ellipse. The final orbit is always entered near a node.

A detailed study of a different case appears in 37. Semimajor axis of the ellipse was assumed equal to the circular orbit radius, eccentricity and inclination were fixed at 0.3 and 20 deg, respectively, and the angle between the ascending node and pericenter point was 30 deg. Although the results of this numerical study are complex, some tentative conclusions were drawn which may or may not be characteristic of all noncoplanar circle-to-ellipse transfers.

It was found that minimum-impulse transfers do not originate from (or terminate at) an apse of the ellipse, but that arrival should always be at or near a nodal point. These conclusions are in agreement with the results of 56. As would be expected, ΔV increased with inclination, although size and shape of the transfer orbit were insensitive to variations in inclination. Both ΔV and transfer time were found to decrease when eccentricity of the ellipse was decreased.

In a recent study (35), transfer between a circle and a nearby, noncoplanar, elliptic orbit was considered. Although the specific orbit configuration is circle-to-ellipse transfer, the analysis applies to the special category of neighboring orbits, considered elsewhere herein. What is of importance is that, while most optimal transfers involve two impulses, some optimal three-impulse transfers do exist. Thus, it is possible that three-impulse transfers are also optimal in the general circle-to-ellipse case.

Another indication that three-impulse transfers exist in the general problem is provided by study of the coaxial case. In 105 and 299, coaxial circle-to-ellipse transfers utilizing two or three impulses were studied. Depending on orbit geometry, optimal two- and three-impulse transfers were found, as well as transfers through infinity. (Similar solutions in 35 are referred to as the nodal type).

Coaxial Ellipse Transfer

The problem of optimal, time-open transfer between coaxial, elliptical orbits has received considerable attention because it is a special case of elliptic orbit transfer which can be solved. Nevertheless, neither the problem nor its solution is trivial and the results provide some insight into the general case. Coaxial ellipses are defined as ellipses whose major axes are co-linear, either aligned (pericenters on the same side) or opposing (pericenters on opposite sides).

The major axes are not equal, although this special case is not trivial when the orbits are noncoplanar. Coplanar, coaxial orbit transfer will be considered first.

Coplanar Orbits

If it is assumed that the coplanar, coaxial orbit transfer be performed by two impulses, some simple, predictable results are obtained. The optimum transfers are always of the Hohmann type, i.e., with tangential impulses applied at opposing apses. However, there are always two possible transfer ellipses, as shown in Fig. 10. Early investigators, e.g. 260, suggested numerical calculation of ΔV, and direct comparison to determine the better choice. It was subsequently discovered that when the axes are aligned it is always better to use the transfer which includes the most distant apse as one terminal point and the opposing pericenter as the other (Type I in Fig. 10a). This result was obtained in 237 for the case of equal-eccentricity ellipses, and for arbitrary, coplanar, coaxial ellipses with axes aligned, in 170, 192, 193, and 299. The generalization was extended in 170 to cover the case of intersecting orbits, regardless of axis orientation. If the axes are opposed and the orbits are non-intersecting, neither transfer ellipse can be excluded because, depending on the eccentricities of the orbits, either type can have the lower ΔV.

Recent work (192, 199, 193, 299) has added another transfer mode which is often optimal for coplanar, coaxial orbit transfer. Following Marchal (192) this transfer is referred to as transfer through infinity. As seen in Fig. 11, there are two finite tangential impulses, applied at the pericenters of the terminal orbits, and one or more impulses of negligibly small magnitude applied at an infinite distance to connect the parabolic transfer orbits.

Considerable analytical work has been done by Marchal (192, 193, 196) to derive conditions which predict the optimum transfer mode for various coaxial orbit configurations. The results are summarized in the following paragraphs. Winn has computed ΔV's for optimal coaxial transfers (299, 300, 301) and provided diagrams summarizing the results of these calculations.

Axes Aligned

In this case the following conditions apply whether or not the orbits intersect.

If $P_2/P_1 > 11.938$: transfer is through infinity

If $9.0 \leq P_2/P_1 \leq 11.938$: transfer is either through infinity or Hohmann, depending on the magnitudes of P_2/P_1 and the larger of A_1 and A_2.

If $P_2/P_1 < 9.0$: transfer is always Hohmann using the larger of A_1 and A_2.

A single diagram (taken from 192) suffices to summarize the regions in which transfers through infinity and Hohmann transfers are optimal. If the orbits are tangent, the transfer is always by a single impulse if the tangency point is at the pericenters, and may be either through infinity or by one-impulse if the tangency point is at the apocenters (193). In the latter case, Fig. 11 can be used to determine whether the transfer through infinity is the proper choice.

Axes Opposed

Three transfer modes are possible when the axes of the terminal orbits are opposed:

1. Through infinity
2. Two-impulse, apocenter-apocenter
3. Two-impulse, pericenter-pericenter

It was shown in 237 that if the terminal eccentricities are equal, the progression of transfer modes is pericenter-pericenter, apocenter-apocenter, pericenter-pericenter as the ratio a_1/a_2 increases from 0 to a value greater than 10.0.

Intersecting Orbits

When the orbits intersect and the axes are opposed, the pericenter-pericenter transfer is never optimal. Some conditions which help in determining the optimal transfer mode have been derived by Marchal (192, 193, 196).

If $e_1, e_2 \leq 0.5$: transfer is apocenter-apocenter

If $e_1 + e_2 > 1.07067$: transfer is never apocenter-apocenter

If $e_1, e_2 > 0.5$ and $e_1 + e_2 \leq 1.07067$: transfer can be either through infinity or apocenter-apocenter

If $e_1 + e_2 < 0.845 + 0.31 \cdot \min(P_1, P_2)$: transfer is apocenter-apocenter.

Nonintersecting Orbits

For nonintersecting ellipses with axes opposed a convenient diagram which can be used to determine the optimal transfer mode has been conceived by Marchal (196). This diagram, reproduced herein as Fig. 12, is divided into five regions, in each of which conditions on P_1/P_2 and P_1/A_1 determine which of the three transfer modes is optimal. If $x = P_1/P_2$, $y = P_1/A_1$, and the ordinate refers to either of these, then the following conditions apply:

x in Zone I or II: transfer is apocenter-apocenter

x in Zone III:
- y in Zone I: transfer is apocenter-apocenter
- y in Zone II: transfer is apocenter-apocenter, or pericenter-pericenter
- y in Zone III: transfer is pericenter-pericenter

x in Zone IV:
- y in Zone I: transfer is through infinity or apocenter-apocenter
- y in Zone II, III, IV: transfer is through infinity

x in Zone V: transfer is through infinity

The equations of the curves in Fig. 12 are given in 196. Some concise results which summarize the conditions in the diagram are as follows:

If $e_1 + e_2 > 1.024$: the transfer is through infinity

If $e_1 > 3e_2/(3 + e_2)$: the transfer <u>is never</u> apocenter-apocenter

If $e_2 > 1.726\, e_1/(1 + e_1)$: the transfer <u>is never</u> pericenter-pericenter

If $A_2/P_1 < 8.7967$: the transfer <u>is never</u> through infinity

(The first orbit is assumed to be the smaller; i.e., $P_1 \leq A_1 < P_2 \leq A_2$.)

Tangent Orbits

The orbits must be tangent at one apocenter and at the other pericenter. Either a transfer through infinity or a one-impulse transfer is possible. The two-impulse transfers in the intersecting and nonintersecting cases degenerate to one-impulse transfers when the orbits are tangent. An equation which describes the condition for which the two types yield equal ΔV is provided in 193.

$$\sqrt{2(1+e_2)} = \left(1 + \sqrt{\frac{P_2}{P_1}}\right) \cdot \left(1 - \frac{\sqrt{\frac{P_2}{P_1}} - 1}{\sqrt{\frac{P_2}{P_1}} + 1}\right)$$

where $P_1/P_2 < 1$, and $A_1 = P_2$.

If P_1/P_2 is larger than the value predicted by this equation for a given e_2, a one-impulse transfer is optimal.

Noncoplanar Orbits

As in the coplanar problem, all impulses should be apsidal and circumferential (but not tangential if there is an out-of-plane component). A basic difference between coplanar and noncoplanar transfers lies in the existence of optimal, finite, three-impulse transfers in the noncoplanar case (192, 299). Three transfer types occur: Hohmann-type two-impulse, finite three-impulse, and three- or four-impulse transfers "through infinity", in which only two impulses are finite. All three types occur whether the axes are aligned or opposed. More than three finite impulses are never used (299). In a transfer through infinity the entire plane change is always made at infinity with negligible ΔV expense, so that the total ΔV expense for all such transfers is identical to that in the two-dimensional case. Fractioning of the plane change among the impulse points for the Hohmann and finite three-impulse transfers, as well as the location of the intermediate impulse in the latter case, must be determined.

Axes Aligned

When the axes are aligned it is possible to show the boundaries which separate regions describing the optimal type of transfer between given orbits in terms of three parameters (192, 195, 196): inclination of the orbit planes, i_2; the ratio of the minimum to maximum pericenter radius, $\min(P_1, P_2)/\max(P_1, P_2)$; and the ratio of the minimum pericenter to the maximum apocenter radius, $\min(P_1, P_2)/\max(A_1, A_2)$. Figure 13 shows the diagram presented in 196 to summarize transfers between aligned, coaxial orbits. Important points on the boundary curves can be located from the data in Table II. Optimal transfer modes in each region of Fig. 13 are described in Table III. Following Marchal (196) the three-dimensional Hohmann-type transfers will be called "Generalized Hohmann" solutions.

The Generalized Hohmann transfers always connect the higher apocenter and the opposite pericenter. When three-impulse transfers are optimal, the intermediate ellipses are also coaxial and the common apocenter always exceeds the apocenter of either the initial or final orbit. The transfer orbits are joined to the terminal orbits at their pericenters, the first impulse always being an acceleration and the third a deceleration. Most of the plane change is effected at the common apocenter. In 195 it is shown that the intermediate inclination change is always at least 73.8% of the total change. Therefore, the assumption that all plane change be performed at this intermediate point does not affect the results significantly (106). Several important facts about noncoplanar, coaxial orbit transfer can be deduced from Fig. 13 and Table III. First of all, note that for inclination angles greater than a certain value (60.185 deg) only transfers through infinity are optimal, whereas for an inclination angle of zero, Generalized Hohmann transfers are used unless $\rho < 0.28942$ (which corresponds to $P_2/P_1 = 11.938$). As was found in the coplanar case, transfers through infinity are optimal for pericenter ratios greater than 11.938.

The boundaries EKJB and E_1LC separate regions of "all through infinity" from "never through infinity" solutions. Above EKJB the transfer through infinity is never optimal, below E_1LC it is always optimal, and in the intervening region it may be optimal. The special case, e_1 or $e_2 \simeq 1.0$ (but not both $\simeq 1.0$), was studied in 192, 195, and 196. It was shown that three-impulse transfers are never optimal when one of the eccentricities is near unity. Since this condition usually results in a very small value for ρ, it is clear from Fig. 13 that a transfer through infinity is often optimal. However, Generalized Hohmann transfers are also possible.

The optimal transfer modes in each of the regions of Fig. 13 are summarized in Table III. Depending on the locations of the ends M and N of the vectors $\rho = OM$ and $\sigma = ON$, the optimal transfer mode can often be determined exactly, and can usually be restricted to two of the three possible transfer types.

As an example of the utility of Fig. 13, consider the case of transfer from a low-altitude, circular, parking orbit in the plane of an Earth surface launch site to a stationary, equatorial orbit. If the initial orbit altitude is 100 n mi, and the final orbit altitude is 19,040 n mi; then the required parameters are calculated as follows:

$$\rho = \sqrt{\frac{\min\,(P_1\,,\,P_2)}{\max\,(P_1\,,\,P_2)}} = \sqrt{\frac{4060}{23000}} = 0.42$$

$$\sigma = \sqrt{\frac{\min\,(P_1\,,\,P_2)}{\max\,(A_1\,,\,A_2)}} = \sqrt{\frac{4060}{23000}} = 0.42$$

For a launch from the Atlantic Missile Range the initial orbit inclination is 28 deg. With this inclination angle i_2, and the above values for ρ and σ, it is apparent in Fig. 13 that the Generalized Hohmann transfer is optimal (Point F). On the other hand, launch from a higher latitude, e.g., a launch site in the Soviet Union, with latitude 60 deg, places the transfer in the "through infinity" region (point F_2).

Numerical data concerning ΔV of noncoplanar coaxial transfers is provided in 301 for a range of inclinations and initial orbit eccentricities. The special case where one orbit is circular is treated in 105. The diagrams of 301 show the trend toward transfers through infinity and away from three-impulse transfers as inclination approaches 60 deg.

Axes Opposed

Considerably less is known when the axes of noncoplanar, coaxial orbits are opposed. The same three classes of optimal transfer can occur; however, their characters are altered somewhat due to the different orientation. The Generalized Hohmann transfer is either from apocenter to apocenter or from pericenter to pericenter. The three-impulse transfer is either from one pericenter to the opposite apocenter, or vice versa. The transfer through infinity is a four-impulse transfer with two finite and two infinitesimal impulses.

The pericenter-pericenter transfer of the Generalized Hohmann type can occur only if the condition $\min(A_1, A_2) \leq \max(P_1, P_2)$ is satisfied, i.e., if the larger pericenter distance exceeds the smaller apocenter distance. When a three-impulse transfer is optimal, the path may be one of two types: (1) from one pericenter to an intermediate apocenter higher than the apocenter of the initial orbit, and then to the apocenter of the final orbit, or (2) from one apocenter to an apocenter higher than the final orbit apocenter, and then to the pericenter of the final orbit (195).

A useful result for transfers through infinity is that if such a transfer is optimal for aligned, coaxial orbits, a transfer through infinity is also optimal for the same orbits when their axes are opposed (195). The transfer through infinity is not optimal if $\cos i_2/2 \geq 1/\sigma^3 \, [\sqrt{2} - (1 - (3/4)\,\sigma^2)\,\sqrt{2(1 + \sigma^2)}]$.

Special Cases

Congruent Orbits

Coplanar

Transfer between two congruent coplanar ellipses is a problem first treated by Lawden (152), who used arguments of symmetry to arrive at the optimum two-impulse solution. Although the optimal solution cannot be obtained in closed form the equations are not difficult to solve numerically and results appear in several papers (237, 170, 195). An especially good summary of transfer orbit data is presented in 170, and data describing true anomaly of the impulse point can be found in 56. The symmetric solution was shown to be optimum in 81, wherein it was also determined that the symmetric transfer ellipse is the limiting member of an entire family of coplanar transfer orbits.

Since the congruent orbits always intersect in the coplanar case, a one-impulse transfer is possible, but it has been shown that the one-impulse transfer never improves on the symmetric two-impulse solution (152, 237). Transfers through infinity are sometimes optimal, however. The boundary separating two-impulse and through infinity solutions was described in 195 and 196 and is shown in Fig. 14, a diagram which summarizes optimal, coplanar, congruent orbit transfers. The ordinate in Fig. 14 is eccentricity of the initial and final orbits, and the abscissa is the angle between their pericenters. Contours of constant ΔV normalized by $\sqrt{\mu/a_o}$ are shown as dashed lines meeting at the boundary which separates the two transfer modes.

It is apparent from Fig. 14 that the two-impulse symmetric solution is always optimal below an eccentricity of 0.53533 (196). The ΔV for transfer through infinity is independent of the rotation angle. Its magnitude is such that this mode is never optimal for small rotation angles, as seen on the diagram. It was pointed out in 61 that the transfer through infinity between congruent ellipses is a case where the number of impulses exceeds the maximum number permitted in the linearized case. This fact suggests that multiple-impulse transfers hold promise in nonlinear orbit transfer problems.

Noncoplanar

When the congruent orbits are noncoplanar their configuration can take many forms, depending on the orientation of the line of nodes. Three different configurations, depicted in Fig. 15, have been treated in the literature. In each case, the view in which the line of nodes appears as a point is also shown. The orbits are designated by the letter O with subscript T for transfer and 1 or 2 for the initial or final orbit, and the impulses are designated by p.

The configuration for which the line of nodes is perpendicular to the axis of symmetry (bisector of the angle formed by the major axes in Fig. 15a) was analyzed in 54 assuming the transfer orbit was symmetric, as in the sketch. The analysis was carried to the point where solution of two simultaneous equations in two unknowns describes a two-impulse transfer. Although no conclusions or numerical data were presented in 54, a special case of this configuration, in which the angle ω in Fig. 15a is zero, was treated in detail in 56.

Both one- and two-impulse transfers were considered in 56 and it was concluded that one-impulse transfers are superior only if the eccentricity of the initial and final orbits is small. Details of the symmetric two-impulse solution, including graphs of true anomaly of the impulse point and total ΔV, plotted against orbit inclination, were presented. However, these symmetric solutions are probably not the optimal two-impulse solution as indicated by the limiting case of 180-deg inclination, which can be improved upon. A one-impulse solution is superior in the range bounded by an eccentricity of 0.38 when i = 180 deg, decreasing to 0.0 when i = 0 deg.

In the configuration depicted in Fig. 15b, the line of nodes is coincident with the axis of symmetry. The transfer shown is the three-impulse symmetric solution described in 55. In that study, the three-impulse solution was compared with one- and two-impulse transfers and a transfer through infinity. (All plane change in the latter case is made at infinity with negligible ΔV expense.) Numerical data on the three-impulse transfer are given in 55 for an orbit eccentricity of 0.6, rotation angles, ω, from 0 to 45 deg, and inclination angles, i, from 0 to 60 deg. True anomaly of the impulse points and total ΔV are plotted against inclination.

It is shown in 55 that one-impulse transfers are optimal only for small angles, ω, and inclinations between 0 and approximately 55 deg. For small inclinations, two impulses are optimal (unless ω is also small). Three-impulse transfers are optimal for inclinations from about 15 deg to 60 deg, regardless of ω. For inclinations above 60 deg, the transfer through infinity is always optimum.

The configuration shown in Fig. 15c allows the line of nodes to be inclined at any angle v to the axis of the initial ellipse. A three-impulse mode for this configuration was described in 227 and is indicated in Fig. 15c. The sequence is: circularize at apocenter of the first ellipse, p_1 (no plane change); change planes at the node, p_2, (plane change only); and establish the second orbit at its apocenter p_3. If the angle v is 90 deg, the configuration is the same as that depicted in Fig. 15a. However, either a one-impulse transfer or the two-impulse symmetric transfer described in connection with that configuration appears to be superior to the three-impulse mode of 227.

If the angle v is different from 90 deg, the ΔV requirement of the three-impulse mode is unchanged, but the one-impulse ΔV decreases. Curves are presented in 227 showing the benefit of the three-impulse mode over a one-impulse transfer for various values of the angle v. No two-impulse data are available for such configurations.

All the configurations in Fig. 15 become identical if the orbits are circular since in this case the major axes are undefined. The case of transfer between equal but inclined circles is covered as a special case in the section on circular orbit transfer. It is shown that two types of optimal solutions are possible, a finite three-impulse transfer and a transfer through infinity. The latter is optimal only for inclinations greater than about 60 deg.

Neighboring Orbits

Transfer between neighboring (or nearby) orbits has received considerable attention because it is a problem which can be analyzed by small-disturbance theories. Therefore considerably more has been achieved in the way of analytical results using linear and second-order models than has been possible in the general case.

There is a theorem (224, 265) which states that the number of impulses in a linearized problem never exceeds the number of state variables specified at the final condition. Thus, orbit transfer may require as many as five impulses in three dimensions, or three impulses in a coplanar problem. The theorem applies only within the linear approximation and is violated (61) by some optimal nonlinear transfers, e.g., the transfer through infinity between congruent ellipses, (192).

A geometrical interpretation of optimal impulsive transfer between coplanar, nearby orbits, first presented in 45, shows that in a state space composed of small changes in energy, angular momentum, and argument of pericenter, the set of reachable states describes a three-dimensional spool-shaped figure. In 45, Contensou described the geometric features of the spool and indicated how it could be used to construct optimum impulsive transfers. Subsequent studies (81, 205) have presented accurate representation of Contensou's spool showing how it evolves from a single plane figure for e = 0, to a complex, self-intersecting surface at e close to 1.0.

Changing Orbit Elements

Small changes in the elements of an orbit may be required for orbit modification and station-keeping of satellites. Some rules for effecting such changes are presented in 59 and 66. If individual elements of a near-circular orbit are to be corrected the following rules apply (59):

To change:

1. major axis - tangential impulses at the apses (Hohmann ellipse)
2. eccentricity - same as 1
3. inclination - normal impulse at nodal crossing
4. position in orbit - transfer to slightly different orbit and wait for moment to return; use tangential impulses
5. argument of the node-normal impulse 90 deg from a node (66)

A good summary of equations for small changes in the elements appears in Chapter III of 66.

If more than one element is to be changed simultaneously the following optimum maneuvers apply (59):

To change:

1. major axis and eccentricity - use Hohmann ellipse
2. major axis, eccentricity, and position - if one of the two impulses in 1, above, is split into two parts, position change can be made without increase ΔV

Circle-to-Circle

Optimum two-impulse transfer between neighboring, coplanar, circular orbits with central angle fixed was considered in 143. In this formulation, flight path angle and radial and circumferential components of velocity at the impulse points were used as parameters. Equal flight path angles at the impulse points was shown to result in a stationary solution, to first order in $r_2/r_1 - 1$. Results for ΔV show good correlations with exact results, even for moderate orbit separations.

It has been shown in 290 that for this same problem, without linearization, an eighth-degree equation in the semi-latus rectum, ℓ, of the transfer orbit must be solved to define the optimum two-impulse solution. However, a Taylor series expansion in $r_2/r_1 - 1$ results in closed-form solutions for both ℓ and ΔV in terms of radius ratio and central angle, if only the linear terms are retained. Comparisons with exact numerical results in 290 indicate that the linear theory accurately represents ΔV up to $r_2/r_1 \simeq 1.5$, and that accuracy is poorest for small central angles, regardless of r_2/r_1. These results agree with those obtained in 143. Improvements obtained by inclusion of second-order terms in the expansion are also presented in 290.

Another important linearized analysis was carried out in 238. In that study optimal two-, three- and four-impulse solutions to the problem of fixed-time rendezvous between coplanar, circular orbits were obtained using Lawden's primer vector theory. The results obtained in 238 are described under the category of Rendezvous. It is significant that the use of a linear approximation permits solution of a complex problem such as fixed-time rendezvous. The importance of linearization lies in the fact that it results in separation of the state and adjoint differential equations, thereby permitting determination of the form of the optimal control separately from solution of the two-point boundary value problem posed by the state equations.

Small-Eccentricity Orbits

Most studies of neighboring orbit transfer in the literature have dealt with the case of small-eccentricity orbits. The reason for this concentration of effort is that while linearization can be performed about an orbit of any eccentricity, linearization about a circular orbit results in the simplest analytical form of the governing equations.

Coplanar

Transfer between neighboring, coplanar, small-eccentricity ellipses was studied in 147, wherein closed-form solutions for ΔV were obtained by series expansion in $\Delta r/r_1$. Two transfer modes were discovered whose use depends on whether or not the orbits intersect. If the orbits do not intersect, all impulses are tangential and may be either accelerating (for outward transfers) or decelerating (for inward transfers). The number of impulses used is arbitrary except that at least two are necessary. Thus a two-impulse transfer always suffices. (A higher-order theory would resolve the question of how many impulses, but to first order the effect on ΔV is not detectable.) Decelerating impulses occur on a radial line in the direction of the largest change in radius during the transfer, as defined in 147. Accelerations are applied 180 deg from this line. (These transfers are referred to as "spiral-limited" in 81.)

For intersecting orbits, tangential impulses are again optimal, but they are alternately accelerations or decelerations. The decelerations occur on a line in the direction of maximum change in radius, and the accelerations, 180 deg away. Only two impulses can be applied per revolution and the first impulse may be of either type. (These transfers are referred to as "symmetric-limited" in 81.)

Noncoplanar

The noncoplanar, neighboring orbit transfer problem has been solved in 200, 81, 62, and 301. Although the methods used in these studies are slightly different the same results are obtained. It has been shown that transfers with more than two impulses are not required in this linearized problem (200), and that two nondegenerate transfer modes exist. The first of these is a "nodal" (62) transfer with two nodal impulses (180 deg apart). The second is a general two-impulse mode for which symmetry does exist, but for which the impulses are in general neither nodal nor apsidal. A singular (81) or degenerate (200) mode also exists for which thrust direction is defined everywhere, but for which the location of the impulses is arbitrary (62). This mode also admits nonimpulsive thrusts, but two-impulse transfers are not improved upon by inclusion of additional impulses or other thrusting periods. The extension of these results to time-open rendezvous is considered in 201.

A number of useful summary diagrams are presented in 62 to describe all such linear transfers in terms of increments in the orbital elements. Nondegenerate two-impulse transfers are the most prevalent type. Some three-dimensional primer locus diagrams are also presented in 62 to differentiate the solution modes.

The existence of optimizing three-impulse solutions has been demonstrated by higher-order theories in 200 and 35. In 200 the linearization is performed about a noncircular (small eccentricity) orbit and the degeneracy of the singular solutions disappears. One-, two-, and three-impulse solutions take their place. Transfer between a circle and a nearby ellipse was considered in 35, wherein a second-order theory was developed and a special "transition" analysis was performed near the boundary of the singular solutions. Two-impulse nodal and nondegenerate solutions occur in the second-order model, the large impulse always preceding the smaller in going from the circle to the ellipse. It is interesting that in the nonlinear case, three-impulse transfers occupy a considerable region of the parameter space (35).

Multi-Impulse Transfer

Although optimizing four-impulse transfers have not been found in any nonlinear orbit transfer problems (except those transfers involving parabolic arcs) a linearized study of fixed-time transfers between neighboring coplanar orbits (302) has revealed the existence of such solutions. In a sense, the analysis of 302 overlaps rendezvous because of the equivalence of time and central angle in the linearized case.

Linearization is performed about an orbit of arbitrary eccentricity in 302 so that near circularity is not an assumption of the analysis, although departure from the initial orbit is always assumed to be at pericenter. Data for nominal eccentricities from $0 \leq e_o \leq 0.7$ are presented in 302. Transfer angles up to 540 deg and as low as 180 deg or less were found to result in minimizing solutions. In the $e_o = 0$ case, the second and third impulses became symmetric with respect to the first and fourth impulses. Thus, for short times, the four-impulse transfer becomes a two-impulse transfer. When $e_o = 0$ and the time is short, the third and fourth impulses merge, producing a three-impulse transfer.

Cotangential Transfer

Cotangential transfer is a transfer in which orbits are joined at tangency points by tangential impulses. Only the magnitude of the velocity vector is changed in such a maneuver and therefore only coplanar transfers are possible. Although this definition does not preclude three- or more-impulse transfers, only one- and two-impulse transfers have been studied. The first application of a cotangential transfer was the Hohmann transfer (109) between circular orbits. Some time later, Lawden (153) observed that tangential impulses give near-optimum performance when the orbits are elliptical, and more recently general studies of orbit transfer (192, 193, 212) have placed well-defined limits on the angle between the thrust and velocity vectors, thus demonstrating the near optimality of cotangential transfer for a wide class of problems. Hyperbolic terminal orbits (296) and free-orientation ellipses (51) have also been treated but these cases are covered elsewhere in this report.

Tangency Conditions

The condition that the transfer orbit be tangent to both the initial and final orbits restricts the class of potential transfer orbits. The condition can be applied in various ways (153, 296, 66, 24, 287, 148, 214). Although for the time-open case such orbits can always be found, cotangential transfer is not always possible when time is fixed (153).

If the terminal orbits intersect, the vacant focus of the transfer orbit (288) and the tangency points (153) describe a hyperbolic locus. The transfer orbit may be elliptical, parabolic, or hyperbolic (296). For nonintersecting orbits, the locus of the tangency points (153) and the vacant focus (288) is an ellipse and the transfer orbit is also an ellipse (296).

No completely analytic solutions have been derived for arbitrary terminal orbits but the equations can be reduced to a relatively simple form for numerical studies (214). Data presented in 24 indicates that ΔV is quite sensitive to departure point when the terminal orbits intersect, but that it is insensitive for nonintersecting orbits. However, these observations are based on a few numerical cases and are not conclusive.

Nearly Tangent Orbits

The case of nearly tangent, coplanar ellipses has been studied in 26, 27, 28, and 204 to determine whether one-impulse transfers can be optimal for "shallowly intersecting" orbits. Data presented in 28 indicate that one-impulse transfers can be superior to two-impulse transfers but that the superiority exists in a very narrow range of orbit orientations near tangency. This narrow region was also noted in 45, 193, et al.

Small-Eccentricity Orbits

Cotangential transfers are truly optimizing only if the orbits are coaxial. Since that includes circular orbits it is not surprising that the assumption of small eccentricities leads to near-optimal cotangential transfers (153, 260, 214, 288, 287). Furthermore, the impulse points are very nearly apsidal on the transfer ellipse (153, 288).

It was shown in 288 that, even if only one orbit has a small eccentricity, cotangential transfer is near-optimal. If both orbits have eccentricities less than 0.2, cotangential ΔV's are only 1% greater than the optimum (214).

Ascent

Ascent to orbit is not usually treated as an impulsive transfer problem because the launch phase cannot generally be compressed into a short enough time span relative to the total time of ascent to warrant assumption of impulsive thrusts. Also, atmospheric effects and staging considerations make the minimization of impulsive ΔV a questionable criterion in the ascent problem. However, there are some cases for which impulsive ascent to orbit is meaningful. The results which are summarized in this section are taken from papers in which ascent to orbit was the intended application. There are other papers in the sections on disorbit, rendezvous, and orbit transfer which may also be considered relevant to ascent but they are not covered here.

There are two basic groupings of the ascent problem, one according to whether the model is coplanar or noncoplanar, and the other by the nature of the final orbit. Initial conditions usually consist of a radius and velocity vector, the latter often being prescribed as zero. The coplanar case is treated first, with circular, elliptical, and hyperbolic final orbital conditions taken up in succession. In all the papers cited, transfer time is unspecified and, in the coplanar case, transfer angle is always free as well.

Coplanar

Ascent to Circular Orbit

When transfer angle is left open, the optimum solution consists of a "minimum-energy" trajectory, i.e., the ellipse of smallest major axis (151). Therefore, entry into the circular orbit is always tangential. Two impulses generally suffice, one to enter the transfer ellipse and the second to establish the circular orbit.

If initial conditions consisting of a radius and velocity vector are assumed, the optimum transfer orbit is one with apocenter equal to the radius of the final circle (114). The second impulse is tangential, and the transfer orbit is tangent to the circle. The magnitude and direction of the first impulse depends on initial conditions. If the initial speed is zero, it has been shown (12) that a horizontal impulse is optimal. If the orbit-to-planet radius ratio exceeds 11.94 and the initial velocity is horizontal (or zero) a three-impulse ascent using tangential apsidal impulses is optimal (66).

Ascent to Elliptic Orbit

If the terminal orbit is an ellipse with unspecified orientation, the results concerning circular orbits apply, entry into the ellipse occurring at its apocenter (114). The required number of impulses is not more than three (195). The case of a two-impulse ascent starting from zero velocity, where the jet speeds of the impulses are unequal, is treated in 278. The critical parameters are the ratio of the planet radius to pericenter radius, r_1/P, and the ratio of the jet speeds of the first and the second impulses. If the I_{sp} ratio is less than r_1/P, the first impulse is not horizontal but the apocenter of the transfer ellipse is tangent to the pericenter of the final orbit. If the I_{sp} ratio is greater than r_1/P, the transfer ellipse is of the Hohmann type and both impulses are horizontal. However, the terminal ellipse may be entered either at apocenter or pericenter depending on its size and shape.

Noncoplanar

Only circular terminal orbits have been treated in the impulsive, noncoplanar ascent problem. There are two basic modes of ascent in the time-open noncoplanar case, direct and indirect. The latter makes use of a "parking" orbit and is referred to in 66 as "interrupted ascent". A good discussion of the advantages and disadvantages of parking orbits, as well as the consequences of nonequatorial launch from a rotating planet, is provided in Chapters I and III of 66.

In the treatment of direct ascent carried out in 178, fuel is minimized in a two-impulse ascent in which the first stage is jettisoned before orbital injection, but structural masses are neglected. The initial condition is zero velocity at the planet's surface, but planetary rotation is included. The second impulse is always applied at apocenter of the transfer ellipse. Results show that if the stage I_{sp} ratio is above a certain value, a horizontal first impulse in the direction of planetary rotation is optimal. Below this value a vertical launch ΔV component is necessary. If the stage I_{sp}'s are equal, horizontal launch is always optimal. It is of interest that fuel consumption is a maximum at a finite radius in this problem. The reason is that the speed increment required to reach orbit altitude always increases with altitude, while the speed increment to establish the orbit decreases with altitude.

A similar study, performed in 303, considered a two-impulse ascent from the standpoint of minimum impulse. As in 178, initial conditions were zero velocity at a prescribed radius, but planetary rotation was neglected. The second impulse was assumed to be at apocenter of the transfer ellipse. Data are presented in 303 for transfer to various circular orbits as specified by their radius and the inclination angle between the launch point and the circular orbit plane, measured along an arc normal to the orbital plane. It is shown that a transfer angle of 90 deg results in the smallest inclination between the transfer orbit and the circular orbit but that minimum impulse does not occur at this condition.

A complete optimization of the transfer ellipse for two-impulse ascent from a zero velocity condition, with a nonrotating planet, was performed in 42 and 314. It was found that entry into the circular orbit at apocenter of the transfer ellipse is not a good assumption for minimum ΔV if the radius ratio, r_1/r_2, is small. When the initial inclination angle (as defined above) is large, a transfer angle (angle traversed on the transfer ellipse) of 90 deg is optimal. For zero inclination the optimum angle is 180 deg, unless $r_1 = r_2$, in which case $\theta_{opt} = 70.529$ deg. Comparison with a nonoptimal three-impulse ascent, in which launch into a circular parking orbit is followed by an inclined Hohmann transfer, shows that direct two-impulse ascent is superior for small r_2/r_1, and inferior if r_2/r_1 is large. Further explanation of these effects can be found in 42 and 314.

Terminal-to-Terminal Transfer

In a sense, terminal-to-terminal transfer bridges the gap between orbit-to-orbit transfer and rendezvous. A terminal is merely a specific point on a specific orbit. Thus, transfer between two terminals is also a transfer between the orbits which they designate, but it is not generally the optimum transfer between those orbits because optimum orbit transfer requires selection of the optimal terminals of arrival and departure. If transfer time is fixed, terminal-to-terminal transfer becomes identical to rendezvous between bodies which happen to occupy the given terminals at the appropriate departure and arrival times. Thus, optimal solution of the terminal-to-terminal transfer problem is a step toward solving associated orbit transfer and rendezvous problems.

Time-Free

All published results on time-free terminal-to-terminal transfer involve two-impulse transfers.

Coplanar Terminals

In the first study of the coplanar problem (84), a formulation in terms of chordal and radial components of velocity as parameters resulted in an eleventh-order equation, but use of hodograph parameters in 6 reduced the solution equation to eighth order. A real positive root of the octic equation designates the optimal transfer ellipse.

Noncoplanar Terminals

Three-dimensional formulations of the terminal-to-terminal transfer problem appear in 272, 7, 174, and 311. It is concluded in 272 that the apses of the transfer orbit do not coincide with the terminals, as was also predicted in 84 for the coplanar case.

The first thorough analysis of the two-impulse case was performed in 174, and some important results were obtained. Using Stark's approach (263), an eighth-order polynominal solution equation was derived and extraneous solutions identified and discarded. From the real roots it was shown that two relative minima can occur in the terminal-to-terminal problem, i.e., there are two different transfer orbits which are locally minimizing. One of these required a lower ΔV and is the absolute optimum; however, if trip time is a consideration and the secondary optimum entails an appreciably shorter time, it may be the preferable transfer. Another significant result obtained in 174 is a demonstration of the existence of hyperbolic transfer orbits in some terminal-to-terminal transfers. These results were affirmed in 311 under completely general terminal conditions, including retrograde motion during the transfer. Conditions under which motion in the transfer ellipse opposes that of the terminals are indicated in 311. Also included in 311 is a thorough analysis of the multiple optima discovered in 174.

Time-Fixed

Solutions to the time-fixed problem have been obtained only under the assumptions of a reduced gravitational field. The field-free case was first investigated in 150 and the optimum solution was shown to be impulsive. Further work on the field-free problem (90, 70) yielded the result that intermediate impulses never reduce ΔV, i.e., only terminal impulses are required. These same results were shown to hold when the gravity field is uniform in direction and strength (90). Thus two-impulse transfers suffice in these reduced forms of the problem.

By studying the fixed-time coplanar, terminal-to-terminal problem in terms of a parameter which measures field strength, the conditions for which terminal impulses are optimum were derived in 90. A linearized analysis was used so that only small impulses and neighboring orbits are allowed. Approaching the centrally directed, inverse-square field case by steadily increasing field strength led to the conclusion that intermediate impulses do occur in an inverse-square field. Therefore, multi-impulse transfers can provide ΔV reductions for time-fixed terminal-to-terminal transfer, a result which has important implications for orbit rendezvous. Further evidence of the existence of multiple-impulse solutions has been demonstrated in 183.

Ellipse-to-Ellipse Transfer

It has only been in recent years that significant progress has been made toward solution of the time-open, ellipse-to-ellipse transfer problem, and almost all progress has been confined to the coplanar case. Several concepts which emerged in parallel studies performed in the USA and in France (45, 34, 192, 193, 212, 81) have contributed to rapid progress in this field.

The first important concept was the use of ΔV to replace time as an independent variable in the variational formulation of the optimum transfer problem (45, 34). Next came the designation of a "useful angle" (192, 193, 212) within which thrust is always applied, and the "switching" laws determining the optimal transfer mode. Categorization of transfer arcs (81), and study of the "maneuverability" (45, 192, 81, 205) as defined in 45 have also contributed to a more thorough understanding of this basic problem.

Two recent survey papers (195, 196) have compiled extensive documentation of current knowledge on ellipse-to-ellipse transfer. However, English translations of these surveys have not yet appeared. Much of the discussion which follows is based on the results presented in 192, 193, 195, and 196. The interested reader is referred to these excellent papers for further information and for details of the analyses.

Coplanar

Useful Angle

The concept of the "useful angle" was first proposed in 192, wherein it was shown that thrust is always applied in a rather narrow angular range which always lies between the local horizontal and the local tangent directions (Fig. 16). The useful angle is always smaller than 12.5 deg and has an upper limit ϕ_s, and a lower limit, ϕ_i. Only the limiting values are used when the thrust is impulsive. As shown in Fig. 16, the useful angle includes both forward and rearward thrust directions. It is apparent that the limits are 180 deg apart and that no loss of generality results from considering forward thrust only, the opposite case being understood.

Expressions for ϕ_i and ϕ_s have been obtained by series expansions in 192, 193, 195, and 196 for $e \approx 0$ and $e \approx 1.0$(only e is required to designate the orbit since the major axis does not affect the results). Extensive data on these limits appear in 212 and 47; the latter contains a particularly good summary. Some general information appears in 195 in the form of bounds on the useful angle, namely:

$$|\phi_s - \phi_i| \leq 12.5 \text{ deg}$$

$$|\phi_s - \phi_i|/\gamma < 0.2$$

$$|\phi_s| < 26.2 \text{ deg}$$

The significance of ϕ_i and ϕ_s lies in the fact that they are related to switching points on a given transfer orbit. At each point, v_1, on a given transfer orbit there is an angle ϕ_{i1} or ϕ_{s1} which is the optimal thrust direction for an impulse and, associated with that point, is another point, v_2, on the same orbit with corresponding values, ϕ_{i2} and ϕ_{s2}. If the orbit is entered at v_1, it must be by an impulse with thrust angle ϕ_{i1} or ϕ_{s1}, and it must be departed from at v_2 by an impulse with thrust angle ϕ_{i2} or ϕ_{s2}. A complete summary of all angles ϕ_{i1} and ϕ_{s1} and the angles ϕ_{i2} and ϕ_{s2} associated with them appears in 47. Although there is a continuous change in the "domain of maneuverability" (45) with changes in e for most values of e starting from zero, a distinct change in character occurs at e = 0.925 (193). For e < 0.925 there are forward and backward useful angles for all true anomalies, v. However, for e > 0.925, some values of v have no useful angles. This is related to the occurence of three-impulse transfers for high eccentricities.

A switch (or in the terminology of 192, 193, 195, and 196 a "commutation") ϕ_i always follows an accelerating impulse and precedes a deceleration (192). A switch ϕ_s always follows a small acceleration and precedes a larger one (or follows a large deceleration and precedes a smaller one). It is important to realize that these angles are defined relative to the transfer or coasting ellipse. When a ϕ_i switch applies, the impulses are always on opposite sides of the major axis of the transfer orbit (193). When a ϕ_s switch applies the impulses are always on the same side. This effect was noted in 81, wherein the two types of transfer orbits were referred to as spiral-limited and symmetric-limited transfers because each type is delimited by a Hohmann-type, 180-deg transfer at one extreme, and a "Lawden spiral" or a "symmetric" (congruent orbits) transfer, respectively, at the other extreme. It should be pointed out that the determination of whether or not a particular elliptical arc can be a segment of an optimal time-open, coplanar trajectory can be made (81). The inverse problem, namely to determine the optimal arc or arcs which connect two given orbits, is much more difficult and has yet to be accomplished, except by successive approximation as in 195, or by extensive computation as in 202 and 203.

Transfer Modes

There are four possible optimal transfer modes between coplanar ellipses: one-impulse, two-impulse, three-impulse, and through infinity (192). One-impulse transfers (no switches) are rare because, (1) the orbits must intersect or be tangent, and (2) thrust must be within the useful angle with respect to both orbits (192). When a one-impulse transfer applies it always occurs at the intersection closer to the focus (193). A complete study of one-impulse transfers was conducted in 212 and much useful information is presented there. Since every segment of a multi-impulse transfer is optimal in itself, the individual impulses are optimal for transfer between the orbits they join. Thus, optimal one-impulse transfers are not uncommon. Two-impulse transfers (one switch) are the most common type. If one impulse of a two-impulse transfer is an acceleration, it is always first. The abbreviations F for a forward (accelerating) thrust and R for a rearward (decelerating) thrust will be used from this point on. Two-impulse transfers can be of the FF, FR, and RR types, but RF never occurs.

Three-impulse, time-open transfers with finite radii (two switches) occur but are rare because some rather limiting conditions must be satisfied for their optimality. The following conditions must be fulfilled for a three-impulse transfer to be optimal (193):

$$\left(\frac{\sqrt{P_1}}{a_1} + \frac{\sqrt{P_2}}{a_2} \right) \cdot \max \left(\sqrt{P_1} , \sqrt{P_2} \right) < 0.2873$$

$$0 < |\omega_2| < 22 \text{ deg}$$

$$\frac{9}{25} < \frac{P_1}{P_2} < \frac{25}{9}$$

The first condition is equivalent to $e_1 + e_2 > 1.712$. Another useful condition is that if the ratio $\max(A_1 , A_2)/\min(P_1 , P_2) < 21$, a three-impulse transfer is never optimal.

With regard to the sequence of impulses on a three-impulse trajectory, both ϕ_1- and ϕ_s-type switches occur so that one impulse is always different from the other two (192). In fact, the first impulse is always an acceleration and the last is always a deceleration (193). Thus, types FFR and FRR can occur. A condition which differentiates these modes is as follows (196): If $a_1^2 e_1^7 < a_2^2 e_2^7$, type FRR does not occur; if $a_1^2 e_1^7 > a_2^2 e_2^7$, type FFR never occurs; and if $a_1^2 e_2^7 = a_2^2 e_2^7$ neither type of three-impulse transfer can be optimal.

Transfers through infinity are always optimal if $P_2/P_1 > 11.938$ (or $P_1/P_2 > 11.938$). Also, if a transfer through infinity is optimal for coaxial orbits (ω_2 = 0 or 180 deg), it is optimal for the same orbits when $\omega_2 \neq 0$ or 180 deg.

The preceding discussion concerning the "useful angle" and conditions required for the various transfer modes applies regardless of the configuration of initial and final orbits.

Intersecting Orbits

There are two possible cases under the category of intersecting orbits, one in which the orbits intersect for any orientation, ω_2, and one in which the orbits intersect only for a limited range of values of ω_2. The always-intersecting case is defined by the expression, $A_1 \geq A_2 \geq P_2 \geq P_1$. This relationship implies that $e_1 \geq e_2$. (Since the choice of which orbit is the larger is arbitrary, it will be assumed that $P_2 \geq P_1$ always applies.)

It was shown in 195 that three transfer modes can occur: FR, FRR, and through infinity. If $P_1 > P_2$, type FRR is replaced by FFR. If $e_1 + 0.579\, e_2 \leq 0.845$, the FR mode is always used, and if $8/9\ (\sqrt{2} - 1)\ (1 - \cos \omega_2) \geq (1 - e_1)/e_2$, transfer through infinity is always used (195). Further information on this case is provided in 196.

The case where the orbits intersect for some orientations but not for others is described by $A_1 \geq A_2 \geq P_2 \geq P_1$, or $A_2 > A_1 > P_2 > P_1$. In the first instance, there are three possible transfer modes: FR, FRR, and through infinity (192). A test which helps to determine which is the optimal mode was proposed in 192. This test appears in Fig. 17. When the test is positive, the two-impulse mode FR is always optimal. If the test is negative, and if $8/9\ (\sqrt{2} - 1)(1 - \cos \omega_2) \geq (1 - e_1)/e_2$ as well, the transfer is always through infinity. Otherwise any of the three modes can be optimal, except that the three-impulse mode can occur only if e_1 and e_2 are in the region above the mixed dashed line in Fig. 17 (193).

The second case, $P_1 < P_2 < A_1 < A_2$, is more general and admits six possible modes: F, FF, FR, FFR, FRR, and through infinity. In this case, if the test is positive one of the two-impulse modes is optimal. They must be compared in each instance to determine the better, as explained in 195. If the test is negative, the transfer through infinity is used if $(\sqrt{2}-1)(1 - \cos \omega_2) \geq \min\left((1 - e_1)/e_2,\ (1-e_2)/e_1\right)$. Otherwise any of the six modes can occur, except that for the three-impulse mode to be optimal, e_1 and e_2 must be to the right of the dahsed line in Fig. 17 (193). The data in 212 determine the trace of this curve more precisely. When the three-impulse mode fulfills these conditions, the choice between FFR and FRR depends upon an additional condition. If $a_1^2 e_1^7 < a_2^2 e_2^7$, the FRR mode cannot occur, and if $a_1^2 e_1^7 > a_2^2 e_2^7$ the FFR mode cannot occur (195). As explained previously, when $a_1^2 e_1^7 = a_2^2 e_2^7$, the three-impulse mode cannot apply.

Nonintersecting Orbits

There are three possible transfer modes when the orbits are nonintersecting: FF, FFR, and through infinity. If the orbits never intersect for any orientation, i.e., if $A_2 > P_2 > A_1 > P_1$, then the FFR mode does not occur, and the choice between FF and through infinity rests on two conditions. If $A_2/P_1 \leq 8.7967$, the FF mode is used, and if $(\sqrt{2} - 1)(1 - \cos \omega_2) \geq \min((1 - e_1)/e_2, (1 - e\ )/e_1)$, the transfer is through infinity (195). Some further conditions on this case are provided in 196.

If the orbits are nonintersecting, but only for certain orientations, all three modes can occur. A summary of this case appears in Fig. 18 (193). Some further conditions which were presented in 195 are also helpful. If $A_1 + P_1 \leq 6.32\ P_2$ the transfer is never three-impulse (FFR), and if $A_2/P_1 \leq 8.7967$, the transfer is always two-impulse (FF).

Some general observations concerning transfer between coplanar ellipses have been made with regard to orbit orientation, ω_2. It was pointed out in 192 that ΔV increases monotonically with $|\omega_2|$ for $-180 \leq \omega_2 \leq 180$ deg. Furthermore, ω_2 varies monotonically (increasing or decreasing) during a transfer consisting of a succession of arcs (193). It was observed in 24, 138, and 193 that ΔV is very sensitive to changes in ω_2 for intersecting orbits, but rather insensitive for nonintersecting orbits.

Noncoplanar

Compared to the coplanar case very little in the way of general results can be listed for noncoplanar transfers. The linearized (see Neighboring Orbits section) case does not admit more than five impulses, a result which may or may not hold in the general case. Also, it has been pointed out before that if a transfer through infinity is optimal for a pair of coplanar orbits it is also optimal when they are inclined. The existence of optimal one-impulse transfers between certain inclined orbits was indicated in 203. Two-impulse transfers also exist.

Some numerical methods have been advanced for studying the general ellipse-to-ellipse transfer problem. The method described in 202 and further documented in 203 appears to handle particular two-impulse problems satisfactorily. The contour plots presented in these studies provide the only available information of a general nature on noncoplanar transfers.

RENDEZVOUS TRAJECTORIES

The term rendezvous has been applied to a variety of problems treated in the literature. In the sense used herein, a rendezvous problem begins with a vehicle performing a prescribed motion as the initial condition and ends with the vehicle performing a time-related prescribed motion as the final condition. This definition is fairly general and encompasses problems in such categories as terminal-phase rendezvous, orbital rendezvous, and direct launch to rendezvous with an orbiting spacecraft. Terminal-phase rendezvous is specifically concerned with the relative motion between the target and the rendezvous vehicle. The equations of motion are generally linearized about the target body and, while impulses can be employed to accomplish rendezvous, they are implied rather than explicitly provided for. Further, the linearized equations of motion are generally restricted to small displacements about the target. Thus, terminal-phase rendezvous is a subject more closely related to guidance or navigation than to impulsive trajectories, and is not considered here.

Direct launch to orbit encompasses a variety of logistics and operational considerations such as launch site selection, launch windows, and characteristics of launch vehicles. Again, the connection with impulsive trajectories is not close enough to warrant inclusion in this study.

This leaves what is ordinarily referred to as orbital rendezvous as the basic impulsive trajectory problem in the rendezvous category. General surveys of the overall rendezvous problem are found in 53, 116, and 289. However, an examination of these surveys reveals that the vast majority of papers deal with terminal-phase rendezvous.

Rendezvous time is generally measured from the instant a rendezvous command is given to the moment rendezvous occurs. As such, it may consist of two parts: (1) waiting time in the original orbit and (2) actual transfer time. Within orbit rendezvous it is useful to define three categories or classes of problems: (1) time-open, (2) time-limited, and (3) time-fixed. The time-open rendezvous problem places no limit on the total rendezvous time and therefore degenerates to the problem of optimum orbit transfer with an appropriate waiting period. The time-limited problem has application whenever the time-open solution requires an excessive time and it is desirable to investigate alternative methods of achieving rendezvous in a reduced time period. Institution of a time limit usually involves a tradeoff study between ΔV and t, and consideration of alternative transfer modes. It is not necessary for a well-defined upper limit on rendezvous time to exist in a time-limited problem. The rendezvous time is generally a result of the analysis.

Time-fixed rendezvous is distinguished from the other two categories in that the total rendezvous time is specified in advance, and the required result is the optimum rendezvous path for that time. Use of these time constraint categories provides a convenient and sensible means of classifying rendezvous problems. This breakdown is also indicative of the specialized areas in which past research has been applied and leads to a better picture of how future research should be directed.

As pointed out earlier in the section on terminal-to-terminal transfer, rendezvous and terminal-to-terminal transfer are closely related problems. If the orbital positions of the target and pursuer are specified in a rendezvous problem, i.e., if the terminals are prescribed, the results described under terminal-to-terminal transfer apply.

Time-Open Rendezvous

Time-open rendezvous degenerates to the problem of optimum orbit transfer coupled with an appropriate waiting period in the initial orbit. It is shown in 25, 57, and 282 that the minimum expense for rendezvous is equivalent to the minimum ΔV for time-open orbit transfer. For the general case it is shown in 25 and 57 that phase angle or time constraints may be handled by an impulse splitting technique with no net expense in ΔV. For the case of rendezvous between nonintersecting, coplanar orbits (circle-to-ellipse, ellipse-to-circle, or axially aligned ellipse-to-ellipse), an expression for the lowest rendezvous time which can be achieved by any scheme which also minimizes ΔV is derived in 282. It is also shown in 57 that time-open rendezvous provides an upper bound on time for rendezvous between general orbits. For some configurations the required waiting time to achieve proper phasing for a rendezvous which utilizes the optimum transfer path may become excessive. For example, it is shown in 250 that the maximum waiting time for rendezvous between coplanar circular orbits using the Hohmann transfer path is equal to the synodic period.

Time-Limited Rendezvous

Due to the preponderance of information available for the circle-to-circle rendezvous case and the rather sparse information available for noncircular orbits, the time-limited rendezvous problem may be further classified according to the initial and final orbit configuration: (1) circle-to-circle and (2) general orbits. The circle-to-circle category can be further broken down into sections on coplanar and noncoplanar results, although this is not convenient for the general orbit class.

Circle-to-Circle

There are basically three transfer techniques which have been applied to rendezvous between circular orbits: (1) Hohmann, (2) bi-elliptic, and (3) semi-tangential. The Hohmann and bi-elliptic transfers are well known; the semi-tangential maneuver utilizes a transfer ellipse tangent to one of the original orbits, and intersecting, i.e., nontangential to, the other. It is shown in 207 that semitangential transfers tangent to the inner orbit are more economical in terms of both ΔV and t for time-fixed transfers between coplanar circular orbits. Consequently, only semitangential transfers tangent to the inner orbit need be considered for rendezvous. Each of these transfer maneuvers may be extended by considering parking in one of the transfer ellipses or parking in intermediate ellipses obtained through the various impulse splitting techniques.

Coplanar

Figure 19 describes the orbital geometry and phase angles for rendezvous between coplanar circular orbits. The phase angle, β, is defined as the angle between the rendezvous vehicle and the target location at the inception of rendezvous, measured positive in the direction of motion. The angle β_H is the initial phase angle required for rendezvous via a Hohmann transfer ellipse and $\Delta\beta$ is defined as the difference between the actual phase angle and the Hohmann phase angle. In the discussion following it will be assumed that the interceptor is initially in a circular orbit of radius r_1 and that $r_2 > r_1$ unless otherwise noted. The results, of course, may be applied to the inverse case, $r_2 < r_1$, by proper interpretation.

The use of the Hohmann transfer ellipse to achieve rendezvous is investigated in 31, 231, and 250. The required phase angle, β_H, for a given radius ratio, k, is derived in 231 and a plot of phase angle vs. radius ratio is given. It is shown that, for $r_2 > r_1$, the target is always rendezvoused with on the first revolution of the target body; however, for $r_2 < r_1$, multiple revolutions of the target body must be allowed for small k. It is shown in 31 that β_H is limited to the range $0 \le \beta_H \le \pi\,[1 - (\frac{1}{2})^{3/2}\,] \simeq 116.36$ deg.

The required delay time to make the Hohmann trip, for a given phase angle, is derived in 231, and it is shown that for fixed r_1 and r_2 the total rendezvous time increases linearly with increasing $\Delta\beta$. The maximum delay for a Hohmann transfer is equal to the synodic period, and therefore the rendezvous time for the Hohmann maneuver increases without bound for $r_1 \to r_2$, necessitating the use of other transfer methods (250).

As an alternative to the Hohmann maneuver, the bi-elliptic transfer is investigated in 17, 31, and 250. All the analytical results obtained are contained in 31, although many were derived earlier. It is shown that full $\Delta\beta$ coverage, i.e., $0 \le \Delta\beta \le 2\pi$, is available only for outer bi-elliptic transfers ($r_1 > r_2$). For inner bi-elliptic transfers ($r_1 < r_2$) the upper limit occurs for $r_1 \to 0$ and is given as a function of r_1 and r_2 in 31. For $r_1 \ge r_1$, $\Delta\beta_{max}$ is slightly less.

The bi-elliptic rendezvous time decreases linearly with increasing $\Delta\beta$ from $t_H + 3T_2/2$ to $t_H + T_2/2$ for the outer bi-elliptic, and from $t_H + T_1/2$ to a lower limit determined by the value of $\Delta\beta_{max}$ for the inner bi-elliptic. A comparison of the Hohmann and bi-elliptic rendezvous times shows that the Hohmann maneuver is always faster than the bi-elliptic maneuver for $r_1 < r_1 < r_2$. For $r_1 < r_1$ the bi-elliptic maneuver is faster than the Hohmann; the break-even point, i.e. the point where the Hohmann and bi-elliptic rendezvous maneuvers require equal times, for the inner bi-elliptic maneuver can be shown to correspond to $\Delta\beta = \pi\,[1 - k^{-3/2}]$ and therefore the inner bi-elliptic is faster than the Hohmann for all $\Delta\beta \ge \pi\,(1 - k^{-3/2})$. For $r_1 > r_2$, the Hohmann maneuver requires less time than the bi-elliptic for $\Delta\beta \ge 3\pi\,[1 - k^{-3/2}]$; it requires less time for all $\Delta\beta$, $0 \le \Delta\beta \le 2\pi$, for $k \ge 9^{1/3} \simeq 2.08$. Coupling this to orbit transfer results (107), it is seen that the Hohmann rendezvous maneuver requires less time and a greater total ΔV than the outer bi-elliptic maneuver if $k > 15.582$. It is shown in 31 and 250 that no benefits may be gained (either in ΔV or t) by waiting before initiating a bi-elliptic rendezvous maneuver. Further, a ΔV penalty is associated with waiting before initiating the transfer maneuver, if $r_1/r_1 < 15.582$.

The question of parking in ellipse I or II (i.e., the first or second transfer ellipse of the bi-elliptic transfer) is considered in 31. Equations for the rendezvous time are derived for parking in ellipse II, and break-even points in $\Delta\beta$ (points where Hohmann time is equal to bi-elliptic time) are expressed as functions of r_1, r_2, and n, the number of revolutions in the parking ellipse. It is shown that the case of parking in ellipse I does not yield convenient equations describing the rendezvous. However, parking in ellipse II is more desirable in terms of both ΔV and t.

For parking in ellipse II with $r_1 < r_2$ the Hohmann maneuver requires less time for $\Delta\beta \le (2n + 1)\,\pi\,[1 - k^{-3/2}]$; with $r_1 > r_2$ the Hohmann is faster for $\Delta\beta \le (2n + 3)\,\pi\,[1 - k^{-3/2}]$. If $k \ge [(2n+3)/(2n+1)]^{2/3}$, then the Hohmann is faster than the outer bi-elliptic for all β, i.e., $0 \le \beta \le 2\pi$. This expression gives a convenient way of determining the maximum number of revolutions in the parking ellipse for the bi-elliptic maneuver with parking to be competitive with the Hohmann maneuver. Convenient plots of $\Delta\beta$ vs. t (31), with lines of constant ΔV and n for the Hohmann and bi-elliptic maneuvers, are shown to be useful in performing trade-off studies for a given rendezvous mission. The bi-elliptic maneuver with parking in ellipse II is also shown to be an effective method of reducing t for a given ΔV capability.

A further possible rendezvous mode is the semitangential transfer. Only inner semitangential transfers need be considered (207). To define this transfer mode it is necessary to specify the velocity at pericenter of the transfer ellipse, V_P. Equations describing the total ΔV, the true anomaly on the transfer ellipse at both intersection points with the final orbit, the phase angle $\Delta\beta$, and the rendezvous time, are developed as functions of r_1, r_2, V_P, and the number of revolutions in the transfer orbit. It is shown that the least effective mode for minimizing t for a given ΔV capability is the semitangential transfer. Rendezvous at the second intersection point is shown to be more desirable than at the first. Semitangential maneuvers are shown to be effective in reducing the total rendezvous time for a given $\Delta\beta$ at the expense of ΔV. For higher values of ΔV, the semitangential maneuver with parking offers a ΔV saving over the simple bi-elliptic transfer at the expense of rendezvous time.

A sub-case of rendezvous between coplanar circular orbits, the case where $r_1 = r_2$, may be considered separately. Although the previous results apply in the limit as $r_2 \rightarrow r_1$, this specific case of equal orbits has been studied in 29, 142, and 308.

The variational equations are developed for two-impulse rendezvous and one-impulse intercept, as functions of the number of revolutions in the transfer orbit, and it is shown (142) that the intercept and rendezvous trajectories for this case are equivalent. It is further shown that the use of tangential, apsidal impulses constitutes a nonoptimal singular solution to the necessary equations for a given number of revolutions. However, for most cases, these conditions are near optimal, and an analytic solution may be obtained using a small parameter expansion about the singular solution. It is shown that the thrust angle, ϕ, in any case, is small (29, 142).

Noncoplanar

Rendezvous between noncoplanar circular orbits has received less attention than the coplanar problem due to the increased difficulty associated with consideration of the out-of-plane motion. Three transfer methods are proposed in 250: (1) Hohmann with plane change (at nodal points), (2) bi-elliptic, initiated at nodal points, with plane change at r_{max}, and (3) the modified Hohmann (in-plane Hohmann to r_2, followed by plane change impulse at line of nodes).

The definition of $\Delta\beta$ is analogous to that in the coplanar case; here $\beta = \Omega_2 - \Omega_1$, where Ω_i (i = 1, 2) is the angular displacement from the line of nodes for the target and interceptor, respectively, and $\Delta\beta = \Delta\beta$ (mod 2π). For the Hohmann maneuver, $\Delta\beta$ must be zero at the point where the interceptor crosses the line of nodes, and therefore would lead to excessively long waiting times for most cases. The relationship between $\Delta\beta$ and $\Delta\beta_\Omega$ for any given configuration is given in 250. The velocity requirement for bi-elliptic rendezvous, with the plane change occurring at r_{max} for any fixed orbit inclination, is given in 250. It is shown that the modified Hohmann results in definite penalties over the Hohmann for even small changes in the phase angle.

The equations for optimal plane change and the total velocity requirement for bi-elliptic rendezvous are given as functions of r_1, r_1 (or $\Delta\beta$), r_2 and i in 17. It is shown that for $r_1 = r_2$, the equations reduce to those for the Hohmann transfer with plane change. Under the assumption that the interceptor is at the line of nodes when the rendezvous command is given for the bi-elliptic maneuver, and at the line of nodes when rendezvous is achieved for the modified Hohmann -- which could lead to errors in rendezvous time as large as $\frac{1}{2}$ T_1 for the bi-elliptic maneuver, and $\frac{1}{2}$ T_2 for the modified Hohmann--generalized results are given in 17.

For $k \approx 1$, the outer bi-elliptic transfer with optimum plane change yields slightly lower ΔV than the Hohmann transfer with optimum plane change for $1 \leq r_1/r_1 \leq 2$; the range of k increases for increasing plane change, i. The modified Hohmann ΔV is always greater than or equal to the Hohmann ΔV (equal for $i = 0$). Within the assumptions made, the same break-even analysis on the phase angles and the regions of interest hold in the noncoplanar case as in the coplanar case.

Figures 20 and 21 summarize the comparison of the bi-elliptic rendezvous maneuver with the modified Hohmann maneuver for both the coplanar and noncoplanar cases. Note that the modified Hohmann maneuver, as defined for the noncoplanar case, is equivalent to the Hohmann for $i = 0$.

Figure 20 is a plot of r_1/r_1 vs $\Delta\beta$ for constant k. The regions discussed in the text are illustrated on the figure. Figure 21 illustrates ΔV for the modified Hohmann and outer bi-elliptic rendezvous maneuvers with equal rendezvous times normalized with respect to V_{c1}, as a function of k, $\Delta\beta$ and plane change angle, i. The regions where the bi-elliptic requires less ΔV than the modified Hohmann is also illustrated. Note that this region is bounded on the right by the vertical line through $\Delta\beta = 360$ deg.

<u>General Orbits</u>

The problem of rendezvous between general orbits is exceedingly more complex than circle-to-circle rendezvous. Several methods have been proposed for rendezvous between general orbits. One of the earliest was a four-impulse method suggested in 308. The scheme involves: (1) plane change to target orbit plane, (2) transfer orbit is chosen to be tangential to the target orbit at a point (a numerical method is outlined for generating a family of such orbits), (3) period of interceptor orbit is altered so that rendezvous occurs at the tangency point in some arbitrary time, and (4) tangential ΔV is applied at the rendezvous point to achieve rendezvous. A general injection technique, similar to the above, is discussed in 116. The phasing technique (chasing, looping, epoch changing, or impulse splitting) is discussed in 108, 116, and 267. This consists of using period-changing impulses, at apocenter or pericenter, to satisfy the time constraint.

The problem of rendezvous between a circle and a nonintersecting, coplanar ellipse (or ellipse-to-circle), using N tangential impulses, is considered in 282. It is shown that the ΔV for this case is equal to the Hohmann transfer ΔV and that the number of impulses may be reduced to, at most, three. Necessary and sufficient conditions that rendezvous be possible at any given time are derived, and an expression is given for the lowest rendezvous time which can be achieved by any scheme which also results in minimum ΔV.

The use of the impulse function (see Appendix I) is examined in 25 and 202 for applications to the general orbit rendezvous problem. Optimum rendezvous trajectories in the vicinity of the optimum transfer are determined in 25 by this method. The use of the impulse-splitting technique is also examined as a means of extending the range of β in which rendezvous may be accomplished, using the ΔV required for optimum two-impulse transfer. Various other numerical studies have been performed for interplanetary rendezvous; however, no general conslusions may be drawn from them.

Time-Fixed Rendezvous

By definition, the category of fixed-time rendezvous encompasses a narrower field than the time-limited case. The desire here is to determine the optimum rendezvous trajectory between orbiting bodies when time is prescribed to be a fixed value. There is some overlap between the time-fixed and time-limited cases. There are a number of fixed-time rendezvous papers which include trade-off studies of ΔV and t, and as such could be considered time-limited. Most of these, however, are for particular problems and general conclusions can not be deduced from the results. Time-fixed rendezvous problems are subclassified according to (1) neighboring orbits and (2) general orbits.

Neighboring Oribts

An important, recent development in the solution of this problem is described in 238. The approach in 238 is geometric in nature and consists of applying Lawden's theory of the primer vector to the equations of motion, linearized about an intermediate orbit. Specifically, the problem treated is rendezvous between coplanar circular orbits with the equations of motion linearized about an intermediate circular reference orbit. For this case the linearized equations of motion and the adjoint equations are uncoupled and analytically simple. The solution to the primer vector equations is given by:

$$\lambda = A(\cos \tau + 2B)$$

$$\mu = A(-2\sin \tau - 3B\tau + C)$$

where λ and μ are the radial and circumferential components of the primer, respectively, τ is nondimensional time, and A, B, and C are arbitrary integration constants. These equations are a parametric description of the primer locus in the λ - μ plane. For B = 0, the locus is an ellipse, first shown by Lawden (172); for B $\neq$ 0 the locus is a multiple-loop, cycloid-like curve. Typical plots of the primer locus are given in 238 for various values of B. The constant C merely determines the relative position of the locus with respect to the λ axis and A is a normalizing constant. Utilizing this locus, methods for generating two-, three-, and four-impulse optimal solutions to the fixed-time rendezvous problem are developed. It is shown that only $B > 0$ need be considered (solutions for $B < 0$ are reciprocal solutions of $B > 0$); in addition, for $B > 2/3$ the locus contains no loops and only two-impulse solutions are possible. In all cases, the boundary value problem is handled separately.

The existence of optimal four-impulse solutions was first demonstrated in 302. In the four-impulse case the intermediate impulse times and thrust angles may be obtained directly from the primer locus. It is shown that: (1) the impulse times are symmetric about the midpoint in time and (2) the first and fourth, and the second and third impulses have equal radial components and tangential components which are the negatives of each other. Optimal four-impulse solutions exist only in the range, $1 < t < 2.5$, where t is the rendezvous time measured in reference orbit periods. There is another group of four-impulse solutions in the range, $0.46 \leq t \leq 1.5$; however, these are shown to be nonoptimal for the circle-to-circle case. It is further shown that terminal coasts are not allowable for four-impulse solutions in the circle-to-circle case.

Optimal three-impulse soltuions lack the symmetry found in the four-impulse solutions. While this makes the intermediate impulse times more difficult to evaluate, it also increases the number of three-impulse primer solutions satisfying the necessary conditions. This results in a large part of the plane of reachable states (Fig. 22) occupied by three-impulse solutions. Terminal-coast three-impulse solutions exist as a subset of optimal four-impulse solutions. As in the case of four-impulse solutions, the thrust angles are obtained directly from the primer locus.

A normalized plot of the reachable final state variations is shown in Fig. 22 (238). Here the time is measured in units of the reference orbit period; $\delta\theta$ and δr correspond to the normalized variation at rendezvous from the reference orbit. The regions for optimal two-impulse, two-impulse with terminal coasts, three-impulse, three-impulse with terminal coasts, and four-impulse trajectories, are shown. The Hohmann with final coast region is also illustrated. The great extent of the region of optimal multiple-impulse solutions is quite evident from the figure.

As an example of the use of this figure, consider rendezvous between coplanar circular orbits of radii $r_1 = 0.95$ and $r_2 = 1.05$. This gives $r_0 = (r_1 + r_2)/2 = 1.0$, and $\delta r = (r_2 - r_1)/r_0 = 0.1$. Assume that, at the instant the rendezvous command is given, $\beta = 0.35\pi$ and that the desired rendezvous time is one reference orbit period. It is shown in 238 that $\delta\theta = \beta - 3\tau\delta r/4$, where τ is the time in radians, for the circular orbit case. This gives $\delta\theta = 0.2\pi$, and $\delta\theta/\delta r = 2\pi$. For $\delta\theta/\delta r = 2\pi$ and $t = 1$, Fig. 22 shows that optimal rendezvous is accomplished using three impulses and that the normalized cost is approximately twice the Hohmann transfer cost. This sample calculation is illustrated in Fig. 22.

General Orbits

The problem of fixed-time rendezvous between general orbits, as for the case of fixed-time transfer, has received little attention beyond numerical studies relating to particular problems of interplanetary rendezvous. Optimal time-fixed rendezvous in the vicinity of the optimum transfer trajectory has been studied in 25 and 202 using the impulse function technique mentioned previously. The possibility of reducing the two-impulse, fixed-time rendezvous requirements by employing intermediate impulses has been investigated in 52, 297, and 298. For a given launch time and transfer time, the two-impulse rendezvous problem is completely determined, whereas in the three-impulse problem, certain variables related to the position and timing of the intermediate impulse must be optimized. In the most general case this is a four-parameter optimization. A method of generating optimal three-impulse rendezvous trajectories, under the assumption that intermediate impulses are tangential, is developed in 297; for this case a two-parameter optimization is required. The previous method is expanded in 298 to consider nontangential impulses, although no out-of-plane component is allowed; for this case a three-parameter optimization is necessary. In both cases, the starting solution is based on the two-impulse rendezvous trajectory, a procedure which breaks down for central angles greater than 2π. For larger central angles a starting guess must be employed.

A general method, i.e., a four-parameter optimization for generating three-impulse rendezvous trajectories, is developed in 52. Starting solutions in this method are based on the low-thrust solution to the problem. A low-thrust analog is shown to be useful in determining the proper number of impulses on an optimal impulsive thrust trajectory. The methods described above were applied to Mars-Earth rendezvous, and three-impulse trajectories were shown to be more economical than two-impulse trajectories for long-time, long-angle transfers. Sensitivity to variations in the launch date was also shown to be less pronounced for three-impulse trajectories.

The concept of the primer vector (172) has been extended to nonoptimal trajectories in 183 and 313 and has resulted in necessary and sufficient conditions for locally optimal two- and multi-impulse trajectories. A method for generating optimal multi-impulse trajectories based on the two-impulse primer vector solution is presented in 129.

DISCUSSION OF RESULTS

One of the objectives of this study was to isolate problem areas, within the subject of impulsive trajectories, in which current knowledge is incomplete and toward which the application of future research should be devoted. In addition it was intended that spaceflight applications would be enumerated in which such research would be beneficial.

Three specific problem areas in which additional research is necessary have emerged in this investigation: fixed-time trajectories, optimal multi-impulse modes, and optimal rendezvous. The applications for which it is felt such research would provide immediate and long-range benefits are: space rescue, operations in near-Earth space, disorbit to a specified impact point, interplanetary probes and landers, and abort from terrestrial and interplanetary missions.

Some of these applications do not in themselves define specific impulsive trajectory problems. For example, space rescue is a general subject within which a variety of specific problems can be defined. One such problem is that of rendezvousing with an orbiting spacecraft by a direct launch from Earth, followed by a return to Earth or subsequent rendezvous with a space station. The same problem can be formulated with both launch from, and return to, a space station as boundary conditions.

Each of the other applications offers a similar degree of flexibility in the definition of specific problems. The point is that these applications merit priority because they are fundamental to space flight. Thus, the problem areas which were to be isolated in this study should relate to these applications in a real sense, or else research in impulsive trajectories will reduce to academic exercises. Some explanation of how the three problem areas cited above relate to the enumerated applications is therefore in order.

In the early phases of the study, it was anticipated that time constraints would constitute an important classification by which papers could be readily differentiated and categorized. But it became evident after a substantial percentage of the papers had been reviewed that, in the overwhelming majority of papers, time is left entirely open, i.e., no time constraint is imposed whatsoever. Therefore, major categories of time-fixed and time-free constraints could not be used.

One characteristic of the five space flight applications listed earlier is that in each case time is a significant if not a dominant consideration. It is difficult to conceive of a space rescue problem in which at least an upper limit on time is not imposed. In the abort, orbital operations, and interplanetary applications, time constraints are likely if meaningful problems are to be defined. Only in the disorbit application do time-open problems play an important role, and even here the desirability of a time limit is not without foundation.

Use of multi-impulse modes in the solution of trajectory problems is a topic of considerable current interest. Some multi-impulse modes, such as the bi-elliptic transfer, are well documented and easily understood. More recent developments in this area are not as universally known, but have been shown to yield benefits in some applications which indicate the desirability of additional research.

The characteristic which most effectively describes multi-impulse modes is flexibility. Use of multiple impulses invariably affords the opportunity to optimize a trajectory as opposed to determining a trajectory which merely satisfies boundary conditions in a given problem. Widening of launch opportunities and reductions in fuel consumption are the results, and since these are objectives common to all space flight problems, further research in this area is clearly desirable.

It was pointed out in the section on rendezvous, that in terms of the number of papers devoted to this problem alone, rendezvous by impulses is not a well documented subject. Optimal rendezvous remains virtually unexplored. But several of the applications noted above necessarily involve rendezvous, namely, rescue, space station operations, and interplanetary landers; and rendezvous may also play an important part in the remaining applications. In particular, it should be stressed that fixed-time, multi-impulse rendezvous is perhaps the most fruitful area of research in impulsive trajectories.

The basic recommendation that has come out of this investigation then, is that the promotion of research in three specific areas can lead to solution of meaningful problems in priority applications within space flight. These problem areas are fixed-time trajectories, optimal multi-impulse modes, and optimal rendezvous. The difficulty associated with pursuit of such problems should not be minimized. Indeed it is this very fact which has discouraged past research in the areas in question. However, by drawing upon the body of existing knowledge, which as evidenced in the survey is considerable, and by utilizing the promising techniques described in Appendix I (or by devising new ones), progress can undoubtedly be made.

As an additional recommendation, optimal solutions to space flight problems, especially in the sense of minimizing fuel, should be pursued. The literature is full of heuristic schemes, the chief virtues of which are simplicity in analysis and implementation. Conditions of optimality, even when they result in seemingly academic solutions, can provide important information, and should be applied. A good case in point is the use of parabolic transfer arcs which occur so frequently in time-open problems. Even though the solution itself is completely impractical, the knowledge that infinite time is optimal, which is not the case in all time-open problems, constitutes useful information. Furthermore, the lower bound on fuel consumption, which can be readily calculated for such a solution, is also useful. In other cases, where the optimal solution is not impractical, the tradeoff between optimal performance and complexity in implementation should be ascertained before a nonoptimal approach is adopted.

APPENDIX I

Methods of Analysis

This appendix is devoted to a brief exposition of the methods of analysis which have been used in optimal impulsive trajectory problems. These methods are taken up in a roughly chronological order, the last few being the most recent methods and the most promising approaches in the solution of future problems.

Some of the simpler impulsive trajectory problems have been successfully analyzed by application of the theory of ordinary maxima and minima. By successive differentiation of the system of algebraic equations which describes a problem it is sometimes possible to determine the optimal placement, direction and magnitude of impulses (272). In complex problems, however, this approach breaks down. It was shown in 170 that, for time-open transfer between coplanar orbits, a system of 7n - 3 equations (n is the number of impulses) in as many unknowns must be solved. No numerical evidence of the solution of these equations has been presented.

Another technique which has been applied with limited success is the hodograph method, as described, for example, in 2. The velocity and acceleration hodographs provide a unique tool for understanding the dynamics of impulsive trajectories (22). However, results obtained by this method have thus far been confined to time-open, two-impulse transfer between orbits or terminals (6, 7, 269).

The indirect methods of the calculus of variations or Pontryagin's Maximum Principle can be applied to optimal impulsive trajectory problems. An analytic formulation of the two-impulse transfer problem, including necessary conditions for optimality, was derived in 154, and the lengthy system of solution equations was reduced in 54. An extension was made in 235 to present additional constants of the motion so that an iterative numerical method can be more easily implemented.

One numerical method which has proved successful in obtaining solutions to the time-open, two-impulse transfer problem is the adaptive steepest descent program described in 202. Here an "impulse function", I, is defined and minimization of ΔV is accomplished by taking steps in the gradient direction in the three space describing the two-impulse, time-open problem, i.e., the steepest local path to decrease I, until a relative minimum is reached. Results obtained by this approach appear in 202 and 203. Another numerical method, quasi linearization, has been used successfully in solving a difficult two-point boundary value problem (315). The application in 315 was a three body problem involving a two-impulse trajectory.

A significant step in the direction of achieving analytical solutions to impulsive trajectory problems was taken in 45 and 34. By replacing time with impulse as the independent variable in a variation-of-parameters formulation, some useful new results were obtained concerning the coasting arcs which join impulse points on an optimal impulsive trajectory. This technique has been utilized in several subsequent studies of time-open orbit transfer (35, 212, 213, 62, et al), and significant progress has been achieved in each case. A concise description of both the variation-of-parameters formulation and adaptive steepest descent appears in 148.

The variation-of-parameters approach has also led to some geometric concepts which provide new insight into optimal impulsive trajectories. The "spools" first described by Contensou in 45 have been further studied in 81, 200, and 205. A similar geometric interpretation of optimal impulsive trajectories was utilized in 40 with respect to time-open disorbit.

A method which has seen much recent use can be described as primer vector maximization. The concept of the primer was first proposed by Lawden in 156, and is described in detail in Chapters 3 and 5 of 172. Conditions on the primer, which is the adjoint vector associated with the velocity components, can be used to describe an optimal trajectory, either with finite thrust periods or impulses. The technique was applied to the time-open, coplanar, ellipse-to-ellipse transfer problem in 81, to the two-impulse noncoplanar problem in 185, and has since found application in optimal, multi-impulse, transfer and rendezvous analyses. In particular, extension of the primer concept to nonoptimal trajectories in 183 has resulted in the establishment of necessary conditions for the inclusion of additional impulses on a trajectory. An iterative, numerical application in which results for a fixed-time rendezvous problem were obtained is presented in 129.

A sufficiency test for fixed-time impulsive transfer trajectories was developed in 313. This test shows that two-impulse transfers which satisfy the primer vector conditions are locally optimal. However, multiple-impulse trajectories must satisfy additional conditions given in 313 to be locally optimal. These additional conditions are related to the Jacobi test of the classical calculus of variations.

APPENDIX II

The Impulsive Approximation

The impulsive approximation consists of replacing finite-thrust powered phases of finite duration by instantaneous changes of velocity. The results of this study are valid, for practical application to real problems, only to the extent that the impulsive approximation itself is valid. The nonexactness of this approximation in realistic cases, and in particular the performance penalty associated with the use of finite thrust instead of impulses has been shown (243) to be due to the effects of: (1) gravity gradient, and (2) nonconstant thrust direction during thrusting periods.

Lawden (163) has considered the problem of escape from circular orbit using high thrust. He applies the perturbation equations about the optimal impulsive solution to the problem, assuming constant thrust acceleration, a, and derives the solution through second order in λ, where $\lambda = 1/a$. The total ΔV penalty, to second order in λ, is $0.001615\ \lambda^2 V_c$, for an optimal thrust angle program. The problem of escaping from a circular orbit with finite velocity at infinity is treated in 184 in a similar manner. Here the ΔV penalty is derived as a function of V_∞. In 292 and 306, similar methods are applied to Hohmann-type transfer maneuvers. An examination of the changes in the transfer orbit elements due to finite thrusting time for various thrust levels is undertaken in 306 and it is concluded that any error analysis based on the impulsive approximation is of doubtful value.

Numerical examinations of the effect of finite burn time for coplanar maneuvers using non-optimal steering programs are found in 87 and 96; a graphical display of the results is included in both. A definition of high-, intermediate-, and low-thrust systems is given in 87 in terms of the change in V_∞ for escape or capture problems utilizing finite-thrust with the same ΔV expenditure as the impulsive soltuion.

The most complete analysis of the impulsive approximation is found in 243. In that study, an improved approximation, allowing discontinuities in both position and velocity, is developed for the time open case and shown to be useful in evaluating the ΔV penalty for finite-thrust solutions relative to the impulsive solution. A method for determining whether the impulsive approximation is valid, for a given tolerable performance penalty, in terms of the burning time and the Schuler frequency (μ/r^3) is presented. An upper bound on the performance penalty is also given. This method has been successfully applied and shown to be quite useful for error analyses.

It may be concluded that impulsive trajectory analysis provides a good estimate to the fuel requirements of most missions for a wide range of thrust-to-weight ratios for high-thrust chemical and nuclear rockets. However, any error analysis based on impulsive trajectories is of doubtful value. The generalized impulsive approximation developed in 243 has been found to be useful in providing a more accurate estimate of the trajectory parameters and fuel requirements and should be used for detailed trajectory analysis.

APPENDIX III

Singular Arcs

by T. N. Edelbaum

Singular arcs are a mathematical curiosity which have recently aroused considerable interest because of possible practical implications for space flight. A minimum-fuel trajectory containing singular arcs has subarcs where the thrust assumes values in between its maximum and minimum values. Until recently, it was believed probable that such subarcs do not occur in minimum-fuel rocket trajectories. However, Robbins has recently demonstrated (244) that singular arcs are minimizing for some end conditions for fixed-time coplanar trajectories in an inverse-square field. These trajectories may be approximated arbitrarily closely with a finite number of impulses (225). However, the number of impulses required to obtain near-minimum fuel consumption is not known. This Appendix will briefly review the known results on singular arcs as well as their significance.

The theory of singular arcs becomes quite simple if the gravitational field is linear. In such fields, singularity corresponds to non-uniqueness and represents cases where the minimum fuel consumption may be realized by many different trajectories with different numbers of impulses or with finite thrust arcs. It is shown in 224 and 265 that in such fields the minimum fuel consumption for the singular case can be realized with a number of impulses no larger than the number of specified terminal conditions.

One important example of a linear field is field-free space far from any massive body (70, 177). Here singular arcs arise when the terminal position is unconstrained. These singular arcs may be replaced by a single impulse with the same fuel consumption. The other important case of singular arcs in linear gravitational fields occurs for time-open transfers in the close vicinity of a circular orbit. In this case, there is a coplanar singular arc treated in 206, and a noncoplanar singular arc treated in 35, 81, and 200. Breakwell has recently shown that a slightly nonlinear version of the latter problem requires no more than three impulses (35).

In nonlinear fields, singular arcs no longer correspond to nonuniqueness and the theory becomes much more complicated (93, 137, 225). For example, 137 demonstrates that junctions between singular and nonsingular arcs for minimum-fuel rocket problems ideally require an infinite number of closely spaced thrusting periods and coasting periods.

All the known results for nonlinear fields are for coplanar inverse-square fields. The original result which stimulated most of the work reported herein was Lawden's discovery and analytic integration of the time-open singular arc generally called the Lawden spiral (168, 169, 172). This singular arc has many interesting properties. One of these is that it is a locus of infinitesimal two-impulse transfers having zero coasting arc-length (34, 47, 81, 212).

A fair amount of new research in optimal control theory has been devoted to proving that the Lawden spiral is not minimizing (88, 135, 136, 137, 144, 242). An incorrect version of such work is given in 134.

While this new work has shown the Lawden spiral to be nonoptimal, the fixed-time coplanar singular arcs considered in 172 remain for consideration. One of these arcs corresponding to free central angle can be integrated analytically (72), as can the Lawden spiral. In general, numerical methods must be used (242, 78). Reference 242 demonstrates that some of these singular arcs are actually minimum-fuel trajectories. Included among these is a portion of the angle-open arc duscussed in 72.

In summary, a minimum-fuel rocket trajectory may ideally require either a singular arc or an infinite number of impulses (61). The number of impulses required in practice to approximate the fuel consumption of these trajectories remains to be determined.

GLOSSARY OF IMPULSIVE TRANSFER TERMS

1. apocenter - point on an orbit which is most distant from the force center
2. circumferential - radial component is zero
3. coaxial ellipses - ellipses whose major axes are co-linear, either aligned (pericenters on the same side) or opposing (pericenters on opposite sides)
4. conjunction ratio - ratio of the radius of the conjunction point of bi-elliptic transfer ellipses to the initial radius
5. cotangential - the condition in which an orbit is tangent to two connected orbits simultaneously
6. disorbit - maneuver in which a body is removed from an orbital condition and intercepts a final radius (usually coincident with the upper limit of the sensible atmosphere)
7. domain of maneuverability - region in state space which can be reached by optimal application of a control parameter
8. down-range angle - great circle arc traversed in the plane of initial motion
9. entry angle - path angle at moment of atmospheric entry
10. exterior conjunction - bi-elliptic transfer with conjunction ratio greater than the final to initial radius ratio; also known as outer bi-elliptic
11. generalized Hohmann or tilted Hohmann - three-dimensional version of the Hohmann transfer using circumferential, apsidal impulses
12. interior conjunction - bi-elliptic transfer with conjunction ratio less than the final to initial radius ratio; also known as inner bi-elliptic
13. interior impulse - impulse not located at a boundary point of the trajectory, i.e., in the interior of the trajectory

14. lateral range angle - great circle arc traversed normal to the plane of initial motion

15. path angle - measured between velocity vector and local horizontal, positive outward from focus

16. pericenter - point on an orbit which is closest to the force center

17. phase angle - instantaneous angular separation between an interceptor and target vehicle, measured positive in the direction of motion

18. primer vector - a vector formed of components which are adjoint variables associated with the components of the velocity vector; the concept was introduced by Lawden in his variational formulation of the optimum rocket trajectory problem

19. semitangential - designates a transfer orbit tangent to only one of two connected orbits

20. switching point - point on an optimal trajectory composed of subarcs at which a control parameter exhibits a "jump" or discontinuity

21. terminal coast (initial or final) - coasting arc which forms an extension of the trajectory before the initial impulse or beyond the final impulse; an initial coast is equivalent to a delayed departure and a final coast is equivalent to an early arrival

22. thrust angle - measured between thrust vector and local horizontal, positive outward from focus

23. transfer through infinity - a transfer involving parabolic intermediate conditions; escape to infinity on a parabola permits changes in direction to be performed by one or more infinitesinal impulses at infinity; (sometimes referred to as "bi-parabolic" transfers)

24. useful angle - angular range between tangent and local horizontal directions within which thrust must be directed. (see Fig. 16)

LIST OF SYMBOLS

r	Radius
V	Velocity
ΔV	Characteristic velocity
t	Time
a	Semi-major axis
e	Eccentricity
ℓ	Semi-Latus rectum
i	Inclination
ω	Argument of pericenter
Ω	Argument of the ascending node
v	True anomaly
θ	Central Angle
ϕ	Thrust angle
γ	Path angle
β	Phase angle
T	Orbital period
k	Radius ratio r_2/r_1
P	Radius of pericenter
A	Radius of apocenter
Γ	Turning angle between hyperbolic asymptotes

x	Ratio of pericenter radii, P_1/P_2
y	Ratio of pericenter to apocenter, P_1/A_1
ρ	$\sqrt{\dfrac{\min(P_1, P_2)}{\max(P_1, P_2)}}$
σ	$\sqrt{\dfrac{\min(P_1, P_2)}{\max(A_1, A_2)}}$
I_{sp}	Specific impulse
μ	Gravitational parameter

Subscripts

o	Reference Orbit
1	Initial Condition
2	Final Condition
∞	At Infinity
esc	Escape
i or ii	Intermediate
c	Circular
opt	Optimum
H	Hohmann
S	Synodic Period
p	Pericenter

TABLE I

Bibliography Entries Not Included in Survey

Not Acquired:	13, 15, 23, 43, 48, 49, 98, 99, 103, 131, 139, 140, 141 161, 175, 182, 222, 223, 229, 230, 246, 254, 258, 271, 305
Reviewed, But Not Referenced in Text:	1, 3, 4, 5, 10, 19, 21, 30, 41, 44, 60, 65, 68, 73, 76, 85, 91, 95, 118, 119, 120, 122, 128, 130, 133, 149, 157, 159, 160, 164, 165, 171, 173, 189, 191, 208, 210, 216, 218, 221, 226, 228, 232, 233, 245, 252, 255, 257, 262, 268, 284, 307, 309
Not Available in Translation:	39, 46, 102, 195, 196, 197, 198, 200, 201
Terminal Phase Rendezvous:	14, 63, 64, 69, 89, 110, 111, 181, 256, 261, 279, 280, 281
Correction Maneuvers:	33, 71, 86, 145, 158, 166, 167, 236, 254, 264, 265
Interplanetary Applications:	36, 67, 117, 132, 162, 190, 211, 247, 248, 285
Computational Techniques:	16, 21, 36, 44, 85, 190, 247

TABLE II

Location of Points in Figure 13

Point	ρ	i_2, deg	slope	on	curve
E	1.0	60	radial		EB
E_1	1.0	60.185	radial		E_1 L
B	0.33333	0	circumferential		EB
C	0.28942	0	circumferential		LC
A	1.0	0	circumferential		AQ
L	0.388	37.54			
Q	0.37194	34.043			
K	0.66023	55.6			
curve AL	0.534	36.68			
	0.447	36.88			

Point	σ	i_2, deg
E_1'	1.0	60.185
Q'	0.37194	34.043
A	1.0	0
curve ARO	0.707	47.92
	0.529	54.00
	0.3089	58.488
	0	60

Note: point Q' is symmetric with respect to point Q
point E_1' is symmetric with respect to point E_1
curve AQ' is symmetric with respect to curve AJQ
curves E_1 L and LC are tangent at L

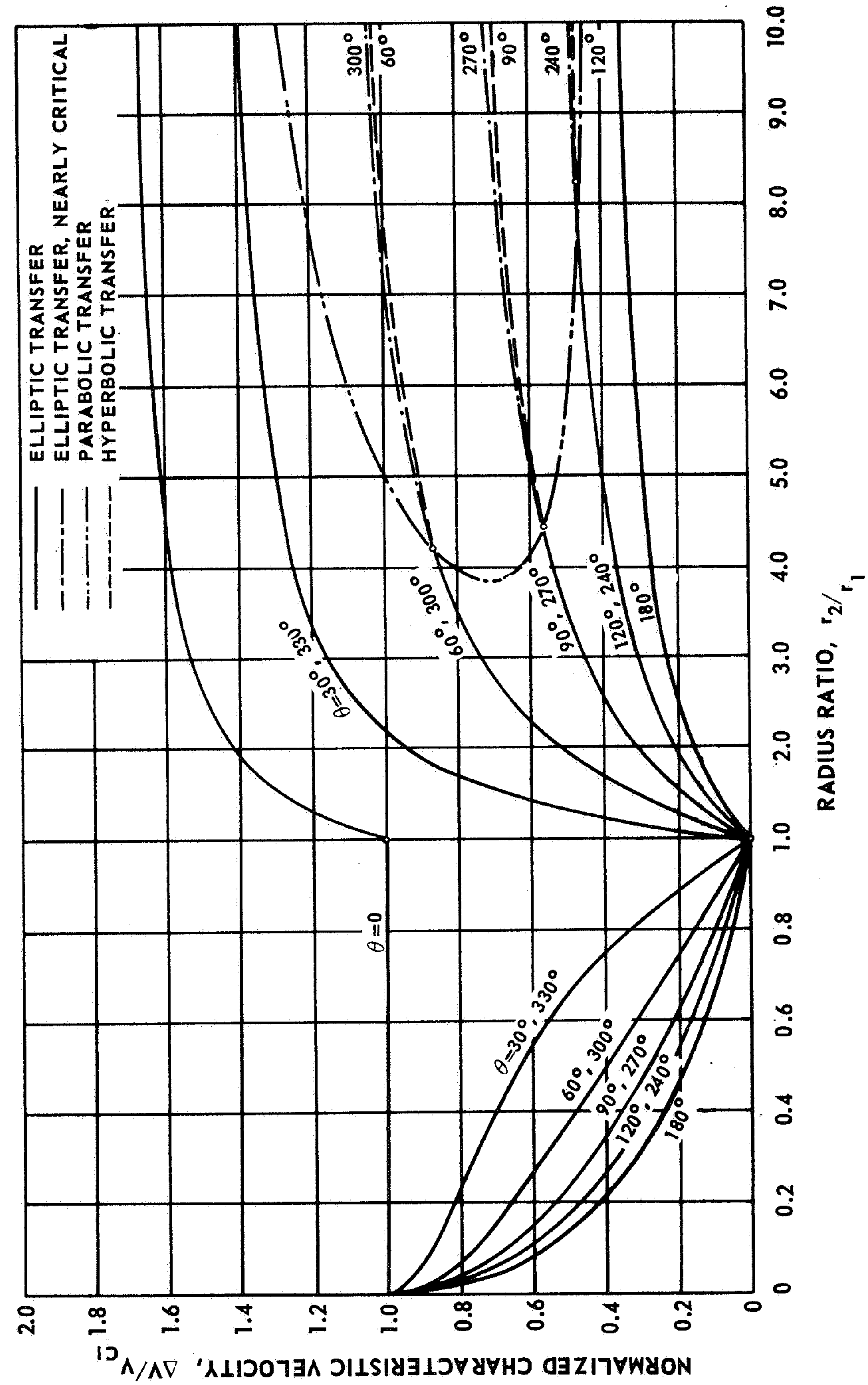
TABLE III

Optimal Transfer Modes in Fig. 13

M in zone	Optimal Mode
AJB	Generalized Hohmann
AJE	Either Generalized Hohmann or Three-Impulse
BJQLC	Either Generalized Hohmann or through Infinity
JQLK	Any mode can be optimal
EE_1LK	Either Three-Impulse or through Infinity
below E_1LC	Through Infinity

N in zone	Optimal Mode
$AQ'CB$	Generalized Hohmann
$Q'AR$	Either Generalized Hohmann or Three-Impulse
RAE_1'	Three-Impulse
$CQ'O$	Either Generalized Hohmann or through Infinity
$Q'RO$	Any mode can be optimal
$ORE_1'D$	Either Three-Impulse or through Infinity
below ODE_1'	Through Infinity

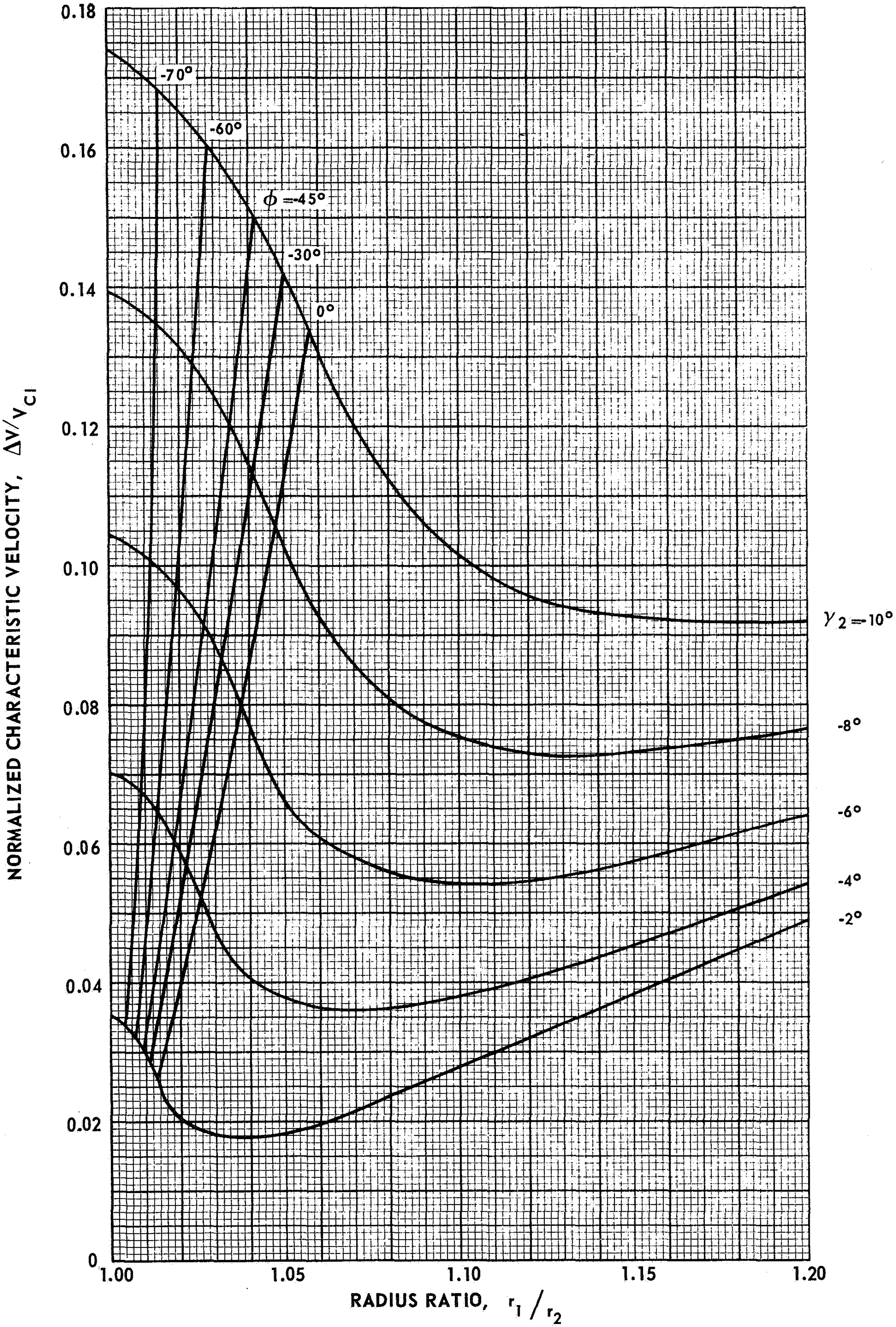
FIG. 1



COPLANAR CIRCLE-TO-POINT INTERCEPT

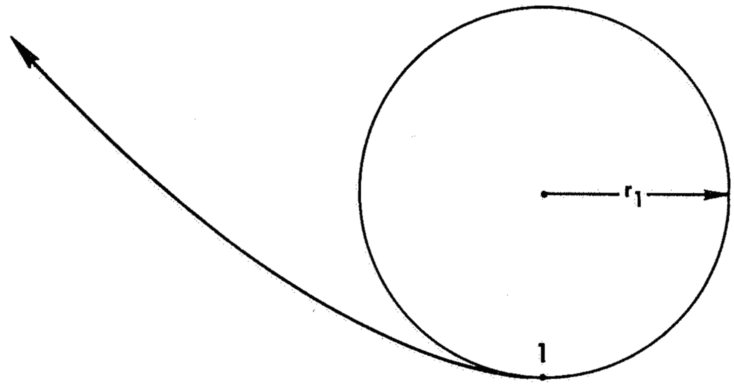
FIG. 2

DISORBIT TO SMALL ENTRY ANGLE

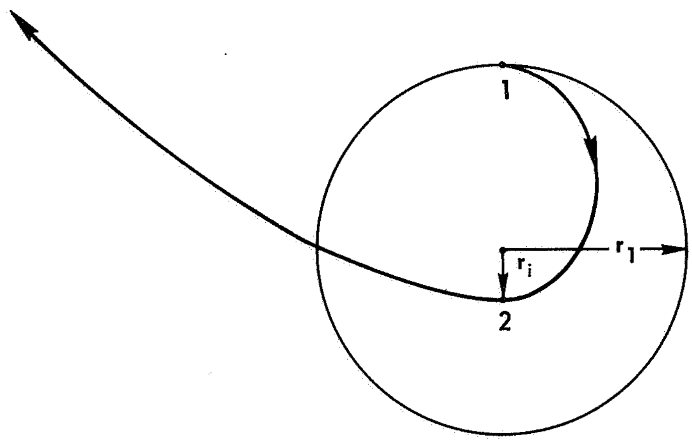


G-910557-11 FIG. 3

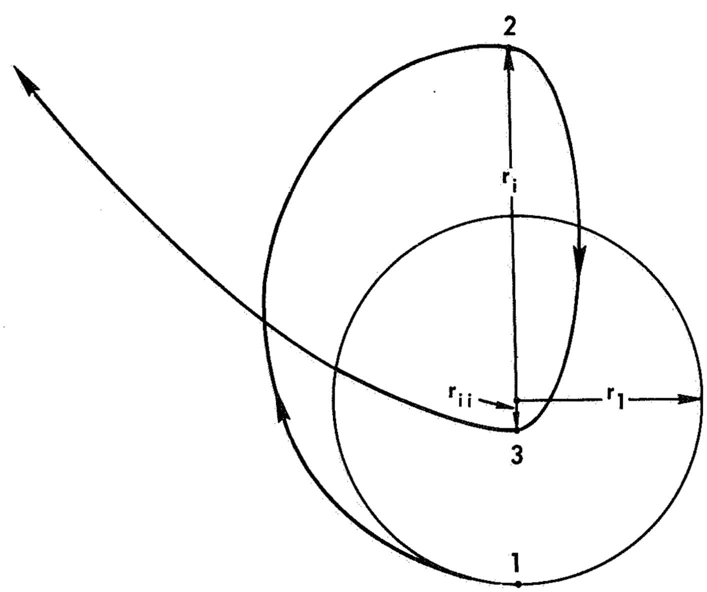
ESCAPE MANEUVERS FROM CIRCULAR ORBITS



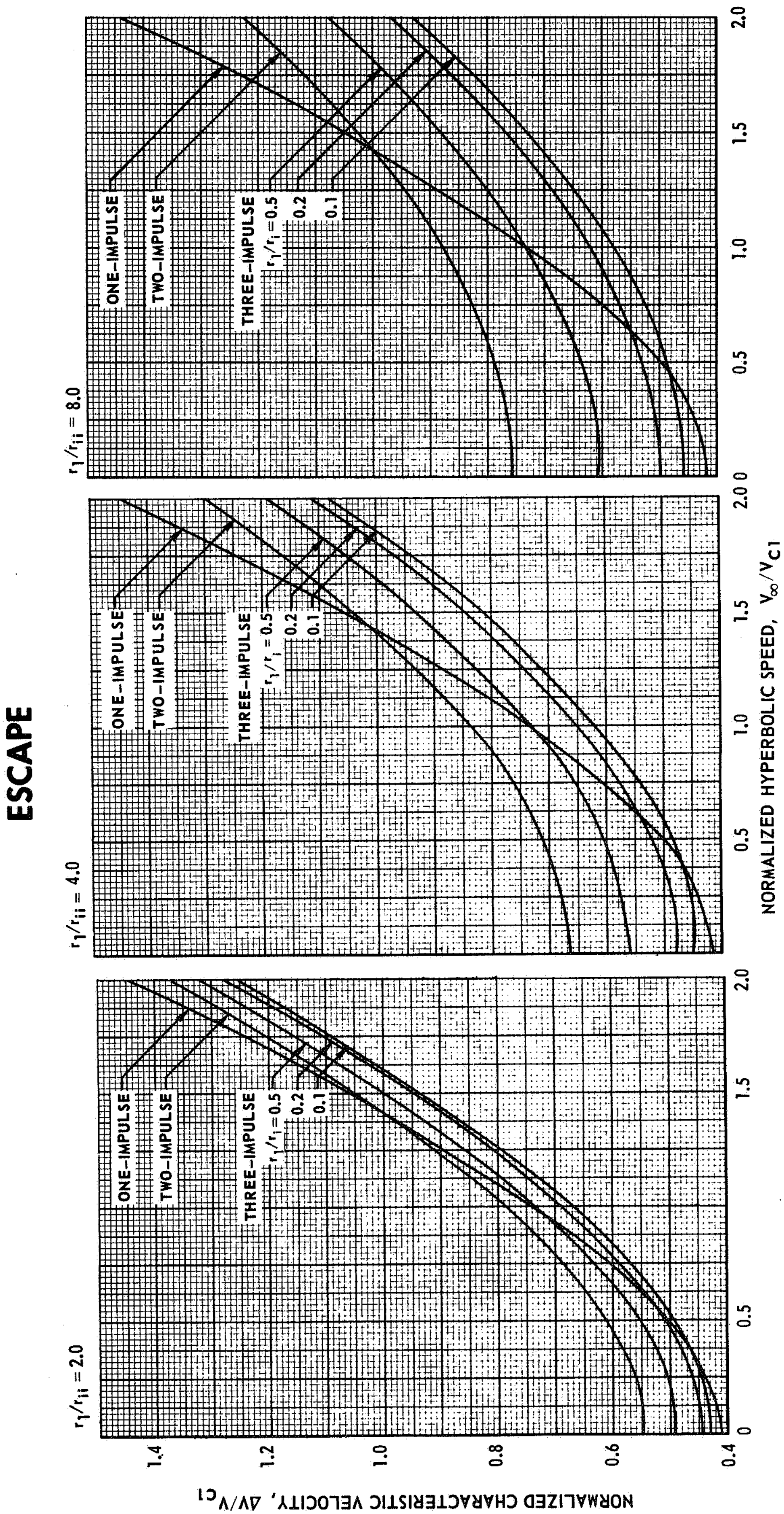
(a) ONE-IMPULSE ESCAPE



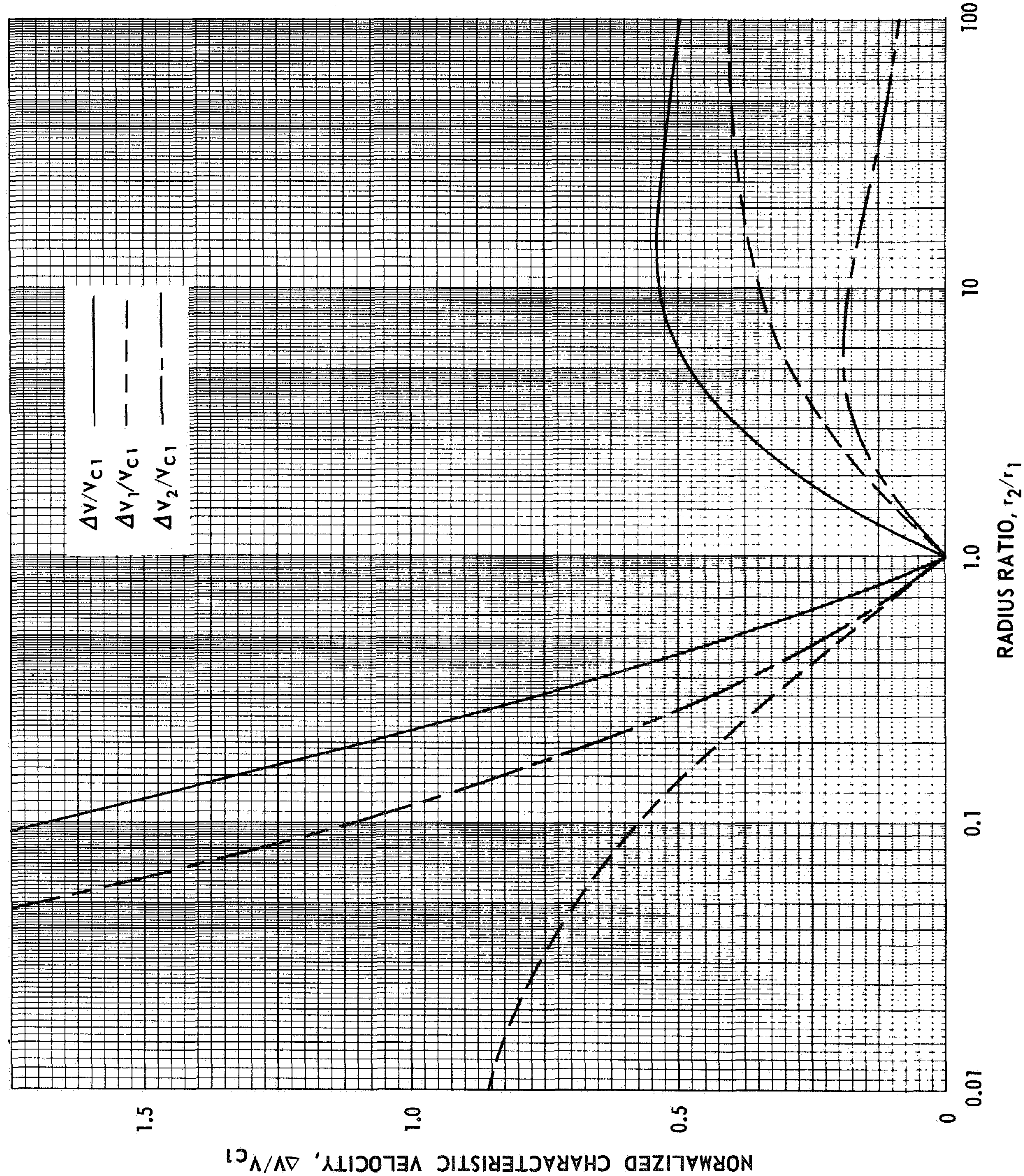
(b) TWO-IMPULSE ESCAPE



(c) THREE-IMPULSE ESCAPE


ESCAPE
$r_1/r_{ii} = 2.0$
$r_1/r_{ii} = 4.0$
$r_1/r_{ii} = 8.0$
ONE–IMPULSE
TWO–IMPULSE
THREE–IMPULSE
$r_1/r_i = 0.5$
0.2
0.1
0
0.5
1.0
1.5
2.0
0.4
0.6
0.8
1.0
1.2
1.4
NORMALIZED HYPERBOLIC SPEED, V_∞/V_{C1}
NORMALIZED CHARACTERISTIC VELOCITY, $\Delta V/V_{C1}$

SUMMARY OF HOHMANN TRANSFERS



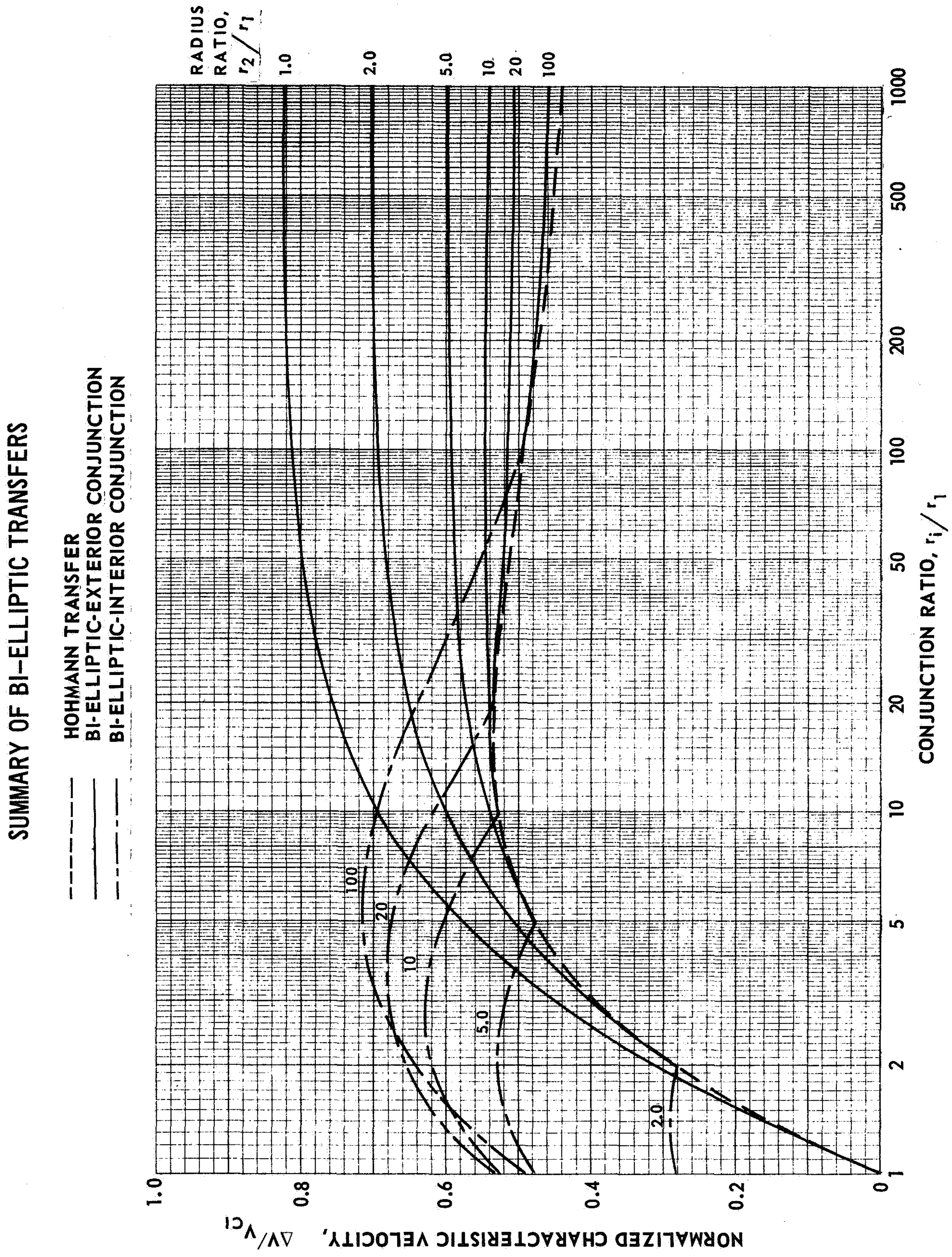
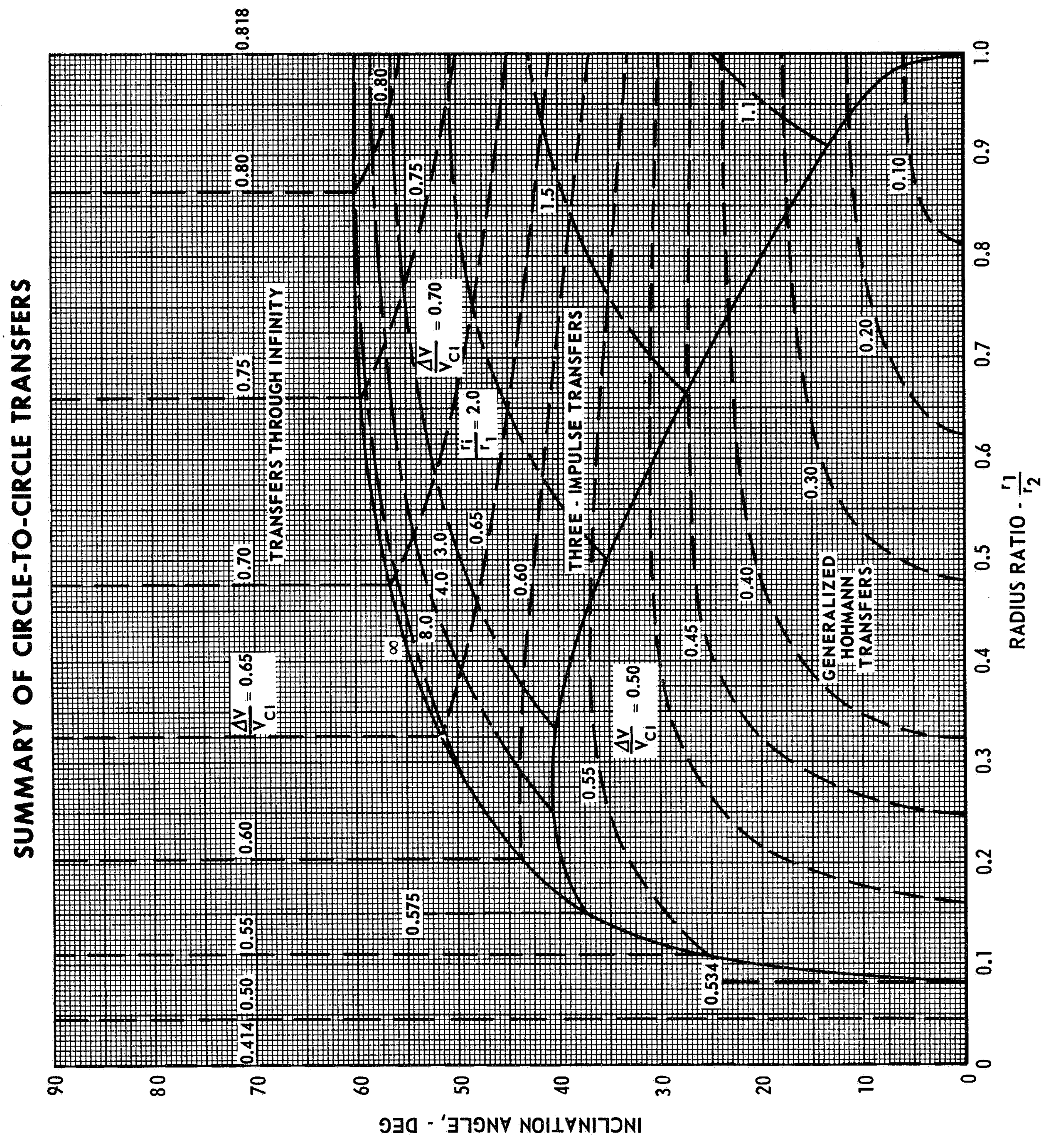
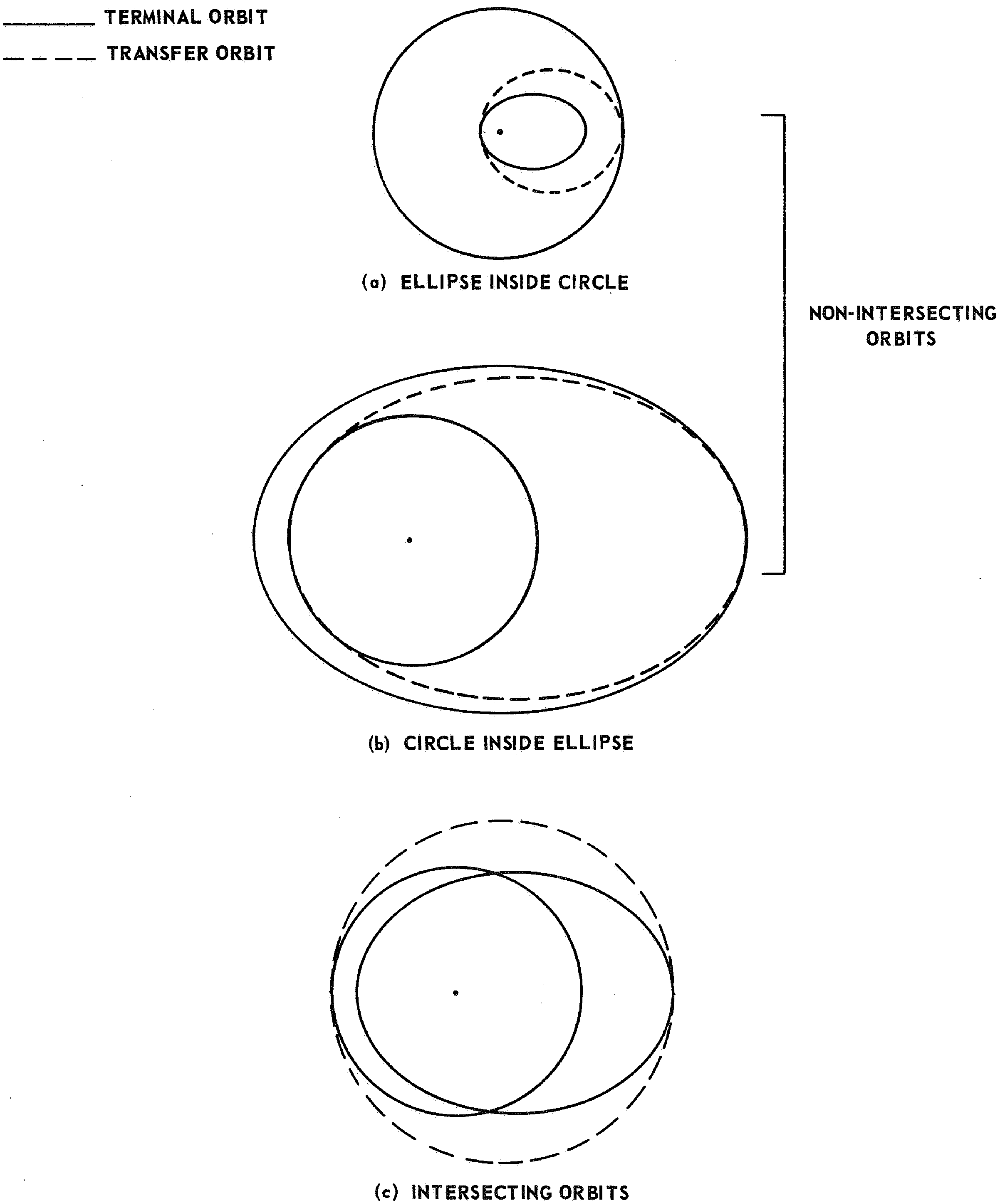

SUMMARY OF BI-ELLIPTIC TRANSFERS
HOHMANN TRANSFER
BI-ELLIPTIC-EXTERIOR CONJUNCTION
BI-ELLIPTIC-INTERIOR CONJUNCTION
RADIUS RATIO, r_2/r_1
1.0
2.0
5.0
10
20
100
CONJUNCTION RATIO, r_i/r_1
1
2
5
10
20
50
100
200
500
1000
NORMALIZED CHARACTERISTIC VELOCITY, $\Delta V/V_{c1}$
0
0.2
0.4
0.6
0.8
1.0

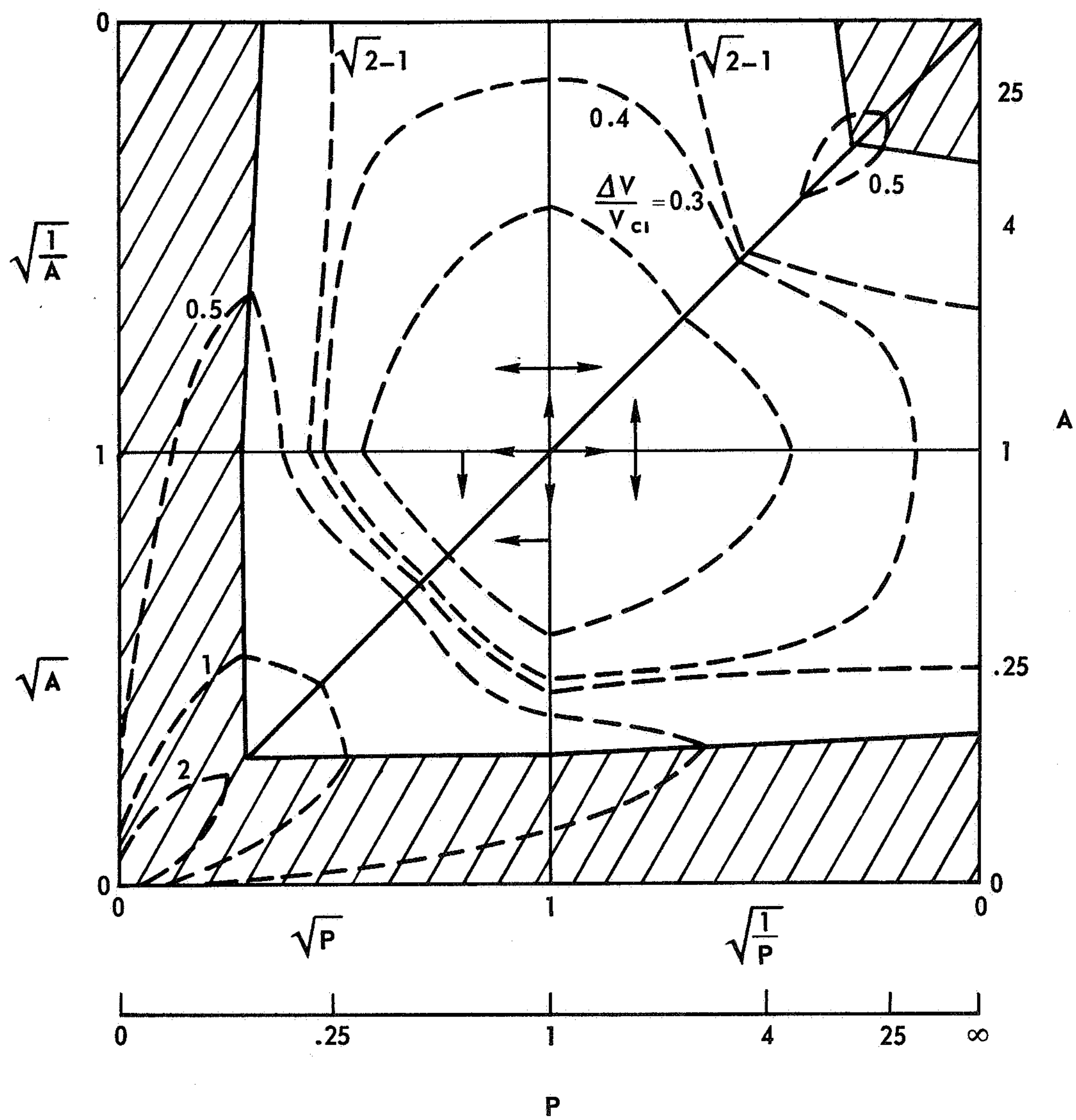
FIG. 7


SUMMARY OF CIRCLE-TO-CIRCLE TRANSFERS
TRANSFERS THROUGH INFINITY
THREE - IMPULSE TRANSFERS
GENERALIZED HOHMANN TRANSFERS
$\frac{\Delta V}{V_{CI}} = 0.65$
$\frac{\Delta V}{V_{CI}} = 0.70$
$\frac{\Delta V}{V_{CI}} = 0.50$
$\frac{r_i}{r_1} = 2.0$
0.818
0.80
0.75
0.70
0.60
0.575
0.55
0.534
0.50
0.414
0.65
0.45
0.40
0.30
0.20
0.10
1.1
1.5
3.0
4.0
8.0
∞
RADIUS RATIO - $\frac{r_1}{r_2}$
INCLINATION ANGLE, - DEG
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
10
20
30
40
50
60
70
80
90

CIRCLE-TO-ELLIPSE TRANSFERS



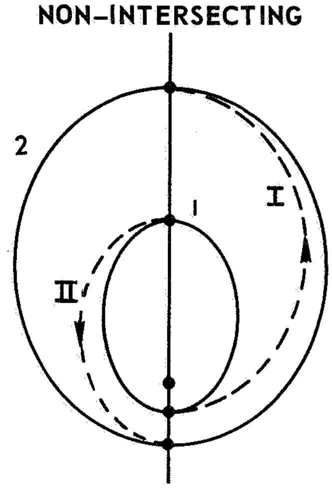
CIRCLE–TO–ELLIPSE TRANSFER



TWO-IMPULSE COPLANAR COAXIAL ORBIT TRANSFER

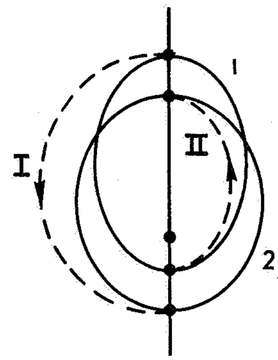
a. AXES ALIGNED

NON-INTERSECTING



USE TYPE I

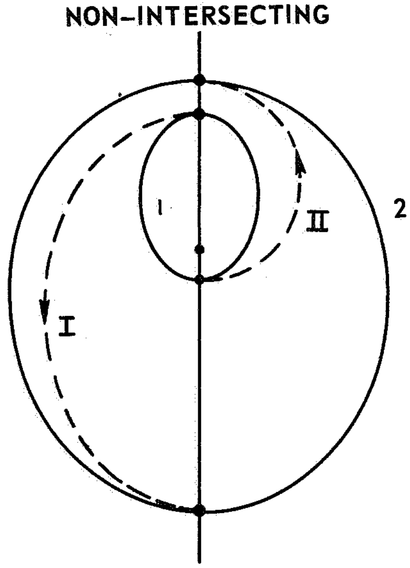
INTERSECTING



USE TYPE I

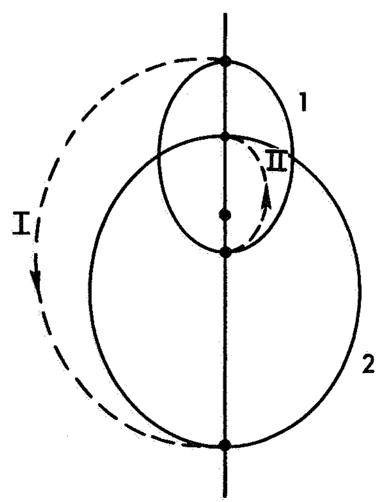
b. AXES OPPOSED

NON-INTERSECTING



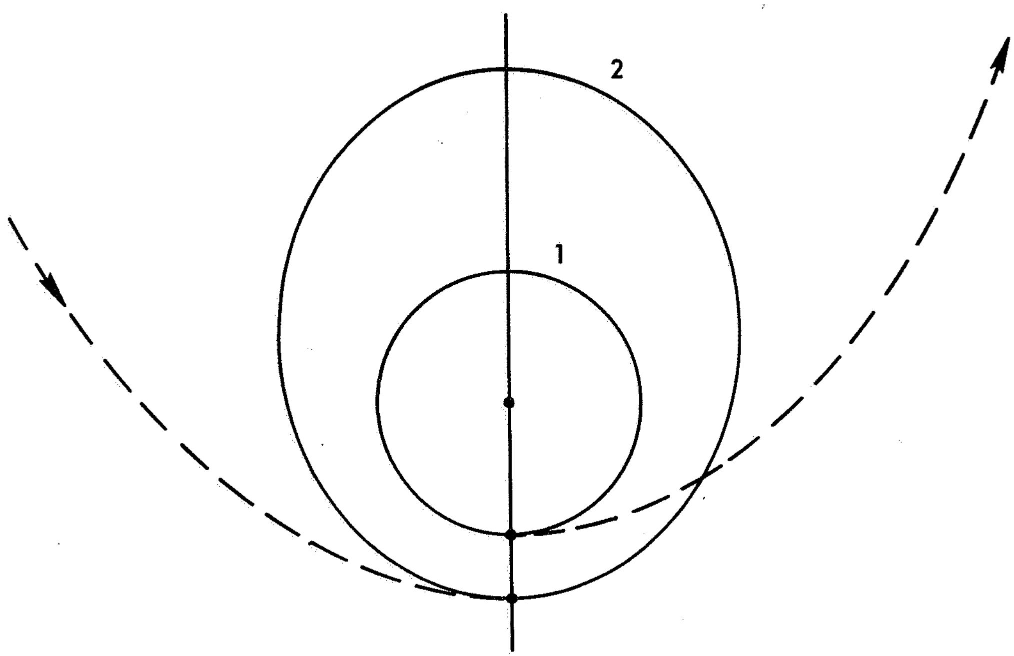
USE EITHER TYPE I OR II

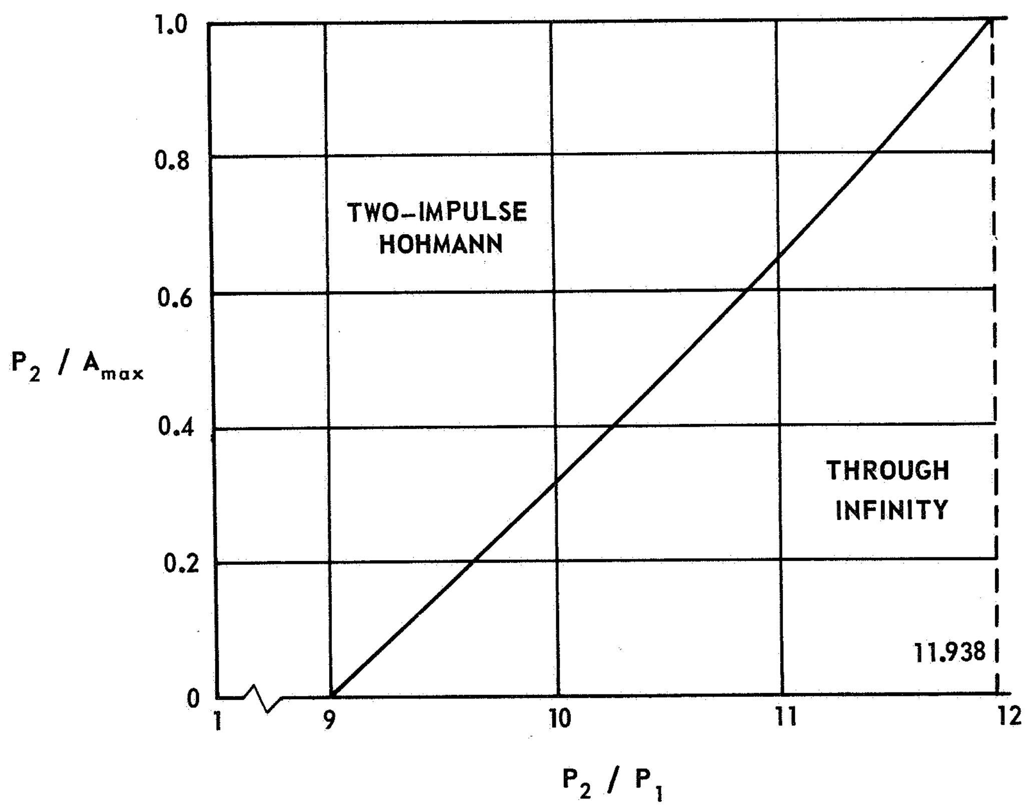
INTERSECTING



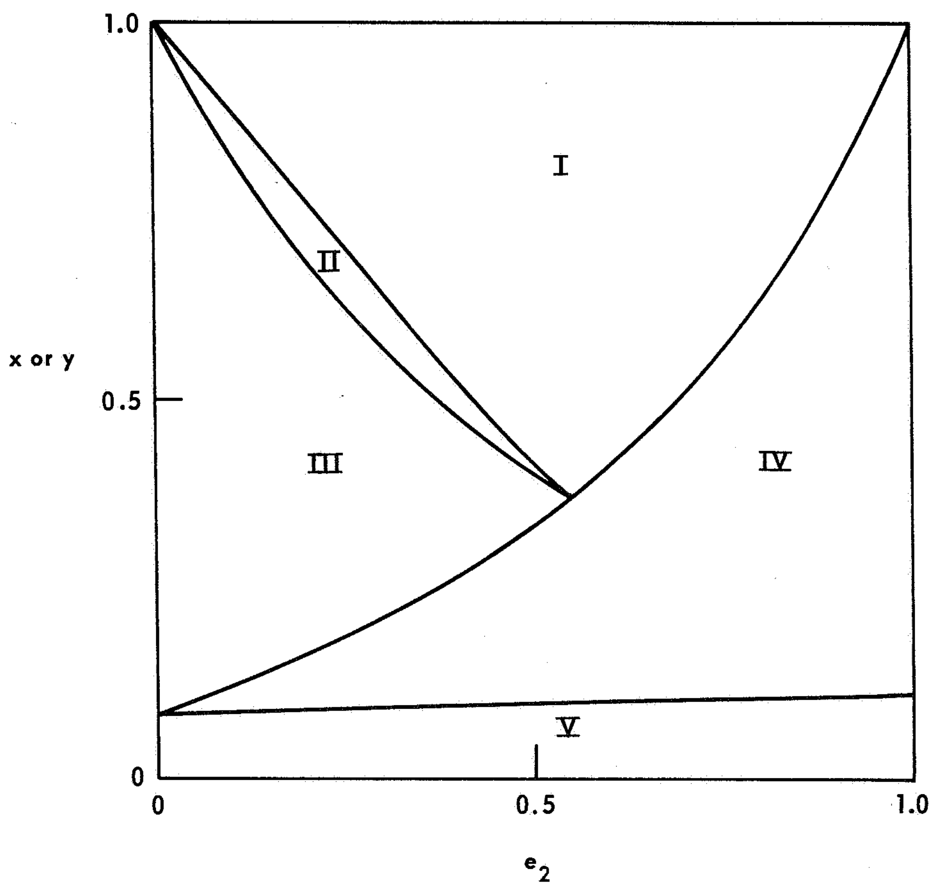
USE TYPE I

COPLANAR ALIGNED COAXIAL ORBITS

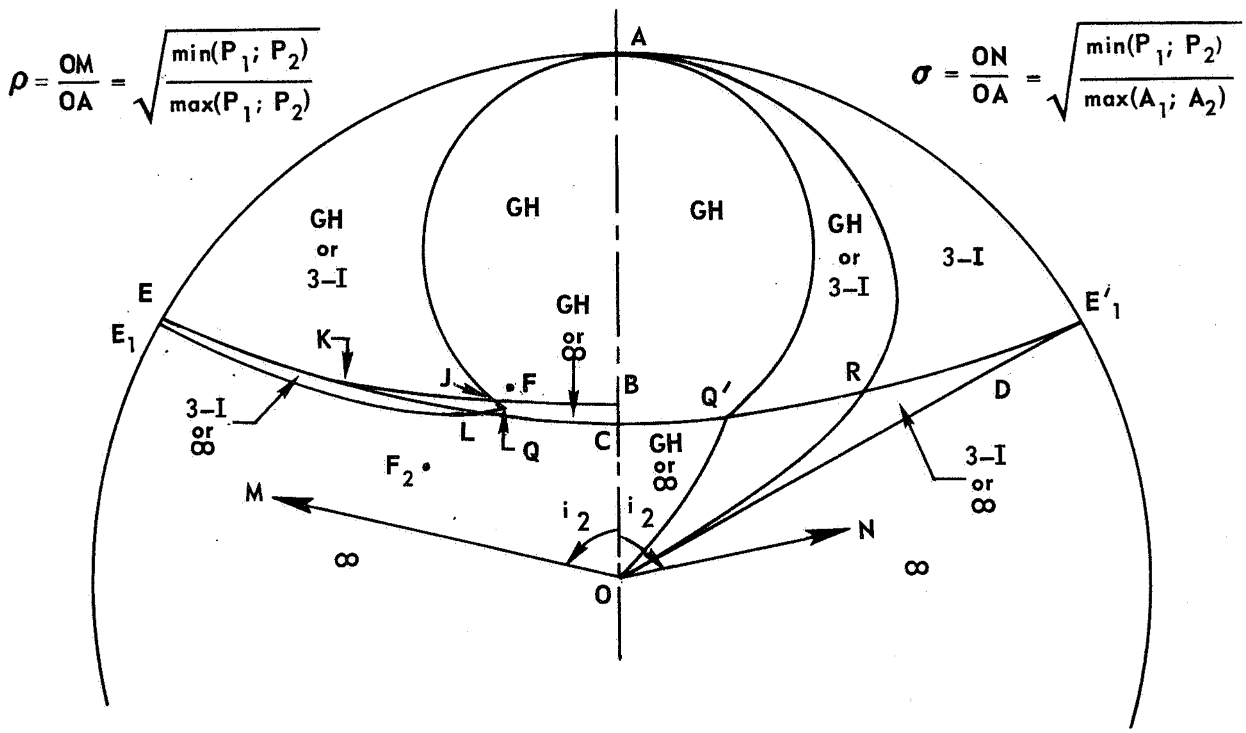




COPLANAR NONINTERSECTING COAXIAL ELLIPSES WITH AXES OPPOSED



DETERMINATION OF THE OPTIMAL MODE OF TRANSFER BETWEEN ALIGNED COAXIAL ORBITS



NOTATION

GH GENERALIZED HOHMANN
3-I FINITE THREE-IMPULSE
∞ THROUGH INFINITY

OPTIMUM TRANSFER BETWEEN COPLANAR CONGRUENT ELLIPSES

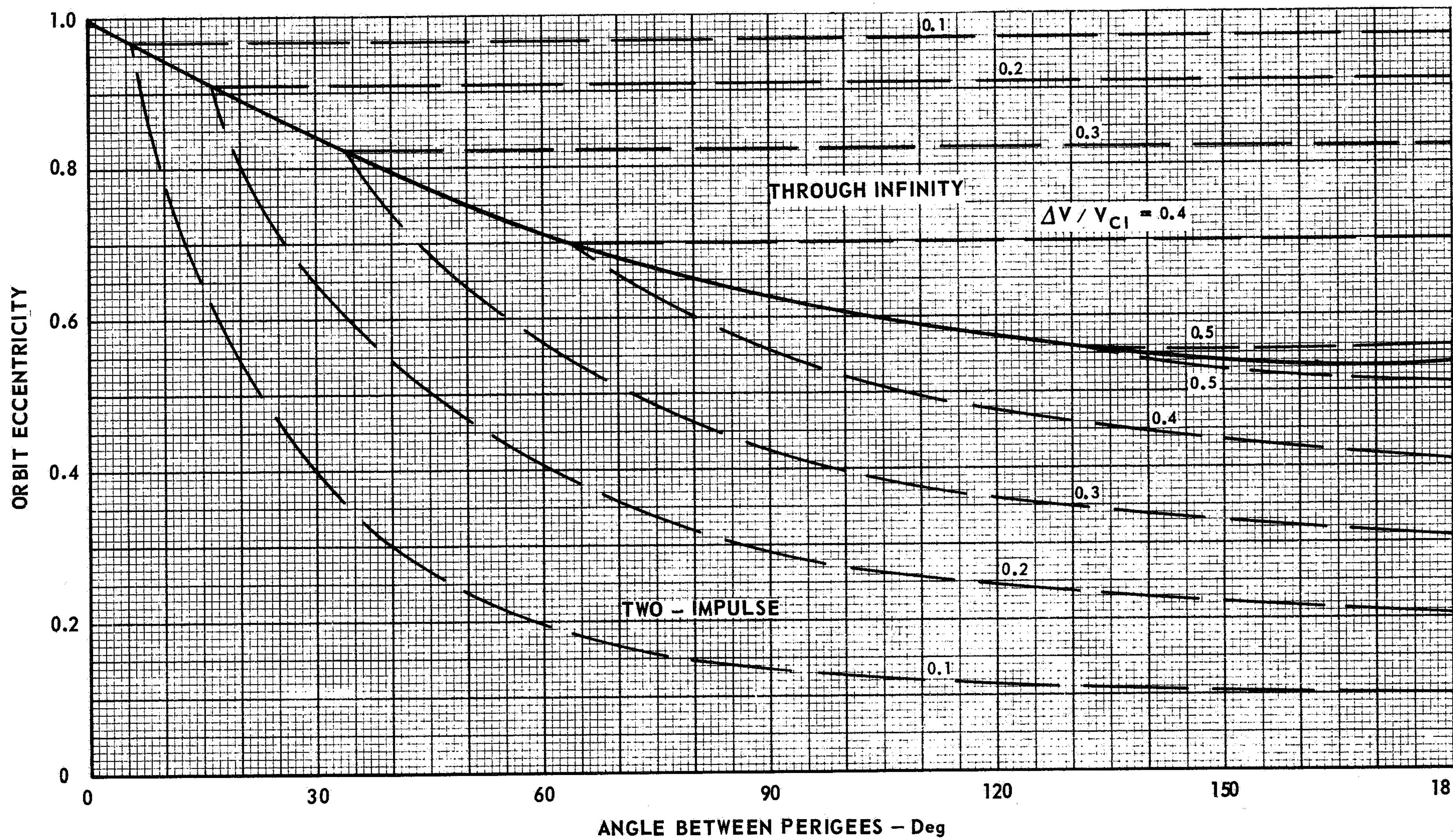
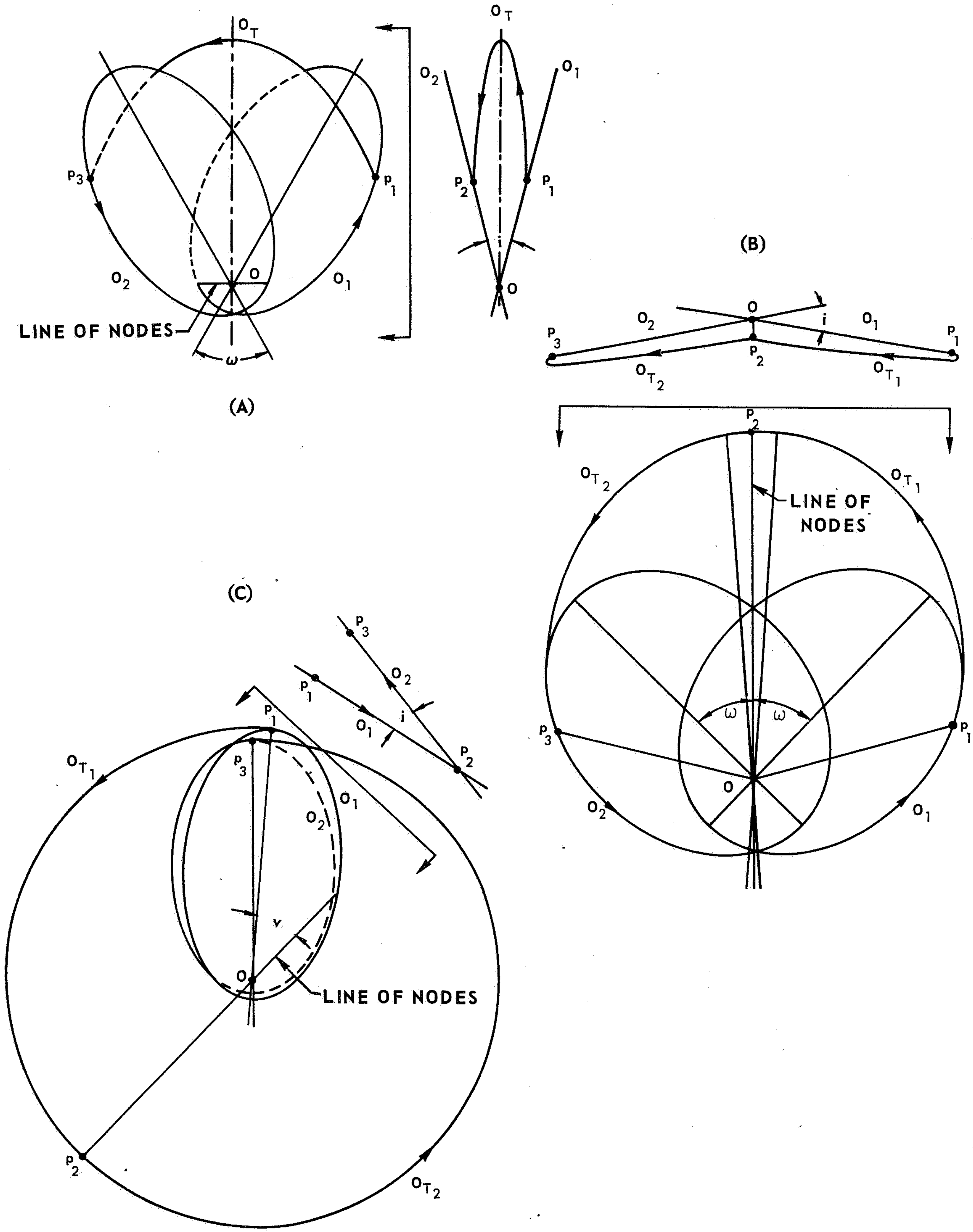
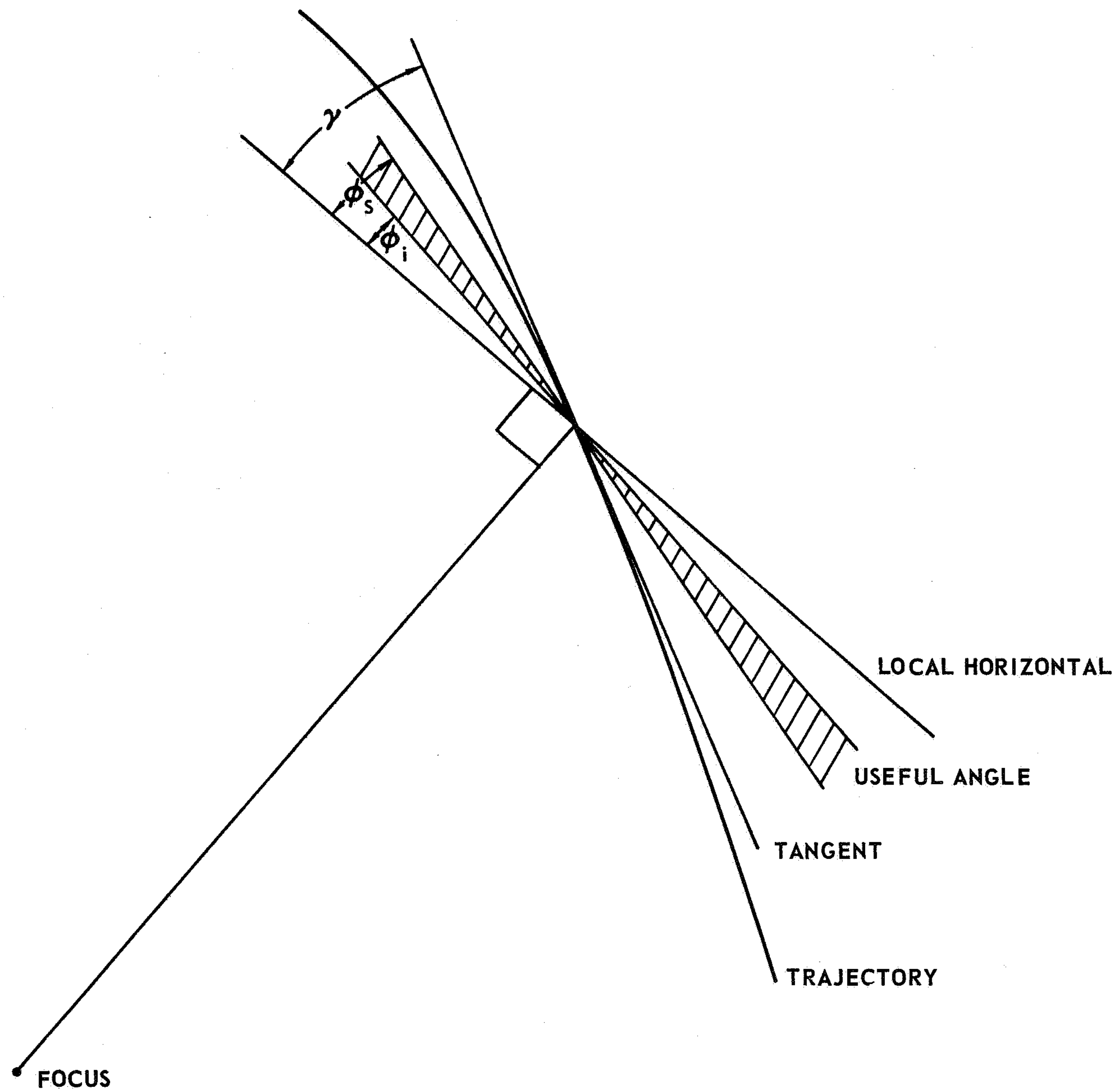


FIG. 14

NONCOPLANAR CONGRUENT ORBITS

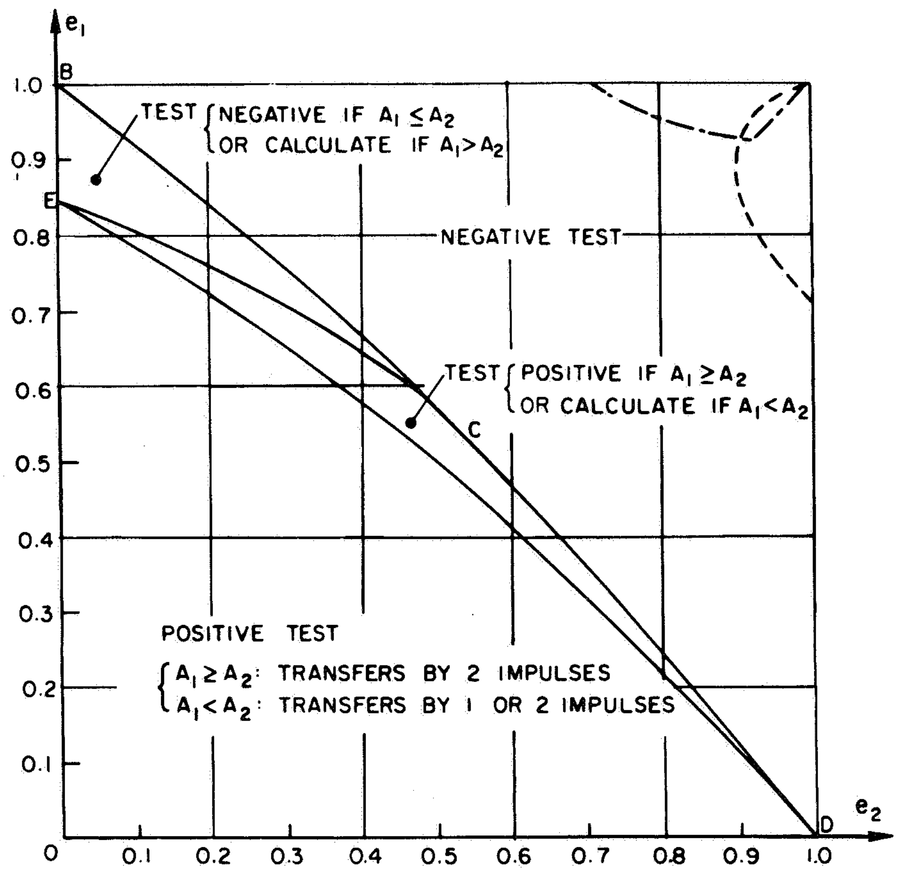


USEFUL ANGLE



TRANSFERS BETWEEN INTERSECTING COPLANAR ORBITS

$P_1 \leq P_2 < A_1 \lesseqgtr A_2$



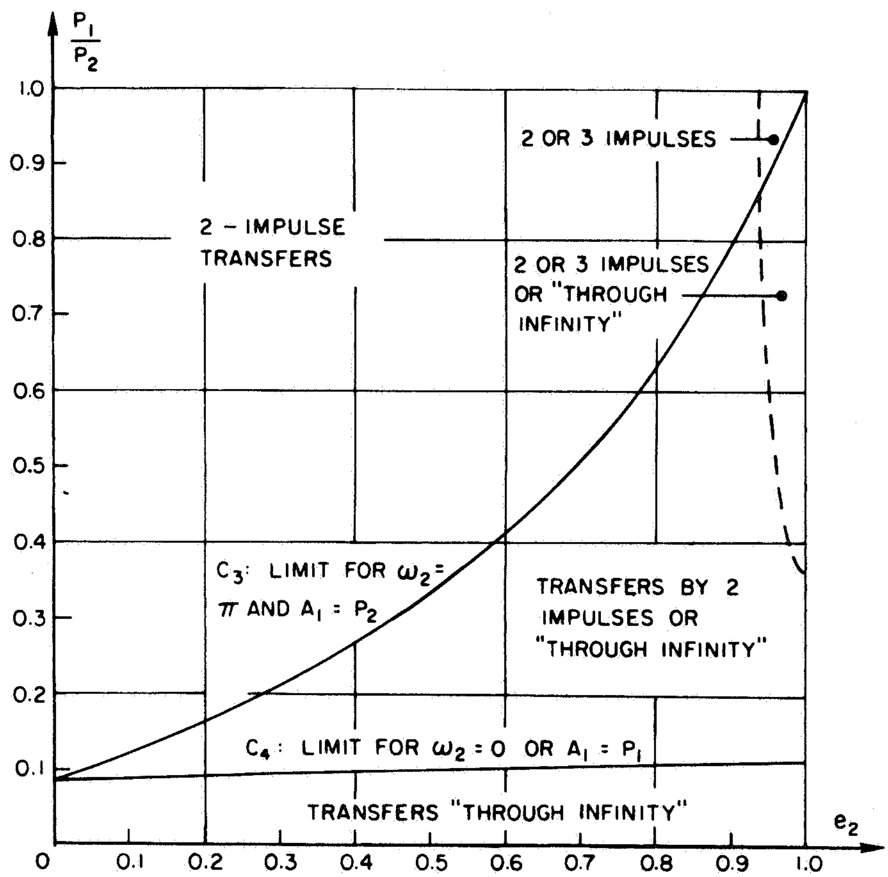
TEST: $Z_1\sqrt{A_2} + Z_2\sqrt{A_1} - \sqrt{A_1 + A_2} \lesseqgtr 0$

$$Z(e) = \frac{\sqrt{1+e} - e\sqrt{2}}{\sqrt{1-e}}$$

IF $|\omega_2| \geq 22°$ THE OPTIMAL TRANSFER CANNOT BE "BY 3 IMPULSES."

TRANSFERS BETWEEN NON-INTERSECTING COPLANAR ORBITS

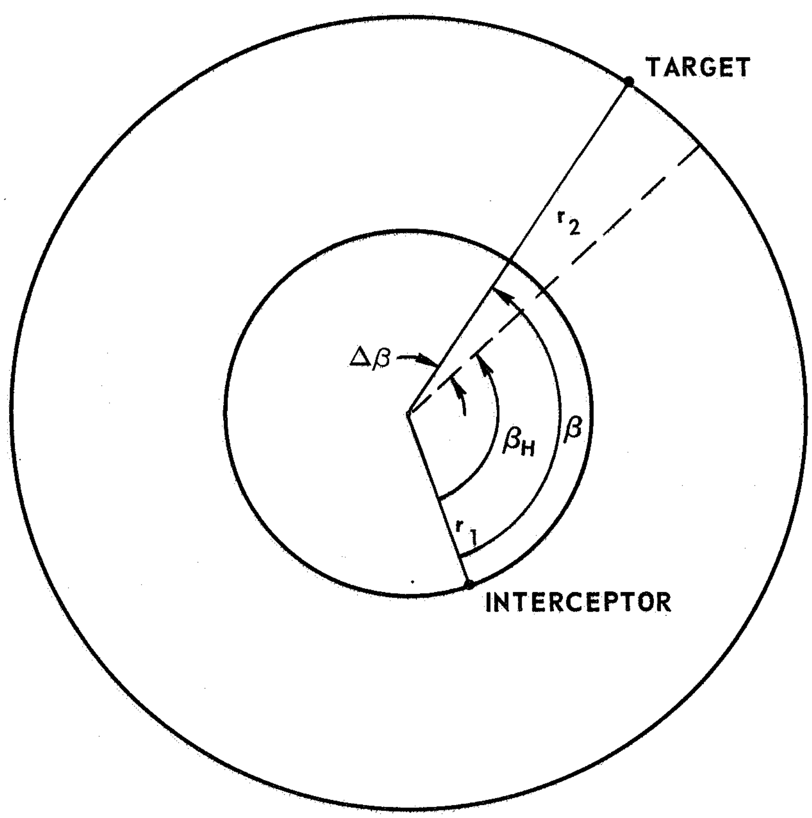
$P_1 \leq A_1 < A_2$, $P_1 < P_2 \leq A_2$



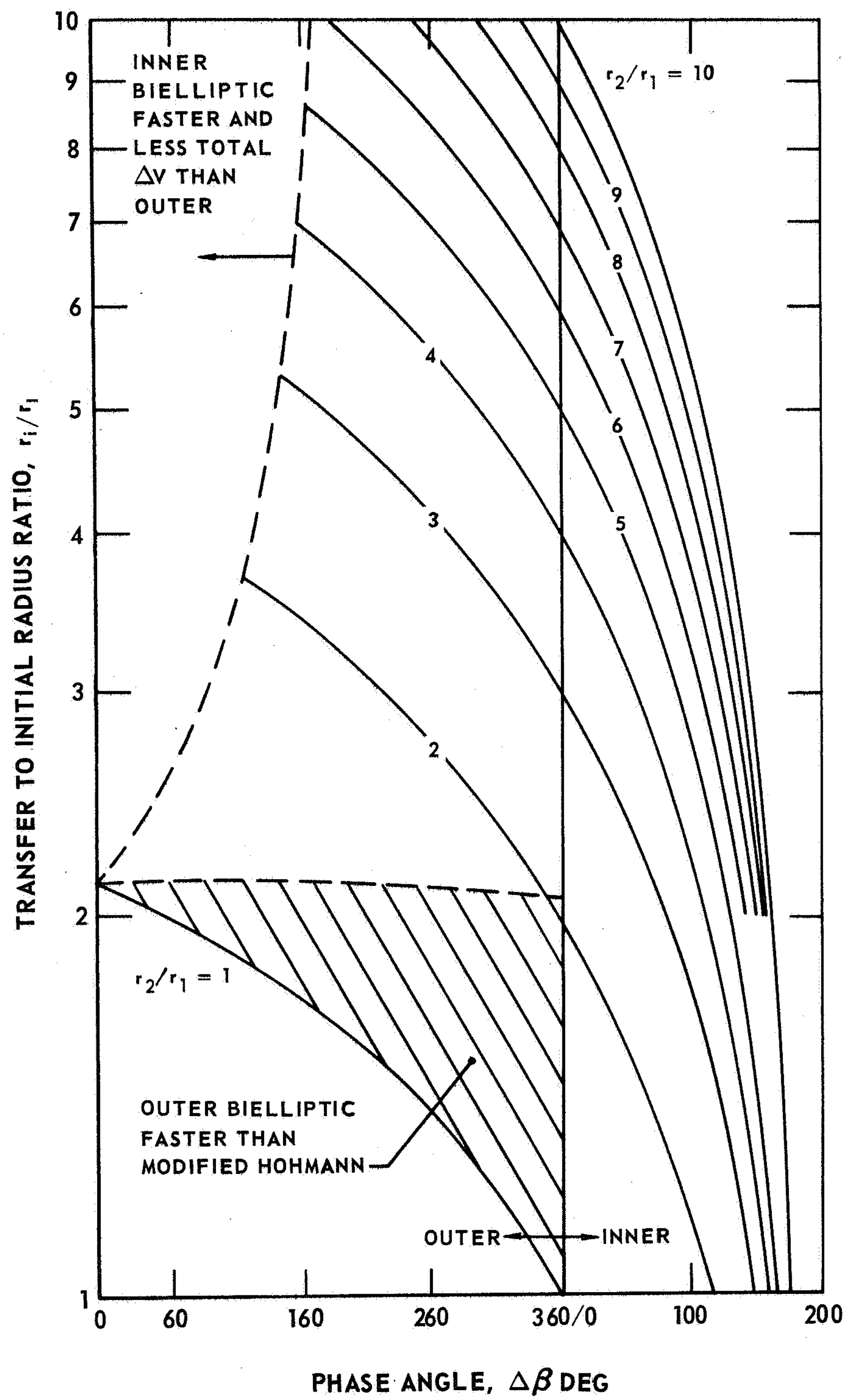
THREE IMPULSE TRANSFER IS: ACCELERATION, ACCELERATION, BRAKE.

IF $A_1 \leq P_1 < (= 2a_1) \leq 6.32\ P_2$ TRANSFER IS NEVER THREE-IMPULSE.

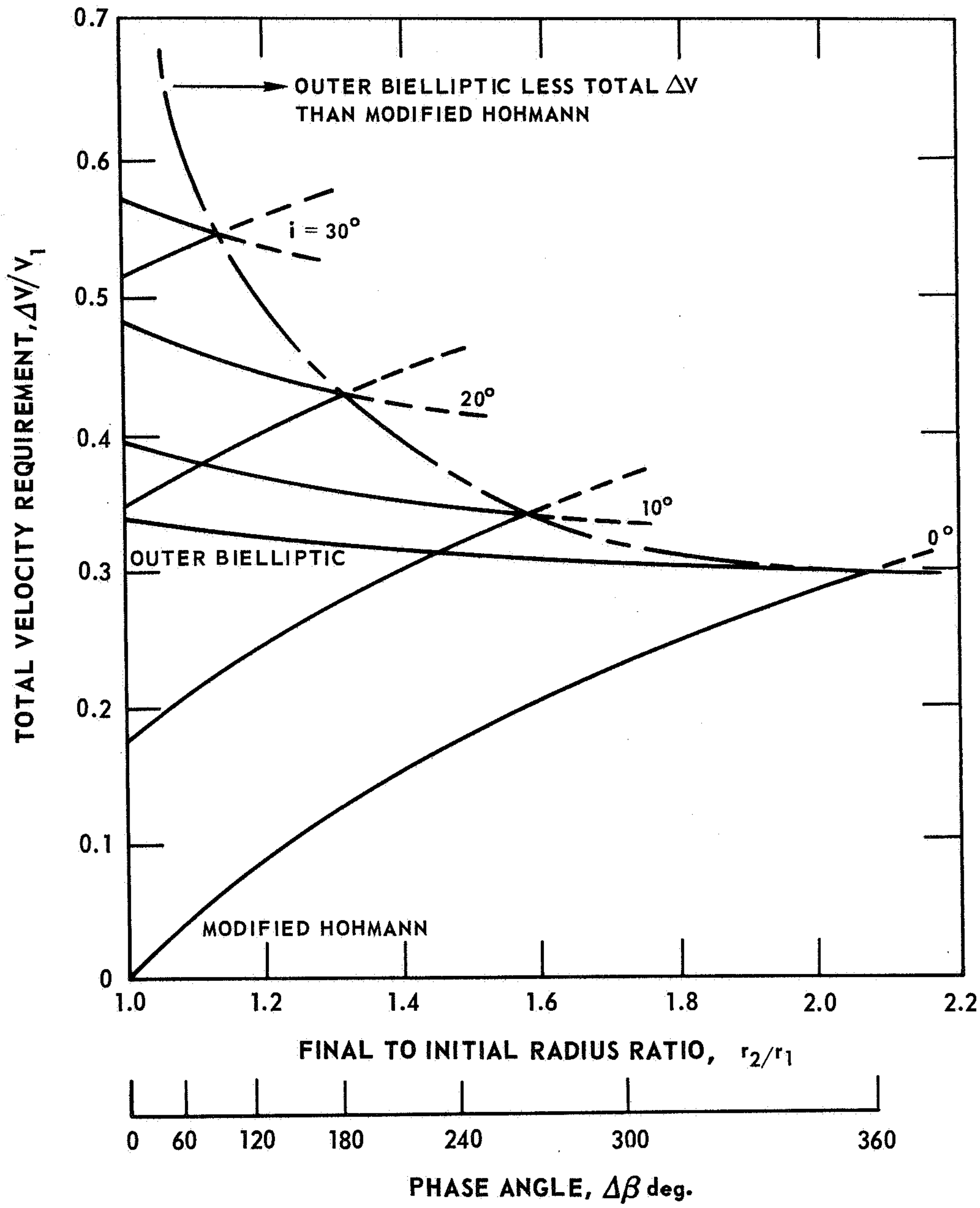
PHASE ANGLES FOR RENDEZVOUS



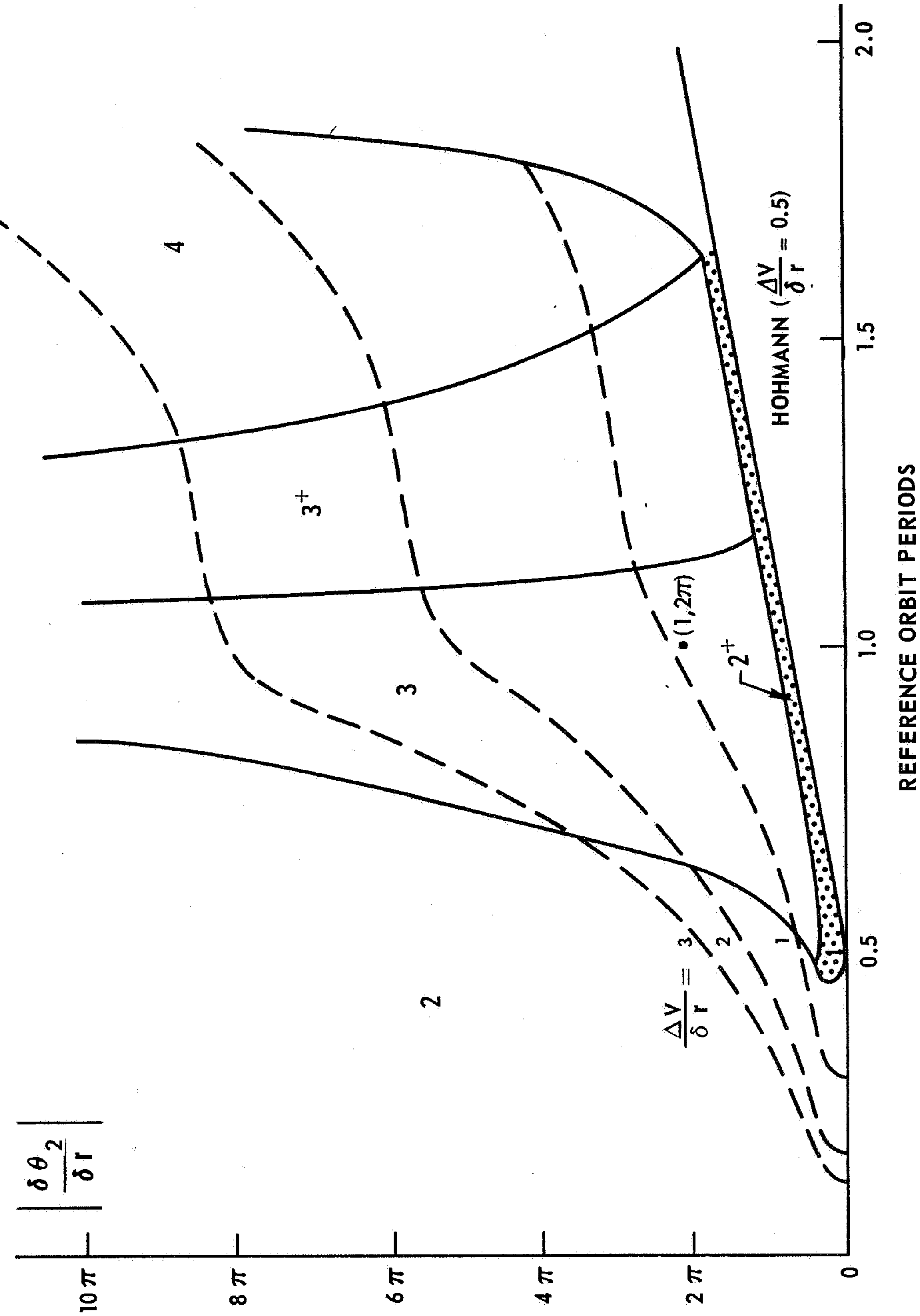
BIELLIPTIC TRANSFER RADIUS REQUIRED FOR RENDEZVOUS



EQUAL TIME RENDEZVOUS



REACHABLE FINAL STATE VARIATIONS. OPTIMAL MULTIPLE-IMPULSE SOLUTIONS.



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